ERGODIC INVARIANT STATES AND IRREDUCIBLE REPRESENTATIONS OF CROSSED PRODUCT C^*-ALGEBRAS

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ABSTRACT. Motivated by reformulating Furstenberg's $\times p, \times q$ conjecture via representations of a crossed product C^*-algebra, we show that in a discrete C^*-dynamical system (A, Γ), the space of (ergodic) Γ-invariant states on A is homeomorphic to a subspace of (pure) state space of $A \rtimes \Gamma$. Various applications of this in topological dynamical systems and representation theory are obtained. In particular, we prove that the classification of ergodic Γ-invariant regular Borel probability measures on a compact Hausdorff space X is equivalent to the classification a special type of irreducible representations of $C(X) \rtimes \Gamma$.

1. INTRODUCTION

Assume that p, q are two positive integers greater than 1 with $\frac{\log p}{\log q}$ irrational.

H. Furstenberg gives the classification of closed $\times p, \times q$-invariant subsets of the unit circle $\mathbb{T}$, which says such a set is either finite or $\mathbb{T}$ [? , Theorem IV.1.]. He also gives the following conjecture concerning the classification of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$.

Conjecture. [Furstenberg's $\times p, \times q$ conjecture]

An ergodic $\times p, \times q$-invariant Borel probability measure on $\mathbb{T}$ is either finitely supported or the Lebesgue measure.

Furstenberg's conjecture is the simplest case of conjectures concerning classifications of invariant measures, and there are vast literatures

about its general versions and their applications in number theory. See [**?**] for a survey.

For Furstenberg's conjecture, the best known result is the following theorem, which is proven by D. J. Rudolph under the assumption that p, q is coprime [**?** , Theorem 4.9.], later improved by A. S. A. Johnson [**?** , Theorem A].

Theorem. [Rudolph-Johnson's Theorem]

If μ is an ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$, then either $h_\mu(T_p) = h_\mu(T_q) = 0$ or μ is the Lebesgue measure.

Here $h_\mu(T_p)$ and $h_\mu(T_q)$ stand for the measure-theoretic entropy of $\times p$ and $\times q$ with respect to μ respectively. See [**?** , Chapter 4] for the definition of entropy for measure preserving maps.

For a $\times p, \times q$-invariant measure on $\mathbb{T}$, denote the two isometries on $L^2(\mathbb{T}, \mu)$ induced by continuous maps $\times p, \times q : \mathbb{T} \to \mathbb{T}$ by V_p, V_q.

By Rudolph-Johnson's Theorem, to classify ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$, it suffices to classify such ergodic measures with zero entropy for T_p or T_q.

J. Cuntz notices that when $h_\mu(T_p) = h_\mu(T_q) = 0$, the operators V_p and V_q are two commuting unitary operators on $L^2(\mathbb{T}, \mu)$ [**?** , Corollary 4.14.3].

For the unitary operator $M_z : L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \mu)$ given by $M_z f(z) = zf(z)$ for all $f \in L^2(\mathbb{T}, \mu)$ and $z \in \mathbb{T}$, one have $V_p M_z = M_z^p V_p$ and $V_q M_z = M_z^q V_q$. So a $\times p, \times q$ invariant measure μ with zero entropy gives rise to a representation π_μ of the universal unital C^*-algebra $C^*(s, t, z)$ generated by three unitaries s, t and z with relations

$$st = ts, \quad sz = z^p s, \quad tz = z^q t$$

in the following way:

$$\pi_\mu(s) = V_p, \quad \pi_\mu(t) = V_q, \quad \pi_\mu(z) = M_z.$$

With the above observation, Cuntz suggests that one can consider ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ via representations of $C^*(s, t, z) \cong C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$, where the two generators of $\mathbb{Z}^2$ acts on $C^*(\mathbb{Z}[\frac{1}{pq}])$ by automorphisms induced by $\times p, \times q$ maps on $[\frac{1}{pq}]$, and the isomorphism $\Phi : C^*(s, t, z) \to C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ is given by $\Phi(s) = a, \Phi(t) = b$ and $\Phi(z) = 1$. Here $a = (1, 0)$ and $b = (0, 1)$ are in $\mathbb{Z}^2$ and 1 is in $\mathbb{Z}[\frac{1}{pq}]$ [**?**].

Motivated by Cuntz's observation, firstly one have to answer the following question:

what kind of representation of $C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ is induced by a $\times p, \times q$-invariant measure on $\mathbb{T}$?

Denote the dual of $\mathbb{Z}[\frac{1}{pq}]$ by S_{pq}, the pq-solenoid [?, A.1]. The $\times p, \times q$ isomorphisms on $\mathbb{Z}[\frac{1}{pq}]$ give rise to $\times p, \times q$ isomorphisms on S_{pq}.

We answer the above question in the following way.

Firstly the space of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ is homeomorphic to the space of ergodic $\times p, \times q$-invariant measures on S_{pq}, hence the classification of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ amounts to classification of ergodic $\times p, \times q$-invariant measures on S_{pq}. Secondly ergodic $\times p, \times q$-invariant measures on S_{pq} 1-1 corresponds to irreducible representations of $C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ whose restriction to $\mathbb{Z}^2$ contains the trivial representation.

Moreover in a more general context, we prove the following which briefly shows how the problem of invariant states relates to crossed product C^*-algebras.

Assume that a discrete group Γ acts on a unital C^*-algebra A as automorphisms. Denote this action by α, which is, a group homomorphism from Γ to the automorphism group $Aut(A)$ of A.

A state φ on A is Γ-**invariant** if $\varphi(\alpha_s(a)) = \varphi(a)$ for all s in Γ and a in A. An extreme point of the set of Γ-invariant states on A [1] are called **ergodic**.

Denote by $A \rtimes \Gamma$ the full crossed product of the C^*-dynamical system (A, Γ, α).

Theorem ??. The space of (ergodic) Γ-invariant states on A is homeomorphic to the space of (pure) states on $A \rtimes \Gamma$ whose restriction to Γ is the trivial character.

We give some applications of Theorem **??** to topological dynamical systems and representation theory.

Suppose a discrete group Γ acts on a compact Hausdorff space X as homeomorphisms (this is the same as Γ acting on the unital C^*-algebra $C(X)$, the space of continuous functions on X, by automorphisms). For

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[1]This is a closed convex set when equipped with weak-$*$ topology, hence when nonempty, the set of extreme points is also nonempty.

a representation $\pi : C(X) \rtimes \Gamma \to B(H)$, denote the space of Γ-invariant vectors in H by H_Γ.

Theorem ??. Every irreducible representation π of $C(X) \rtimes \Gamma$ on a Hilbert space H satisfies that $\dim H_\Gamma \leq 1$. When $\dim H_\Gamma = 1$, the representation π is uniquely induced by an ergodic Γ-invariant regular Borel probability measure μ on X.

A special case of Theorem **??** is the following.

Corollary ??. Suppose a discrete group Γ acts on a discrete abelian group G by group automorphisms.

Every irreducible unitary representation π of $G \rtimes \Gamma$ on a Hilbert space H satisfies that $\dim H_\Gamma \leq 1$.

When $\dim H_\Gamma = 1$, the representation π is uniquely induced by an ergodic Γ-invariant regular Borel probability measure μ on the Pontryagin dual $\widehat{G}$ of G.

The paper is organized as follows.

In the preliminary section, we recall some background of crossed product C^*-algebras. The proof of Theorem **??** is given in section 3.2. At the end of this section, we include two immediate applications of Theorem **??**, namely, Proposition **??** and Proposition **??**, to C^*-dynamical systems. In section 3.3, we show Theorem **??** and Theorem **??**. In the last section we prove Theorem **??** which enables us to reformulate Furstenberg's $\times p, \times q$ problem in terms of representation theory of the semidirect product group $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$.

2. Preliminary

In this section, we list some background for C^*-dynamical systems.

Within this article Γ stands for a discrete group and A stands for a unital C^*-algebra whose state space and pure state space are denoted by $S(A)$ and $P(A)$ respectively.

Denote the GNS representation of A with respect to a $\varphi \in S(A)$ by $\pi_\varphi : A \to B(L^2(A, \varphi))$ where $L^2(A, \varphi)$ stands for the Hilbert space corresponding to π_φ. Let $I_\varphi = \{a \in A | \varphi(a^*a) = 0\}$. Denote $a + I_\varphi$ by $\hat{a}$ for all $a \in A$.

Definition 2.1. An action of Γ on A as automorphisms is a group homomorphism $\alpha : \Gamma \to Aut(A)$, where $Aut(A)$ stands for the set of $*$-isomorphisms from A to A (this is a group under composition). We call (A, Γ, α) a dynamical system.

A Γ-**invariant state** is a state φ on A such that $\varphi(\alpha_s(a)) = \varphi(a)$ for all $s \in \Gamma$ and $a \in A$ [?]. Denote the set of Γ-invariant states on A by $S_\Gamma(A)$. It is clear that $S_\Gamma(A)$ is a convex closed set under weak-$*$ topology. If $S_\Gamma(A)$ is nonempty, then it contains at least one extreme point. We call an extreme point of $S_\Gamma(A)$ an **ergodic Γ-invariant state** on A. The set of ergodic Γ-invariant states on A is denoted by $E_\Gamma(A)$.

A representation of a C^*-algebra B on a Hilbert space H is a $*$-homomorphism $\pi : B \to B(H)$ and it is called irreducible if the commutant $C(\pi(B))$ consisting of elements in $B(H)$ commuting with every element in $\pi(B)$ are scalar multiples of identity operator.

A **covariant** representation (π, U, H) of a dynamical system (A, Γ, α) consists of a representation π of A and a unitary representation U of Γ on a Hilbert space H such that

$$\pi(\alpha_s(a)) = \mathrm{U}_s\pi(a)\mathrm{U}_s^*$$

for all $a \in A$ and $s \in \Gamma$.

Let $C_c(\Gamma, A)$ be the space of finitely supported A-valued functions on Γ. For $f, g \in C_c(\Gamma, A)$, the product $f * g$ is given by

$$f * g(t) = \sum_{s_1 s_2 = t} f(s_1) \alpha_{s_1}(g(s_2))$$

and f^* is given by

$$f^*(t) = \alpha_t(f(t^{-1})^*)$$

for every $t \in \Gamma$. Then $C_c(\Gamma, A)$ is a $*$-algebra . Given a covariant representation (π, U, H) of a dynamical system (A, Γ, α), one can construct a representation of $C_c(\Gamma, A)$ on H.

Definition 2.2. For a dynamical system (A, Γ, α), the **crossed product C^*-algebra** $A \rtimes \Gamma$ is the completion of $C_c(\Gamma, A)$ under the norm $\|f\| = \sup \|\tilde{\pi}(f)\|$ for $f \in C_c(\Gamma, A)$ where the supreme is taken over all representations of $C_c(\Gamma, A)$. Denote by u_s the unitary in $A \rtimes \Gamma$ corresponding to an $s \in \Gamma$.

There is a one-one correspondence between representations of $A \rtimes \Gamma$ and covariant representations of (A, Γ, α).

We refer readers to [? , Chapter VIII] for more about discrete crossed products.

3. MAIN RESULTS

3.1. **Covariant representations of (A, Γ, α) induced by invariant states.**

If $\varphi \in S_\Gamma(A)$, then there is a unitary representation (the Koopman representation) U_φ of Γ on $L^2(A, \varphi)$ given by

$$\mathrm{U}_\varphi(s)(\hat{a}) = \widehat{\alpha_s(a)}$$

for all $s \in \Gamma$ and $a \in A$ [? ? ?]. We give details below for completeness.

The representation U_φ is well-defined since

(1) For every $a \in I_\varphi$, we have

$$\varphi(\alpha_s(a)^* \alpha_s(a)) = \varphi(\alpha_s(a^*a)) = \varphi(a^*a) = 0$$

for all $s \in \Gamma$.

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(2) For every $s \in \Gamma$, the map $U_\varphi(s)$ is surjective since its image is dense in $L^2(A, \varphi)$, and $U_\varphi(s)$ is an isometry since for all $a \in A$,

$$\langle \widehat{\alpha_s(a)}, \widehat{\alpha_s(a)} \rangle = \varphi(\alpha_s(a)^* \alpha_s(a)) = \varphi(\alpha_s(a^*a)) = \varphi(a^*a) = \langle \hat{a}, \hat{a} \rangle.$$

So $U_\varphi(s)$ is a unitary.

(3) $U_\varphi(st)(\hat{a}) = \widehat{\alpha_{st}(a)} = U_s(U_t(\hat{a}))$ for all $s, t \in \Gamma$ and $a \in A$.

Hence U_φ is a unitary representation of Γ on $L^2(A, \varphi)$.

Furthermore we have the following.

Lemma 3.1. Given a Γ-invariant state φ on A, the triple $(\pi_\varphi, U_\varphi, L^2(A, \varphi))$ gives a covariant representation of (A, Γ, α). So there is a representation of $A \rtimes \Gamma$ on $L^2(A, \varphi)$, which we denote by ρ_φ, given by

$$\rho_\varphi(\sum_{s \in \Gamma} a_s u_s) = \sum_{s \in \Gamma} \pi_\varphi(a_s) U_\varphi(s)$$

for any $\sum_{s \in \Gamma} a_s u_s \in C_c(\Gamma, A)$.

Proof. For all $a, b \in A$ and $s \in \Gamma$, we have

$$U_\varphi(s)(\pi_\varphi(a))U_\varphi(s^{-1})(\hat{b}) = U_\varphi(s)(\pi_\varphi(a))((\alpha_{s^{-1}}(b))^\wedge)$$
$$= U_\varphi(s)((a\alpha_{s^{-1}}(b))^\wedge) = \alpha_s((a\alpha_{s^{-1}}(b))^\wedge) = (\alpha_s(a)\alpha_{ss^{-1}}(b))^\wedge$$
$$= (\alpha_s(a)b)^\wedge = \pi_\varphi(\alpha_s(a))(\hat{b}).$$

This completes the proof. $\qquad\square$

3.2. Γ-invariant states on A and states on $A \rtimes \Gamma$.

Denote $\{\varphi \in S(A \rtimes \Gamma) \mid \varphi(u_s) = 1 \text{ for all } s \in \Gamma\}$ by $S^1(A \rtimes \Gamma)$ and $\{\psi \in P(A \rtimes \Gamma) \mid \psi(u_s) = 1 \text{ for all } s \in \Gamma\}$ by $P^1(A \rtimes \Gamma)$.

We have the following.

Theorem 3.2. When equipped with weak-$*$ topologies, the restriction maps $R : S_\Gamma(A) \to S^1(A \rtimes \Gamma)$ and $R : E_\Gamma(A) \to P^1(A \rtimes \Gamma)$ are homoemorphisms.

To prove this theorem, we first prove the following lemma which says that for every φ in $S^1(A \rtimes \Gamma)$, the restriction $\varphi|_A$ belongs to $S_\Gamma(A)$.

Lemma 3.3. For any state φ on $A \rtimes \Gamma$ such that $\varphi(u_s) = 1$ for every $s \in \Gamma$, we have $\varphi(u_s a u_t) = \varphi(a)$ for all $a \in A$ and $s, t \in \Gamma$. Consequently the restriction $\varphi|_A$ is a Γ-invariant state on A.

Proof. The proof follows from [**?** , Proposition 1.5.7.] since by assumption every u_s is contained in the multiplicative domain of φ.

For convenience of readers, we give a direct proof which is also based on Cauchy-Schwarz inequality like [**?** , Proposition 1.5.7.].

For all $a \in A$ and $s, t \in \Gamma$, we have

$$|\varphi(u_s a u_t) - \varphi(a)| \leq |\varphi(u_s a u_t) - \varphi(a u_t)| + |\varphi(a u_t) - \varphi(a)|$$
$$= |\varphi((u_s - 1) a u_t)| + |\varphi(a(u_t - 1))|$$
$$= |\langle a u_t, (u_s - 1)^* \rangle_{L^2(A \rtimes \Gamma, \varphi)}| + |\langle u_t - 1, a^* \rangle_{L^2(A \rtimes \Gamma, \varphi)}|$$

(Cauchy-Schwarz Inequality)

$$\leq [\varphi((u_t)^* a^* a u_t)]^{\frac{1}{2}} [\varphi((u_s - 1)(u_s - 1)^*)]^{\frac{1}{2}} + [\varphi((u_t - 1)^*(u_t - 1))]^{\frac{1}{2}} [\varphi(a a^*)]^{\frac{1}{2}}$$

($\varphi(u_t) = 1$ implies that $\varphi((u_t - 1)(u_t - 1)^*) = 0$ and $\varphi((u_t - 1)^*(u_t - 1)) = 0$.)

$$= 0.$$

$\square$

It follows from Lemma **??** that for a Γ-invariant state φ, there is a representation ρ_φ of $A \rtimes \Gamma$ on $L^2(A, \varphi)$ given by

$$\rho_\varphi(\sum_{s \in \Gamma} a_s u_s) = \sum_{s \in \Gamma} \pi_\varphi(a_s) U_\varphi(s)$$

for every $\sum_{s \in \Gamma} a_s u_s \in C_c(\Gamma, A)$.

Proof. [Proof of Theorem **??**.]

By Lemma **??**, the restriction map $R : S^1(A \rtimes \Gamma) \to S_\Gamma(A)$ given by $R(\varphi) = \varphi|_A$ for every $\varphi \in S^1(A \rtimes \Gamma)$, is well-defined. Since A is a C^*-subalgebra of $A \rtimes \Gamma$, the map R is continuous under weak-$*$ topology.

If $R(\varphi_1) = R(\varphi_2)$ for $\varphi_1, \varphi_2 \in S^1(A \rtimes \Gamma)$, then $\varphi_1(a) = \varphi_2(a)$ for all $a \in A$. By Lemma **??**,

$$\varphi_1(a u_s) = \varphi_2(a u_s),$$

for all $a \in A$ and $s \in \Gamma$. Since every element in $C_c(\Gamma, A)$ is a linear combination of $a u_s$ and $C_c(\Gamma, A)$ is a dense subspace of $A \rtimes \Gamma$. Hence $\varphi_1 = \varphi_2$ and R is injective.

Moreover given a $\varphi \in S_\Gamma(A)$. By Lemma **??**, φ gives a representation ρ_φ of $A \rtimes \Gamma$ on $L^2(A, \varphi)$. Let $\tilde{\varphi}$ be the state of $A \rtimes \Gamma$ given by $\tilde{(\varphi)}(b) = \langle \rho_\varphi(b)(\hat{1}), \hat{1} \rangle$ for all $b \in A \rtimes \Gamma$. By the definition of ρ_φ, we see that $\tilde{\varphi} \in S^1(A \rtimes \Gamma)$ and $\tilde{\varphi}|_A = \varphi$. This shows the surjectivity of R.

Above all R is a bijective continuous map between two compact Hausdorff spaces $S^1(A \rtimes \Gamma)$ and $S_\Gamma(A)$. Therefore R is a homeomorphism.

Note that R is an affine map between two convex spaces $S^1(A \rtimes \Gamma)$ and $S_\Gamma(A)$, so the set of extreme points of $S^1(A \rtimes \Gamma)$ is homeomorphic to $E_\Gamma(A)$.

Suppose that φ is an extreme point of $S^1(A \rtimes \Gamma)$ and $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ for two states φ_1, φ_2 on $A \rtimes \Gamma$ and some $0 < \lambda < 1$. Then $1 = \varphi(u_s) = \lambda\varphi_1(u_s) + (1 - \lambda)\varphi_2(u_s)$ for every $s \in \Gamma$. It follows that $\varphi_1(u_s) = \varphi_2(u_s) = 1$, that is, $\varphi_1, \varphi_2 \in S^1(A \rtimes \Gamma)$. Hence $\varphi = \varphi_1 = \varphi_2$ and φ is a pure state. $\qquad \square$

For a character ξ on Γ (a group homomorphism from Γ to $\mathbb{T}$), denote $\{\varphi \in S(A \rtimes \Gamma)|\varphi(u_s) = \xi(s)\,\text{for all}\,s \in \Gamma\}$ by $S^\xi(A \rtimes \Gamma)$ and $\{\varphi \in P(A \rtimes \Gamma)|\varphi(u_s) = \xi(s)\,\text{for all}\,s \in \Gamma\}$ by $P^\xi(A \rtimes \Gamma)$.

For a representation of $A \rtimes \Gamma$ on a Hilbert space H, define $H_\xi = \{x \in H|\pi(u_s)(x) = \xi(s)x\,\text{for all}\,s \in \Gamma\}$.

We have the following improvement of Theorem **??**.

Corollary 3.4. Let ξ be a character on Γ. When equipped with weak-$*$ topologies, $S_\Gamma(A) \cong S^\xi(A \rtimes \Gamma)$ and $E_\Gamma(A) \cong P^\xi(A \rtimes \Gamma)$.

Proof. By Theorem **??**, $S_\Gamma(A) \cong S^1(A \rtimes \Gamma)$. Note that $S^1(A \rtimes \Gamma) \cong S^\xi(A \rtimes \Gamma)$ $(P^1(A \rtimes \Gamma) \cong P^\xi(A \rtimes \Gamma)$ follows) and the homeomorphism is induced by the isomorphism $\Lambda : A \rtimes \Gamma \to A \rtimes \Gamma$ given by $\Lambda(\sum_{s \in \Gamma} a_s u_s) = \sum_{s \in \Gamma} \xi(s)a_s u_s$ for all $\sum_{s \in \Gamma} a_s u_s \in C_c(\Gamma, A)$. $\qquad \square$

For a representation $\pi : A \rtimes \Gamma \to B(H)$, denote $\{x \in H|\pi(u_s)(x) = x\,\text{for all}\,s \in \Gamma\}$ by H_Γ.

Proposition 3.5. For a C^*-dynamical system (A, Γ, α), the following are equivalent.

(1) The set $S_\Gamma(A)$ is nonempty.

(2) The canonical homomorphism $C^*(\Gamma) \to A \rtimes \Gamma$ is an embedding.

(3) There exists a representation $\pi : A \rtimes \Gamma \to B(H)$ such that $H_\Gamma \neq 0$, or equivalently, there exists a covariant representation (π, U, H) of (A, Γ, α) such that U contains the trivial representation of Γ.

Proof. $(1) \Longrightarrow (2)$.

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Take a $\varphi \in S_\Gamma(A)$. Let Γ act on $\mathbb{C}$ trivially. By the invariance, the map $\varphi : A \to \mathbb{C}$ is a Γ-equivariant contractive completely positive map. By [? , Exercise 4.1.4], there exists a contractive completely positive map $\tilde{\varphi} : A \rtimes \Gamma \to C^*(\Gamma)$ such that $\tilde{\varphi}(\sum_{s \in \Gamma} a_s u_s) = \sum_{s \in \Gamma} \varphi(a_s) u_s$. Immediately one can check that the composition of maps

$$C^*(\Gamma) \to A \rtimes \Gamma \underset{\tilde{\varphi}}{\to} C^*(\Gamma)$$

is the identity map. Hence the canonical homomorphism $C^*(\Gamma) \to A \rtimes \Gamma$ is an embedding.

$(2) \Longrightarrow (1)$.

By Theorem ??, it suffices to show $S^1(A \rtimes \Gamma)$ is nonempty.

Let $\pi_0 : \Gamma \to \mathbb{C}$ be the trivial unitary representation of Γ on $\mathbb{C}$. Then π_0 is a state on $C^*(\Gamma)$ such that $\pi_0(u_s) = 1$ for every $s \in \Gamma$. Note that $C^*(\Gamma)$ is a Banach subspace of $A \rtimes \Gamma$. By the Hahn-Banach Theorem, one can extend π_0 to a bounded linear functional φ on $A \rtimes \Gamma$ without changing its norm [? , Corollary 6.5]. Hence $\|\varphi\| = \|\pi_0\| = 1 = \pi_0(1) = \varphi(1)$. So φ is a state on $A \rtimes \Gamma$ [? , Theorem 4.3.2], and satisfies that $\varphi(u_s) = 1$ for all $s \in \Gamma$.

$(1) \Longrightarrow (3)$.

This is guaranteed by Lemma ??.

$(3) \Longrightarrow (1)$.

Take a unit vector x in H_Γ and define a state φ on $A \rtimes \Gamma$ by $\varphi(b) = \langle \pi(b)x, x \rangle$ for all $b \in A \rtimes \Gamma$. It follows that $\varphi \in S^1(A \rtimes \Gamma)$. □

Remark 3.6. The equivalence of (1) and (2) may be well-known. When A is commutative and Γ is locally compact, this is mentioned in [? , Remark 7.5]. To the best of our knowledge, it does not appear elsewhere in the literature.

Notice that U_φ gives rise to an action of Γ on $B(L^2(A, \varphi))$, also denoted by α for convenience, defined by

$$\alpha_s(T) = U_\varphi(s)TU_\varphi(s^{-1})$$

for every $T \in B(L^2(A, \varphi))$ and $s \in \Gamma$.

Denote $\langle T(\hat{1}), \hat{1}\rangle$ by $\varphi(T)$ for all $T \in B(L^2(A, \varphi))$. When φ is a Γ-invariant state on A, it is also a Γ-invariant state on $B(L^2(A, \varphi))$ since

$$\varphi(\alpha_s(T)) = \varphi(\mathrm{U}_\varphi(s)T\mathrm{U}_\varphi(s^{-1})) = \langle \mathrm{U}_\varphi(s)T\mathrm{U}_\varphi(s^{-1})(\hat{1}), \hat{1}\rangle$$
$$= \langle T\mathrm{U}_\varphi(s^{-1})(\hat{1}), \mathrm{U}_\varphi(s^{-1})(\hat{1})\rangle = \langle T(\hat{1}), \hat{1}\rangle = \varphi(T)$$

for all $s \in \Gamma$.

We can also see that

(3.A) $$\alpha_s(\pi_\varphi(a)) = \pi_\varphi(\alpha_s(a))$$

for every $a \in A$ and $s \in \Gamma$.

We call a $T \in B(L^2(A, \varphi))$ is Γ-**invariant** if $\alpha_s(T) = T$ for all $s \in \Gamma$. Denote the set of Γ-invariant operators in $B(L^2(A, \varphi))$ by $B(L^2(A, \varphi))_\Gamma$. Let

$$\pi_\varphi(A)' = \{T \in B(L^2(A, \varphi)) \,|\, T\pi_\varphi(a) = \pi_\varphi(a)T \text{ for all } a \in A\}.$$

Proposition 3.7. A Γ-invariant state φ on A is ergodic iff $\varphi(T^*T) = |\varphi(T)|^2$ for every $T \in B(L^2(A, \varphi))_\Gamma \cap \pi_\varphi(A)'$.

Proof. Recall that $R : S^1(A \rtimes \Gamma) \to S_\Gamma(A)$ is the restriction map. For $\varphi \in S_\Gamma(A)$, denote $R^{-1}(\varphi)$ by ψ. By Theorem **??**, ψ is in $S^1(A \rtimes \Gamma)$.

Observe that $B(L^2(A, \varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$.

Again by Theorem **??**, the state φ on A is an ergodic Γ-invariant state iff ψ is a pure state on $A \rtimes \Gamma$ iff $\pi_\psi(A \rtimes \Gamma)' = \mathbb{C}$. The "only if" part follows immediately.

Now suppose $\varphi(T^*T) = |\varphi(T)|^2$ for every $T \in B(L^2(A, \varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$. A straightforward calculation shows that $T(\hat{1}) = \varphi(T)\hat{1}$. Then for every $a \in A$, we have

$$T(\hat{a}) = T\pi_\varphi(a)(\hat{1}) = \pi_\varphi(a)T(\hat{1})$$
$$= \pi_\varphi(a)(\varphi(T)\hat{1}) = \varphi(T)\hat{a}.$$

This means $T = \varphi(T)$, a scalar multiple of the identity operator. Hence $\pi_\psi(A \rtimes \Gamma)' = \mathbb{C}$ and ψ is a pure state. $\square$

Remark 3.8. The key observation $B(L^2(A, \varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$ in the proof is pointed out to us by Sven Raum.

3.3. Ergodic Γ-invariant states on A and irreducible representations of $A \rtimes \Gamma$.

We say a representation $\pi_1 : B \to B(H_1)$ of a C^*-algebra B is **unitarily equivalent** to a representation $\pi_2 : B \to B(H_2)$ if there exists a surjective isometry $U : H_1 \to H_2$ such that $U\pi_1(b)(x) = \pi_2(b)U(x)$ for all $b \in B$ and $x \in H_1$.

Theorem 3.9. A representation $\pi : A \rtimes \Gamma \to B(H)$ is unitarily equivalent to ρ

$$A \rtimes \Gamma$$

Without loss of generality we can assume $N > 0$. There exist nonnegative integers i, j, K, M such that $KN = Mp^i q^j$ with M coprime to pq. Then $\pi(z^{Mp^i q^j})y = \pi(z^{KN})y = y$.

Note that $\mathbb{Z}^2$ acts on $\mathbb{Z}[\frac{1}{pq}]$ by $\times p, \times q$, that is, $(m, n) \cdot z^k = z^{kp^m q^n}$ for all $m, n \in \mathbb{Z}$ and every $k \in \mathbb{Z}[\frac{1}{pq}]$. Since y is in $H_{\mathbb{Z}^2}$,

we have $\pi((i, j))\pi(z^M)y = \pi(z^{Mp^i q^j})\pi((i, j))y = \pi(z^{Mp^i q^j})y = y$, which implies $\pi(z^M)y = y$.

Secondly we prove that $\pi(\mathbb{Z}[\frac{1}{pq}])y = \pi(\mathbb{Z})y$.

For all nonnegative integers i, j, there exists an integer l such that $lp^i q^j = rM + 1$ since M is coprime to pq. Hence for every positive integer k, we have

$$\pi(z^{\frac{k}{p^i q^j}})y = \pi(z^{\frac{k(lp^i q^j - rM)}{p^i q^j}})y = \pi(z^{kl})\pi(z^{\frac{-krM}{p^i q^j}})y$$
$$(\pi(z^M)y = y \text{ and } y \in H_{\mathbb{Z}^2})$$
$$= \pi(z^{kl})y.$$

This shows that $\pi(\mathbb{Z}[\frac{1}{pq}])y \subseteq \pi(\mathbb{Z})y$.

Lastly we prove that $\mathrm{Span}(\pi(\mathbb{Z})y)$ is finite dimensional.

Every $k \in \mathbb{Z}$ can be written as $k = lN + r$ for some $l \in \mathbb{Z}$ and $0 \le r < N$. Hence $\pi(z^k)y = \pi(z^{lN+r})y = \pi(z^r)y$. This implies that

$$\pi(\mathbb{Z})y \subset \mathrm{Span}\{\pi(z^i)y\}_{i=0}^{N-1}.$$

So $\dim H \le N$. We finish proof of the claim. $\qquad \square$

Define a state ψ on $C(S_{pq}) \rtimes \mathbb{Z}^2$ by $\psi(b) = \langle \pi(b)y, y \rangle$ for every $b \in C(S_{pq}) \rtimes \mathbb{Z}^2$. We have $\psi \in P^1(C(S_{pq}) \rtimes \mathbb{Z}^2)$ since π is irreducible and $y \in H_{\mathbb{Z}^2}$. By Theorem ??, $\nu = R(\psi) = \psi|_{C(S_{pq})}$ is an ergodic $\times p, \times q$-invariant measure on S_{pq}.

By Theorem ?? and Theorem ??, we have $\pi \sim \rho_\nu$. Of course $H \cong L^2(S_{pq}, \nu)$. From the claim, $L^2(S_{pq}, \nu)$ is finite dimensional.

Hence ν is finitely supported in S_{pq}. Let $\mu = R(\nu) = \nu|_{C(S_{pq})}$. As before, ρ_ν is unitarily equivalent to π_μ. Hence $L^2(\mathbb{T}, \mu) \sim L^2(S_{pq}, \nu)$ is finite dimensional. Hence μ is a finitely supported ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$ and $\pi \sim \pi_\mu$. $\qquad \square$

Consequently we have the following.

Corollary 4.3. Furstenberg's conjecture is true iff there is a unique irreducible unitary representation $U : \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2 \to B(H)$ such that $H_{\mathbb{Z}^2} \neq 0$ and $\pi(z^k)x \neq x$ for every nonzero integer k and nonzero $x \in H_{\mathbb{Z}^2}$.

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