A GENERALIZED FEJÉR'S THEOREM FOR LOCALLY COMPACT GROUPS

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ABSTRACT. The classical Fejér's theorem is a criterion for pointwise convergence of Fourier series on the unit circle. We generalize it to locally compact groups.

1. INTRODUCTION

Let $\mathbb{T}$ be the unit circle. When necessary, identify $\mathbb{T}$ with $[0,1)$ or $\mathbb{R}/\mathbb{Z}$. Denote $e^{2\pi inx}$ by $e(nx)$ for a real number x and an integer n. For an f in $L^1(\mathbb{T})$, its Fourier coefficients $\hat{f}(n)$ for $n \in \mathbb{Z}$ is given by $\hat{f}(n) = \int_{\mathbb{T}} f(x)e(-nx)\,dx$ and define $S_N(f)(x)$ as $\sum_{n=-N}^{N} \hat{f}(n)e(nx)$ for every nonnegative integer N. The Fourier series of f is $\sum_{n\in\mathbb{Z}} \hat{f}(n)e(nx)$.

In classical Fourier analysis, an important question is pointwise convergence of $S_N(f)$ for f in $L^1(\mathbb{T})$. By Carleson-Hunt theorem [C66, Thm. (c)] [H68, Thm. 1], the Fourier series $S_N(f)(x)$ of an f in $L^p(\mathbb{T})$ with $1 < p < \infty$ converges to $f(x)$ almost everywhere. But in general the pointwise convergence does not hold even when f is continuous. Cf. [K04, Chap. II, Sec. 2].

If the Fourier series is replaced by its average, called the Cesàro mean or the Fejér mean,

$$\sigma_N(f,x) = \frac{1}{N+1}[S_0(f)(x) + \cdots + S_N(f)(x)] = K_N * f(x),$$

then one have a much better convergence. Here $K_N(x)$ is the Fejér's kernel (see section 2 for the definition).

In 1900, L. Fejér came up with an explicit criteria which tells us when and to which value $\sigma_N(f)$ converges. Cf. [F00][K04, Chap. I, Thm. 3.1(a)][G14, Thm. 3.4.1]. [1]

[1]For the history of Fejér's theorem, see [K06], wherein some applications and continuations of Fejér's theorem are also mentioned.

Theorem 1.1. [Fejér's theorem]

For an f in $L^1(\mathbb{T})$, if both the left and the right limit of $f(x)$ exist at some x_0 in $\mathbb{T}$ (denoted by $f(x_0+)$ and $f(x_0-)$ respectively), then

$$\lim_{N\to\infty} K_N * f(x_0) = \frac{1}{2}[f(x_0+) + f(x_0-)].$$

In particular, when f is continuous $\sigma_N(f,x)$ converges to $f(x)$ for every x in $\mathbb{T}$.

Note that the left and right limits of f at x_0 can be interpreted in terms of the finite partition $\{(0,\frac{1}{2}), [\frac{1}{2},1)\}$ of $\mathbb{T} = [0,1)$ (up to measure 0 since $(0,\frac{1}{2}) \cup [\frac{1}{2},1) = (0,1) = [0,1) \setminus \{0\}$): [2]

$$f(x_0-) = \lim_{\substack{y\to 0 \\ y\in(0,\frac{1}{2})}} f(x_0 - y), \quad f(x_0+) = \lim_{\substack{y\to 0 \\ y\in[\frac{1}{2},1)}} f(x_0 - y).$$

Moreover for the approximate identity (cf. Definition 2.1) $\{K_n\}_{n=1}^{\infty}$ of $L^1(\mathbb{T})$, one have

$$\int_{[0,\frac{1}{2})} K_n(t)\,dt = \int_{[\frac{1}{2},1)} K_n(t)\,dt = \frac{1}{2}$$

for all $n \geq 0$.

This observation motivates the following generalization of Fejér's theorem to locally compact groups.

Let G be a locally compact group with the unit e_G and a fixed left Haar measure μ.

A finite collection $\{A_1, A_2, \cdots, A_k\}$ of Borel subsets of G is called a **local partition** (at e_G) if the following are true:

(1) $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq k$,

(2) $\mu(G \setminus \bigcup_{i=1}^{k} A_i) = 0$,

(3) each $A_j \cap \mathcal{N} \neq \emptyset$ for any neighborhood $\mathcal{N}$ of e_G.

Theorem 1.2. [A generalized Fejér's theorem]

Consider a locally compact group G with a fixed left Haar measure μ. Let $\{F_\theta\}_{\theta\in\Theta}$ be an approximate identity of $L^1(G)$. Assume that there exists a local partition $\{A_1, A_2, \cdots, A_k\}$ of G such that $\lim_\theta \int_{A_j} F_\theta(y)\,d\mu(y) = \lambda_j$ for every $1 \leq j \leq k$.

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[2]Note that $y \to 0$ when $y \in [\frac{1}{2}, 1)$ makes sense since we identify 0 with 1.

For an f in $L^\infty(G)$, if there exists x in G such that $\lim\limits_{\substack{y \to e_G \\ y \in A_j}} f(y^{-1}x)$ (denoted by $f(x, A_j)$) exists for every $1 \le j \le k$, then

$$\lim_\theta F_\theta * f(x) = \sum_{j=1}^k \lambda_j f(x, A_j).$$

Moreover if $\lim\limits_\theta \sup\limits_{y \in \mathcal{N}^c} |F_\theta(y)| = 0$ for any neighborhood $\mathcal{N}$ of e_G, then for every f in $L^1(G)$ (or $L^\infty(G)$) such that each $f(x, A_j)$ exists for some x in G, we have

$$\lim_\theta F_\theta * f(x) = \sum_{j=1}^k \lambda_j f(x, A_j).$$

The paper is organized as follows.

In section 2, after some preliminaries, we prove Theorem 1.2 and its variant Corollary 2.2. To give some applications, various special cases (either abelian or non-abelian groups) of Theorem 1.2 are discussed in section 3.

ACKNOWLEDGEMENTS

2. THE MAIN THEOREM

Within this article G stands for a locally compact group with a fixed left Haar measure μ. Let e_G be the identity of G. Denote by $L^1(G)$ the space of integrable functions (with respect to μ) on G and by $L^\infty(G)$ the space of essentially bounded functions (with respect to μ) on G.

The **convolution** $f * g$ for f and g in $L^1(G)$ is given by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,d\mu(y)$$

for every $x \in G$.

Definition 2.1. [Approximate identity] [G14, Defn. 1.2.15.]

An **approximate identity** is a family of functions $\{F_\theta\}_{\theta\in\Theta}$ in $L^1(G)$ such that

(1) $\|F_\theta\|_{L^1(G)} \leq C$ for all θ.

(2) $\int_G F_\theta(x)\,d\mu(x) = 1$ for all θ.

(3) $\lim_\theta \int_{\mathcal{N}^c} |F_\theta(x)|\,d\mu(x) = 0$ for any neighborhood $\mathcal{N}$ of e_G.

There always exists an approximate identity in $L^1(G)$ [F95, Chap.2, Prop. 2.42].

Now we are ready to prove the main theorem.

Proof of Theorem 1.2.

Suppose f is in $L^\infty(G)$ such that each $f(x, A_j)$ exists for some x in G.

For an arbitrary $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of e_G such that $|f(y^{-1}x) - f(x, A_j)| < \varepsilon$ for every $1 \leq j \leq k$ whenever y is in $\mathcal{N} \cap A_j$.

Then

$$F_\theta * f(x) - \sum_{j=1}^k \lambda_j f(x, A_j) = \int_G F_\theta(y)f(y^{-1}x)\,d\mu(y) - \sum_{j=1}^k \lambda_j f(x, A_j)$$

$$= (\int_{\mathcal{N}} + \int_{\mathcal{N}^c}) F_\theta(y)f(y^{-1}x)\,d\mu(y) - \sum_{j=1}^k \lambda_j f(x, A_j).$$

First

$$\int_{\mathcal{N}} F_\theta(y)f(y^{-1}x)\,d\mu(y) - \sum_{j=1}^k \lambda_j f(x, A_j)$$

$$= \sum_{j=1}^k \int_{\mathcal{N}\cap A_j} F_\theta(y)f(y^{-1}x)\,d\mu(y) - \sum_{j=1}^k \lambda_j f(x, A_j)$$

$$= \sum_{j=1}^k \int_{\mathcal{N}\cap A_j} F_\theta(y)(f(y^{-1}x) - f(x, A_j))\,d\mu(y) + \sum_{j=1}^k (\int_{\mathcal{N}\cap A_j} F_\theta(y)\,d\mu(y) - \lambda_j)f(x, A_j).$$

So we have

$$\limsup_\theta |\int_{\mathcal{N}} F_\theta(y)f(y^{-1}x)\,d\mu(y) - \sum_{j=1}^k \lambda_j f(x, A_j)|$$

$$\leq \sum_{j=1}^{k} \limsup_\theta \int_{\mathcal{N} \cap A_j} |F_\theta(y)||f(y^{-1}x) - f(x, A_j)| \, d\mu(y)$$

$$+ \sum_{j=1}^{k} \limsup_\theta |\int_{\mathcal{N} \cap A_j} F_\theta(y) \, d\mu(y) - \lambda_j| \, |f(x, A_j)|$$

$$\leq \limsup_\theta \sum_{j=1}^{k} \varepsilon \int_{\mathcal{N} \cap A_j} |F_\theta(y)| \, d\mu(y) + \sum_{j=1}^{k} \limsup_\theta |\int_{\mathcal{N} \cap A_j} F_\theta(y) \, d\mu(y) - \lambda_j| \, |f(x, A_j)|.$$

Note that for every θ

$$\sum_{j=1}^{k} \int_{\mathcal{N} \cap A_j} |F_\theta(y)| \, d\mu(y) = \int_{\mathcal{N}} |F_\theta(y)| \, d\mu(y) \leq \|F_\theta\|_{L^1(G)} \leq C,$$

and it follows from $\lim_\theta \int_{\mathcal{N}^c} |F_\theta(y)| \, d\mu(y) = 0$ that

$$\lim_\theta \int_{\mathcal{N} \cap A_j} F_\theta(y) \, d\mu(y) = \lim_\theta \int_{A_j} F_\theta(y) \, d\mu(y) = \lambda_j$$

for every $1 \leq j \leq k$.

Therefore

$$\limsup_\theta |\int_{\mathcal{N}} F_\theta(y) f(y^{-1}x) \, d\mu(y) - \sum_{j=1}^{k} \lambda_j f(x, A_j)| \leq C\varepsilon.$$

Moreover

$$\limsup_\theta |\int_{\mathcal{N}^c} F_\theta(y) f(y^{-1}x) \, d\mu(y)| \leq [\limsup_\theta \int_{\mathcal{N}^c} |F_\theta(x)| \, d\mu(x)]\|f\|_{L^\infty(G)} = 0.$$

Hence

$$\limsup_\theta |F_\theta * f(x) - \sum_{j=1}^{k} \lambda_j f(x, A_j)| \leq C\varepsilon$$

for any $\varepsilon > 0$. This proves the first part of the theorem.

Now assume that $\limsup_\theta \sup_{y \in \mathcal{N}^c} |F_\theta(y)| = 0$ for any neighborhood $\mathcal{N}$ of e_G and f is in $L^1(G)$ such that each $f(x, A_j)$ exists for some x in G.

As before, we have

$$\limsup_\theta |\int_{\mathcal{N}} F_\theta(y) f(y^{-1}x) \, d\mu(y) - \sum_{j=1}^{k} \lambda_j f(x, A_j)| \leq C\varepsilon$$

for every $\varepsilon > 0$.

Moreover

$$\limsup_\theta |\int_{\mathcal{N}^c} F_\theta(y) f(y^{-1}x) \, d\mu(y)| \leq [\limsup_\theta \sup_{y \in \mathcal{N}^c} |F_\theta(y)|] \|f\|_{L^1(G)} = 0.$$

This completes the proof. $\square$

Although Theorem 1.2 requires that a local partition $\{A_1, \cdots, A_k\}$ satisfies that every $\lim_\theta \int_{A_j} F_\theta(y) \, d\mu(y)$ exists, this assumption could be easily satisfied when one considers subnets of $\{F_\theta\}$. Note that

$$|\int_{A_j} F_\theta(y) \, d\mu(y)| \leq \int_{A_j} |F_\theta(y)| \, d\mu(y) \leq \|F_\theta\|_{L^1(G)} \leq C$$

for all θ and j, so for any given approximate identity $\{F_\theta\}_{\theta \in \Theta}$ of $L^1(G)$ and local partition $\{A_1, \cdots, A_k\}$ of G, there always exists a subnet Θ_1 of Θ such that every $\lim_{\theta \in \Theta_1} \int_{A_j} F_\theta(y) \, d\mu(y)$ (denoted by $\lambda_j(\Theta_1)$) exists.

The argument goes as follows.

Consider the net of bounded complex numbers $\{\int_{A_1} F_\theta(y) \, d\mu(y)\}_{\theta \in \Theta}$. There is a subnet Θ' of Θ such that $\lim_{\theta \in \Theta'} \int_{A_1} F_\theta(y) \, d\mu(y)$ exists and equals some λ_1. Then consider the net of bounded complex numbers $\{\int_{A_2} F_\theta(y) \, d\mu(y)\}_{\theta \in \Theta'}$, as before, there exists a subnet of Θ'' of Θ' such that $\lim_{\theta \in \Theta''} \int_{A_2} F_\theta(y) \, d\mu(y)$ exists and equals some λ_2 (also note that $\lim_{\theta \in \Theta''} \int_{A_1} F_\theta(y) \, d\mu(y) = \lambda_1$). Repeat this procedure. After finite steps, we can find a subnet Θ_1 of Θ such that every $\lim_{\theta \in \Theta_1} \int_{A_j} F_\theta(y) \, d\mu(y)$ exists.

So we have the following variant of Theorem 1.2.

Corollary 2.2. Given any approximate identity $\{F_\theta\}_{\theta \in \Theta}$ of $L^1(G)$ and local partition $\{A_1, \cdots, A_k\}$ of G, if for an f in $L^\infty(G)$, every $f(x, A_j)$ exists, then there exists a subnet Θ_1 of Θ such that every $\lim_{\theta \in \Theta_1} \int_{A_j} F_\theta(y) \, d\mu(y)$ exists (denoted by $\lambda_j(\Theta_1)$) and

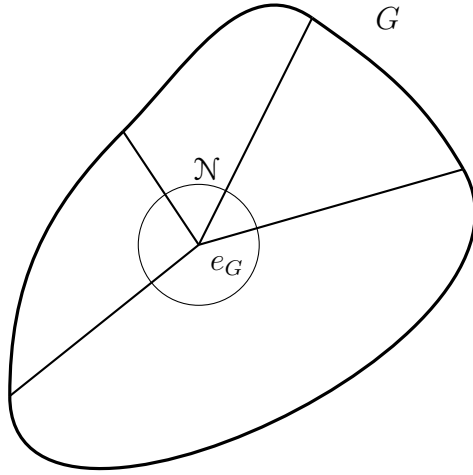$$\lim_{\theta \in \Theta_1} F_\theta * f(x) = \sum_{j=1}^k \lambda_j(\Theta_1) f(x, A_j).$$

Moreover if $\lim\limits_{\theta \in \Theta_1} \sup\limits_{y \in \mathcal{N}^c} |F_\theta(y)| = 0$ for every neighborhood $\mathcal{N}$ of e_G, then for every f in $L^1(G)$ (or $L^\infty(G)$) such that every $f(x, A_j)$ exists for some x in G, we have

$$\lim_{\theta \in \Theta_1} F_\theta * f(x) = \sum_{j=1}^{k} \lambda_j(\Theta_1) f(x, A_j).$$

3. Some special cases

In this section we consider some concrete examples of Theorem 1.2 including both abelian and non-abelian groups.

A local partition of a locally compact group G looks as follows:

3.1. **d-torus.**

The **Fejér kernel** for $\mathbb{T}$ is given by

$$K_n(t) = \sum_{j=-n}^{n} (1 - \frac{|j|}{n+1}) e(jt) = \frac{\sin^2 (n+1)\pi t}{(n+1) \sin^2 \pi t}$$

for every nonnegative integer n.

The Fejér kernel has many nice properties. Below we list some of them which would be frequently used throughout the paper. See [G14, p.181, Prop. 3.1.10] and [G14, p.205, (3.4.3)] for a proof.

Proposition 3.1. [Properties of K_n]

    (1) $K_n(t) \geq 0$ for every t in $\mathbb{T}$ and $n \geq 0$.

    (2) $\int_{\mathbb{T}} K_n(t)\, dt = 1$ for all $n \geq 0$.

    (3) For $\lambda \in (0, \frac{1}{2})$, we have $\displaystyle\sup_{t \in [\lambda, 1-\lambda]} K_n(t) \leq \frac{1}{(n+1)\sin^2(\pi\lambda)}$, hence $\displaystyle\lim_{n\to\infty} \int_\lambda^{1-\lambda} K_n(t)\, dt =$
    0.

So $\{K_n(t)\}_{n=1}^\infty$ is an approximate identity of $L^1(\mathbb{T})$. In addition, $\displaystyle\lim_{n\to\infty}\sup_{t\in\mathcal{N}^c} |K_n(t)| = 0$ for every neighborhood $\mathcal{N}$ of 0.

In the unit circle, there is some "strange-looking" local partition. For instance, choose two sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ such that

    • $0 < a_{n+1} < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1 = \frac{1}{2}$;

    • $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

Let $A_1 = (\frac{1}{2}, 1)$, $A_2 = \displaystyle\bigcup_{n=1}^\infty (a_n, b_n)$ and $A_3 = \displaystyle\bigcup_{n=1}^\infty (b_{n+1}, a_n)$. Then $\{A_1, A_2, A_3\}$ is a local partition which is completely different from the local partition $\{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$ used in the classical Fejér's theorem.

So it is worthy of mentioning the following generalized Fejér's theorem for the unit circle.

Corollary 3.2. Given a local partition $\{A_1, A_2, \cdots, A_k\}$ of $\mathbb{T}$. Assume that $\displaystyle\lim_{n\to\infty} \int_{A_j} K_n(x)\, dx = \lambda_j$ for every $1 \leq j \leq k$. For f in $L^1(\mathbb{T})$, if each $f(x_0, A_j)$ exists at some x_0 in $\mathbb{T}$, then

$$\lim_{n\to\infty} K_n * f(x_0) = \sum_{j=1}^k \lambda_j f(x_0, A_j).$$

Now consider d-torus for $d \geq 2$.

We identify $\mathbb{T}^d$ with $[0,1)^d$ or $\mathbb{R}^d/\mathbb{Z}^d$ when necessary. The inner product $x \cdot y$ of $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$ in $\mathbb{R}^d$ is given by $x_1 y_1 + \cdots + x_d y_d$.

The **square Fejér kernel** for $\mathbb{T}^d$ is defined by

$$K_n^d(x_1, \cdots, x_d) = \prod_{j=1}^d K_n(x_j) = \sum_{m \in \mathbb{Z}^d, |m_j| \leq n} (1 - \frac{|m_1|}{n+1}) \cdots (1 - \frac{|m_d|}{n+1}) e(m \cdot x)$$

for every nonnegative integer n.

Proposition 3.3. [Properties of K_n^d]

(1) $K_n^d(x) \geq 0$ for every x in $\mathbb{T}^d$ and $n \geq 0$.

(2) $\int_{\mathbb{T}^d} K_n^d(x)\,dx = 1$ for all $n \geq 0$.

Proof. All these properties of K_n^d are induced by properties of K_n. See [G14, p.181, Prop. 3.1.10 & p.205, (3.4.3)].

(1) $K_n^d \geq 0$ since $K_n \geq 0$.

(2) $\int_{\mathbb{T}^d} K_n^d(x)\,dx = \prod_{j=1}^d \int_{\mathbb{T}} K_n(x_j)\,dx_j = 1$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Also $\{K_n^d\}_{n=1}^\infty$ is an approximate identity of $L^1(\mathbb{T}^d)$.

For $k = (k_1, \cdots, k_d)$ in $\{0,1\}^d$, define

$$I_k = \prod_{j=1}^d I_{k_j}$$

with $I_0 = [0, \frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1)$. Note that $\int_0^{\frac{1}{2}} K_n(t)\,dt = \int_{\frac{1}{2}}^1 K_n(t)\,dt = \frac{1}{2}$ for all $n \geq 0$. So $\{I_k\}_{k \in \{0,1\}^d}$ is a local partition of $\mathbb{T}^d$ $(= [0,1)^d)$ such that

$$\int_{I_k} K_n^d(x)\,dx = \prod_{j=1}^d \int_{I_{k_j}} K_n(x_j)\,dx_j = \frac{1}{2^d}$$

for all $k \in \{0,1\}^d$ and $n \geq 0$.

For f in $L^1(\mathbb{T}^d)$, define

$$f(x, I_k) = \lim_{\substack{y \to 0 \\ y \in I_k}} f(x - y)$$

for every $k \in \{0,1\}^d$.

Corollary 3.4. Let $d \geq 2$. For f in $L^\infty(\mathbb{T}^d)$ and x in $\mathbb{T}^d$, if each $f(x, I_k)$ exists, then

$$\lim_{n \to \infty} K_n^d * f(x) = \frac{1}{2^d} \sum_{k \in \{0,1\}^d} f(x, I_k).$$

Proof. Apply the first part of Theorem 1.2 to the approximate identity $\{K_n^d\}_{n=0}^\infty$ of $L^1(\mathbb{T}^d)$ and the local partition $\{I_k\}_{k \in \{0,1\}^d}$ of $\mathbb{T}^d$. $\qquad\square$

3.2. **Euclidean spaces.**

The **Wigner semicircle kernel** $W_\theta(t)$ is given by

$$W_\theta = \begin{cases} \frac{2}{\pi\theta^2}\sqrt{\theta^2 - t^2} & \text{when } -\theta \le t \le \theta, \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in \mathbb{R}$ and $\theta > 0$.

For a λ in $(0,1)$, by shifting the graph of $W_\theta(t)$ along the t-axis, we get a new function $W_{\theta,\lambda}(t)$ satisfying

$$\int_{(-\infty,0)} W_{\theta,\lambda}(t)\, dt = \lambda.$$

Also note that every $W_{\theta,\lambda}(t)$ is compactly supported in a closed interval shrinking to $\{0\}$ as θ goes to 0.

Define $W_{\theta,\lambda}^d(x) = \prod_{j=1}^{d} W_{\theta,\lambda}(x_j)$ for any positive integer d, then $\{W_{\theta,\lambda}^d(x)\}_{\theta>0}$ is an approximate identity of $L^1(\mathbb{R}^d)$ and satisfies that $\lim_{\theta\to 0} \sup_{x\in\mathcal{N}^c} |W_{\theta,\lambda}^d(x)| = 0$ for every neighborhood $\mathcal{N}$ of 0 in $\mathbb{R}^d$.

Corollary 3.5. Fix a λ in $(0,1)$. For an f in $L^1(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$, if every $f(x, J_k)$ exists for some $x \in \mathbb{R}^d$, then

$$\lim_{\theta\to 0} W_{\theta,\lambda}^d * f(x) = \sum_{k\in\{0,1\}^d} \prod_{j=1}^{d} \lambda^{1-k_j}(1-\lambda)^{k_j} f(x, J_k).$$

Proof. Apply Theorem **??** to Consider the local partition $\{J_k\}_{k\in\{0,1\}^d}$ of $\mathbb{R}^d$ and the approximate identity $\{W_{\theta,\lambda}^d\}_{\theta>0}$ of $L^1(\mathbb{R}^d)$. Note that $\lim_{\theta\to 0} \sup_{x\in\mathcal{N}^c} |W_{\theta,\lambda}^d(x)| = 0$ for every neighborhood $\mathcal{N}$ of 0 in $\mathbb{R}^d$. Furthermore

$$\int_{J_k} W_{\theta,\lambda}^d(x)\, dx = \prod_{l=1}^{d} \int_{J_{k_l}} W_{\theta,\lambda}(x_l)\, dx_l = \prod_{j=1}^{d} \lambda^{1-k_j}(1-\lambda)^{k_j}.$$

Applying Theorem 1.2 finishes the proof. □

Remark 3.6. In [FW06], H. G. Feichtinger and F. Weisz prove some theorems about convergence for convolutions of some special types of approximate identities with f in $L^1(\mathbb{R}^d)$ or $L^1(\mathbb{T}^d)$ at a Lebesgue point of f. Cf. [FW06, Thm. 4.6 & Thm. 7.2]). Corollaries 3.4 and 3.5 have some overlaps with, but are not covered by those in [FW06] since the points satisfying the assumptions of Corollaries 3.4 and 3.5 are not necessarily Lebesgue points.

A **Lebesgue point** of an f in $L^1(\mathbb{R}^d)$ is a point x in $\mathbb{R}^d$ such that

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dm(y) = 0.$$

Here m is the Lebesgue measure of $\mathbb{R}^d$ and $B_r(x)$ is the open ball centered at x with radius $r > 0$ [R87, 7.6].

A point $x \in \mathbb{R}^d$ so that every $f(x, J_k)$ exists is not necessarily a Lebesgue point.

For instance, consider $f = \mathbf{1}_{(0,1)}$, the characteristic function of $(0, 1)$, which is in $L^1(\mathbb{R})$. Then $f(0-) = 0$ and $f(0+) = 1$. However 0 is not a Lebesgue point [3] since

$$\lim_{r \to 0} \frac{1}{2r} \int_{(-r,r)} |f(y) - f(0)| dy = \lim_{r \to 0} \frac{1}{2r} \int_{(0,r)} |1 - 0| dy = \frac{1}{2}.$$

3.3. The Heisenberg group.

The continuous **Heisenberg group** $\mathbb{H}$ is the group of 3 by 3 upper triangular real matrices with diagonal entries 1, that is,

$$\mathbb{H} = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) | a, b, c \in \mathbb{R}\}.$$

As a topological space $\mathbb{H}$ is homeomorphic to $\mathbb{R}^3$ and a left Haar measure of $\mathbb{H}$ is $da\, db\, dc$.

Define

$$W_\theta^3(a, b, c) = W_\theta(a) W_\theta(b) W_\theta(c)$$

for every $\theta > 0$ and a, b, c in $\mathbb{R}$. Then it is easy to see that $\{W_\theta^3\}_{\theta > 0}$ is an approximate identity of $L^1(\mathbb{H})$ and

$$\lim_{\theta \to 0} \sup_{(a,b,c) \in \mathcal{N}^c} |W_\theta^3(a, b, c)| = 0$$

for every neighborhood $\mathcal{N}$ of $e_{\mathbb{H}} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ in $\mathbb{H}$.

For $k = (k_1, k_2, k_3)$ in $\{0, 1\}^3$, define $J_k = \prod_{l=1}^{3} J_{k_l}$ with $J_0 = (-\infty, 0)$ and $J_1 = [0, \infty)$. Then $\{J_k\}_{k \in \{0,1\}^3}$ is a local partition of $\mathbb{H}$ at $e_{\mathbb{H}}$ such that $\int_{J_k} W_\theta^3(a, b, c) da\, db\, dc = \frac{1}{8}$ for every $k \in \{0, 1\}^3$.

We get the following.

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[3] We can change the value of $f(0)$, but this does not affect the fact that 0 is not a Lebesgue point.

Corollary 3.7. For an f in $L^1(\mathbb{H})$ or $L^\infty(\mathbb{H})$, if every $f(x, J_k)$ exists for some x in $\mathbb{H}$, then

$$\lim_{\theta \to 0} W_\theta^3 * f(x) = \frac{1}{8} \sum_{k \in \{0,1\}^3} f(x, J_k).$$

Proof. Applying Theorem 1.2 to the local partition $\{J_k\}_{k \in \{0,1\}^3}$ of $\mathbb{H}$ gives the proof. $\square$

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ERRATUM OF "A GENERALIZED FEJÉR'S THEOREM FOR LOCALLY COMPACT GROUPS"

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ABSTRACT. As pointed out by Hanfeng Li, we fix a gap in Theorem 1.2.

There are three issues in the paper.

(1) Let G be a locally compact group with the unit e_G and a fixed left Haar measure μ. Let ν be the right Haar measure given by $\nu(A) = \mu(A^{-1})$ for every Borel subset A of G.

Theorem 1.2 is stated as follows.

Theorem. [A generalized Fejér's theorem]

Consider a locally compact group G with a fixed left Haar measure μ and the corresponding right Haar measure ν. Let $\{F_\theta\}_{\theta \in \Theta}$ be an approximate identity of $L^1(G)$. Assume that there exists a local partition $\{A_1, A_2, \cdots, A_k\}$ of G such that $\lim_\theta \int_{A_j} F_\theta(y) \, d\mu(y) = \lambda_j$ for every $1 \leq j \leq k$.

For an f in $L^\infty(G)$, if there exists x in G such that $\lim_{\substack{y \to e_G \\ y \in A_j}} f(y^{-1}x)$ (denoted by $f(x, A_j)$) exists for every $1 \leq j \leq k$, then

$$\lim_\theta F_\theta * f(x) = \sum_{j=1}^k \lambda_j f(x, A_j).$$

Moreover if $\lim_\theta \sup_{y \in \mathcal{N}^c} |F_\theta(y)| = 0$ for any neighborhood $\mathcal{N}$ of e_G, then for every f in $L^1(G, \nu)$ (or $L^\infty(G)$) such that each $f(x, A_j)$ exists for some x in G, we have

$$\lim_\theta F_\theta * f(x) = \sum_{j=1}^k \lambda_j f(x, A_j).$$

In the proof of Theorem 1.2, there are two corrections: