ON FANO MANIFOLDS OF PICARD NUMBER ONE WITH BIG AUTOMORPHISM GROUPS

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ABSTRACT. Let X be an n-dimensional smooth Fano complex variety of Picard number one. Assume that the VMRT at a general point of X is smooth irreducible and non-degenerate (which holds if X is covered by lines with index $> (n+2)/2$). It is proven that $\dim \mathfrak{aut}(X) > n(n+1)/2$ if and only if X is isomorphic to $\mathbb{P}^n, \mathbb{Q}^n$ or $\mathrm{Gr}(2,5)$. Furthermore, the equality $\dim \mathfrak{aut}(X) = n(n+1)/2$ holds only when X is isomorphic to the 6-dimensional Lagrangian Grassmannian $\mathrm{Lag}(6)$ or a general hyperplane section of $\mathrm{Gr}(2,5)$.

1. Introduction

For a projective complex variety X of dimension n, the dimension of its automorphism group $\mathrm{Aut}(X)$ can be arbitrarily large with respect to its dimension. Even if we restrict to klt Fano varieties of dimension n, there is no upper bound of $\dim \mathrm{Aut}(X)$ in terms of n: let $\bar{\mathbb{F}}_m$ be the contraction of the negative section of the Hirzebruch surface $\mathbb{F}_m$. Then $\bar{\mathbb{F}}_m$ is a klt Fano surface of canonical index m with $\dim \mathrm{Aut}(\bar{\mathbb{F}}_m) = m+5$. On the other hand, by Birkar's solution to the BAB conjecture ([B]), the dimensions of automorphism groups of ϵ-klt Fano varieties of dimension n are bounded from above by some constant $C(n,\epsilon)$ depending only on n and ϵ. A natural question is to find out the optimal constant $C(n,\epsilon)$.

In the case of n-dimensional Fano manifolds of Picard number one, Hwang proved in [H, Corollary 2] that the dimensions of their automorphism groups are bounded from above by $n \times \binom{2n}{n}$. In this case, an optimal upper bound is proposed by the following:

Conjecture 1.1 ([HM2, Conjecture 2])**.** Let X be an n-dimensional Fano manifold of Picard number one. Then $\dim \mathfrak{aut}(X) \leq n^2 + 2n$, with equality if and only if $X \simeq \mathbb{P}^n$.

In [HM2, Theorem 1.3.2], this conjecture is proven under the assumption that the variety of minimal rational tangents (VMRT for short, cf. Definition 3.1) at a general point of X is smooth irreducible non-degenerate and linearly normal. The purpose of this note is to push further the ideas of [HM2], combined with the recent results in [FH1] and [FH2], to prove the following

Theorem 1.2. *Let X be an n-dimensional Fano manifold of Picard number one. Assume that the VMRT at a general point of X is smooth irreducible and non-degenerate. Then we have*

(a) $\dim \mathfrak{aut}(X) > n(n+1)/2$ *if and only if X is isomorphic to* $\mathbb{P}^n, \mathbb{Q}^n$ *or* $\mathrm{Gr}(2,5)$.
(b) *The equality* $\dim \mathfrak{aut}(X) = n(n+1)/2$ *holds only when X is isomorphic to* $\mathrm{Lag}(6)$ *or a general hyperplane section of* $\mathrm{Gr}(2,5)$.

Remark 1.3. As proved in [HM2, Corollary 1.3.3], the assumption on the VMRT of X is satisfied if there exists an embedding $X \subset \mathbb{P}^N$ such that X is covered by lines with index $> \frac{n+2}{2}$.

Recall that $\dim \mathfrak{aut}(\mathbb{P}^n) = n^2 + 2n$ and $\dim \mathfrak{aut}(\mathbb{Q}^n) = \dim \mathfrak{so}_{n+2} = \frac{(n+1)(n+2)}{2}$. The previous theorem indicates that there may exist big gaps between the dimensions of automorphism groups of Fano manifolds of Picard number one.

To prove Theorem 1.2, we first show the following result, which could be of independent interest.

Theorem 1.4. *Let $n \geq 2$ be an integer. Let $X \subsetneq \mathbb{P}^n$ be an irreducible and non-degenerate subvariety of codimension $c \geq 1$, which is not a cone. Let $G_n^X = \{g \in \mathrm{PGL}_{n+1}(\mathbb{C}) | g(X) = X\}$. Then*

(a) $\dim G_n^X \leq \frac{n(n+1)}{2} - \frac{(c-1)(c+4)}{2}$.

(b) $\dim G_n^X = \frac{n(n+1)}{2}$ *if and only if X is a smooth quadratic hypersurface.*

(c) *if X is smooth and is not a quadratic hypersurface, then* $\dim G_n^X \leq \frac{n(n+1)}{2} - 3$.

The idea of the proof of Theorem 1.2 is similar to that in [HM2]: the dimension of $\mathfrak{aut}(X)$ is controlled by

$$n + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)}.$$

In this expression, $\mathcal{C}_x$ is the VMRT of X at a general point, and $\hat{\mathcal{C}}_x$ is the affine cone of the projective variety $\mathcal{C}_x$. The term $\mathfrak{aut}(\hat{\mathcal{C}}_x)$ is the Lie algebra of infinitesimal automorphisms of $\hat{\mathcal{C}}_x$ while $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)}$ is the first prolongation of $\mathfrak{aut}(\hat{\mathcal{C}}_x)$ (cf. Definition 3.4). By Theorem 1.4, we have an optimal bound for $\dim \mathfrak{aut}(\hat{\mathcal{C}}_x)$, which gives the bound for $\dim \mathfrak{aut}(X)$ in the case when $\dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} = 0$. For the case when $\dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$, we have a complete classification of all such embeddings $\mathcal{C}_x \subset \mathbb{P}T_xX$ by [FH1] and [FH2]. Then a case-by-case check gives us the bound in Theorem 1.2. Finally we apply Cartan-Fubini extension theorem of Hwang-Mok ([HM1]) and the result of Mok ([M]) to recover the variety X from its VMRT.

Convention: For a projective variety X, we denote by $\mathfrak{aut}(X)$ its Lie algebra of automorphism group, while for an embedded variety $S \subset \mathbb{P}V$, we denote by $\mathfrak{aut}(\hat{S})$ the Lie algebra of infinitesimal automorphisms of $\hat{S}$, which is given by

$$\mathfrak{aut}(\hat{S}) := \{g \in \mathrm{End}(V) | g(\alpha) \in T_\alpha(\hat{S}), \text{for any smooth point } \alpha \in \hat{S}\}.$$

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2. Automorphism group of embedded varieties

For each positive integer n, we let

$$G_n = \mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C}).$$

If $X \subseteq \mathbb{P}^n$ is a subvariety, we denote by $G_n^X \subseteq G_n$ the subgroup of elements g such that $g(X) = X$. Note that if $X \subset \mathbb{P}^n$ is non-degenerate, then $G_n^X \subset \mathrm{Aut}(X)$. For a subvariety $X \subset \mathbb{P}^n$, the vertex set of X is

$$C_X := \{x \in X | \ X \text{ is a cone with vertex } x\}.$$

It is well-known that C_X is a linear subspace (see for example [R, Proposition 1.3.3]).

The goal of this section is to prove the following theorem, which is more general than Theorem 1.4.

Theorem 2.1. *Assume that $n \geq 2$. Let $X \subsetneq \mathbb{P}^n$ be an irreducible and non-degenerate subvariety of codimension $c \geq 1$. Set $r_X := -1$ if $C_X = \emptyset$ and $r_X := \dim C_X$ otherwise. Then we have*

$$\dim G_n^X \leq \frac{(n-r_X-1)(n-r_X)}{2} - \frac{(c-1)(c+4)}{2} + (r_X+1)(n+1).$$

Note that Theorem 1.4 (a) is a special case of Theorem 2.1 when $C_X = \emptyset$.

The idea of the proof is to cut X by a general hyperplane, and then use induction on n to conclude. To this end, we will first introduce the following notation. For a hyperplane H in $\mathbb{P}^n$, we may choose a coordinates system $[x_0 : x_1 : \cdots : x_n]$ such that H is defined by $x_0 = 0$. For every $g \in G_n$, it has a representative $M \in \mathrm{GL}_{n+1}(\mathbb{C})$, such that its action on $\mathbb{P}^n$ is given by $g([x_0 : x_1 : \cdots : x_n]) = [y_0 : y_1 : \cdots : y_n]$, where

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Then $g \in G_n^H$ if and only if it can be represented by a matrix of the shape

$$\left(\begin{array}{c|ccc} a_0 & 0 & \cdots & 0 \\ \hline a_1 & & & \\ a_2 & & A & \\ \vdots & & & \\ a_n & & & \end{array}\right).$$

There is a natural morphism $r_H : G_n^H \to \mathrm{Aut}(H) \cong \mathrm{PGL}_n(\mathbb{C})$. Then an element g is in the kernel $\mathrm{Ker}\, r_H$ if and only if it can be represented by a matrix of the shape

$$\left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline a_1 & & & \\ a_2 & & \mathrm{Id}_n & \\ \vdots & & & \\ a_n & & & \end{array}\right).$$

For such $g \in \mathrm{Ker}\, r_H$, we call λ the special eigenvalue of g. The action of g on the normal bundle of H is then the multiplication by λ. We see that this is independant of the choice of representatives of g in $\mathrm{GL}_{n+1}(\mathbb{C})$. We also note that if g, h are two elements in $\mathrm{Ker}\, r_H$, with special eigenvalues λ and μ respectively, then the special eigenvalue of gh is equal to $\lambda\mu$. This gives a homomorphism $\chi_H : \mathrm{Ker}\, r_H \to \mathbb{C}^*$.

Before giving the proof of Theorem 1.4, we will first prove several lemmas.

Lemma 2.2. *Let H be a hyperplane in $\mathbb{P}^n$, and let $X \subseteq \mathbb{P}^n$ be any subvariety. Then*

$$\dim G_n^X \leq \dim (G_n^H \cap G_n^X) + n.$$

Proof. This lemma follows from the fact that $\dim G_n = \dim G_n^H + n$. $\square$

We also need the following Bertini type lemma.

Lemma 2.3. *Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a non-degenerate irreducible subvariety of positive dimension which is not a cone. Then for a general hyperplane H, the intersection $X \cap H$ is still non-degenerate in H and is not a cone.*

Proof. Since $X \subseteq \mathbb{P}^n$ is irreducible and non-degenerate, the intersection of X and a general hyperplane H is non-degenerate in H.

Let $V \subset X \times (\mathbb{P}^n)^*$ be the subset of pair (x, H) such that $H \cap X$ is a cone with vertex x. Set $\pi_1 : V \to X$ and $\pi_2 : V \to (\mathbb{P}^n)^*$ the projections to the first and the second factors. If π_2 is not surjective, we concludes the proof. So we may assume that π_2 is surjective. Hence $\dim V \geq n$. Set $Y := \pi_1(V)$.

We first assume that there is $x \in Y$ such that $\dim \pi_1^{-1}(x) \geq 1$. Since any non trivial complete one-dimensional family of hyperplanes in $\mathbb{P}^n$ covers the whole $\mathbb{P}^n$, this condition implies that for every point $x' \in X \backslash \{x\}$, there is some hyperplane H containing x and x' such that $H \cap X$ is a cone with vertex x. Therefore, the line joining x and x' is contained in $H \cap X$ and hence in X. This shows that X is a cone with vertex x. We obtain a contradiction.

So the morphism $\pi_1 : V \to Y$ is finite. Then we get

$$n \leq \dim V = \dim Y \leq \dim X < n,$$

which is a contradiction. This concludes the proof. □

In the following lemmas, we will show that the kernel of $G_n^X \cap G_n^H \to G_{n-1}^{X \cap H}$ is a finite set if H is a general hyperplane. Note that this kernel is nothing but $G_n^X \cap \operatorname{Ker} r_H$, as $X \cap H$ is non-degenerate by Lemma 2.3. We will discuss according to the special eigenvalue of an element inside. We will first study the case when it is equal to 1.

Lemma 2.4. *Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be an irreducible subvariety which is not a cone. Let H be a hyperplane in $\mathbb{P}^n$ such that $X \not\subseteq H$. Let $g \in \operatorname{Ker} r_H \cap G_n^X$. If the special eigenvalue of g is 1, then g is the identity in G_n. In other words, the map $\chi_H : \operatorname{Ker} r_H \cap G_n^X \to \mathbb{C}^*$ is injective.*

Proof. Assume by contradiction that g is not the identity in G_n. We choose a homogeneous coordinates system $[x_0 : x_1 : \cdots : x_n]$ such that H is defined by $x_0 = 0$, and that $g([x_0 : x_1 : \cdots : x_n]) = [y_0 : y_1 : \cdots : y_n]$, where

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline a_1 & & & \\ a_2 & & \mathrm{Id}_n & \\ \vdots & & & \\ a_n & & & \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 + a_1 x_0 \\ x_2 + a_2 x_0 \\ \vdots \\ x_n + a_n x_0 \end{pmatrix}$$

By assumption, g is not the identity in G_n. Therefore, the a_i are not all equal to zero. Let $p \in \mathbb{P}^n$ be the point with homogeneous coordinates $[0 : a_1 : \cdots : a_n]$. For any point $x \in X \backslash H$ with homogeneous coordinates $[1 : x_1 : \cdots : x_n]$, the point $g^k(x)$ has coordinates

$$g^k(x) = [1 : x_1 + k a_1 : \cdots : x_n + k a_n].$$

This shows that all $g^k(x)$ are on the unique line $L_{p,x}$ passing through p and x. Since the $g^k(x)$ are pairwise different, this implies that $L_{p,x}$ has infinitely many intersection points with X. Therefore, $L_{p,x} \subseteq X$.

Since X is irreducible, every point $y \in X \cap H$ is a limit of points in $X \backslash H$. Hence by continuity, for each point $x \in X \backslash \{p\}$, the line $L_{p,x}$ is contained in X. This implies that X is a cone with vertex p. We obtain a contradiction. □

Now we will look at the case when the special eigenvalue is different from 1.

Lemma 2.5. *Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a subvariety. Then there is a number $d(X)$ such that if a line L intersects X at more than $d(X)$ points, then $L \subseteq X$.*

Proof. Assume that X is defined as the common zero locus of homogeneous polynomials $P_1, ..., P_k$. Let $d(X)$ be the maximal degree of them. Assume that a line L intersects X at more than $d(X)$ points, then L intersects the zero locus of each P_i at more than $d(X)$ points. By degree assumption, this shows that L is contained in the zero locus of each P_i. Hence $L \subseteq X$. $\square$

Lemma 2.6. *Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be an irreducible subvariety which is not a cone. Let H be a hyperplane in $\mathbb{P}^n$ such that $X \nsubseteq H$. Let $g \in \mathrm{Ker}\,(r_H) \cap G_n^X$. Then the special eigenvalue λ of g is a root of unity. Moreover, its order is bounded by the number $d(X)$ from above.*

Proof. We may assume that λ is different from 1. Then g is diagonalizable in this case. We may choose homogeneous coordinates $[x_0 : x_1 : \cdots : x_n]$ of $\mathbb{P}^n$ such that H is defined by $x_0 = 0$ and that

$$g([x_0 : x_1 : \cdots : x_n]) = [\lambda x_0 : x_1 : \cdots : x_n].$$

Assume by contradiction that the order of λ is greater than $d(X)$ (by convention, if λ is not a root of unity, then its order is $+\infty$). Let p be the point with homogeneous coordinates $[1 : 0 : \cdots : 0]$. For any point $x \in X \backslash (H \cup \{p\})$ with homogeneous coordinates $[1 : x_1 : \cdots : x_n]$, the point $g^k(x)$ has coordinates

$$g^k(x) = [\lambda^k : x_1 : \cdots : x_n].$$

This shows that all of the $g^k(x)$ are on the line $L_{p,x}$ passing through p and x. Moreover, we note that the cardinality of

$$\{g^k(x) \mid k \in \mathbb{Z}\}$$

is exactly the order of λ. By Lemma 2.5, we obtain that the line $L_{p,x}$ is contained in X. By the same continuity argument as in the proof of Lemma 2.4, this implies that for any point $x \in X \backslash \{p\}$, the line $L_{p,x}$ is contained in X. Hence X is a cone, and we obtain a contradiction. $\square$

Lemma 2.7. *Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^n$ be an irreducible subvariety which is not a cone. Let H be a hyperplane in $\mathbb{P}^n$ such that $X \nsubseteq H$. Then $\mathrm{Ker}\,(r_H) \cap G_n^X$ is a finite set. As a consequence, we have*

$$\dim\, G_n^X \leq \dim G_{n-1}^{X \cap H} + n.$$

Proof. By Lemma 2.4, the map $\chi_H : \mathrm{Ker}\, r_H \cap G_n^X \to \mathbb{C}^*$ is injective, while Lemma 2.6 implies that its image has bounded order, hence $\mathrm{Ker}\,(r_H) \cap G_n^X$ is finite. By Lemma 2.2, we obtain that

$$\dim\, G_n^X \leq \dim G_{n-1}^{X \cap H} + n.$$

$\square$

Now we can conclude Theorem 1.4.

Proof of Theorem 1.4. By Lemma 2.3, we can apply Lemma 2.7 inductively to get

$$\dim G_n^X \leq \dim G_{n-1}^{X \cap H} + n \leq \cdots \leq \dim G_{c+1}^{X \cap H^{n-c-1}} + n + (n-1) + \cdots + (c+2),$$

where $X \cap H^k$ means the intersection of X with k general hyperplanes of $\mathbb{P}^n$. Then $C = X \cap H^{n-c-1} \subset \mathbb{P}^{c+1}$ is a curve. As $C \subset \mathbb{P}^{c+1}$ is non-degenerate by Lemma 2.3, we have an inclusion $G^C_{c+1} \subset \mathrm{Aut}(C)$, while the latter has dimension at most 3. This gives that

$$\dim G^X_n \leq 3 + \sum_{j=c+2}^{n} j = \frac{n(n+1)}{2} - \frac{(c-1)(c+4)}{2},$$

which proves (a).

For (b), if $\dim G^X_n = \frac{n(n+1)}{2}$, then $c = 1$ by (a), i.e. $X \subset \mathbb{P}^n$ is a hypersurface. As X is not a cone, it must be smooth if it is quadratic. Therefore it remains to show that if X is a hypersurface of degree at least 3, then $\dim G^X_n \leq \frac{n(n+1)}{2} - 1$. We prove it by induction on the dimension of X. When $\dim X = 1$, pick a general line H in $\mathbb{P}^2$. By Lemma 2.7 , we have

$$\dim G^X_2 \leq \dim G^{X\cap H}_1 + 2.$$

In this case, $X \cap H$ is a set of $\deg X \geq 3$ points, hence $\dim G^{X\cap H}_1 = 0$. Thus, it follows that $\dim G^X_2 \leq 2 = \frac{2(2+1)}{2} - 1$. When $\dim X = n - 1 \geq 2$, pick a general hyperplane H in $\mathbb{P}^n$. By Lemma 2.7 , we have

$$\dim G^X_n \leq \dim G^{X\cap H}_{n-1} + n.$$

Then by induction hypothesis we have

$$\dim G^X_n \leq \dim G^{X\cap H}_{n-1} + n \leq \frac{n(n-1)}{2} - 1 + n = \frac{n(n+1)}{2} - 1.$$

For (c), if we assume further that X is a smooth hypersurface of degree greater than 2, then it is a classical result that $\dim G^X_n = 0$ (see for example [MM, Theorem 1]). As a consequence, if X is smooth and is not a quadratic hypersurface, then $\dim G^X_n \leq \frac{n(n+1)}{2} - 3$. If the codimension c of X is at least 2, then the result follows from part (a). This completes the proof of the theorem. □

Proof of Theorem 2.1. If $C_X = \emptyset$, then we conclude the proof by Theorem 1.4. Now assume that $C_X \neq \emptyset$, which is a linear subspace. For simplicity, we set $r = r_X = \dim C_X$.

Pick a coordinates system of $\mathbb{P}^n$ such that C_X is defined by $x_0 = \cdots = x_{n-r-1} = 0$. Let V be the subspace of $\mathbb{P}^n$ defined by $x_{n-r} = \cdots = x_n = 0$ and we identify it with $\mathbb{P}^{n-r-1}$. We let $\pi : \mathbb{P}^n \setminus C_X \to V \cong \mathbb{P}^{n-r-1}$ be the projection

$$\pi : [x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-r-1} : 0 : \cdots : 0].$$

Denote the image $\pi(X \setminus C_X)$ by Y. Then we have $X \setminus C_X = \pi^{-1}(Y)$, and $Y = X \cap V$. Moreover, Y is not a cone and it is non-degenerate in $\mathbb{P}^{n-r-1}$. Since C_X is preserved by G^X_n, we see that $G^X_n \subseteq G^{C_X}_n$.

Each element g in $G^{C_X}_n$ can be represented by a matrix of the shape

$$\left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right)$$

such that A and C are square matrices of dimension $n-r$ and $r+1$ respectively. We may now define an action of $G^{C_X}_n$ on V as follows. For each

$$y = [a_0 : \cdots : a_{n-r-1} : 0 : \cdots : 0] \in V,$$

the new action $g * y$ of g on y is defined as

$$g * y = \pi(g.y),$$

where $g.y$ represents the standard action of G_n on $\mathbb{P}^n$. With the representative above, this action is just defined as

$$g * [a_0 : \cdots : a_{n-r-1} : 0 : \cdots : 0] = [b_0 : \cdots : b_{n-r-1} : 0 : \cdots : 0],$$

where

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-r-1} \end{pmatrix} = A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-r-1} \end{pmatrix}.$$

Thanks to this action, and by identifying V with $\mathbb{P}^{n-r-1}$, we obtain a group morphism $\rho : G_n^{C_X} \to G_{n-r-1}$.

On the one hand, we note that an element $g \in G_n^{C_X}$ belongs to G_n^X if and only if $\rho(g) \in G_{n-r-1}^Y$. On the other hand, an element $g \in G_n^{C_X}$ belongs to $\operatorname{Ker}\rho$ if and only if it can be represented by a matrix of the shape

$$\left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline B & C \end{array}\right)$$

Hence $\dim \operatorname{Ker}\rho = (r+1)(n+1)$. As we can see that $\operatorname{Ker}\rho \subseteq G_n^X$, we obtain that

$$\dim G_n^X = \dim G_{n-r-1}^Y + (r+1)(n+1).$$

Finally, by applying Theorem 1.4 to $Y \subseteq V$, we get

$$\dim G_n^X \leq \frac{(n-r-1)(n-r)}{2} - \frac{(c-1)(c+4)}{2} + (r+1)(n+1).$$

□

3. Proof of the main theorem

Definition 3.1. Let X be a uniruled projective manifold. An irreducible component $\mathcal{K}$ of the space of rational curves on X is called *a minimal rational component* if the subscheme $\mathcal{K}_x$ of $\mathcal{K}$ parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by $\mathcal{K}$ will be called *minimal rational curves.* Let $\rho : \mathcal{U} \to \mathcal{K}$ be the universal family and $\mu : \mathcal{U} \to X$ the evaluation map. The tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$ is defined by $\tau(u) = [T_{\mu(u)}(\mu(\rho^{-1}\rho(u)))] \in \mathbb{P}T_{\mu(u)}(X)$. The closure $\mathcal{C} \subset \mathbb{P}T(X)$ of its image is the *VMRT-structure* on X. The natural projection $\mathcal{C} \to X$ is a proper surjective morphism and a general fiber $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is called the VMRT at the point $x \in X$. The VMRT-structure $\mathcal{C}$ is *locally flat* if for a general $x \in X$, there exists an analytical open subset U of X containing x with an open immersion $\phi : U \to \mathbb{C}^n, n = \dim X$, and a projective subvariety $Y \subset \mathbb{P}^{n-1}$ with $\dim Y = \dim \mathcal{C}_x$ such that $\phi_* : \mathbb{P}T(U) \to \mathbb{P}T(\mathbb{C}^n)$ maps $\mathcal{C}|_U$ into the trivial fiber subbundle $\mathbb{C}^n \times Y$ of the trivial projective bundle $\mathbb{P}T(\mathbb{C}^n) = \mathbb{C}^n \times \mathbb{P}^{n-1}$.

Examples 3.2. An irreducible Hermitian symmetric space of compact type (IHSS for short) is a homogeneous space $M = G/P$ with a simple Lie group G and a maximal parabolic subgroup P such that the isotropy representation of P on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at x. The following table (see [FH1, Section 3.1]) collects basic information on these varieties.

Type	I.H.S.S. M	VMRT S	$S \subset \mathbb{P}T_x(M)$	$\dim \mathfrak{aut}(M)$	$\dim \mathfrak{aut}(S)$
I	$\mathrm{Gr}(a, a+b)$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	Segre	$(a+b)^2 - 1$	$a^2 + b^2 - 2$
II	$\mathbb{S}_r$	$\mathrm{Gr}(2, r)$	Plücker	$r(2r-1)$	$r^2 - 1$
III	$\mathrm{Lag}(2r)$	$\mathbb{P}^{r-1}$	Veronese	$r(2r+1)$	$r^2 - 1$
IV	$\mathbb{Q}^r$	$\mathbb{Q}^{r-2}$	Hyperquadric	$(r+1)(r+2)/2$	$(r-1)r/2$
V	$\mathbb{OP}^2$	$\mathbb{S}_5$	Spinor	78	45
VI	$E_7/(E_6 \times U(1))$	$\mathbb{OP}^2$	Severi	133	78

Lemma 3.3. *Let M be an IHSS of dimension n different from $\mathbb{P}^n$ and $S \subset \mathbb{P}^{n-1}$ its VMRT at a general point. Then*

(1) $\dim \mathfrak{aut}(M) \leq \frac{n(n+1)}{2}$ *unless $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ or the natural embedding of $\mathbb{Q}^{n-2} \subset \mathbb{P}^{n-1}$.*

(2) *The equality holds if and only if $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the second Veronese embedding of $\mathbb{P}^2$.*

Proof. For Type (I), we have $M = \mathrm{Gr}(a, a+b)$, $n = \dim M = ab$ and $\dim \mathfrak{aut}(M) = (a+b)^2 - 1$. As M is not a projective space, we may assume $b \geq a \geq 2$. Then the inequality $(a+b)^2 - 1 \geq \frac{ab(ab+1)}{2}$ is equivalent to $3ab + 2 \geq (a^2-2)(b^2-2)$, which holds if and only if $(a, b) = (2, 2)$ or $(2, 3)$. In both cases, the inequality is strict.

For Type (II), we have $M = \mathbb{S}_r$, $n = \dim M = r(r-1)/2$ and $\dim \mathfrak{aut}(M) = r(2r-1)$. We may assume $r \geq 5$ as $\mathbb{S}_4 \simeq \mathbb{Q}^6$. Then one checks that $\dim \mathfrak{aut}(M) < \frac{n(n+1)}{2}$.

For Type (III), we have $M = \mathrm{Lag(2r)}$, $n = \dim M = r(r+1)/2$ and $\dim \mathfrak{aut}(M) = r(2r+1)$. We may assume $r \geq 3$ as $\mathrm{Lag}(4) \simeq \mathbb{Q}^3$. Then one checks that $\dim \mathfrak{aut}(M) \leq \frac{n(n+1)}{2}$, with equality if and only if $r = 3$. In this case, $S \subset \mathbb{P}^5$ is the second Veronese embedding of $\mathbb{P}^2$.

For type (IV), we have $M = \mathbb{Q}^r$ and $\dim \mathfrak{aut}(M) = (r+1)(r+2)/2$, which does not satisfy $\dim \mathfrak{aut}(M) \leq \frac{r(r+1)}{2}$.

For types (V) and (VI), it is obvious that $\dim \mathfrak{aut}(M) \leq \frac{n(n+1)}{2}$. □

Definition 3.4. Let V be a complex vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie subalgebra. The *k-th prolongation* (denoted by $\mathfrak{g}^{(k)}$) of $\mathfrak{g}$ is the space of symmetric multi-linear homomorphisms $A : \mathrm{Sym}^{k+1} V \to V$ such that for any fixed $v_1, \cdots, v_k \in V$, the endomorphism $A_{v_1,\ldots,v_k} : V \to V$ defined by

$$v \in V \mapsto A_{v_1,\ldots,v_k,v} := A(v, v_1, \cdots, v_k) \in V$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(k)} = \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V) \cap \mathrm{Hom}(\mathrm{Sym}^k V, \mathfrak{g})$.

It is shown in [HM2, Theorem 1.1.2] that for a smooth non-degenerate variety $C \subsetneq \mathbb{P}^{n-1}$, the second prolongation satisfies $\mathfrak{aut}(\hat{C})^{(2)} = 0$.

Examples 3.5. Fix two integers $k \geq 2, m \geq 1$. Let Σ be an $(m+2k)$-dimensional vector space endowed with a skew-symmetric 2-form ω of maximal rank. The symplectic Grassmannian $M = \mathrm{Gr}_\omega(k, \Sigma)$ is the variety of all k-dimensional isotropic subspaces of Σ, which is not homogeneous if m is odd. Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Let $\mathbf{t}$ be the tautological line bundle over $\mathbb{P}W$. The VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ of $\mathrm{Gr}_\omega(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$ over $\mathbb{P}W$ with the projective embedding given by the complete linear system

$$H^0(\mathbb{P}W, (Q \otimes \mathbf{t}^*) \oplus (\mathbf{t}^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \mathrm{Sym}^2 W^*.$$

By [FH1, Proposition 3.8], we have $\mathfrak{aut}(\hat{\mathcal{C}}_x) \simeq (W^* \otimes Q) \rtimes (\mathfrak{gl}(W) \oplus \mathfrak{gl}(Q))$ and $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \simeq \mathrm{Sym}^2 W^*$. This gives that

$$\dim \mathfrak{aut}(\hat{\mathcal{C}}_x) = m^2 + k^2 + km \text{ and } \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} = k(k+1)/2.$$

For the VMRT-structure $\mathcal{C}$ on X, we denote by $\mathfrak{aut}(\mathcal{C}, x)$ the Lie algebra of infinitesimal automorphisms of $\mathcal{C}$, which consists of germs of vector fields whose local flow preserves $\mathcal{C}$ near x. Note that the action of $\mathrm{Aut}^0(X)$ on X sends minimal rational curves to minimal rational curves, hence it preserves the VMRT structure, which gives a natural inclusion $\mathfrak{aut}(X) \subset \mathfrak{aut}(\mathcal{C}, x)$ for $x \in X$ general.

The following result is a combination of Propositions 5.10, 5.12, 5.14 and 6.13 in [FH1].

Proposition 3.6. *Let X be an n-dimensional smooth Fano variety of Picard number one. Assume that the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is smooth irreducible and non-degenerate. Then*

$$\dim \mathfrak{aut}(X) \leq n + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)}.$$

The equality holds if and only if the VMRT structure $\mathcal{C}$ is locally flat, or equivalently if and only if X is an equivariant compactification of $\mathbb{C}^n$.

For the 10-dimensional spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$, it is covered by a family of $\mathbb{P}^4$'s parameterized by its dual variety. A $\mathbb{P}^4$-general linear section of $\mathbb{S}_5$ is $\mathbb{S}_5 \cap L$, where L is a general linear subspace of $\mathbb{P}^{15}$ containing a $\mathbb{P}^4$ on $\mathbb{S}_5$. By [FH2, Corollary 2.8, Proposition 2.9], $\mathbb{P}^4$-general linear sections of $\mathbb{S}_5$ of codimension ≤ 3 are smooth.

We also recall the following result of [FH2, Theorem 7.5].

Theorem 3.7. *Let $S \subsetneq \mathbb{P}V$ be an irreducible smooth non-degenerate variety such that $\mathfrak{aut}(\widehat{S})^{(1)} \neq 0$. Then $S \subset \mathbb{P}V$ is projectively equivalent to one in the following list.*

(1) *The VMRT of an irreducible Hermitian symmetric space of compact type of rank ≥ 2.*
(2) *The VMRT of a symplectic Grassmannian.*
(3) *A smooth linear section of* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ *of codimension ≤ 2.*
(4) *A $\mathbb{P}^4$-general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of codimension ≤ 3.*
(5) *Biregular projections of (1) and (2) with nonzero prolongation, which are completely described in Section 4 of* [FH1].

By using this classification, we prove the following proposition.

Proposition 3.8. *Let $S \subsetneq \mathbb{P}V$ be an irreducible smooth non-degenerate variety such that $\mathfrak{aut}(\widehat{S})^{(1)} \neq 0$. Let $n = \dim V$. Then*

(a) *we have* $\dim \mathfrak{aut}(\hat{S}) + \dim \mathfrak{aut}(\hat{S})^{(1)} \leq \frac{n(n-1)}{2}$ *unless $S \subset \mathbb{P}V$ is projectively equivalent to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ or the natural embedding of $\mathbb{Q}^{n-2} \subset \mathbb{P}^{n-1} (n \geq 3)$.*
(b) *The equality holds if and only if $S \subset \mathbb{P}V$ is projectively equivalent to the second Veronese embedding of $\mathbb{P}^2$ or a general hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.*

Proof. Consider case (1) in Theorem 3.7. Let M be an IHSS and $S \subset \mathbb{P}V$ its VMRT at a general point. As the VMRT structure is locally flat, we have $\dim \mathfrak{aut}(\hat{S}) + \dim \mathfrak{aut}(\hat{S})^{(1)} = \dim \mathfrak{aut}(M) - \dim M = \dim \mathfrak{aut}(M) - n$ by Proposition 3.6. Now the claim follows from Lemma 3.3.

Consider case (2) in Theorem 3.7. By Example 3.5, we have

$$\dim \mathfrak{aut}(\hat{S}) + \dim \mathfrak{aut}(\hat{S})^{(1)} = m^2 + k^2 + km + k(k+1)/2$$

with $k \geq 2$ and $n = km + k(k+1)/2$. Note that $n \geq km + 3$. Assume first that $m \geq 2$, then we have $m^2 + k^2 < (km+3)km/2 \leq n(n-3)/2$, which gives the claim. Now assume $m = 1$, then it is easy to check that $1 + k^2 \leq n(n-3)/2$ with equality if and only if $(k, m) = (2, 1)$. By [FH1, Lemma 3.6], this implies that $S \subset \mathbb{P}V$ is projectively equivalent to a general hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

Consider case (3) in Theorem 3.7. If S is the hyperplane section of $\mathrm{Gr}(2,5)$, then we have $\dim \mathfrak{aut}(\hat{S}) = 16$ and $\dim \mathfrak{aut}(\hat{S})^{(1)} = 5$ by Proposition 3.11 and Proposition 3.12 of [FH1]. Now assume that S is a codimension 2 linear section of $\mathrm{Gr}(2,5)$, then $\dim \mathfrak{aut}(\hat{S}) = 9$ and $\dim \mathfrak{aut}(\hat{S})^{(1)} = 1$ by [BFM, Lemma 4.6]. The claim follows immediately.

Consider case (4) in Theorem 3.7. If S is the hyperplane section of $\mathbb{S}_5$, then $\dim \mathfrak{aut}(\hat{S}) = 31$ and $\dim \mathfrak{aut}(\hat{S})^{(1)} = 7$ by Proposition 3.9 and Proposition 3.10 of [FH1]. Now assume that S_k is a $\mathbb{P}^4$-general linear section of $\mathbb{S}_5$ of codimension $k = 2, 3$. By Proposition 4.7 and Proposition 4.11 of [BFM], we have $\dim \mathfrak{aut}(\hat{S_2}) = 19$ and $\dim \mathfrak{aut}(\hat{S_3}) = 12$. By [HM2, Theorem 1.1.3], we have $\dim \mathfrak{aut}(\hat{S})^{(1)} \leq \dim V^*$, hence $\dim \mathfrak{aut}(\hat{S_2})^{(1)} \leq 14$. By a similar argument as [BFM, Lemma 4.6], we have $\dim \mathfrak{aut}(\hat{S_3})^{(1)} = 1$. Now the claim follows immediately.

The case (5) will follow from Proposition 3.9 below. □

Proposition 3.9. *Let $S \subsetneq \mathbb{P}V$ be an irreducible linearly-normal non-degenerate smooth variety such that $\mathfrak{aut}(\widehat{S})^{(1)} \neq 0$. Let $L \subset \mathbb{P}V$ be a linear subspace such that the linear projection $p_L : \mathbb{P}V \dashrightarrow \mathbb{P}(V/L)$ maps S isomorphically to $p_L(S)$. Assume that $\mathfrak{aut}(p_L(S))^{(1)} \neq 0$. Then $\dim \mathfrak{aut}(\widehat{p_L(S)}) + \dim \mathfrak{aut}(\widehat{p_L(S)})^{(1)} < \frac{\ell(\ell-1)}{2}$, where $\ell = \dim(V/L)$.*

Proof. By [FH1, Section 4], $S \subset \mathbb{P}V$ is one of the followings: VMRT of IHSS of type (I), (II), (III) or VMRT of the symplectic Grassmanninans. We will do a case-by-case check based on the computations in [FH1, Section 4].

Consider the case of VMRT of IHSS of type (I). Let A, B be two vector spaces of dimension $a \geq 2$ (resp. $b \geq 2$). Then $S \simeq \mathbb{P}A^* \times \mathbb{P}B \subset \mathbb{P}\mathrm{Hom}(A, B)$. For a linear subspace $L \subset \mathrm{Hom}(A, B)$, we define $\mathrm{Ker}(L) = \cap_{\phi \in L} \mathrm{Ker}(\phi)$ and $\mathrm{Im}(L) \subset B$ the linear span of $\cup_{\phi \in L} \mathrm{Im}(\phi)$. By [FH1, Proposition 4.10], we have $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \simeq \mathrm{Hom}(B/\mathrm{Im}(L), \mathrm{Ker}(L))$. As p_L is an isomorphism from S to $p_L(S)$, $\mathbb{P}L$ is disjoint from $\mathrm{Sec}(S)$, hence elements in L have rank at least 3. This implies that $\dim \mathfrak{aut}(\widehat{p_L(S)})^{(1)} \leq (b-3)(a-3)$ and $a, b \geq 4$. As $\mathfrak{aut}(\widehat{p_L(S)}) \subset \mathfrak{aut}(\hat{S}) = \mathfrak{gl}(A^*) \oplus \mathfrak{gl}(B)$, we obtain

$$\dim \mathfrak{aut}(\widehat{p_L(S)}) + \dim \mathfrak{aut}(\widehat{p_L(S)})^{(1)} < a^2 + b^2 + (b-3)(a-3).$$

Note that $\ell = \dim \mathrm{Hom}(A, B) - \dim L \geq 3(a+b) - 9$ by [FH1, Proposition 4.10]. Now it is straightforward to check that $a^2 + b^2 + (b-3)(a-3) < \ell(\ell-1)/2$ since $a + b \geq 8$.

Consider the case of VMRT of IHSS of type (II), then $S = \mathrm{Gr}(2, W)$, where W is a vector space of dimension $r \geq 6$. Let $L \subset \wedge^2 W$ be a linear subspace. We denote by $\mathrm{Im}(L)$ the linear span of all $\cup_{\phi \in L} \mathrm{Im}(\phi)$, where $\phi \in L$ is regarded as an element in $\mathrm{Hom}(W^*, W)$. By [FH1, Proposition 4.11], we have $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \simeq \wedge^2(W/\mathrm{Im}(L))^*$. As p_L is biregular on S, the rank of an element in L is at least 5, hence $\dim \mathfrak{aut}(\widehat{p_L(S)})^{(1)} \leq (r-5)(r-6)/2$. On the other hand, we have $\ell = \dim \wedge^2 W - \dim L \geq 6r - 11$ by [FH1, Proposition 4.11]. As $\dim \mathfrak{aut}(\widehat{p_L(S)}) < r^2$, we check easily that $r^2 + (r-5)(r-6)/2 < \ell(\ell-1)/2$.

Consider the case of VMRT of IHSS of type (III), then $S = \mathbb{P}W$, where W is a vector space of dimension $r \geq 4$. By [FH1, Proposition 4.12], we have $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \simeq \mathrm{Sym}^2(W/\mathrm{Im}(L))^*$, which has dimension at most $(r-2)(r-3)/2$. On the other hand, we have $\ell \geq 3r-3$ by *loc. cit.*, hence we have $r^2 + (r-2)(r-3)/2 < \ell(\ell-1)/2$.

Now consider the VMRT of symplectic Grassmannians. We use the notations in Example 3.5 with $V = (W \otimes Q)^* \oplus \mathrm{Sym}^2 W^*$, where $\dim W = k \geq 2$ and $\dim Q = m$. By [FH1, Lemma 4.19], we have $k \geq 3$, hence $k + m \geq 4$. Let $L \subset V$ be a linear subspace, then by [FH1, Proposition 4.18], we have $\ell \geq 3(k+m) - 3$ and $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \simeq \mathrm{Sym}^2(W/\mathrm{Im}_W(L))^*$ which has dimension at most $k(k+1)/2$. As $\dim \mathfrak{aut}(\hat{S}) = k^2 + m^2 + km$, we have

$$\dim \mathfrak{aut}(\widehat{p_L(S)}) + \dim \mathfrak{aut}(\widehat{p_L(S)})^{(1)} < k^2 + m^2 + km + k(k+1)/2.$$

Put $s = k+m$, then $k^2+m^2+km+k(k+1)/2 = s^2 - km + k(k+1)/2 < 3s^2/2 - km < \ell(\ell-1)/2$ since $k + m \geq 4$. □

Now we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.6, we have

$$\dim \mathfrak{aut}(X) \leq n + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)}.$$

If $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} = 0$, then $\mathcal{C}_x \subset \mathbb{P}^{n-1}$ is not a hyperquadric, which implies that $\dim \mathfrak{aut}(\hat{\mathcal{C}}_x) \leq n(n-1)/2 - 2$ by Theorem 1.4. This gives that

$$\dim \mathfrak{aut}(X) \leq n + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) \leq n + n(n-1)/2 - 2 < n(n+1)/2.$$

Now assume $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$. If $\mathcal{C}_x = \mathbb{P}T_xX$ for general $x \in X$, then X is isomorphic to $\mathbb{P}^n$ by [CMSB]. If $\mathcal{C}_x \subset \mathbb{P}T_xX$ is a hyperquadric $\mathbb{Q}^{n-2}$, then $X \simeq \mathbb{Q}^n$ by [M]. If $\mathcal{C}_x \subset \mathbb{P}T_xX$ is isomorphic to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, then $X \simeq \mathrm{Gr}(2,5)$ by [M]. Now assume $\mathcal{C}_x \subset \mathbb{P}T_xX$ is not one of the previous varieties, then by Proposition 3.8 we have $\dim \mathfrak{aut}(X) \leq n + n(n-1)/2 = n(n+1)/2$, which proves claim (a) in Theorem 1.2.

Now assume the equality $\dim \mathfrak{aut}(X) = n(n+1)/2$ holds, then by Proposition 3.6, the VMRT structure is locally flat. By Proposition 3.8, $\mathcal{C}_x \subset \mathbb{P}T_xX$ is either the second Veronese embedding of $\mathbb{P}^2$ or a general hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2$. Note that these are also the VMRT of $\mathrm{Lag}(6)$ and a general hyperplane section of $\mathrm{Gr}(2,5)$, which have locally flat VMRT structure. This implies that X is isomorphic to $\mathrm{Lag}(6)$ or a general hyperplane section of $\mathrm{Gr}(2,5)$ respectively by the Cartan-Fubini extension theorem of [HM1]. □

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