DYNAMICAL MORDELL-LANG CONJECTURE FOR BIRATIONAL POLYNOMIAL MORPHISMS ON $\mathbb{A}^2$

XIE JUNYI

Abstract. We prove the dynamical Mordell-Lang conjecture for birational polynomial morphisms on $\mathbb{A}^2$.

1. Introduction

The Mordell-Lang conjecture proved by Faltings [9] and Vojta [21] says that if V is a subvariety of a semiabelian variety G defined over $\mathbb{C}$ and Γ is a finitely generated subgroup of $G(\mathbb{C})$, then $V(\mathbb{C})\bigcap\Gamma$ is a union of at most finitely many translates of subgroups of Γ.

The following dynamical analogue of the Mordell-Lang conjecture was proposed by Ghioca and Tucker.

Dynamical Mordell-Lang Conjecture ([13]). Let X be a quasiprojective variety defined over $\mathbb{C}$, let $f : X \to X$ be an endomorphism, and V be any subvariety of X. For any point $p \in X(\mathbb{C})$ the set $\{n \in \mathbb{N}|\ f^n(p) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions.

An arithmetic progression is a set of the form $\{an + b|\ n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$ possibly with $a = 0$.

Observe that this conjecture implies the classical Mordell-Lang conjecture in the case $\Gamma \simeq (\mathbb{Z}, +)$.

The Dynamical Mordell-Lang conjecture has been proved by Denis [6] for automorphisms of projective spaces and was later generalized by Bell [2] to the case of automorphisms of affine varieties. In [3], Bell, Ghioca and Tucker proved it for étale maps of quasiprojective varieties. The conjecture is also known in the case where $f = (F(x_1), G(x_2)) : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ where F, G are polynomials and the subvariety V is a line ([14]), and in the case $f = (F(x_1), \cdots , F(x_n)) : \mathbb{A}^n_K \to \mathbb{A}^n_K$ where $F \in K[t]$ is an indecomposable polynomial defined over a number field K which has no periodic critical points other than the point at infinity and V is a curve ([4]).

Our main result can be stated as follows.

Theorem A. *Let K be any algebraically closed field of characteristic 0, and $f : \mathbb{A}^2_K \to \mathbb{A}^2_K$ be any birational polynomial morphism defined over K. Let C be any curve in $\mathbb{A}^2_K$, and p be any point in $\mathbb{A}^2(K)$. Then the set $\{n \in \mathbb{N}|\ f^n(p) \in C\}$ is a union of at most finitely many arithmetic progressions.*

Date: October 13, 2015.

The author is supported by the ERC-starting grant project Nonarcomp no.307856.

In the case the map is an automorphism of $\mathbb{A}^2_K$ of Hénon type (see [11]) then this result follows from [3]. Our proof provides however an alternative approach and does not rely on the construction of p-adic invariant curves.

Recall that the *algebraic degree* of a polynomial transformation $f(x,y) = (f_1(x,y), f_2(x,y))$ is defined by $\deg f := \max\{\deg f_1, \deg f_2\}$. The limit $\lambda(f) := \lim_{n\to\infty}(\deg f^n)^{1/n}$ exists and we refer to it as the *dynamical degree* of f (see [7, 8]). Our proof shows that when $\lambda(f) > 1$, then Theorem A holds for fields of arbitrary characteristic.

Note however that our Theorem A does not hold when $\mathrm{char} K > 0$ and $\lambda(f) = 1$ (see [2, Proposition 6.1] for a counter-example).

To explain our strategy, we fix a birational polynomial morphism $f : \mathbb{A}^2_K \to \mathbb{A}^2_K$. By some reduction arguments, we may assume that $K = \overline{\mathbb{Q}}$.

We may compactify $\mathbb{A}^2$ by [10] to a smooth projective surface, such that f extends to a birational transformation on X fixing a point Q in $X \setminus \mathbb{A}^2$, and f contracts all curves at infinity to Q (see [10] and Section 6.1).

The key idea of our proof is to take advantage of this attracting fixed point and to apply the following local version of the Dynamical Mordell-Lang conjecture.

Theorem 1.1. *Let X be a smooth projective surface over an arbitrary valued field $(K, |\cdot|)$ and $f : X \dashrightarrow X$ be a birational transformation defined over K. Let C be any curve in X. Pick any K-point p such that $f^n(p) \in X \setminus I(f)$ for all integers $n \geq 0$, and $f^n(p)$ tends to a fixed K-point $Q \in I(f^{-1}) \setminus I(f)$ with respect to a projective metric induced by $|\cdot|$ on X.*

If the set

$$\{n \in \mathbb{N} |\ f^n(p) \in C\}$$

is infinite, then either $f^n(p) = Q$ for some $n \geq 0$ or C is fixed.

To complete the proof of Theorem A we now rely on a global argument. When the curve C is passing through the fixed point Q in X, we cover the $\overline{\mathbb{Q}}$-points of the curve C by the basin of attraction of Q with respect to all absolute values on $\overline{\mathbb{Q}}$. If the point p belongs to one of these attracting basins, then the local dynamical Mordell-Lang applies and we are done. Otherwise it is possible to bound the height of p and Northcott theorem shows that it is periodic.

Finally when neither the curve C nor its iterates contain the fixed point Q, we are in position to apply the next result which allows us to conclude.

Theorem 1.2. *Let X be a smooth projective surface over an algebraically closed field, $f : X \dashrightarrow X$ be an algebraically stable birational transformation and C be an irreducible curve in X such that f^n does not contract C for any n.*

If $f^n(C) \bigcap I(f) \neq \emptyset$ for all n, then C is periodic.

We show mention that our approach seems difficult to deal with arbitrary endomorphisms of surfaces. The key point of our proof is to take advantage of an attracting fixed point in some suitable model. But such a point does not exist for a general surface endomorphism.

The article is organized in 8 sections. In Section 2 we give background informations on birational surface maps and metrics on projective varieties defined over a valued field. In Section 3 we prove Theorem 1.2, which is a criterion for a curve to be periodic. In Section 4 we prove some basic properties for the maps satisfying the conclusion of dynamical Mordell-Lang conjecture. In Section 5 we prove Theorem 1.1. In Section 6 we prove Theorem A in the case the dynamical degree $\lambda(f) = 1$. In Section 7 we prove a technical lemma which gives a upper bound on height when $\lambda(f) > 1$. In Section 8 we prove Theorem A.

Acknowledgement

I would like to thank C. Favre for his support and his direction during the writing of this article.

2. Notations and basics

2.1. **Basics on birational maps on surfaces.** See [5, 7, 10] for details.

In this section a variety is defined over an algebraically closed field k. Recall that the resolution of singularities exists for surfaces over any algebraically closed field (see [1]).

Let X be a smooth projective surface. We denote by $N^1(X)$ the Néron-Severi group of X i.e. the group of numerical equivalence classes of divisors on X and write $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$. Let $\phi : X \to Y$ be a morphism of smooth projective surfaces. It induces a natural map $\phi^* : N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. Since $\dim X = 2$, one has a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N^1(X)_{\mathbb{R}} \to \mathbb{R},\ \ (\delta, \gamma) \to (\delta \cdot \gamma) \in \mathbb{R}$$

induced by the intersection form. We denote by $\phi_* : N^1(X)_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$ the dual operator of ϕ^*.

Let X, Y be two smooth projective surfaces and $f : X \dashrightarrow Y$ be a birational map. We denote by $I(f) \subseteq X$ the indeterminacy set of f. For any curve $C \subset X$, we write

$$f(C) := \overline{f(C \setminus I(f))}$$

the strict transform of C.

Let $f : X \dashrightarrow X$ be a birational transformation and Γ be a desingularization of its graph. Denote by $\pi_1 : \Gamma \to X, \pi_2 : \Gamma \to X$ the natural projections. Then the diagram

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "f"] & & X \end{tikzcd} \tag{$*$}$$

is commutative and we call it a *resolution* of f.

Proposition 2.1 ([15]). *We have the following properties.*

(i) *The morphisms π_1, π_2 are compositions of point blowups.*
(ii) *For any point $Q \notin I(f)$, there is a Zariski open neighborhood U of p in X and an injective morphism $\sigma : U \to \Gamma$ such that $\pi_1 \circ \sigma = \mathrm{id}$.*

Then we define the following linear maps
$$f^* = \pi_{1*}\pi_2^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}},$$
and
$$f_* = \pi_{2*}\pi_1^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}.$$
Observe that $f_* = f^{-1*}$. Note that in general we have $(f \circ g)^* \neq g^* f^*$.

For any big and nef class $\omega \in N^1_{\mathbb{R}}(X)$, we set
$$\deg_\omega(f) := (f^*\omega \cdot \omega),$$
the limit $\lim_{n\to\infty} \deg_\omega(f^n)^{1/n}$ exists and does not depend on the choice of ω (see [7, 8]). We denote this limit by $\lambda(f)$ and call it the *dynamical degree* of f.

Definition 2.2 (see [7]). Let $f : X \dashrightarrow X$ be a birational transformation on a smooth projective surface. Then f is said to be *algebraically stable* if and only if there is no curve $V \subseteq X$ such that $f^n(V) \subseteq I(f)$ for some integer $n \geq 0$.

In the case $X = \mathbb{P}^2$, f is algebraically stable if and only if $\deg(f^n) = (\deg f)^n$ for any $n \in \mathbb{N}$.

Theorem 2.3 ([7]). *Let $f : X \dashrightarrow X$ be a birational transformation of a smooth projective surface. Then there exists a smooth projective surface $\widehat{X}$, and a proper modification $\pi : \widehat{X} \to X$ such that the lift of f to $\widehat{X}$ is an algebraically stable map.*

By a *compactification* of $\mathbb{A}^2$, we mean a smooth projective surface X admitting a birational morphism $\pi : X \dashrightarrow \mathbb{P}^2$ that is an isomorphism above $\mathbb{A}^2 \subseteq \mathbb{P}^2$, see [10].

The theorem follows from [10, Proposition 2.6] and [10, Theorem 3.1], and provides us with a good compactification of $\mathbb{A}^2$.

Theorem 2.4 ([10]). *Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a birational polynomial transformation with $\lambda(f) > 1$. Then there exists a compactification X of $\mathbb{A}^2$ satisfying the following properties.*

(i) *The map f extends to an algebraically stable map $\widetilde{f}$ on X.*
(ii) *There exists a $\widetilde{f}$-fixed point $Q \in X \setminus \mathbb{A}^2$ such that $d\widetilde{f}^2(Q) = 0$.*
(iii) *There exists an integer $n \geq 1$ such that $\widetilde{f}^n(X \setminus \mathbb{A}^2) = Q$.*

2.2. **Branches of curves on surfaces.** [12, 16] Let X be a smooth projective surface over an algebraically closed field k. Let C be an irreducible curve in X and p be a point in C.

Definition 2.5. A *branch* of C at p is a point in the normalization of C whose image is p.

Let $I_{C,p}$ be the prime ideal associated to C in the local function ring $O_{X,p}$ at p and $\widehat{I_{C,p}}$ be the completion of $I_{C,p}$ in the completion of local function ring $\widehat{O_{X,p}}$.

Let $i : \widetilde{C} \to C$ is a normalization of C and $\widetilde{p}$ a point in $i^{-1}(p)$. Let s be the branch of C at p defined by the point $\widetilde{p}$. The morphism $i : \widetilde{C} \to C$ induces a morphism $i^*:\widehat{\mathcal{O}_{X,p}}\to\widehat{\mathcal{O}_{\widetilde{C},\widetilde{p}}}$ between the completions of local function rings. The map $s\mapsto\mathfrak{p}_s:=\ker i^*$ gives us a one to one correspondence between the set of branches of C at p and the set of prime ideals of $\widehat{O_{X,p}}$ with height 1 which contains $\widehat{I_{C,p}}$.

Given any two different branches s_1 and s_2 at a point $p\in X$, the intersection number is denoted by

$$(s_1\cdot s_2):=\dim_k\widehat{\mathcal{O}_{X,p}}/(\mathfrak{p}_{s_1}+\mathfrak{p}_{s_2}).$$

For convenience, we set $(s_1\cdot s_2):=0$ if s_1 and s_2 are branches at different points.

Let Z be a smooth projective surface and $f:X\dashrightarrow Z$ be a birational map. If f does not contract C then we denote by $f(s)$ the branch of $f(s)$ defined by the point $\widetilde{p}$ in the normalization $f\circ i:\widetilde{C}\to f(C)$ and call it the strict transform of s. Observe that $f(s)$ is a branch of $f(C)$ and when $p\notin I(f)$, we have that $f(s)$ is a branch of curve at $f(p)$.

If f is regular at p, we write

$$f_*s=\begin{cases} f(s), & \text{when } f \text{ does not contract } C;\\ 0, & \text{otherwise.}\end{cases}$$

Let Y be another smooth projective surface and $\pi:Y\to X$ be a birational morphism. Denote by $\pi^\# s:=\pi^{-1}(s)$ the strict transform of s. Let E_i, $i=1,\cdots,m$ be the exceptional curves of π. There is a unique sequence of non negative integers $(a_i)_{0\le i\le m}$ such that for any irreducible curve D in Y different from $\pi^\# C$, we have $(s\cdot\pi_*D)=(\pi^\# s+\sum_{i=1}^m a_iE_i\cdot D)$. Denote by $\pi^*s:=\pi^\# s+\sum_{i=1}^m a_iE_i$ and call it the pull back of s.

Proposition 2.6. *We have the following properties.*

(i) *We have $\pi_*\pi^*s=s$.*

(ii) *For any irreducible curve (resp. any branch of curve) D in Y different from $\pi^\# C$ (resp. $\pi^\# s$), we have*

$$(\pi^*s\cdot D)=(s\cdot\pi_*D).$$

(iii) *For any curve (resp. any branch of curve) D in X different from C (resp. s) then we have*

$$(s\cdot D)=(\pi^\# s\cdot\pi^*D).$$

2.3. **Metrics on projective varieties defined over a valued field.** A field with an absolute value is called a valued field.

Definition 2.7. Let $(K,|\cdot|_v)$ be a valued field. For any integer $n\ge1$, we define a metric d_v on the projective space $\mathbb{P}^n(K)$ by

$$d_v([x_0:\cdots:x_n],[y_0:\cdots:y_n])=\frac{\max_{0\le i,j\le n}|x_iy_j-x_jy_i|_v}{\max_{0\le i\le n}|x_i|_v\max_{0\le j\le n}|y_j|_v}$$

for any two points $[x_0:\cdots:x_n],[y_0:\cdots:y_n]\in\mathbb{P}^n(K)$.

Observe that when $|\cdot|_v$ is archimedean, then the metric d_v is not induced by a smooth riemannian metric. However it is equivalent to the restriction of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{P}^n(\mathbb{R})$ to $\mathbb{P}^n(K)$ induced by σ_v.

More generally, for a projective variety X defined over K, if we fix an embedding $\iota : X \hookrightarrow \mathbb{P}^n$, we may restrict the metric d_v on $\mathbb{P}^n(K)$ to a metric $d_{v,\iota}$ on $X(K)$. This metric depends on the choice of embedding ι in general, but for different embeddings ι_1 and ι_2, the metrics d_{v,ι_1} and d_{v,ι_2} are equivalent. Since we are mostly intersecting in the topology induced by these metrics we shall usually write d_v instead of $d_{v,\iota}$ for simplicity.

3. A criterion for a curve to be periodic

Our aim in this section is to prove Theorem 1.2 from the introduction. Let us recall the setting:

(i) X is a smooth projective surface over an algebraically closed field;
(ii) $f : X \dashrightarrow X$ is an algebraically stable birational transformation;
(iii) C is an irreducible curve in X such that f^n does not contract C and $f^n(C) \bigcap I(f) \neq \emptyset$ for all n.

Our aim is to show that C periodic. Let us begin with the following special case.

Lemma 3.1. *Let x be a point in $I(f) \bigcap C$. If there exists a branch s of C at x such that $f^n(s)$ is again a branch at x for all $n \geq 0$, then C is fixed by f.*

Proof of Lemma 3.1. Since f is birational, we may chose a resolution of f as in the diagram $(*)$ in Section 2.1.

If C is not fixed, we have $f(s) \neq s$ so that $A := (s \cdot f(s))_x < \infty$. By Proposition 2.1, π_2 is invertible on a Zariski neighbourhood of x. Let F_x be the fiber of π_1 over x.

For any $m \geq 0$, we have,

$$\begin{aligned}((f^m(s) \cdot f^{m+1}(s))_x =& \sum_{y \in F_x} (\pi_1^{\#} f^m(s) \cdot \pi_1^{*} f^{m+1}(s))_y \\ \geq& (\pi_1^{\#} f^m(s) \cdot \pi_1^{*} f^{m+1}(s))_{\pi_2^{-1}(x)} \\ =& (\pi_1^{\#} f^m(s) \cdot \pi_1^{\#} f^{m+1}(s))_{\pi_2^{-1}(x)} + (\pi_1^{\#} f^m(s) \cdot F_x)_{\pi_2^{-1}(x)} \\ =& (f^{m+1}(s) \cdot f^{m+2}(s))_x + (\pi_1^{\#} f^m(s) \cdot F_x)_{\pi_2^{-1}(x)} \\ \geq& (f^{m+1}(s) \cdot f^{m+2}(s))_x + 1.\end{aligned}$$

It follows that $A = (s \cdot f(s))_x \geq (f^m(s) \cdot f^{m+1}(s))_x + m \geq m$ for all $m \geq 0$ which yields a contradiction. □

We now treat the general case.

Proof of Theorem 1.2. Recall that f^n does not contract C and $f^n(C) \bigcap I(f) \neq \emptyset$ for all n. By Lemma 3.1, it is sufficient to find a point $x \in I(f) \bigcap C$ such that the image by f^n of the branch of C at x is again a branch of a curve at x for all $n \geq 0$. By contradiction we suppose that C is not periodic.

To do so, we introduce the set

$$P(f) = \{x \in I(f) |\ \text{ there is } n_1 > n_2 \geq 0 \text{ such that } f^{-n_1}(x) = f^{-n_2}(x)\}$$

and the set

$$O(f) = \{f^{-n}(x) |\ x \in P(f) \text{ and } n \geq 0\}.$$

By definition, $O(f)$ is finite. Since f is algebraically stable, $O(f) = O(f^n)$ for all $n \geq 1$. Replacing f by f^l for a suitable $l \geq 1$, we may assume that $O(f) = P(f)$. Set $N(f) = I(f) \setminus P(f)$.

First, we prove

Lemma 3.2. *For all $n \geq 0$, $f^n(C) \bigcap O(f) \neq \emptyset$.*

Proof of Lemma 3.2. We assume that $I(f) = \{p_1, \cdots, p_m\}$ and define the map

$$F = (f^{-1}, \cdots, f^{-1}) : X^m \dashrightarrow X^m.$$

Denote by π_i the projection onto the i-th factor and set

$$D = \bigcup_{i=1}^{m} \pi_i^{-1}(C).$$

Pick a point $q = (p_1, \cdots, p_m) \in X^m$. Since $f^n(C) \bigcap I(f) \neq \emptyset$ for all $n \geq 0$ by assamption, we have $F^n(q) \in D$ for all $n \geq 0$. Let Z' be the Zariski closure of $\{F^n(q) | n \geq 0\}$. Then we have $Z' \subseteq D$. Let Z be the union of all irreducible components of Z' of positively dimension. If Z is empty, then p_i is f^{-1}-preperiodic for all i and we conclude.

Otherwise since $\{F^n(q) | n \geq 0\} \bigcap I(F) = \emptyset$, the proper transformation of Z by F is well defined and satisfies $F(Z) = Z$, hence all irreducible components of Z are periodic. Let l be a common period for all components of Z. Observe that any irreducible component of Z is included in some $\pi_i^{-1}(C)$ for $i = 1, \cdots, m$. In other words, there exists $k \geq 0$ and $i \in \{1, \cdots, m\}$ such that $f^{-ln-k}(p_i) \in C$ for all $n \geq 0$. If p_i is not f^{-1}-preperiodic, then C is the Zariski closure of $\{f^{-ln-k}(p_i) | n \geq 0\}$ which is f^{-l}-invariant. This implies C to be periodic which contradicts to our hypothesis. It follows that p_i is f^{-1}-preperiodic.

Repeating the same argument for $f^n(C)$, we have $f^n(C) \bigcap O(f) \neq \emptyset$ for all $n \geq 0$. ☐

Denote by $D(n)$ the number of branches of $f^n(C)$ at points of $O(f)$. Since $f^{-1}(O(f)) \subseteq O(f)$, we have $D(n)$ is decrease and by Lemma 3.2, we have $D(n) \geq 1$. Replace C by $f^M(C)$ for some $M \geq 0$, we may assume that $D(n)$ is constant for $n \geq 0$. It follows that for any branch of curve of $f^n(C)$ at a point in $O(f)$, its image by f is again a branch of $f^{n+1}(C)$ at a point of $O(f)$. Set

$$S = \{x \in O(f) |\ \text{there are infinitely many } n \geq 0 \text{ such that } x \in f^n(C)\}.$$

By the finiteness of $O(f)$, we may suppose that

$$f^n(C) \bigcap O(f) = f^n(C) \bigcap S$$

for all integer $n \geq 0$.

We claim that

Lemma 3.3. *Replacing f by a positive iterate, there exists a point $x \in C \bigcap S$ for which there is a branch s of C at x such that $f^n(s)$ is again a branch of curve at x for all $n \geq 0$.*

According to Lemma 3.1, we conclude. $\square$

Proof of Lemma 3.3. Pick a resolution of f as in the diagram $(*)$ in Section 2.1. For any point $x \in S$, denote by F_x the fibre of π_1 over x and $E_x = \pi_2(F_x) \bigcap S$.

We have $E_x \neq \emptyset$. Otherwise, there exists $n \geq 0$ for which $x \in f^n(C)$ and a branch s of $f^n(C)$ at x. The assumption $E_x = \emptyset$ implies that $f(s)$ is not a branch at any point in S. This shows that $D(n+1) < D(n)$ and we get a contradiction.

On the other hand, let x_1, x_2 be two different points in S. If $E_{x_1} \bigcap E_{x_1} \neq \emptyset$, there exists $y \in S$ such that $y \in \pi_2(F_{x_1}) \bigcap \pi_2(F_{x_2})$. By Zariski's main theorem, $\pi_2^{-1}(y)$ is a connected curve meeting F_{x_1} and F_{x_2}. So $\pi_1(\pi_2^{-1}(y))$ is a curve and it is contracted by f to $y \in S \subseteq I(f)$. This contradicts the fact that f is algebraically stable. So we have

$$E_{x_1} \bigcap E_{x_1} = \pi_2(F_{x_1}) \bigcap \pi_2(F_{x_2}) \bigcap S = \emptyset.$$

Set $T = \coprod_{x \in S} E_x \subseteq S$. Since $\#E_x \geq 1$ for all x, we have $\#T \geq \#S$. It follows that $T = S$ and $\#E_x = 1$ for all $x \in S$. This allows us to define a map $G : S \to S$ sending $x \in S$ to the unique point in E_x. Then G is an one to one map. For all $n \geq M$, f sends a branch of $f^n(C)$ at a point $x \in S$ to a branch of $f^{n+1}(C)$ at the point $G(x)$. By replacing f by $f^{(\#S)!}$, we may assume that $G = \mathrm{id}$. Then for any $x \in S \bigcap C$ and s a branch of C at x, we have $f^n(s)$ is again a branch at x for all $n \geq 0$. $\square$

4. The DML property

For convenience, we introduce the following

Definition 4.1. Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation. We say that the pair (X, f) satisfies the DML property if for any irreducible curve C on X and for any closed point $p \in X$ such that $f^n(p) \notin I(f)$ for all $n \geq 0$, the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is a union of at most finitely many arithmetic progressions.

In our setting the DML property is equivalent to the following seemingly stronger property.

Proposition 4.2. *Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation. The following statements are equivalent.*

1. *The pair (X, f) satisfies the DML property.*
2. *For any curve C on X and any closed point $p \in X$ such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} | f^n(p) \in C\}$ is infinite, then p is preperiodic or C is periodic.*

Proof. Suppose (1) holds. Let C be any curve in X and p be a closed point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$. Assume that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$

is infinite. The DML property of (X, f) implies that there are integers $a > 0$ and $b \geq 0$ such that $f^{an+b}(p) \in C$ for all $n \geq 0$. If p is not preperiodic, the set $O_{a,b} := \{f^{an+b}(p) | \ n \geq 0\}$ is Zariski dense in C and $f^a(O_{a,b}) \subseteq O_{a,b}$. It follows that $f^a(C) \subseteq C$, hence C is periodic.

Suppose (2) holds. If the set $S := \{n \in \mathbb{N} | \ f^n(p) \in C\}$ is finite or p is preperiodic, then there is nothing to prove. We may assume that S is infinite and p is not preperiodic. The property (2) implies that C is periodic. There exists an integer $a > 0$ such that $f^a(C) \subseteq C$. We may suppose that $f^i(C) \nsubseteq C$ for $1 \leq i \leq a-1$. Since p is not preperiodic, there exists $N \geq 0$, such that $f^n(p) \notin (\bigcup_{1 \leq i \leq a-1} f^i(C)) \bigcap C$ for all $n \geq N$. So $S \setminus \{1, \cdots, N-1\}$ takes form $\{an + b | \ n \geq 0\}$ where $b \geq 0$ is an integer, and it follows that (X, f) satisfies the DML property. □

Theorem 4.3. *Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation, then the following properties hold.*

(i) *For any $m \geq 1$, (X, f) satisfies the DML property if and only if (X, f^m) satisfies the DML property.*
(ii) *Suppose U is an open subset of X such that the restriction $f_{|U} : U \to U$ is a morphism. Then (X, f) satisfies the DML property, if and only if $(U, f_{|U})$ satisfies the DML property.*
(iii) *Suppose $\pi : X \to X'$ is a birational morphism between smooth projective surfaces, and $f : X \dashrightarrow X$, $f' : X' \dashrightarrow X'$ are rational maps such that $\pi \circ f = f' \circ \pi$. If the pair (X, f) satisfies the DML property, then (X', f') satisfies the DML property.*
(iv) *Suppose $\pi : X \to X'$ is a birational morphism between smooth projective surfaces, and $f : X \dashrightarrow X$, $f' : X' \dashrightarrow X'$ are* birational *transformations such that $\pi \circ f = f' \circ \pi$. If f' is algebraically stable and the pair (X', f') satisfies the DML property, then (X, f) satisfies the DML property.*

Definition 4.4. Let X be a smooth projective surface defined over an algebraically closed field and $f : X \dashrightarrow X$ be a birational transformation. We say that (X', f') is a birational model of (X, f) if there is a birational map $\pi : X' \dashrightarrow X$ such that

$$f' = \pi^{-1} \circ f \circ \pi.$$

Corollary 4.5. *Let X be a smooth projective surface defined over an algebraically closed field and $f : X \dashrightarrow X$ be an algebraically stable birational transformation such that (X, f) satisfies the DML property. Then all birational models (X', f') of (X, f) satisfy the DML property.*

Proof of Corollary 4.5. Pick Y a desingularization of the graph of f and set π_1, π_2 the projections which make the diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

to be commutative. Since f is algebraically stable, its lift to Y satisfies the DML property by Theorem 4.3 (iv). We conclude that (X', f') satisfies the DML property by Theorem 4.3 (iii). $\square$

Proof of Theorem 4.3. (i). The "only if" part is trivial, so that we only have to deal with the "if" part. We assume that (X, f^m) satisfies the DML property. Let C be a curve in X and p be a point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$. Suppose that the set $\{n \in \mathbb{N}|\ f^n(p) \in C\}$ is infinite. Since

$$\{n \in \mathbb{N}|\ f^n(p) \in C\} = \bigcup_{i=0}^{m-1} \{n \in \mathbb{N}|\ f^{nm}(f^i(p)) \in C\},$$

then for some i, the set $\{n \in \mathbb{N}|\ f^{nm}(f^i(p)) \in C\}$ is also infinite. Since (X, f^m) satisfies the DML property, C is periodic or $f^i(p)$ is preperiodic. It follows that C is periodic or p is preperiodic.

(ii). If (X, f) satisfies the DML property, since $f_{|U} : U \to U$ is a morphism, $(U, f_{|U})$ satisfies the DML property.

Conversely suppose that $(U, f_{|U})$ satisfies the DML property. Let C be an irreducible curve in X, p be a closed point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N}|f^n(p) \in C\}$ is infinite. The set $E = X - U$ is a proper closed subvariety of X. If $p \in U$, then we have that $C \not\subseteq E$. Since $(U, f_{|U})$ satisfies the DML property, we have either p is preperiodic or C is periodic. Otherwise, we may assume that for all $n \geq 0$, $f^n(p) \in E$, then the Zariski closure D of $\{f^n(p)|\ n \geq 0\}$, is contained in E. We assume that p is not preperiodic, then $C \subseteq D$. Since D is fixed, we have that C is periodic.

(iii). It is sufficient to treat the case when π is the blowup at a point $q \in X'$. Let C' be a curve in X', p' be a point in X' such that $(f')^n(p') \notin I(f')$ for all $n \geq 0$ and the set $\{n \in \mathbb{N}|\ (f')^n(p') \in C'\}$ is infinite. We assume that p' is not a periodic point, so that for n large enough, $f'^n(p') \neq q$. Replacing p by $f'^m(p')$ for some m large enough, we may assume that $f'^n(p') \neq q$ for all $n \geq 0$. Set $p = \pi^{-1}(p')$ and $C = \pi^{-1}(C')$, then we have $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set

$$\{n \in \mathbb{N}|\ f^n(p) \in C\}$$

is infinite. This implies C and then C' to be periodic.

(iv). Let $C \subseteq X$ be a curve, p be a point in X such that $f^n(p) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N}|\ f^n(p) \in C\}$ is infinite. We may assume that C is irreducible. Let E be the exceptional locus of π.

Lemma 4.6. *If $C \subseteq E$ and $\pi(C)$ is a point in $I(f')$, then (iv) holds.*

Proof of Lemma 4.6. Set $q := \pi(C) \in I(f')$. Since f' is algebraically stable, we have $q \notin I((f')^{-n})$ and

$$\pi(f^{-n}(C)) = (f')^{-n}(q)$$

for all $n \geq 1$. It follows that $f^{-n}(C)$ is a point or an exceptional curve of π for $n \geq 1$.

If there exists $l \geq 1$ such that $f^{-l}(C)$ is a point, we pick two integers $n_1 > n_2 \geq l$ such that $f^{n_1}(p), f^{n_2}(p) \in C$. Then $f^{n_1-l}(p) = f^{n_2-l}(p)$, which implies p to be preperiodic.

Otherwise $f^{-n}(C)$ is an exceptional curve of π, for all $n \geq 0$. Since there are only finitely many irreducible components of E, we have that C is periodic. □

Denote by $K = \pi^{-1}(I(f'))$.

Lemma 4.7. *If there are infinitely many $n \geq 0$ such that $f^n(p) \in K$, then (iv) holds.*

Proof of 4.7. There is an irreducible component F of K such that the set $\{n \geq 0 | \, f^n(p) \in F\}$ is infinite.

If F is a point, then p is preperiodic.

Otherwise F is a curve, then $F \subseteq E$ and $\pi(F) \subseteq I(f')$. Suppose that p is not preperiodic, Lemma 4.6 shows that F is periodic. Then $F' = \bigcup_{k\geq 0} f^k(F)$ is a curve and $f^n(p) \subseteq F'$ for all $n \geq 0$. If $C \subseteq F'$, then C is periodic. If $C \nsubseteq F'$, then $C \bigcap F'$ is finite, and this shows that p is preperiodic. □

Lemma 4.8. *If $C \subseteq E$, then (iv) holds.*

Proof. By Lemma 4.7, we may assume that there exists an integer $N \geq 0$, such that $f^n(p) \notin K$ for all $n \geq N$.

Set $q := \pi(C)$. By Lemma 4.6, we assume that $q \notin I(f')$. Then we have

$$\pi(f^{N+l}(p)) = f'^l(\pi(f^N(p)))$$

for $l \geq 0$. It follows that there are infinitely many $l \geq 0$, such that $f'^l(\pi(f^N(p))) = q$. Then q is preperiodic and the obit of $f'^N(q)$ does not meet $I(f')$. Since $\pi(f^n(C)) = f'^n(q)$ for all $n \geq 0$, we have $f^n(C) \subseteq \bigcup_{k\geq N} \pi^{-1}(f^k(q))$ for all $n \geq N$. Hence ether C is periodic or for some $n \geq 1$, $f^n(C)$ is a point. In the second case, we conclude that p is preperiodic. □

Let $L = K \bigcup E$.

Lemma 4.9. *If there are infinitely many $n \geq 0$ such that $f^n(p) \in L$, then (iv) holds.*

Proof of Lemma 4.9. There is an irreducible component F of L such that $\{n \geq 0 | \, f^n(p) \in F\}$ is infinite.

If F is a point, then p is preperiodic.

Otherwise F is a curve, then $F \subseteq E$. Suppose that p is not preperiodic, Lemma 4.8 shows that F is periodic. Then $F' = \bigcup_{k\geq 0} f^k(F)$ is a curve and $f^n(p) \subseteq F'$ for all $n \geq 0$. If $C \subseteq F'$, then C is periodic. Otherwise $C \nsubseteq F'$, we have that $C \bigcap F'$ is finite and then p is preperiodic. □

We may assume that there is an integer $M \geq 0$, such that $f^n(p) \notin L$ for all $n \geq M$.

If $C \nsubseteq E$, $\pi(C)$ is a curve. For all $l \geq 0$ we have

$$\pi(f^{M+l}(p)) = f'^l(\pi(f^M(p))) \notin I(f').$$

Since (X', f') satisfies the DML property, either $\pi(C)$ is periodic or $\pi(p)$ is preperiodic. When $\pi(C)$ is periodic, we have C is periodic. Otherwise $\pi(p)$ is preperiodic. For any $l \geq 0$, π is invertible on some Zariski neighborhood of the point $f'^l(\pi(f^M(p)))$ and then we conclude that p is peperiodic. $\square$

5. Local dynamical Mordell Lang theorem

The aim of this section is to prove Theorem 1.1. We are in the following situation:

(i) X is a smooth projective surface defined over an arbitrary valued field $(K, |\cdot|)$.
(ii) $f : X \dashrightarrow X$ is a birational transformation defined over K;
(iii) Q is K-point of X such that $Q \in I(f^{-1}) \bigcap I(f)$ and $f(Q) = Q$;
(iv) p is K-point of X such that $f^n(p) \notin I(f)$ for all $n \geq 0$;
(v) $f^n(p) \to Q$ as $n \to \infty$ with respect to the topology induced by $|\cdot|$;
(vi) C is a curve in X such that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite.

We want to prove that C is fixed by f.

Proof of Theorem 1.1. Pick a resolution of f as in the diagram $(*)$ in Section 2.1. Recall Proposition 2.1. Assume that for all $n \geq 0$, $f^n(p) \neq Q$. There is an infinite sequence $\{n_k\}_{k\geq 0}$ such that $f^{n_k}(p) \in C \setminus \{Q\}$. It follows that $f^{n_k-m}(p) \in f^{-m}(C)$ for k large enough. Setting $k \to \infty$, we get $Q \in f^{-m}(C)$ for all $m \geq 0$.

If $C \neq f^{-1}(C)$, then we have $f^{-m}(C) \neq f^{-m-1}(C)$ for all $m \geq 0$. By computing local intersection at Q, we get

$$(5.1) \quad (f^{-m}(C) \cdot f^{-m-1}(C))_Q = \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^* f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x$$
$$= \sum_{x \in \pi_2^{-1}(Q)} \left(\left(\pi_2^{\#} f^{-m}(C) + \sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i \right) \cdot \pi_2^{\#} f^{-m-1}(C) \right)_x$$

where $E_i, 1 \leq i \leq s$ are irreducible exceptional curves for π_2. Since

$$\mathrm{Supp}(\sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i) = \bigcup_{1 \leq i \leq s} E_i = \pi_2^{-1}(Q),$$

we have

$$(5.1) = \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^{\#} f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x + \left(\left(\sum_{i=1}^{s} v_{E_i}(f^{-m}(C)) E_i \right) \cdot \pi_2^{\#} f^{-m-1}(C) \right)$$
$$\geq \sum_{x \in \pi_2^{-1}(Q)} (\pi_2^{\#} f^{-m}(C) \cdot \pi_2^{\#} f^{-m-1}(C))_x + 1$$
$$= \big(\sigma(f^{-m-1}(C)) \cdot \sigma(f^{-m-2}(C))\big)_{\sigma(Q)} + 1$$
$$= (f^{-m-1}(C) \cdot f^{-m-2}(C))_Q + 1.$$

It follows that

$$0 < (f^{-m}(C)\cdot f^{-m-1}(C))_Q \leq (f^{-m+1}(C)\cdot f^{-m}(C))_Q - 1 \leq \cdots \leq (C\cdot f^{-1}(C))_Q - m$$

for all $m \geq 0$, which yields a contradiction. So we have $C = f^{-1}(C)$ and then $f(C) = C$. $\square$

Observe that our proof of Theorem 1.1 actually gives

Proposition 5.1. *Let X be a projective surface over an algebraically closed field and $f : X \dashrightarrow X$ be a birational map with a fixed point $Q \in I(f^{-1}) \setminus I(f)$. Then all periodic curves passing through Q are fixed.*

6. The case $\lambda(f) = 1$

In this section, we prove Theorem A in the case $\lambda(f) = 1$. Denote by K an algebraically closed field of characteristic 0.

Recall from [7] and [10], that if $\lambda(f) = 1$, then we are in one of the following two cases:

(1) there exists a smooth projective surface X and an automorphism f' on X such that the pair (X, f') is birationally conjugated to $(\mathbb{A}^2, f)$;
(2) in suitable affine coordinates, $f(x, y) = (ax + b, A(x)y + B(x))$ where A and B are polynomials with $A \neq 0$ and $a \in K^*$, $b \in K$.

The case of automorphism has been treated by Bell, Ghioca and Tucker. Theorem A thus follows from [3, Theorem 1.3] in case (1) and in case (2) where $\deg A = 0$. So in this section we suppose that f takes form

$$f(x, y) = (ax + b, A(x)y + B(x)) \qquad (**)$$

with $A, B \in K[x]$, $\deg A \geq 1$, $a \in K^*$ and $b \in K$.

6.1. **Algebraically stable models.** Any map of the form $(**)$ can be made algebraically stable in a suitable Hirzebruch surface $\mathbb{F}_n$ for some $n \geq 0$. It is convenient to work with the presentation of these surfaces as a quotient by $(\mathbb{G}_m)^2$, as in [17]. By definition, the set of closed points $\mathbb{F}_n(K)$ is the quotient of $\mathbb{A}^4(K) \setminus (\{x_1 = 0 \text{ and } x_2 = 0\} \bigcup \{x_3 = 0 \text{ and } x_4 = 0\})$ by the equivalence relation generated by

$$(x_1, x_2, x_3, x_4) \sim (\lambda x_1, \lambda x_2, \mu x_3, \mu/\lambda^n x_4)$$

for $\lambda, \mu \in K^*$. We denote by $[x_1, x_2, x_3, x_4]$ the equivalence class of (x_1, x_2, x_3, x_4). We have a natural morphism $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ given by $\pi_n([x_1, x_2, x_3, x_4]) = [x_1 : x_2]$ which makes $\mathbb{F}_n$ into a locally trivial $\mathbb{P}^1$ fibration.

We shall look at the embedding

$$i_n : \mathbb{A}^2 \hookrightarrow \mathbb{F}_n : (x, y) \mapsto [x, 1, y, 1].$$

Then $\mathbb{F}_n \setminus \mathbb{A}^2$ is union of two lines: one is the fiber at infinity F_∞ of π_n, and the other one is a section of π_n which we denote by L_∞.

Recall that f has the form $(**)$. For each $n \geq 0$, set $d = \max\{\deg A, \deg B - n\}$. By the embedding i_n, the map f extends to a birational transformation

$$f_n : [x_1, x_2, x_3, x_4] \mapsto [ax_1 + bx_2, x_2, A(x_1/x_2)x_2^d x_3 + B(x_1/x_2)x_2^{d+n}x_4, x_2^d x_4]$$

on $\mathbb{F}_n$. For any $n \geq \deg B - \deg A + 1$, we have $d = \deg A$ and

$$I(f_n) = \{[x_1, x_2, x_3, x_4] \in \mathbb{F}_n | x_2 = x_3 = 0\}.$$

The unique curve which is contracted by f_n is $F_\infty = \{x_2 = 0\}$ and its image is $f_n(F_\infty) = [1, 0, 1, 0]$. It implies the following:

Proposition 6.1. *For any integer $n \geq \deg B - \deg A + 1$, f_n is algebraically stable on $\mathbb{F}_n$ and contracts the curve F_∞ to the point $[1, 0, 1, 0]$.*

6.2. **The attracting case.** In the remaining of this section, we fix an integer m such that the extension of f to $\mathbb{F}_m$ is algebraically stable. For simplicity, we write f for the map f_m induced by f on $\mathbb{F}_m$.

Proposition 6.2. *Let $|\cdot|$ be an absolute value on K such that $|a| > 1$. Then $(\mathbb{F}_m, f)$ satisfies the DML property.*

Proof. Since $a \neq 1$, by changing coordinates, we may assume that $f = (ax, A(x)y + B(x))$. Since f contracts the fiber F_∞ to $O := L_\infty \bigcap F_\infty$, the point O is fixed and the two eigenvalues of df at O are $1/a$ and 0. Since $|a| > 1$, there is a neighbourhood U of O, such that $U \bigcap I(f) = \emptyset$, $f(U) \subseteq U$ and $f^n \to O$ uniformly on U.

Let C be an irreducible curve in $\mathbb{P}^2_K$ and p be a point in $\mathbb{A}^2_K$ such that the set $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite. By Lemma 4.3, we may assume that $p \in \mathbb{A}^2_K$ and $C \not\subseteq L_\infty \bigcup F_\infty$.

If $C \bigcap F_\infty = \{O\}$, there is an open set V of $\mathbb{P}^1_K$, such that $[1 : 0] \in V$ and $\pi_m^{-1}(V) \bigcap C \subseteq U$. Since $|a| > 1$, for n large enough, $f^n(p) \in \pi_m^{-1}(V)$. So there is an integer $n_1 > 0$ such that $f^{n_1}(p) \in U$. Theorem 1.1 implies that the curve C is fixed.

We may assume now that $f^n(C) \bigcap F_\infty \neq \{O\}$ for all $n \geq 0$.

If $C \bigcap F_\infty = \emptyset$, then C is a fiber of the rational fibration $\pi_m : \mathbb{F}_m \to \mathbb{P}^1$. Since $\{n \in \mathbb{N} |\ f^n(p) \in C\}$ is infinite, the curve C is fixed.

Finally assume that $f^n(C) \bigcap F_\infty \neq \emptyset$ for all $n \geq 0$. Since f contracts F_∞ to O, we have

$$f^n(C) \bigcap I(f) \neq \emptyset,$$

and we conclude by Theorem 1.2 that C is periodic in this case. $\square$

6.3. **The general case.**

Proposition 6.3. *The pair $(\mathbb{F}_m, f)$ satisfies the DML property.*

Proof. Let C be a curve in $\mathbb{F}_m$, and p be a point in $\mathbb{A}^2_K$ such that the set $\{n \geq 0 | f^n(p) \in C\}$ is infinite. We may assume that the transcendence degree of K is finite, since we can find a subfield of K such that it has finite transcendence degree and f, C and p are all defined over this subfield.

In the case f acts on the base as the identity, the proposition holds trivially. Assume that it is not that case. Let $O = L_\infty \bigcap F_\infty$. As in the proof of Proposition 6.2, we only have to consider the case $C \bigcap F_\infty = O$.

If a is a root of unity, we may replace f by f^n for some integer $n > 0$ and assume that $a = 1$ and $b = 1$. Since the transcendence degree of K is finite, we may embed

K in the field of complex numbers $\mathbb{C}$. Let $|\cdot|$ be the standard absolute value on $\mathbb{C}$. Since f contracts F_∞ to O, there is a neighborhood U of O with respect to the usual euclidian topology such that for all point $q \in U \bigcap \{(x,y) \in \mathbb{C}^2 |\ \mathrm{Re}(x) > 0\}$, we have $\lim_{n\to\infty} f^n(q) = O$. Since $C \bigcap F_\infty = O$, there exists $M > 0$, such that $C \bigcap \{(x,y) |\ \mathrm{Re}(x) > M\} \subseteq U$ and we conclude by using Theorem 1.1 in this case.

If a is an algebraic number over $\mathbb{Q}$ and is not a root of unity, by [20, Theorem 3.8] there exists an absolute value $|\cdot|_v$ (either archimedean or non-archimedean) on $\overline{\mathbb{Q}}$ such that $|a|_v > 1$. This shows that $(\mathbb{F}_m, f)$ satisfies the DML property by Proposition 6.2.

If a is not an algebraic number over $\mathbb{Q}$, we claim that there exists a field embedding $\iota : K \hookrightarrow \mathbb{C}$ such that $|\iota(a)| > 1$, and we may conclude again by using Proposition 6.2.

It thus remains to prove the claim. There is a subring R of K which is finitely generated over $\overline{\mathbb{Q}}$, such that f, C and p are all defined over R. There is an integer $l > 0$, such that $R = \overline{\mathbb{Q}}[t_1, \cdots, t_l]/I$, where I is a prime ideal of $\overline{\mathbb{Q}}[t_1, \cdots, t_l]$. It induces an embedding $\operatorname{Spec} R := V \subseteq \mathbb{A}^l_{\overline{\mathbb{Q}}}$. We set

$$V_{\mathbb{C}} := V \times_{\operatorname{Spec} \overline{\mathbb{Q}}} \operatorname{Spec} \mathbb{C} \subseteq \mathbb{A}^l_{\mathbb{C}}.$$

For any polynomial $F \in \overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I$, we also define $V_F := \{F = 0\}$. Then $V_{\mathbb{C}} \setminus V_F$ is a dense open set in the usual euclidian topology. Since $\overline{\mathbb{Q}}[t_1, \cdots, t_l] \setminus I$ is countable, the set $V_{\mathbb{C}} \setminus (\bigcup_{F \in \overline{\mathbb{Q}}[t_1,\cdots,t_l]\setminus I} V_F)$ is dense. Interpreting a a nonconstant holomorphic function on $V_{\mathbb{C}}$, we see that there exists an open set $W \subseteq V_{\mathbb{C}}$ such that $|a| > 1$ on W.

Pick a closed point $(s_1, \cdots, s_l) \in W \setminus (\bigcup_{F \in \overline{\mathbb{Q}}[t_1,\cdots,t_l]\setminus I} V_F)$ and consider the unique morphism $\iota : R = \mathbb{Q}[t_1, \cdots, t_l]/I \to \mathbb{C}$ sending t_i to s_i. This morphism is in fact an embedding. We may extend it to an embedding of K as required. $\square$

7. Upper bound on heights when $\lambda(f) > 1$

7.1. **Absolute values on fields.** ([20]) Set $\mathcal{M}_{\mathbb{Q}} := \{|\cdot|_\infty$ and $|\cdot|_p$ for all prime p$\}$ where $|\cdot|_\infty$ is the usual absolute value and $|\cdot|_p$ is the p-adic absolute value defined by $|x| := p^{-\mathrm{ord}_p(x)}$ for $x \in \mathbb{Q}$.

Let $K/\mathbb{Q}$ be a number field. The *set of places* on K is denoted by $\mathcal{M}_K$ and consists of all absolute values on K whose restriction to $\mathbb{Q}$ is one of the places in $\mathcal{M}_{\mathbb{Q}}$. Further we denote by $\mathcal{M}_K^\infty$ the set of archimedean places; and by $\mathcal{M}_K^0$ the set of nonarchimedean places.

When v is archimedean, there exists an embedding $\sigma_v : K \hookrightarrow \mathbb{C}$ (or $\mathbb{R}$) such that $|\cdot|_v$ is the restriction to K of the usual absolute value on $\mathbb{C}$ (or $\mathbb{R}$).

Similarly, we introduce the set of places on function fields.

Let C be a a smooth projective curve defined over an algebraically closed field k and $L := k(C)$ be the function field of C. The set of places on L, denoted by $\mathcal{M}_L$ consists of all absolute values of the form:

$$|\cdot|_p : x \mapsto e^{\mathrm{ord}_p(x)}$$

for any $x \in L$ and any closed point $p \in C$.

Let K/L be a finite field extension. The set of places on K is denoted by $\mathcal{M}_K$ and consists of all absolute values on K whose restriction to L is one of the places in $\mathcal{M}_L$. In this case, all the places in $\mathcal{M}_K$ are nonarchimedean. Set $\mathcal{M}_K^0 = \mathcal{M}_K$ and $\mathcal{M}_K^\infty = \emptyset$ for convenience.

Let K/L be a finite field extension where $L = \mathbb{Q}$ or a function field $k(C)$ of a curve C. For any place $v \in \mathcal{M}_K$, denote by $n_v := [K_v : L_v]$ the local degree of v then we have the product formula

$$\prod_{v\in\mathcal{M}_K} |x|_v^{n_v} = 1$$

for all $x \in K^*$.

For any $v \in \mathcal{M}_K$, denote by $O_v := \{x \in K|\ |x|_v \leq 1\}$ the ring of v-integers. In the number field case, we also denote by $O_K := \{x \in K|\ |x|_v \leq 1 \text{ for all } v \in \mathcal{M}_K^0\}$ the ring of integers.

7.2. **Basics on Heights.** We recall some basic properties of heights that are needed in the proof of Theorem A, see [18] or [19] for detail.

In this section, we set $L = \mathbb{Q}$ or $k(C)$ the function field of a curve C defined over an algebraically closed field k. Denote by $\overline{L}$ its algebraic closure.

Proposition-Definition 7.1. Let K/L be a finite field extension. Let $p \in \mathbb{P}^n(K)$ be a point with homogeneous coordinate $p = [x_0 : \cdots : x_n]$ where $x_0, \cdots, x_n \in K$. The *height* of p is the quantity

$$H_{\mathbb{P}^n}(p) := (\prod_{v\in\mathcal{M}_K} \max\{|x_0|_v, \cdots, |x_n|_v\}^{n_v})^{1/[K:L]}.$$

The height $H_{\mathbb{P}^n}(p)$ depends neither on the choice of homogeneous coordinates of p, nor on the choice of a field extension K which contains p.

When $L = k(C)$, we have a geometric interpretation of the height $H_{\mathbb{P}^n}(p)$. Observe that $\mathbb{P}^n_L$ is the generic fiber of the trivial fibration $\pi : \mathbb{P}^n_C := \mathbb{P}^n \times C \to C$. We set $s_p : D \to \mathbb{P}^n_C$ the normalization of the Zariski closure of p in $\mathbb{P}^n_C$. Then we have

$$H_{\mathbb{P}^n}(p) = e^{\deg(s_p^* O_{\mathbb{P}^n_C}(1))/\deg(\pi\circ s_p)}.$$

Proposition 7.2. *Let $f : \mathbb{P}^n_{\overline{L}} \dashrightarrow \mathbb{P}^m_{\overline{L}}$ be a rational map and X be a subvariety of $\mathbb{P}^n_{\overline{L}}$ such that $I(f)\bigcap X$ is empty and the restriction $f|_X$ is finite of degree d onto its image $f(X)$.*

Then there exist $A > 0$ such that for all point $p \in X(\overline{L})$, we have

$$\frac{1}{A} H_{\mathbb{P}^n}(p)^d \leq H_{\mathbb{P}^m}(f(p)) \leq A H_{\mathbb{P}^n}(p)^d.$$

Proposition 7.3 (Northcott Property)**.** *Let $K/\mathbb{Q}$ be a number field, and $B > 0$ be any constant. Then the set*

$$\{p \in \mathbb{P}^n(K)|\ H_{\mathbb{P}^n}(p) \leq B\}$$

is finite.

Remark 7.4. The Northcott Property does not hold in the case $K = k(C)$ when k is not a finite field. For example, the set

$$\{p \in \mathbb{P}^n(k(t))|\ H_{\mathbb{P}^n}(p) = 0\} = \{[x : y]|\ (x, y) \in k^2 \setminus \{(0, 0)\}\}$$

is infinite.

7.3. **Upper bounds on heights.** Let K be a number field or a function field of a smooth curve over an algebraically closed field k'. Let $f : \mathbb{A}^2_{\overline{K}} \to \mathbb{A}^2_{\overline{K}}$ be any birational polynomial morphism defined over $\overline{K}$ and assume that $\lambda(f) > 1$.

According to Theorem 2.4, we may suppose that there exists a compactification X of $\mathbb{A}^2_{\overline{K}}$, a closed point $Q \in X \setminus \mathbb{A}^2_{\overline{K}}$ such that f extends to a birational transformation $\widetilde{f}$ on X which satisfies the following properties:

(i) $\widetilde{f}$ is algebraically stable on X;
(ii) there exists a closed point $Q \in X \setminus \mathbb{A}^2$ fixed by $\widetilde{f}$, such that $d\widetilde{f}(Q) = 0$;
(iii) $\widetilde{f}(X \setminus \mathbb{A}^2) = Q$.

To simplify, we write $f = \widetilde{f}$ in the rest of the paper. We fix an embedding $X \subseteq \mathbb{P}^N_{\overline{K}}$. Let C be an irreducible curve in X whose intersection with $\mathbb{A}^2_{\overline{K}}$ is non empty.

Proposition 7.5. *Suppose that C is not periodic and $C \setminus \mathbb{A}^2_{\overline{K}} = \{Q\}$. Then there exists a number $B > 0$ such that for any point $p \in C(K)$ for which the set $\{n \in \mathbb{N}|\ f^n(p) \in C\}$ is infinite, we have $H_{\mathbb{P}^N}(p) \leq B$.*

Proof. Assume that X, f, C and Q are all defined over K and $Q = [1 : 0 : \cdots : 0] \in \mathbb{P}^N_K$. We can extend f to a rational morphism on $\mathbb{P}^N$ which is regular at Q. Then there exists an element $a \in K^*$ and $F_i \in (x_1, \cdots, x_N)K[x_0, \cdots, x_N]$ for $i = 0, \cdots, N$ such that

$$f([1 : x_1 : \cdots : x_N]) = [a + F_0 : F_1 : \cdots : F_N]$$

for any $[1 : x_1 : \cdots : x_N] \in X$. Since f is regular at Q and $a \neq 0$, there is a finite set $S \subseteq \mathcal{M}^0_K$ such that for any $v \in \mathcal{M}^0_K \setminus S$, we have $|a|_v = 1$ and all coefficients of f are defined in O_v. Recall that we may endow X with a metric d_v, see Section 2.3.

For any $v \in \mathcal{M}^0_K \setminus S$, set $r_v := 1$ and $U_v := \{x \in X(K)|\ d_v(x, Q) < 1\}$. Since $df(Q) = 0$, we see that for all $x \in U_v$, $d_v(f(x), Q) \leq d_v(x, Q)^2$, hence

$$\lim_{n\to\infty} f^n(x) = Q.$$

For any $v \in S$, set $r_v := |a|_v$ and $U_v := \{x \in X(K)|\ d_v(x, q) < r_v\}$. We see that for all $x \in U_v$, $d_v(f(x), Q) \leq d_v(x, Q)^2/r_v$, and again it follows that

$$\lim_{n\to\infty} f^n(x) = Q.$$

For any $v \in \mathcal{M}^\infty_K$, since $df(Q) = 0$, there is $r_v > 0$ such that for any $x \in U_v := \{x \in X(K)|\ d_v(x, q) < r_v\}$ we have $f(x) \subseteq U_v$ and

$$\lim_{n\to\infty} f^n(x) = Q.$$

If $p \in \bigcup_{v\in\mathcal{M}_K} U_v$, Theorem 1.1 shows that C is periodic and this contradicts our assumption. In other words, we need to estimate the height of a given point

$$p \in C(K) \setminus \bigcup_{v\in\mathcal{M}_K} U_v.$$

If C intersects the line at infinity only at the point Q, then we may directly estimate the height of p given by the embedding of C into $\mathbb{P}^N_K$. Since we do not assume that this is the case, we shall work first with a height induced by a divisor on C given by the divisor Q, and then estimate $h_{\mathbb{P}^N}(p)$ using Proposition 7.2. To do so, let $i : \widetilde{C} \to C \subseteq X$ be the normalization of C and pick a point $Q' \in i^{-1}(Q)$. There is a positive integer l such that lQ' is a very ample divisor of $\widetilde{C}$. So there is an embedding $j : \widetilde{C} \hookrightarrow \mathbb{P}^M$ for some $M > 0$ such that

$$Q' = [1 : 0 : \cdots : 0] = H_\infty \bigcap \widetilde{C}$$

where $H_\infty = \{x_M = 0\}$ is the hyperplane at infinity. Let

$$g : \widetilde{C} \to \mathbb{P}^1$$

be a morphism sending $[x_0 : \cdots : x_M] \in \widetilde{C}$ to $[x_0 : x_M] \in \mathbb{P}^1$. It is well defined since $\{x_0 = 0\} \bigcap H_\infty \bigcap \widetilde{C} = \emptyset$. Then g is finite and

$$g^{-1}([1 : 0]) = H_\infty \bigcap \widetilde{C} = [1 : 0 \cdots : 0].$$

By base change, we may assume that $\widetilde{C}, i, j, g$ are all defined over K.

In the function field case, there is a smooth projective curve D such that $K = k'(D)$; and in the number field case, we set $D = \operatorname{Spec} O_K$.

We consider the irreducible scheme $\widetilde{\mathcal{C}} \subseteq \mathbb{P}^M_D$ over D whose generic fiber is $\widetilde{C}$ and the irreducible scheme $\mathcal{X} \subseteq \mathbb{P}^N_D$ over D whose generic fiber is X. Then i extends to a map $\iota : \widetilde{\mathcal{C}} \dashrightarrow \mathcal{X}$ over D birationally to its image. For any $v \in \mathcal{M}^0_K$, let

$$\mathfrak{p}_v = \{x \in O_v |\ v(x) > 0\}$$

be a prime ideal in O_v. There is a finite set T consisting of those places $v \in \mathcal{M}^0_K$ such that ι is not regular along the special fibre $\widetilde{C}_{O_v/\mathfrak{p}_v}$ at $\mathfrak{p}_v \in D$ or $\widetilde{C}_{O_v/\mathfrak{p}_k} \bigcap H_{\infty, O_v/\mathfrak{p}_v} \neq \{[1 : 0 : \cdots : 0]\}$.

For any $v \in \mathcal{M}^0_K \setminus T \bigcup S$, observe that we have

$$\begin{aligned} V_v :=& \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_i|_v < 1, i = 1, \cdots, M\} \\ =& \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_M|_v < 1\} = g^{-1}(\Omega_v) \bigcap \widetilde{C}(K) \end{aligned}$$

with $\Omega_v := \{[1 : x] \in \mathbb{P}^1(K) |\ |x|_v < t_v\}$ and $t_v := 1$.

For any $v \in T \bigcup S \bigcup \mathcal{M}^\infty_K$, by the continuity of i, there is $s_v > 0$ such that

$$i(V_v) \in U_v$$

where $V_v = \{[1 : x_1 : \cdots : x_M] \in \widetilde{C}(K) |\ |x_i|_v < s_v, i = 1, \cdots M\}$. Since $g^{-1}([1 : 0]) = \{[1 : 0 : \cdots : 0]\}$, there exists $t_v > 0$, such that

$$g^{-1}(\Omega_v) \bigcap \widetilde{C} \subseteq V_v$$

where $\Omega_v = \{[1:x] \in \mathbb{P}^1(K) | \ |x|_v < t_v\}$.

We need to find an upper bound for the height of points in $C(K) \setminus \bigcup_{v\in\mathcal{M}_K} U_v$. Since the set $\mathrm{Sing}(C)$ of singular points of C is finite, we only have to bound the height of points in $C(K) \setminus (\mathrm{Sing}(C) \bigcup_{v\in\mathcal{M}_K} U_v)$.

Let p be a point in $C(K) \setminus (\mathrm{Sing}(C) \bigcup_{v\in\mathcal{M}_K} U_v)$. Observe that $i^{-1}(p) \in \widetilde{C}(K)$ and $x := j(i^{-1}(p))$ is also defined over K. We have $x \notin V_v$ hence $y := g(x) \notin \Omega_v$ for all $v \in \mathcal{M}_K$.

For any $y = [y_0 : y_1] \in \mathbb{P}^1(K) \setminus (\bigcup_{v\in\mathcal{M}_K} \Omega_v)$, we have $|y_1/y_0|_v \geq t_v$ for all v. We get the following upper bound

$$\begin{aligned} H_{\mathbb{P}^1}(y)^{[K:\mathbb{Q}]} &= \prod_{v\in\mathcal{M}_K} \max\{|y_0|_v, |y_1|_v\}^{n_v} \\ &\leq \prod_{v\in\mathcal{M}_K} \max\{|y_1|_v/t_v, |y_1|_v\}^{n_v} \\ &= \prod_{v\in\mathcal{M}_K} |y_1|_v^{n_v} \prod_{v\in\mathcal{M}_K} \max\{1, 1/t_v\}^{n_v} \\ &= \prod_{v\in\mathcal{M}_K} \max\{1, 1/t_v\}^{n_v} =: B' < \infty. \end{aligned}$$

By Proposition 7.2 applied to $g : \widetilde{C} \hookrightarrow \mathbb{P}^M$ and $i : \widetilde{C} \to \mathbb{P}^N$, we get $H_{\mathbb{P}^N}(p) \leq B$ for some constant B independent on the choice of p as we require. $\square$

8. Proof of Theorem A

Let C be a curve in $\mathbb{A}^2_K$. We want to show that for any point $p \in \mathbb{A}^2(K)$ such that the set

$$\{n \in \mathbb{N} | \ f^n(p) \in C\}$$

is infinite, then p is preperiodic.

According to Section 6, we may assume that $\lambda(f) > 1$. As in Section 7.3, we use Theorem 2.4 to get a compactification X of $\mathbb{A}^2_K$. For simplicity, we also denote by f the map induced by f on X. There exists $n \geq 1$ such that f^n contracts $X \setminus \mathbb{A}^2_K$ to a superattracting fixed point $Q \in X \setminus \mathbb{A}^2_K$. We extend C to a curve in X. Suppose that C is not periodic. By Theorem 1.2, we may assume that $C(K) \setminus \mathbb{A}^2(K) = \{Q\}$. Finally we fix an embedding $X \hookrightarrow \mathbb{P}^N_K$ for some $N \geq 1$.

We first treat the case $K = \overline{\mathbb{Q}}$.

There is a number field K' such that both f and p are defined over K'. Then $f^n(p) \in \mathbb{A}^2(K')$ for all $n \geq 0$.

Proposition 7.5 and the Northcott Property imply that the set $\{f^n(p) | \ n \geq 0\} \bigcap C$ is finite. Since the set $\{n \in \mathbb{N} | \ f^n(p) \in C\}$ is infinite, there exists $n_1 > n_2 > 0$ such that $f^{n_1}(p) = f^{n_2}(p)$. We conclude that p is preperiodic.

Next we consider the general case of an algebraically closed field K of characteristic 0.

By replacing K by an algebraically closed subfield over which p, C and f are all defined, we may suppose that the transcendence degree tr.d.$K/\mathbb{Q}$ of K over $\mathbb{Q}$ is finite. We argue by induction on tr.d.$K/\mathbb{Q}$.

If tr.d.$K/\mathbb{Q} = 0$, then $K = \overline{\mathbb{Q}}$ and we are due by what precedes.

If tr.d.$K/\mathbb{Q} \geq 1$, then there is an algebraically closed subfield k of K such that tr.d.$k/\mathbb{Q} =$ tr.d.$K/\mathbb{Q} - 1$.

There is a smooth projective curve D over k, such that X, f, p, Q and C are defined over the function field $k(D)$ of D. Observe that $K = \overline{k(D)}$.

We consider the irreducible scheme

$$\pi : \mathcal{X} \subseteq \mathbb{P}^N_D \to D$$

over D whose generic fiber is X and $\mathcal{C} \subseteq \mathbb{P}^N_D$ the Zariski closure of C in $\mathcal{X}$.

The map f extends to a birational map $f' : \mathcal{X} \dashrightarrow \mathcal{X}$ over D. For any $x \in D$, denote by X_x and C_x the fiber of $\mathcal{X}$ and $\mathcal{C}$ at x respectively, and denote by f_x the restriction of map f' to the fiber X_x.

Proposition 7.5 implies that there is a number $M \geq 0$ such that for all $n \geq 0$ either $f^n(p) \notin C$ or $H_{\mathbb{P}^N}(f^n(p)) \leq M$.

A point $s \in X(k(D))$ is associated to its Zariski closure in $\mathcal{X}$ which is a section of $\pi : \mathcal{X} \to D$. For simplicity, we also write s for this section. Then the height of s is

$$H_{\mathbb{P}^N}(s) = e^{(s \cdot L)}$$

where $L := O_{\mathbb{P}^N_D}(1)$.

For any section s, observe that π induces an isomorphism from s to the curve D. We may consider the Hilbert polynomial

$$\chi(L^{\otimes n}, s) = 1 - g(s) + n(s \cdot L) = 1 - g(D) + n \log H(s).$$

It follows that there is a quasi-projective k-variety M_H that parameterizes the sections s of π such that $H_{\mathbb{P}^N}(s) \leq M$ (see [5]).

Let T_1 be the set of points $x \in D$ such that f_x is birational and $I(f_x^{-1}) \bigcap I(f_x) \neq \emptyset$. Observe that T_1 is finite. Let T_2 be the set of the points $x \in D \setminus T_1$, such that C_x is fixed. Since C is not fixed, T_2 is finite. Because k is algebraically closed, $D \setminus (T_1 \bigcup T_2)$ is infinite. For any point $x \in D$, denote by $p_x : M_H \to X_x$ the map sending s to $s(x)$. Pick a sequence of distinct points $\{x_i\}_{i \geq 0} \subseteq D \setminus (T_1 \bigcup T_2)$. For any $l \geq 1$, let

$$p_l = \prod_{i=1}^{l} p_{x_i} : M_H \to \prod_{i=1}^{l} X_{x_i}.$$

Observe that any two points $s_1, s_2 \in M_H$ are equal if and only if $p_i(s_1) = p_i(s_2)$ for all $i \geq 0$.

We claim the following lemma, and prove it later.

Lemma 8.1. *Let X be any reduced quasi-projective variety over an algebraically closed field k. For any $i \geq 1$, let $\pi_i : X \to Y_i$ be a morphism. If for any difference points $x_1, x_2 \in X$, there exists $i \geq 0$, such that $\pi_i(x_1) \neq \pi_i(x_2)$, then for l large* *enough the map*

$$p_l = \prod_{i=1}^{l} \pi_i : X \to \prod_{i=1}^{l} Y_i$$

is finite.

By Lemma 8.1, there is an integer L large enough, such that the map p_L is finite. By Proposition 5.1, C_{x_i} is not periodic for all $i \geq 1$. The set $\mathsf{N} := \{n \geq 0 | f^n(p) \in C\}$ is infinite, enumerate $\mathsf{N} = \{n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots\}$. For any $i \geq 0$, there exists $s_i \in M_H$ such that $s_i = f^{n_i}(p)$. By the induction hypothesis, we know that $s_{n_0}(x_i) = f^{n_0}(p)(x_i)$ is a preperiodic point of f_{x_i} for any $1 \leq i \leq L$. Then the orbit G_i of $p(x_i)$ in X_{x_i} is finite. So the set

$$p_L(\{s_i\}_{i\geq 0}) \subseteq \prod_{i=0}^{L} G_i$$

is finite. Since p_L is finite, then we have $\{s_i\}_{i\geq 0}$ is finite. Then there is $i_1 > i_2$ such that $s_{i_1} = s_{i_2}$, and $f^{n_{i_1}}(p) = f^{n_{i_2}}(p)$. Then p is preperiodic. □

Proof of Lemma 8.1. We prove this lemma by induction on the dimension of X.

If $\dim X = 0$, then the result is trivial.

If $\dim X > 0$, we may assume that X is irreducible. We pick any point $x \in X$, and let F_l be the fiber of p_l which contains x. Observe that

$$F_{l+1} \subseteq F_l,$$

so that there is an integer $L' \geq 1$, such that for any $L \geq L'$,

$$F_L = \bigcap_{l\geq 0} F_l.$$

Since for any point $x_1 \in X - \{x\}$, there exists $i \geq 0$, such that $\pi_i(x_1) \neq \pi_i(x)$, we have

$$F_L = \bigcap_{l\geq 0} F_l = \{x\},$$

so that

$$\dim X - \dim p_L(X) \leq \dim F_L = 0.$$

In particular p_L is generically finite. It means that there exists an open set U of $p_L(X)$, such that $p_L : p_L^{-1}(U) \to U$ is finite. Set $X' = X - p_L^{-1}(U)$, then we have $\dim X' \leq \dim X - 1$.

By the induction hypothesis, there is $L'' \geq L'$, such that for any $L \geq L''$, $p_L|_{X'}$ is finite and then p_L is finite. □

References

[1] Shreeram Shankar Abhyankar. *Resolution of singularities of embedded algebraic surfaces.* Pure and Applied Mathematics, Vol. 24. Academic Press, New York, 1966.

[2] Jason P. Bell. A generalised Skolem-Mahler-Lech theorem for affine varieties. *J. London Math. Soc. (2)*, 73(2):367–379, 2006.

[3] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang problem for étale maps. *Amer. J. Math.*, 132(6):1655–1675, 2010.

[4] Robert L. Benedetto, Dragos Ghioca, Pär Kurlberg, and Thomas J. Tucker. A case of the dynamical Mordell-Lang conjecture. *Math. Ann.*, 352(1):1–26, 2012. With an appendix by Umberto Zannier.

[5] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[6] Laurent Denis. Géométrie et suites récurrentes. *Bull. Soc. Math. France*, 122(1):13–27, 1994.

[7] Jeffrey Diller and Charles Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.

[8] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161(3):1637–1644, 2005.

[9] Gerd Faltings. The general case of S. Lang's conjecture. In *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, volume 15 of *Perspect. Math.*, pages 175–182. Academic Press, San Diego, CA, 1994.

[10] Charles Favre and Mattias Jonsson. Dynamical compactification of $\mathbf{C}^2$. *Ann. of Math. (2)*, 173(1):211–248, 2011.

[11] Shmuel Friedland and John Milnor. Dynamical properties of plane polynomial automorpautomo. *Ergodic Theory Dyn. Syst.*, 9:67–99, 1989.

[12] William Fulton. *Algebraic curves. An introduction to algebraic geometry.* W. A. Benjamin, Inc., New York-Amsterdam, 1969. Notes written with the collaboration of Richard Weiss, Mathematics Lecture Notes Series.

[13] Dragos Ghioca and Thomas J. Tucker. Periodic points, linearizing maps, and the dynamical Mordell-Lang problem. *J. Number Theory*, 129(6):1392–1403, 2009.

[14] Dragos Ghioca, Thomas J. Tucker, and Michael E. Zieve. Intersections of polynomials orbits, and a dynamical Mordell-Lang conjecture. *Invent. Math.*, 171(2):463–483, 2008.

[15] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[16] Ernst Kunz. *Introduction to plane algebraic curves.* Birkhäuser Boston Inc., Boston, MA, 2005. Translated from the 1991 German edition by Richard G. Belshoff.

[17] Antonio Laface and Mauricio Velasco. A survey on Cox rings. *Geom. Dedicata*, 139:269–287, 2009.

[18] J-P. Serre. *Lectures on the Mordell-Weil theorem.* Friedr. Vieweg & Sohn, Braunschweig, 1989. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt.

[19] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[20] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2007.

[21] Paul Vojta. Integral points on subvarieties of semiabelian varieties. I. *Invent. Math.*, 126(1):133–181, 1996.

Centre de Mathématiques Laurent Schwartz École Polytechnique, 91128, Palaiseau Cedex, France

E-mail address: `junyi.xie@math.polytechnique.fr`