# MIXED-SPIN-P FIELDS OF FERMAT POLYNOMIALS

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Abstract. This is the first part of the project toward an effective algorithm to evaluate all genus Gromov-Witten invariants of quintic Calabi-Yau threefolds. In this paper, we introduce the notion of Mixed-Spin-P fields, construct their moduli spaces, and construct the virtual cycles of these moduli spaces.

## 1. Introduction

Explicitly solving all genus Gromov-Witten invariants (in short GW invariants) of Calabi-Yau threefolds is one of the major goals in the subject of Mirror Symmetry. For quintic Calabi-Yau threefolds, the mirror formula of genus-zero GW invariants was conjectured in [CdGP] and proved in [Gi, LLY]. The mirror formula of genus-one GW invariants was conjectured in [BCOV] and proved in [LZ, Zi]. A complete determination of all genus GW invariants based on degeneration is provided in [MP] and plays a crucial role in the proof of the GW/Pairs correspondence [PP]. However, the mirror prediction on genus $g$ GW invariants for $2 \leq g \leq 51$ in [HKQ] is still open, even in the $g = 2$ case.

The mirror prediction in [HKQ] includes both GW invariants of quintic threefolds and FJRW invariants of the Fermat quintic. In this paper, we introduce the notion of Mixed-Spin-P fields (in short MSP fields) of the Fermat quintic polynomial, construct their moduli spaces, and establish basic properties of these moduli spaces. This class of moduli spaces will be employed in the sequel of this paper [CLLL] toward developing an effective theory evaluating all genus GW invariants of quintic threefolds and all genus FJRW invariants of the Fermat quintic.

The theory of MSP fields, for the Calabi-Yau quintic polynomial

$$F_{5,5}(x) = x_1^5 + \cdots + x_5^5,$$

provides a transition between FJRW invariants [FJR] and GW invariants of stable maps with p-fields [CL]. It is known that the FJRW invariants of the Fermat quintic is the LG theory taking values in $[\mathbb{C}^5/\boldsymbol{\mu}_5]$ (via spin fields), and the GW invariants of stable maps with p-fields is the LG theory taking values in the canonical line bundle $K_{\mathbb{P}^4}$ (via P-fields). Our idea is to use the master space technique to study the two GIT quotients $[\mathbb{C}^5/\boldsymbol{\mu}_5]$ and $K_{\mathbb{P}^4}$ of $[\mathbb{C}^6/\mathbb{G}_m]$, which led to the notion of

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MSP fields, providing a geometric transition between the LG theories of the two GIT quotients of $[\mathbb{C}^6/\mathbb{G}_m]$.

In this paper, we will introduce the notion of MSP field for the Fermat polynomial

$$F_{n,r}(x) := x_1^r + \cdots + x_n^r. \tag{1.1}$$

With $F_{n,r}$ understood, an MSP field is a collection

$$\xi = (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu), \tag{1.2}$$

consisting of a pointed twisted curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$, fields $\varphi \in H^0(\mathcal{C}, \mathcal{L}^{\oplus n})$ and $\rho \in H^0(\mathcal{C}, \mathcal{L}^{\vee\otimes r} \otimes \omega_{\mathcal{C}}^{\log})$, and a gauge field $\nu = (\nu_1, \nu_2) \in H^0(\mathcal{C}, \mathcal{L} \otimes \mathcal{N} \oplus \mathcal{N})$. The numerical invariants of $\xi$ are the genus of $\mathcal{C}$, the monodromy $\gamma_i$ of $\mathcal{L}$ at the marking $\Sigma_i^{\mathcal{C}}$ (of $\Sigma^{\mathcal{C}}$), and the bi-degrees $d_0 = \deg(\mathcal{L} \otimes \mathcal{N})$ and $d_\infty = \deg \mathcal{N}$.

For a choice of $g$, $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ and $\mathbf{d} = (d_0, d_\infty)$, we form the moduli $\mathcal{W}_{g,\gamma,\mathbf{d}}$ of equivalence classes of stable MSP fields of numerical data $(g, \gamma, \mathbf{d})$. It is a separated DM stack, locally of finite type, though usually not proper. The stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ admits a $T = \mathbb{C}^*$ action, via

$$t \cdot (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2) = (\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, t\nu_1, \nu_2).$$

It comes with a perfect (relative) obstruction theory, of virtual dimension (in case $\gamma = \emptyset$)

$$\text{vir.}\dim \mathcal{W}_{g,\gamma=\emptyset,\mathbf{d}} = (1+n-r)d_0 + (1-n+r)d_\infty + (4-n)(g-1).$$

**Theorem 1.1.** *The moduli stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ (of stable MSP fields of the Fermat polynomial $F_{n,r}$) is a separated DM stack, locally of finite type. It has a cosection localized $T$-equivariant virtual cycle $[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}$, lying in a proper substack $\mathcal{W}^-_{g,\gamma,\mathbf{d}} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$:*

$$[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_*(\mathcal{W}^-_{g,\gamma,\mathbf{d}})^T.$$

In the sequel [CLLL], for the quintic Fermat polynomial $F_{5,5}$ we apply the virtual localization formula to derive a doubly indexed polynomial relations among the GW invariants of quintic threefolds and the FJRW invariants of the Ferman polynomial $F_{5,5}$. These relations provide an effective algorithm in evaluating all genus GW invariants of quintics in terms of FJRW invariants of $F_{5,5}$ (with the insertion 2/5), and provide a collection of relations among all genus FJRW invariants of $F_{5,5}$ (with the insertion 2/5).

This work is inspired by Witten's vision that the "Landau-Ginzburg looks like the analytic continuation of Calabi-Yau to negative Kahler class." (See [Wi, 3.1].) One interpretation of his proposed transition of theories is that the LG theory of $[\mathbb{C}^5/\mathbb{Z}_5]$ and that of $K_{\mathbb{P}^4}$ differ by a fields version of "wall-crossing". The MSP fields introduced can be viewed as a geometric construction to realize this "wall-crossing".

Around the time of the completion of the first draft of this paper, there have been other approaches for high genus LG/CY correspondence [CK, FJR2]. Since then, there are a few notable further developments based on the theory of MSP developed. In [CLLL], the virtual localization formula of the $\mathbb{C}^*$ equivariant moduli of MSP fields were developed, and the mentioned recursion relations among GW and FJRW of the quintic Fermat polynomial is derived. In [GR], Guo-Ross proved the genus-one Landau-Ginzburg/Calabi-Yau conjecture of Chiodo and Ruan. In [CGLZ], the explicit formula of the genus one GW invariants of the quintic CY threefold has been recovered using the recursion relations. Recently, it is shown that the localization of the genus two MSP theory recovers the genus two BCOV's Feyman rule of the GW invariants of the quintic Calabi-Yau threefolds [CGL]. Along the way, the finite generation conjecture of Yamaguchi-Yau and the holomorphic anomaly equation for $g = 2$ GW invariants of the quintic Calabi-Yau threefolds is establised. (cf. [BCOV, YY, ASYZ, LP].)

This paper is organized as follows. In Section one, we will introduce the notion of Mixed-Spin-P fields of the Fermat quintic polynomial; construct the moduli spaces of stable Mixed-Spin-P fields, and construct the cosection localized virtual cycles of these moduli spaces. These cycles lie in the degeneracy loci of the cosection mentioned. In Section two and three, we will prove that these degeneration loci are proper, separated and of finite type.

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## 2. The moduli of Mixed-Spin-P fields

In this section, we introduce the notion of MSP (Mixed-Spin-P) fields, construct their moduli stacks, and form their cosection localized virtual cycles. We introduce the $T$-structure on it. The proof of the localization formula of cosection localized virtual cycles will appear in [CKL].

2.1. **Twisted curves and invertible sheaves.** We recall the basic notions and properties of twisted curves with representable invertible sheaves on them. The materials are drawn from [ACV, AJ, AF, AGV, Cad].

A prestable twisted curve with $\ell$-markings is a one-dimensional proper, separated connected DM stack $\mathcal{C}$, with at most nodal singularities, together with a collection of disjoint closed substacks $\Sigma_1, \cdots, \Sigma_\ell$ of smooth locus of $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{sm}} - \cup_i \Sigma_i$ is a scheme, and node are balanced.

Here an index $r$ balanced node looks like the following model

$$\mathcal{V}_r := \left[\mathrm{Spec}\big(\mathbb{C}[u,v]/(uv)\big)/\boldsymbol{\mu}_r\right], \quad \zeta\cdot(u,v) = (\zeta u, \zeta^{-1}v).$$

Similarly, an index $r$ marking of a twisted curve looks like the model

$$\mathcal{U}_r := \left[\mathrm{Spec}\,\mathbb{C}[u]/\boldsymbol{\mu}_r\right], \quad \zeta\cdot u = \zeta u.$$

Denote by

$$\pi_r : \mathcal{V}_r \to V_r := \mathrm{Spec}\big(\mathbb{C}[x,y]/(xy)\big) \quad \text{and} \quad \pi_r : \mathcal{U}_r \to U_r := \mathbb{A}^1 \tag{2.1}$$

defined by $x \mapsto u^r$ and $y \mapsto v^r$, the maps to their coarse moduli spaces. Note that $\mathcal{V}_r$ contains two subtwisted curves $\mathcal{V}_{r,1}$ and $\mathcal{V}_{r,2}$, each isomorphic to $\mathcal{U}_r$ in (2.1).

This process $\mathcal{V}_r \mapsto \mathcal{U}_{r,u} \coprod \mathcal{U}_{r,v}$ is called the decomposition of $\mathcal{V}_r$ along its node. The reverse process is called the gluing, which can be defined via a push out.

We comment on out convention on invertible sheaves on a twisted curve $\mathcal{C}$ near a stacky point. In the model case $\mathcal{C} = \mathcal{V}_r$, an invertible sheaf on $\mathcal{C}$ is a $\boldsymbol{\mu}_r$-module $\mathcal{M}_m$, for $0 < m < r$, so that

$$\mathcal{M}_m := u^{-(r-m)}\mathbb{C}[u] \oplus_{[0]} v^{-m}\mathbb{C}[v] := \ker\{u^{-(r-m)}\mathbb{C}[u] \oplus v^{-m}\mathbb{C}[v] \to \mathbb{C}_m\},$$

where the arrow is a homomorphism of $\boldsymbol{\mu}_r$-modules, and $\boldsymbol{\mu}_r$ leaves $1 \in \mathbb{C}[u]$ and $1 \in \mathbb{C}[v]$ fixed and acts on $1 \in \mathbb{C}_m \cong \mathbb{C}$ via $\zeta \cdot 1 = \zeta^m 1$, and both $u^{-(r-m)}\mathbb{C}[u] \to \mathbb{C}_m$ and $v^{-m}\mathbb{C}[v] \to \mathbb{C}_m$ are surjective. When $m = 0$,

$$\mathcal{M}_0 := \mathbb{C}[u] \oplus_{[0]} \mathbb{C}[v] := \ker\{\mathbb{C}[u] \oplus \mathbb{C}[v] \to \mathbb{C}\},$$

where maps are defined similarly as the case of $m \neq 0$. Note that the isomorphism classes are indexed by $m \in \{0, \cdots, r-1\}$. Further, in the convention (2.1)

$$\pi_{r*}\mathcal{M}_m = \mathbb{C}[x] \oplus \mathbb{C}[y]. \tag{2.2}$$

Similarly, invertible sheaves on $\mathcal{C}$ near an index $r$ marking in the model case $\mathcal{C} = \mathcal{U}_r$ looks like the $\boldsymbol{\mu}_r$-module

$$\mathcal{M}_m := u^{-m}\mathbb{C}[u]. \tag{2.3}$$

Let $\zeta_r = \exp(2\pi i/r) \in \boldsymbol{\mu}_r$. Under $u \mapsto \zeta_r u$, the generator $u^{-m}1_u \mapsto \zeta_r^{-m} u^{-(r-m)}1_u$, we call (the inverse of $\zeta_r^{-m}$) $\zeta_r^m$ the monodromy of $\mathcal{M}_m$ at the marking, and call $m$ the monodromy index at the marking. Note that for $\mathcal{M}_m$ over $\mathcal{V}_r$, $\mathcal{M}_m$ restricted to $\mathcal{V}_{r,1}$ and to $\mathcal{V}_{r,2}$ have monodromies $\zeta_r^{-m}$ and $\zeta_r^m$, at their respective stacky points.

**Definition 2.1.** *We call $\mathcal{M}_m$ representable if $(m,r) = 1$.*

**Example 2.2.** *Let $\mathcal{C} = \mathbb{A}^1/\boldsymbol{\mu}_r$ be the obvious global quotient twisted curve; let $p \in \mathbb{A}^1$ be its origin. Then we use $\mathcal{O}_{\mathcal{C}}(\frac{m}{r}p)$ to denote the sheaf* (2.3)*, having monodromy index $m$.*

2.2. **Definition of MSP fields.** We fix a Fermat polynomial (1.1). We denote by $\boldsymbol{\mu}_d \le \mathbb{C}^*$ the subgroup of the $d$-th roots of unity. We let

$$\tilde{\boldsymbol{\mu}}_r^+ = \boldsymbol{\mu}_r \cup \{(1,\rho), (1,\varphi)\}, \quad \text{and} \quad \tilde{\boldsymbol{\mu}}_r = \tilde{\boldsymbol{\mu}}_r^+ - \{1\}.$$

For $\alpha \in \tilde{\boldsymbol{\mu}}_r^+$, we let $\langle\alpha\rangle \le \mathbb{G}_m$ be the subgroup generated by $\alpha$; for the two exceptional element $(1,\rho)$ and $(1,\varphi)$, we agree that $\langle(1,\rho)\rangle = \langle(1,\varphi)\rangle = \langle 1\rangle$.

We fix the Fermat polynomial $F_{n,r}$; we let

$$g \ge 0, \quad \gamma = (\gamma_1, \cdots, \gamma_\ell) \in (\tilde{\boldsymbol{\mu}}_r)^{\times\ell}, \quad \text{and} \quad \mathbf{d} = (d_0, d_\infty) \in \mathbb{Q}^{\times 2},$$

and call the triple $(g, \gamma, \mathbf{d})$ a numerical data (for MSP fields), and call $(g, \gamma, \mathbf{d})$ a broad numerical data if $\gamma \in (\tilde{\boldsymbol{\mu}}_r^+)^{\times\ell}$ instead.

For an $\ell$-pointed twisted nodal curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$, denote $\omega_{\mathcal{C}/S}^{\log} := \omega_{\mathcal{C}/S}(\Sigma^{\mathcal{C}})$, and for $\alpha \in \tilde{\boldsymbol{\mu}}_r^+$, let $\Sigma_\alpha^{\mathcal{C}} = \coprod_{\gamma_i = \alpha} \Sigma_i^{\mathcal{C}}$.

**Definition 2.3.** *Let $S$ be a scheme, $(g,\gamma,\mathbf{d})$ be a numerical data. An $S$-family of MSP-fields (of Fermat type $F_{n,r}$) of type $(g,\gamma,\mathbf{d})$ is a datum*

$$(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$$

*such that*

(1) $\cup_{i=1}^{\ell}\Sigma_i^{\mathcal{C}}=\Sigma^{\mathcal{C}}\subset\mathcal{C}$ *is an $\ell$-pointed, genus $g$, twisted curve over $S$ such that the $i$-th marking $\Sigma_i^{\mathcal{C}}$ is banded by the group $\langle\gamma_i\rangle\le\mathbb{G}_m$;*

(2) *$\mathcal{L}$ and $\mathcal{N}$ are representable invertible sheaves on $\mathcal{C}$, and $\mathcal{L}\otimes\mathcal{N}$ and $\mathcal{N}$ have fiberwise degrees $d_0$ and $d_\infty$ respectively. The monodromy of $\mathcal{L}$ along $\Sigma_i^{\mathcal{C}}$ is $\gamma_i$ when $\langle\gamma_i\rangle\neq\langle 1\rangle$;*

(3) $\nu=(\nu_1,\nu_2)\in H^0(\mathcal{L}\otimes\mathcal{N})\oplus H^0(\mathcal{N})$ *such that $(\nu_1,\nu_2)$ is nowhere vanishing;*

(4) $\varphi=(\varphi_1,\ldots,\varphi_n)\in H^0(\mathcal{L})^{\oplus n}$ *so that $(\varphi,\nu_1)$ is nowhere zero, and $\varphi|_{\Sigma^{\mathcal{C}}_{(1,\varphi)}}=0$.*

(5) $\rho\in H^0(\mathcal{L}^{-r}\otimes\omega^{\log}_{\mathcal{C}/S})$ *such that $(\rho,\nu_2)$ is nowhere vanishing, and $\rho|_{\Sigma^{\mathcal{C}}_{(1,\rho)}}=0$;*

In the future, we call $\varphi$ (resp. $\rho$) the $\varphi$-field (resp. $\rho$-field) of the MSP-field.[1] (Here we abbreviate $\mathcal{L}^{-r}=\mathcal{L}^{\vee\otimes r}$.)

**Definition 2.4.** *In case $(g,\gamma,\mathbf{d})$ is a broad numerical data, a similarly defined $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$ as in Definition 2.3 is called an $S$-family of broad MSP-fields.*

We remark that in this paper we will only be concerned with MSP fields.

**Definition 2.5.** *An arrow*

$$(\mathcal{C}',\Sigma^{\mathcal{C}'},\mathcal{L}',\mathcal{N}',\varphi',\rho',\nu')\longrightarrow(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$$

*from an $S'$-MSP-field to an $S$-MSP-field consists of a morphism $S'\to S$ and a 3-tuple $(a,b,c)$ such that*

(1) $a:(\Sigma^{\mathcal{C}'}\subset\mathcal{C}')\to(\Sigma^{\mathcal{C}}\subset\mathcal{C})\times_S S'$ *is an $S'$-isomorphism of pointed twisted curves;*

(2) $b:a^*\mathcal{L}\to\mathcal{L}'$ *and* $c:a^*\mathcal{N}\to\mathcal{N}'$ *are isomorphisms of invertible sheaves such that the pullbacks of $\varphi_k$, $\rho$ and $\nu_i$ are identical to $\varphi'_k$, $\rho'$ and $\nu'_i$, where the pullbacks and the isomorphisms are induced by $a$, $b$ and $c$.*

We define $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$ to be the category fibered in groupoids over the category of schemes, such that the objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$ over $S$ are $S$-families of MSP-fields, and morphisms are given by Definition 2.5.

**Definition 2.6.** $\xi\in\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ *is* stable *if* $\mathrm{Aut}(\xi)$ *is finite.* $\xi\in\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ *is stable if $\xi|_s$ is stable for every closed point $s\in S$.*

Let $\mathcal{W}_{g,\gamma,\mathbf{d}}\subset\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$ be the open substack of families of stable objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}$. We introduce a $T=\mathbb{G}_m$ action on $\mathcal{W}_{g,\gamma,\mathbf{d}}$ by

$$t\cdot(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,(\nu_1,\nu_2))=(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,(t\nu_1,\nu_2)),\quad t\in T. \tag{2.4}$$

**Theorem 2.7.** *The stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is a DM $T$-stack, locally of finite type.*

[1] We call $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\nu)$ satisfying (1)-(3), and $\Sigma^{\mathcal{C}}_{(1,\varphi)}\subset(\nu_1\neq 0)$ and $\Sigma^{\mathcal{C}}_{(1,\rho)}\subset(\nu_2\neq 0)$, a gauged twisted $S$-curve.

*Proof.* The theorem follows immediately from that the stack $\mathcal{M}^{\rm tw}_{g,\ell}$ of stable twisted $\ell$-pointed curves is a DM stack, and each of its connected components is proper and of finite type (see [AJ, Ol]). □

In this paper, we will reserve the symbol $T = \mathbb{G}_m$ for this action on $\mathcal{W}_{g,\gamma,\mathbf{d}}$.

**Example 2.8** (Stable maps with $p$-fields)**.** *A stable MSP-field* $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}$ *having* $\nu_1 = 0$ *will have* $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$, $\nu_2 = 1$*. Then* $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$ *reduces to a stable map* $f = [\varphi] : \Sigma^{\mathcal{C}} \subset \mathcal{C} \to \mathbb{P}^{n-1}$ *together with a p-field* $\rho \in H^0(f^*\mathcal{O}_{\mathbb{P}^{n-1}}(-r) \otimes \omega^{\log}_{\mathcal{C}})$*. Moduli of genus* $g$ $\ell$*-pointed stable maps with p-fields will be denoted by* $\overline{M}_{g,\ell}(\mathbb{P}^{n-1}, d)^p$*.*

**Example 2.9** ($r$-Spin curves with $p$-fields)**.** *A stable MSP-field* $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}$ *having* $\nu_2 = 0$ *will have* $\mathcal{N} \cong \mathcal{L}^\vee$, $\nu_1 = 1$*. Then* $\xi$ *reduces to a pair of a r-spin curve* $(\Sigma^{\mathcal{C}}, \mathcal{C}, \rho : \mathcal{L}^r \cong \omega^{\log}_{\mathcal{C}})$ *and* $n$ *p-fields* $\varphi_i \in H^0(\mathcal{L})$*. Moduli of r-spin curves with fixed monodromy* $\gamma$ *and* $n$ *p-fields will be denoted by* $\overline{M}^{1/r,np}_{g,\gamma}$*.*

2.3. **Cosection localized virtual cycle.** The DM stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ admits a tautological $T$-equivariant perfect obstruction theory.

Let $\mathcal{D}_{g,\gamma}$ be the stack of triples $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$, where $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ are $\ell$-pointed genus $g$ connected twisted curves (i.e. objects in $\mathcal{M}^{\rm tw}_{g,\ell}$), $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$ such that the monodromy of $\mathcal{L}$ along the marked points are given by $\gamma$. Because $\mathcal{M}^{\rm tw}_{g,\ell}$ is a smooth DM stack, $\mathcal{D}_{g,\gamma}$ is a smooth Artin stack, locally of finite type and of dimension $(3g-3)+\ell+2(g-1) = 5g-5+\ell$, where the automorphisms of $(\Sigma^{\mathcal{C}} \subset \mathcal{C}, \mathcal{L}, \mathcal{N})$ are triples $(a,b,c)$ as in Definition 2.5.

Define

$$q : \mathcal{W}_{g,\gamma,\mathbf{d}} \longrightarrow \mathcal{D}_{g,\gamma} \tag{2.5}$$

to be the forgetful morphism, forgetting $(\varphi, \rho, \nu)$ from points $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}$. The morphism $q$ is $T$-equivariant with $T$ acting on $\mathcal{D}_{g,\gamma}$ trivially. Let

$$\pi : \Sigma^{\mathcal{C}} \subset \mathcal{C} \to \mathcal{W}_{g,\gamma,\mathbf{d}} \quad \text{with} \quad (\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu) \tag{2.6}$$

being the universal family over $\mathcal{W}_{g,\gamma,\mathbf{d}}$. Let $0 \le m_i \le r-1$ be so that $\gamma_i = \zeta_r^{m_i}$. For convenience, we let $\ell_\varphi = \#\{i \,|\, \gamma_i = (1,\varphi)\}$, and let $\ell_o = \#\{i \,|\, \gamma_i \in \boldsymbol{\mu}_r\}$. We abbreviate

$$\mathcal{P} = \mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/\mathcal{W}_{g,\gamma,\mathbf{d}}}.$$

**Proposition 2.10.** *The pair* $q : \mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ *admits a tautological* $T$*-equivariant relative perfect obstruction theory taking the form*

$$\Big(R\pi_*\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus n} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus (\mathcal{L}\otimes\mathcal{N}) \oplus \mathcal{N}\big)\Big)^\vee \longrightarrow L^\bullet_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}}.$$

*The virtual dimension* $\delta(g,\gamma,\mathbf{d}) := \mathrm{vir.dim}\, \mathcal{W}_{g,\gamma,\mathbf{d}}$ *is*

$$(1+n-r)d_0 + (1-n+r)d_\infty + (4-n)(g-1) + \ell + (1-n)\big(\ell_\varphi + \sum_{i=1}^{\ell_o} \frac{m_i}{r}\big).$$

*Proof.* The construction of the obstruction theory is parallel to that in [CL, Prop 2.5], and will be omitted. We compute its virtual dimension. Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \cdots)$ be a closed point in $\mathcal{W}_{g,\gamma,\mathbf{d}}$. Observe that when $\langle\gamma_i\rangle \neq \{1\}$, $\varphi|_{\Sigma_i^{\mathcal{C}}} = 0$. Thus by that $(\varphi, \nu_1)$ is nowhere vanishing, we see that $\nu_1|_{\Sigma_i^{\mathcal{C}}} \neq 0$, and the monodromy of $\mathcal{L} \otimes \mathcal{N}$ along $\Sigma_i^{\mathcal{C}}$ is trivial. Therefore, let $\pi : \mathcal{C} \to C$ be the coarse moduli morphism, the degrees of $\pi_*(\mathcal{L}\otimes\mathcal{N})$, $\pi_*\mathcal{L}$ and $\pi_*\mathcal{N}$ are $d_0$, $d_0 - d_\infty - \sum_{i=1}^{\ell_o} \frac{m_i}{r}$ and $d_\infty - \ell_o + \sum_{i=1}^{\ell_o} \frac{m_i}{r}$, respectively. Since the relative virtual dimension of $\mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ is

$$\chi(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus 5} \oplus \mathcal{L}^{\vee\otimes 5}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \otimes \omega_{\mathcal{C}}^{\log} \oplus \mathcal{L}\otimes\mathcal{N} \oplus \mathcal{N}).$$

Here we insert $\Sigma^{\mathcal{C}}_{(1,\varphi)}$ and $\Sigma^{\mathcal{C}}_{(1,\rho)}$ because of (4) and (5) in Definition 2.3. Using that $\dim \mathcal{D}_{g,\gamma} = 5g - 5 + \ell$, applying Riemann-Roch theorem to $\chi(\mathcal{L}) = \chi(\pi_*\mathcal{L})$, we obtain the formula of $\delta(g, \gamma, \mathbf{d})$, as stated in the proposition. □

The relative obstruction sheaf of $\mathcal{W}_{g,\gamma,\mathbf{d}} \to \mathcal{D}_{g,\gamma}$ is

$$\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} := R^1\pi_*\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus n} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus (\mathcal{L}\otimes\mathcal{N}) \oplus \mathcal{N}\big),$$

and the absolute obstruction sheaf $\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$ is the cokernel of the tautological map $q^*T_{\mathcal{D}_{g,\gamma}} \to \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}}$, fitting into the exact sequence

$$q^*T_{\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}} \longrightarrow 0. \tag{2.7}$$

We define a cosection

$$\sigma : \mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}/\mathcal{D}_{g,\gamma}} \longrightarrow \mathcal{O}_{\mathcal{W}_{g,\gamma,\mathbf{d}}} \tag{2.8}$$

by the rule that at an $S$-point $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S)$, (in the notation $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots)$ as in (1.2)),

$$\sigma(\xi)(\dot\varphi, \dot\rho, \dot\nu_1, \dot\nu_2) = r\rho \sum \varphi_i^{r-1}\dot\varphi_i + \dot\rho \sum \varphi_i^r \in H^1(\omega_{\mathcal{C}/S}) \equiv H^0(\mathcal{O}_{\mathcal{C}})^\vee, \tag{2.9}$$

where

$$(\dot\varphi, \dot\rho, \dot\nu_1, \dot\nu_2) \in H^1\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus n} \oplus \mathcal{P}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \oplus \mathcal{L}\otimes\mathcal{N} \oplus \mathcal{N}\big).$$

(Here $\mathcal{P} = \mathcal{L}^{\vee\otimes 5} \otimes \omega^{\log}_{\mathcal{C}/\mathcal{W}_{g,\gamma,\mathbf{d}}}$.) Note that the term $r\rho\sum\varphi_i^{r-1}\dot\varphi_i + \dot\rho\sum\varphi_i^r$ a priori lies in $H^1\big(\mathcal{C}, \omega^{\log}_{\mathcal{C}/S}(-\Sigma^{\mathcal{C}}_{(1,\rho)})\big)$. However, when $\langle\gamma_i\rangle \neq (1,\rho)$, $\varphi_j|_{\Sigma_i^{\mathcal{C}}} = 0$. Thus it lies in $H^1(\mathcal{C}, \omega_{\mathcal{C}/S})$.

**Lemma 2.11.** *The rule* (2.9) *defines a $T$-equivariant homomorphism $\sigma$ as in* (2.8)*. Via* (2.7) *the homomorphism $\sigma$ lifts to a $T$-equivariant cosection of $\mathcal{O}b_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$.*

*Proof.* The proof that the cosection $\sigma$ lifts is exactly the same as in [CLL], and will be omitted. That the homomorphism $\sigma$ is $T$-equivariant is because $T$ acts on $\mathcal{W}_{g,\gamma,\mathbf{d}}$ via scaling $\nu_1$ and $\sigma$ is independent of $\nu_1$. □

As in [KL], we define the degeneracy locus of $\sigma$ to be

$$\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) = \{\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid \sigma|_\xi = 0\}, \tag{2.10}$$

endowed with the reduced structure. It is a closed substack of $\mathcal{W}_{g,\gamma,\mathbf{d}}$.

**Lemma 2.12.** *The closed points of* $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ *are* $\xi\in\mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ *such that*

$$(\varphi=0)\cup(\varphi_1^r+\cdots+\varphi_n^r=\rho=0)=\mathcal{C}. \tag{2.11}$$

*Proof.* We consider individual terms in (2.9). Taking the term $\rho\varphi_i^{r-1}\dot\varphi_i$, by the vanishing along $\Sigma_i^{\mathcal{C}}$ recalled before the statement of Lemma 2.11, we conclude that

$$\rho\varphi_i^{r-1}\in H^0\big(\mathcal{L}^\vee\otimes\omega_{\mathcal{C}}^{\log}(-\Sigma^{\mathcal{C}})\big)=H^0\big(\mathcal{L}^\vee\otimes\omega_{\mathcal{C}}\big).$$

By Serre duality, when $\rho\varphi_i^{r-1}\neq 0$, there is a $\dot\varphi_i\in H^1(\mathcal{L})$ so that $\rho\varphi_i^{r-1}\cdot\dot\varphi_i\neq 0\in H^1(\omega_{\mathcal{C}})$.

Repeating this argument, we conclude that $\sigma|_\xi=0$ if and only if

$$\rho\varphi_1^{r-1}=\cdots=\rho\varphi_n^{r-1}=\varphi_1^r+\cdots+\varphi_n^r=0.$$

This is equivalent to that $(\varphi=0)\cup(\varphi_1^r+\cdots+\varphi_5^r=\rho=0)=\mathcal{C}$. □

Note that (2.10) makes sense for $\xi\in\mathcal{W}^{\rm pre}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ as well. For convenience, we denote

$$\mathcal{W}^{\rm pre-}_{g,\gamma,\mathbf{d}}(\mathbb{C})=\big\{\xi\in\mathcal{W}^{\rm pre}_{g,\gamma,\mathbf{d}}(\mathbb{C})\mid \text{(2.10) holds for }\xi\big\};$$

Applying [KL, CKL], we obtain the cosection localized virtual cycle

$$[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\rm vir}_{\rm loc}\in A^T_\delta\mathcal{W}^-_{g,\gamma,\mathbf{d}},\quad \delta=\delta(g,\gamma,\mathbf{d}).$$

2.4. **MSP invariants.** Using the universal family (2.6) we define the evaluation maps (associated to the marked sections $\Sigma_i^{\mathcal{C}}$):

$${\rm ev}_i:\mathcal{W}_{g,\gamma,\mathbf{d}}\to X:=\mathbb{P}^n\cup(\boldsymbol{\mu}_r)$$

as follows. In case $\langle\gamma_i\rangle\neq 1$, we define ${\rm ev}_i$ to be the constant map to $\langle\gamma_i\rangle\in\boldsymbol{\mu}_r$; in case $\gamma_i=(1,\varphi)$, we define ${\rm ev}_i(\gamma_i)=1\in\boldsymbol{\mu}_r$. In case $\gamma_i=(1,\rho)$, for $s_i:\mathcal{W}_{g,\gamma,\mathbf{d}}\to\mathcal{C}_{g,\gamma,\mathbf{d}}$ the $i$-th marked section of the universal curve, by (2) of Definition 2.3 we have $s_i^*\rho=0$. Thus $s_i^*\nu_2$ is nowhere vanishing, and $s_i^*\mathcal{N}\cong\mathcal{O}_{\mathcal{W}_{g,\gamma,\mathbf{d}}}$. Therefore, $s_i^*(\varphi,\nu_1)$ is a nonwhere vanishing section of $s_i^*\mathcal{L}^{\oplus(n+1)}$, defining the desired evaluation morphism

$${\rm ev}_i=[s_i^*\varphi_1,\cdots,s_i^*\varphi_n,s_i^*\nu_1]:\mathcal{W}_{g,\gamma,\mathbf{d}}\to\mathbb{P}^n \tag{2.12}$$

such that ${\rm ev}_i^*\mathcal{O}_{\mathbb{P}^n}(1)=s_i^*\mathcal{L}$.

Let $T$ act on $\mathbb{P}^n$ by

$$t\cdot[\varphi_1,\ldots,\varphi_n,\nu_1]=[\varphi_1,\ldots,\varphi_n,t\nu_1],$$

and let $T$ act trivially on $\boldsymbol{\mu}_r$. It makes ${\rm ev}_i$ $T$-equivariant.

We introduce the MSP state space. As a $\mathbb{C}$-vector space, the MSP state space and the $T$-equivariant MSP state space are the cohomology group and the $T$-equivariant cohomology group of the evaluation space $X=\mathbb{P}^n\cup(\boldsymbol{\mu}_r)$:

$$\mathcal{H}^{\rm MSP}=H^*(X;\mathbb{C}),\quad\text{and}\quad \mathcal{H}^{{\rm MSP},T}=H^*_T(X;\mathbb{C}).$$

In terms of generators, we have

$$H^*_T(\mathbb{P}^n;\mathbb{C})=\mathbb{C}[H,\mathfrak{t}]/\langle H^n(H+\mathfrak{t})\rangle,\quad\text{and}\quad H^*_T(\boldsymbol{\mu}_r;\mathbb{C})=\bigoplus_{j=1}^r\mathbb{C}[\mathfrak{t}]\mathbf{1}_{\frac{j}{r}},$$

and the (non-equivariant) MSP state space is by setting $\mathfrak{t}=0$, while the grading is given by

$$\deg H = 2, \quad \deg \mathfrak{t} = 2 \quad \text{and} \quad \deg \mathbf{1}_{\frac{j}{r}} = \frac{2(n-1)}{r} j. \tag{2.13}$$

We formulate the gravitational descendants. Given

$$a_1, \ldots, a_\ell \in \mathbb{Z}_{\geq 0}, \quad \phi_1, \ldots, \phi_\ell \in \mathcal{H}^{\mathrm{MSP}} = H^*(X; \mathbb{C}),$$

we define the MSP-invariants

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP}}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k \in \mathbb{C}. \tag{2.14}$$

where

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma \in (\tilde{\boldsymbol{\mu}}_r)^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Similarly, we define $T$-equivariant genus $g$ MSP-invariants to be

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k \in H^*(BT;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}], \tag{2.15}$$

where $\phi_i \in \mathcal{H}^{\mathrm{MSP},T}$, and

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma \in (\tilde{\boldsymbol{\mu}}_r)^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

(Here we use the same $[\cdot]^{\mathrm{vir}}_{\mathrm{loc}}$ to mean the $T$-equivariant class.) Suppose $\phi_1, \ldots, \phi_\ell$ are homogeneous, and let

$$\begin{aligned} \mathbf{e}(a.,\phi.) := \; & \textstyle\sum_{k=1}^{\ell} \Big( a_k + \frac{\deg \phi_k}{2} \Big) - \\ & - \Big( (1+n-r)d_0 + (1-n+r)d_\ell + (4-n)(g-1) + \ell \Big). \end{aligned}$$

By the formula of the virtual dimension of $\mathcal{W}_{g,\ell,\mathbf{d}}$, we see that

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} \in \mathbb{C}\mathfrak{t}^{\mathbf{e}(a.,\phi.)}.$$

In case $\mathbf{e}(a.,\phi.) < 0$, we have vanishing

$$\Big[ \mathfrak{t}^{-\mathbf{e}(a.,\phi.)} \cdot \langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} \Big]_0 = 0, \tag{2.16}$$

where $[\cdot]_0$ is the dimension 0 part of the pushforward to $H_0(pt)$.

By virtual localization, we will express all genus full descendant MSP invariants in terms of (1): GW invariants of the quintic threefold $Q \subset \mathbb{P}^4$; (2): FJRW invariants of the Fermat quintic; and (3): the descendant integrals on $\overline{\mathcal{M}}_{g,n}$. The invariants in item (1) has been solved in genus zero [Gi, LLY] and genus one for all degrees [LZ, Zi], and in all genus for degree zero, those in item (2) has been solved in genus zero [CR], and those in item (3) have been solved in all genera. One of our goals to introduce MSP invariants is to use vanishing (2.16) to obtain recursive relations to determine (1) and (2) in all genus. This will be addressed in details in the sequel [CLLL].

## 3. Properness of the degeneracy loci

In this section, we will prove that $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is proper over $\mathbb{C}$.

3.1. **The conventions.** In this section, we denote by $\eta_0 \in S$ a closed point in an affine smooth curve, and denote $S_* = S - \eta_0$ its complement.

In using valuative criterion to prove properness, we need to take a finite base change $S' \to S$ ramified over $\eta_0$. By shrinking $S$ if necessary, we assume there is an étale $S \to \mathbb{A}^1$ so that $\eta_0$ is the only point lying over $0 \in \mathbb{A}^1$. This way, we can take $S'$ to be of the form $S' = S \times_{\mathbb{A}^1} \mathbb{A}^1$, where $\mathbb{A}^1 \to \mathbb{A}^1$ is via $t \mapsto t^k$ for some integer $k \geq 2$. This way, $\eta_0' \in S'$ lying over $\eta_0 \in S$ is also the only point lying over $0 \in \mathbb{A}^1$. One particular choice of $S'$ is the degree $r$ base change: $S_r = S \times_{\mathbb{A}^1} \mathbb{A}^1 \to S$, where $\mathbb{A}^1 \to \mathbb{A}^1$ is via $t \mapsto t^r$.

To keep notations easy to follow, for a property P that holds after a finite base change $S' \to S$ of a family $\xi$ over $S$, we will say "after a finite base change, the family $\xi$ has the property P", meaning that we have already done the finite base change $S' \to S$ and then replace $S'$ by $S$ for abbreviation of notations.

In this and the next section, for $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ or $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$, we understand

$$\xi = \big(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu\big) \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}. \tag{3.1}$$

Similarly, we will use subscript "$*$" to denote families over $S_*$. Hence $\xi_* \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S_*)$ will be of the form

$$\xi_* = \big(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \mathcal{L}_*, \mathcal{N}_*, \varphi_*, \rho_*, \nu_*\big). \tag{3.2}$$

We first prove a simple version of the extension result we need.

**Proposition 3.1.** *Let $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be such that $\rho_* = 0$. Then after a finite base change, $\xi_*$ extends to a $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$.*

*Proof.* Since $\rho_* = 0$, $\nu_{2*}$ is nowhere vanishing and $\mathcal{N}_* \cong \mathcal{O}_*$. Thus $(\varphi_*, \nu_{1*})$ is a nowhere vanishing section of $H^0(\mathcal{L}_*^{\oplus n} \oplus \mathcal{L}_*)$, and induces a morphism $f_*$ from $(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*)$ to $\mathbb{P}^n$, such that $(f_*)^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}_*$. By the stability assumption of $\xi_*$, this morphism is an $S_*$-family of stable maps. By the properness of moduli stack of stable maps, after a finite base change we can extend $(\Sigma^{\mathcal{C}_*}, \mathcal{C}_*)$ to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ and extend $f_*$ to an $S$-family of stable maps $f$ from $(\Sigma^{\mathcal{C}}, \mathcal{C})$ to $\mathbb{P}^n$. Let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$, which is an extension of $\mathcal{L}_*$. Then $f$ is provided by a section $(\varphi, \nu_1) \in H^0(\mathcal{L}^{\oplus n} \oplus \mathcal{L})$, extending $(\varphi_*, \nu_{1*})$. Since $[f, \Sigma^{\mathcal{C}}, \mathcal{C}]$ is stable, the central fiber $\mathcal{C}_0$ is a connected curve with at worst nodal singularities. Define $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ and $\nu_2$ to be the isomorphism $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ extending $\nu_{2*}$. Define $\rho = 0$. Then $\xi = \big(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \varphi, \rho, \nu\big)$ is a desired extension. □

The case involving $\varphi_* = 0$ over some irreducible components is technically more involved. We will treat this case by first studying the case $\mathcal{C}_*$ is smooth. For this, we characterize stable objects in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(\mathbb{C})$. We say that an irreducible component $\mathcal{E} \subset \mathcal{C}$ is a rational curve if it is smooth and its coarse moduli is isomorphic to $\mathbb{P}^1$.

**Lemma 3.2.** *Let $p_1 \neq p_2 \in \mathbb{P}^1$ be two distinct closed points, $G \leq \mathrm{Aut}(\mathbb{P}^1)$ be the subgroup fixing $p_1$ and $p_2$, and $L$ be a $G$-linearized line bundle on $\mathbb{P}^1$ such that $G$ acts trivially on $L|_{p_1}$. Then the following holds:*

(1) *any invariant $s \in H^0(L)^G$ with $s(p_1) = 0$ must be the zero section;*
(2) *suppose $G$ acts trivially on $L|_{p_2}$, then $L \cong \mathcal{O}_{\mathbb{P}^1}$.*

*Proof.* Both are well-known. □

**Lemma 3.3.** *Let $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$. It is unstable if and only if one of the following holds:*

(1) *$\mathcal{C}$ contains a rational curve $\mathcal{E}$ such that $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains two points, and $\mathcal{L}^{\otimes r}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$;*
(2) *$\mathcal{C}$ contains a rational curve $\mathcal{E}$ such that $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains one point, and either $\mathcal{L}|_{\mathcal{E}} \cong \mathcal{N}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ or $\rho|_{\mathcal{E}}$ is nowhere vanishing;*
(3) *$\mathcal{C}$ is a smooth rational curve with $\Sigma^{\mathcal{C}} = \emptyset$, $d_0 = d_\infty = 0$.*
(4) *$\mathcal{C}$ is irreducible, $g = 1$, $\Sigma^{\mathcal{C}} = \emptyset$, and $\mathcal{L}^{\otimes r} \cong \mathcal{O}_{\mathcal{C}}$ and $\mathcal{L}^\vee \cong \mathcal{N}$.*

*Proof.* We first prove the necessary part. Let $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ be unstable. For each irreducible $\mathcal{E} \subset \mathcal{C}$, let $\mathrm{Aut}_{\mathcal{E}}(\xi)$ be the subgroup of $\mathrm{Aut}(\xi)$ mapping $\mathcal{E}$ to itself. There exists an $\mathcal{E}$ such that $\mathrm{Aut}_{\mathcal{E}}(\xi)$ is of infinite order. If the image of $\mathrm{Aut}_{\mathcal{E}}(\xi) \to \mathrm{Aut}(\mathcal{E})$ is finite, then for a finite index subgroup $G' \leq \mathrm{Aut}_{\mathcal{E}}(\xi)$, $G'$ leaves $\mathcal{C}$ fixed, thus $G'$ acts on $\xi$ by acting on the line bundles $\mathcal{L}$ and $\mathcal{N}$ via scaling. However, by Definition 2.3, that $G'$ leaves $(\varphi, \rho, \nu)$ invariant implies that the image of $G' \to \mathrm{Aut}(\mathcal{L}) \times \mathrm{Aut}(\mathcal{N})$ is finite. Since this arrow is injective, it contradicts to the fact that $G'$ is infinite. Thus the group $G = \mathrm{im}(\mathrm{Aut}_{\mathcal{E}}(\xi) \to \mathrm{Aut}(\mathcal{E}))$ is of infinite order.

We now consider the case where $\mathcal{E}$ has arithmetic genus zero (thus smooth). We divide it into several cases. The first case (when $g_a(\mathcal{E}) = 0$) is when $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains one point, say $p \in \mathcal{E}$. Suppose $\rho|_{\mathcal{E}} = 0$, then $\nu_2|_{\mathcal{E}}$ is nowhere vanishing, implying $\mathcal{N}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$. Thus $(\varphi, \nu_1)|_{\mathcal{E}}$ is a nowhere vanishing section of $H^0(\mathcal{L}^{\oplus(n+1)}|_{\mathcal{E}})$. Since $G$ is infinity and $(\varphi, \nu_1)|_{\mathcal{E}}$ is $G$-equivariant, this is possible only if $\mathcal{L}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ and $(\varphi, \nu_1)|_{\mathcal{E}}$ is a constant section. This is Case (2).

The other case is when $\rho|_{\mathcal{E}} \neq 0$. We argue that $\rho|_{\mathcal{E}}$ is nowhere vanishing. Otherwise, $\nu_2|_{\mathcal{E}} \neq 0$, and then $\deg \mathcal{N}|_{\mathcal{E}} \geq 0$. Since $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$, we have $\varphi|_{\mathcal{E}} = 0$, thus $\nu_1|_{\mathcal{E}}$ is nowhere vanishing and $\mathcal{L}^\vee|_{\mathcal{E}} \cong \mathcal{N}|_{\mathcal{E}}$. Because $\rho|_{\mathcal{E}} \neq 0$ and $\deg \omega^{\log}_{\mathcal{C}}|_{\mathcal{E}} = -1$, we must have $\deg \mathcal{L}|_{\mathcal{E}} < 0$. Thus $\nu_2 \in H^0(\mathcal{N}|_{\mathcal{E}}) = H^0(\mathcal{L}^\vee|_{\mathcal{E}})$ must vanish at some point. Let $p_1$ and $p_2 \in \mathcal{E}$ be such that $\rho(p_1) = 0 = \nu_2(p_2)$. Since $(\rho, \nu_2)$ is nowhere vanishing, we have $p_1 \neq p_2$. Furthermore, since $G$ fixes $p$, $p_1$ and $p_2$, and is infinite, $p = p_1$ or $p_2$.

Suppose $p = p_1$, a similar argument shows that $\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$, contradicting to $\rho \neq 0$ and vanishing somewhere. Suppose $p = p_2$, we conclude that $\deg \mathcal{L}|_{\mathcal{E}} = 0$, contradicting to $\deg \mathcal{L}|_{\mathcal{E}} < 0$. Combined, we proved that if $\rho|_{\mathcal{E}} \neq 0$, then $\rho|_{\mathcal{E}}$ is nowhere vanishing. This is Case (2).

The second case is when $\mathcal{E} \cap (\Sigma^{\mathcal{C}} \cup \mathcal{C}_{\mathrm{sing}})$ contains two points, say $p_1$ and $p_2 \in \mathcal{E}$. Then $G$ fixes both $p_1$ and $p_2$. A parallel argument shows that $G$ acts trivially on $\mathcal{L}^r|_{p_1}$ and $\mathcal{L}^r|_{p_2}$. Applying Lemma 3.2, we conclude that $\mathcal{L}^r \cong \mathcal{O}_{\mathcal{E}}$. This is Case

(1). The third case is when $\mathcal{E}\cap(\Sigma^{\mathcal{C}}\cup\mathcal{C}_{\text{sing}})=\emptyset$. A parallel argument shows that in this case we must have $\mathcal{L}\cong\mathcal{N}\cong\mathcal{O}_{\mathcal{C}}$. This conclude the study of the case $g_a(\mathcal{E})=0$.

The remaining case is when $g_a(\mathcal{E})=1$, then $\mathcal{E}\cap\Sigma^{\mathcal{C}}=\emptyset$, and a similar argument shows that it must belong to Case (4). Combined, this proves that if $\xi$ is unstable, then one of (1)-(4) holds.

We now prove the other direction that whenever there is an $\mathcal{E}\subset\mathcal{C}$ that satisfies one of (1)-(4), then $\xi$ is unstable. Most of the cases can be argued easily, except a sub-case of (2) when $\rho|_{\mathcal{E}}$ is nowhere vanishing, which we now prove. Since $\mathcal{E}\cap(\Sigma_{\mathcal{C}}\cup\mathcal{C}_{\text{sing}})$ contains one point, say $p\in\mathcal{C}$, we have $\deg\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}=-1$. Since $\rho|_{\mathcal{E}}$ is nowhere vanishing, we have $\deg\mathcal{L}^r|_{\mathcal{E}}=-1$. Thus $p$ must be a stacky point. Hence $\mathcal{E}\cong\mathbb{P}_{1,r}$ as stacks. Let $\mathbb{P}_{1,r}=\text{Proj}(k[x,y])$ where $\deg x=1$ and $\deg y=r$. Then $p$ corresponds to the point $[0,1]$. Let $G=\mathbb{C}$ (the additive group) acts on $\mathbb{P}_{1,r}$ via $x\to x$, $y\to\lambda x^r+y$ for $\lambda\in G$. The $G$-action on $\mathbb{P}_{1,r}$ lifts to an action of $\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}$ as well as $\mathcal{L}^\vee|_{\mathcal{E}}\cong\mathcal{O}_{\mathbb{P}_{1,r}}(1)$. One can check via local calculations that $G$ acts trivially on $\big(\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}\big)|_p$ as well as $\big(\mathcal{L}^{-1}|_{\mathcal{E}}\big)|_p$, thus trivially on $\big(\mathcal{L}^{-r}\otimes\omega_{\mathcal{C}}^{\log}\big)|_p$. Since $\mathcal{L}^{-r}\otimes\omega_{\mathcal{C}}^{\log}|_{\mathcal{E}}\cong\mathcal{O}_{\mathcal{E}}$, and since $\mathbb{G}_a=\mathbb{C}$ has no non-trivial characters, by Prop.1.4 in [FMK], $G$ acts on $\big(\mathcal{L}^{-r}\otimes\omega_{\mathcal{C}}^{\log}\big)|_{\mathcal{E}}$ trivially as well. Hence $G$ acts trivially on $\rho|_{\mathcal{E}}$. Therefore the group $G$ is a subgroup of the automorphism group of $\xi|_{\mathcal{E}}$. □

**Corollary 3.4.** *Let $\xi\in\mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(\mathbb{C})$. Let $\pi:\tilde{\mathcal{C}}\to\mathcal{C}$ be the normalization of $\mathcal{C}$, let $\Sigma^{\tilde{\mathcal{C}}}=\pi^{-1}(\Sigma^{\mathcal{C}}\cup\mathcal{C}_{\text{sing}})$, and let $(\tilde{\mathcal{L}},\tilde{\mathcal{N}},\tilde{\varphi},\tilde{\rho},\tilde{\nu})$ be the pullback of $(\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$ via $\pi^*$. Write $\tilde{\mathcal{C}}=\coprod_a\tilde{\mathcal{C}}_a$ the connected component decomposition, and let $\tilde{\xi}_a$ be $(\Sigma^{\tilde{\mathcal{C}}}\cap\tilde{\mathcal{C}}_a,\tilde{\mathcal{C}}_a)$ paired with $(\tilde{\mathcal{L}},\tilde{\mathcal{N}},\tilde{\varphi},\tilde{\rho},\tilde{\nu})|_{\tilde{\mathcal{C}}_a}$. Then $\xi$ is stable if and only if all $\tilde{\xi}_a$ are stable.*

*Proof.* If $\xi$ is unstable, then it contains an irreducible $\mathcal{E}$ satisfying one of (1)-(4) in Lemma 3.3. This $\mathcal{E}$ (or its normalization) will appear in one of $\tilde{\xi}_a$, making it unstable. The other direction is the same. This proves the Corollary. □

## 3.2. **The baskets.**

We will first studying a special case.

**Special type**: *Let $\xi_*\in\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be of the form (3.2) such that $\mathcal{C}_*$ is smooth (and connected), $\varphi_*=0$, $\nu_{2*}\neq0$ and $\rho_*\neq0$.*

**Proposition 3.5.** *Let $\xi_*$ over $S_*$ be of special type. Then after a finite base change,*

- **(a1)** *$(\Sigma^{\mathcal{C}_*},\mathcal{C}_*)$ extends to a pointed twisted curve $(\Sigma^{\mathcal{C}},\mathcal{C})$ over $S$ such that $\mathcal{C}-\Sigma^{\mathcal{C}}$ is a scheme, $\mathcal{C}$ is smooth, and the central fiber $\mathcal{C}_0$ is reduced with at worst nodal singularities and smooth irreducible components;*
- **(a2)** *$\mathcal{L}_*$ and $\mathcal{N}_*$ extend to invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ respectively on $\mathcal{C}$ so that $\nu_*$ extends to a surjective $\nu=(\nu_1,\nu_2):\mathcal{L}^\vee\oplus\mathcal{O}_{\mathcal{C}}\to\mathcal{N}$;*
- **(a3)** *$\rho_*$ extends to a $\rho\in\Gamma(\mathcal{L}^{\vee\otimes5}\otimes\omega^{\log}_{\mathcal{C}/S}(\mathcal{D}))$ for a divisor $\mathcal{D}\subset\mathcal{C}$ contained in the central fiber $\mathcal{C}_0$ such that $\rho$ restricting to every irreducible component of $\mathcal{C}_0$ is non-trivial;*
- **(a4)** *$\overline{(\rho_*=0)}\cap\overline{(\nu_{2*}=0)}=\emptyset$, $\overline{(\rho_*=0)}$ and $\overline{(\nu_{2*}=0)}$ intersect $\mathcal{C}_0$ transversally.*

*Proof.* First, possibly after a finite base change, we can assume that $\overline{(\rho_* = 0)} \cup \overline{(\nu_{2*} = 0)}$ is a union of disjoint sections of $\mathcal{C}_* \to S_*$, and that if we let $\Sigma_*^{\mathrm{ex}}$ be the union of those sections of $\overline{(\rho_* = 0)} \cup \overline{(\nu_{2*} = 0)}$ that are not contained in $\Sigma^{\mathcal{C}_*}$, then $\Sigma_*^{\mathrm{ex}}$ is disjoint from $\Sigma^{\mathcal{C}_*}$. If $(\Sigma^{\mathcal{C}_*} \cup \Sigma_*^{\mathrm{ex}}, \mathcal{C}_*)$ is a stable pointed curve, let $\Sigma_*^{\mathrm{au}} = \emptyset$. Otherwise, let $\Sigma_*^{\mathrm{au}}$ be some extra sections of $\mathcal{C}_* \to S_*$, disjoint from $\Sigma^{\mathcal{C}_*} \cup \Sigma_*^{\mathrm{ex}}$, so that after letting $\Sigma_*^{\mathrm{comb}} = \Sigma^{\mathcal{C}_*} \cup \Sigma_*^{\mathrm{ex}} \cup \Sigma_*^{\mathrm{au}}$, the pair $(\Sigma_*^{\mathrm{comb}}, \mathcal{C}_*)$ is stable.

Since $(\Sigma_*^{\mathrm{comb}}, \mathcal{C}_*)$ is stable, possibly after a finite base change, it extends to an $S$-family of stable twisted curves $(\Sigma^{\mathrm{comb}}, \mathcal{C}')$ such that all singular points of its central fiber $\mathcal{C}'_0$ are non-stacky. Thus after blowing up $\mathcal{C}'$ along the singular points of $\mathcal{C}'_0$ if necessary, taking a finite base change, and followed by a minimal desingularization, we can assume that the resulting family $(\Sigma^{\mathcal{C}}, \mathcal{C})$ is a family of pointed twisted curves with smooth $\mathcal{C}$ satisfying Condition (a1). Condition (a4) is satisfied due to the construction.

Since $\varphi_*$ is identically zero, $\nu_{1*}$ is an isomorphism. We can extend $\mathcal{N}_*$ to an invertible sheaf $\mathcal{N}$ on $\mathcal{C}$ so that $\nu_{2*}$ extends to a section $\nu_2$ of $\mathcal{N}$. Let $\mathcal{L} \cong \mathcal{N}^\vee$. We extend $\nu_{1*}$ to an isomorphism $\nu_1 \colon \mathcal{L}^\vee \to \mathcal{N}$, and extend $\rho_*$ to a section $\rho$ satisfying (a3), for a choice of $\mathcal{D}$. This proves the proposition. □

We will work with the coarse moduli $C$ of $\mathcal{C}$.

**Definition 3.6.** *An $S$-family of pre-stacky pointed nodal curves is a flat $S$-family $(\Sigma, C)$ of pointed nodal curves (i.e. not twisted curves) so that each marked-section $\Sigma_i$ (of $\Sigma$) is either assigned pre-stacky or assigned regular. We call it a good family if in addition $C$ is smooth, and all irreducible components of the central fiber $C_0 = C \times_S \eta_0$ are smooth.*

Note that if we let $C$ be the coarse moduli of the $\mathcal{C}$ in Proposition 3.5, let $\Sigma_i \subset C$ be the image of $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$ under $\mathcal{C} \to C$, and call $\Sigma_i$ pre-stacky if $\Sigma_i^{\mathcal{C}}$ is stacky, and call it regular otherwise, then $(\Sigma, C)$ with this assignment is a good $S$-family of pre-stacky pointed nodal curves.

Given an $S$-family of pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$ so that the only non-scheme points of $\mathcal{C}$ are possibly along $\Sigma^{\mathcal{C}}$, applying the procedure described, we obtain a pre-stacky pointed nodal curve $(\Sigma, C)$. We call this procedure un-stacking. Conversely, applying the root construction (cf. [AGV, Cad]) to the $S$-family of pre-stacky pointed nodal curves $(\Sigma, C)$ obtained, we recover the original family $(\Sigma^{\mathcal{C}}, \mathcal{C})$; we call the later the stacking of $(\Sigma, C)$.

**Definition 3.7.** *Let $(\Sigma, C)$ be a good $S$-family of pre-stacky pointed nodal curves, and let $D_i$, $i \in \Lambda$, be irreducible components of $C_0$. A pre-basket of $(\Sigma, C)$ is a data*

$$\mathcal{B} = \Big(B + \sum\nolimits_{i\in\Lambda} l_i D_i, A + \sum\nolimits_{i\in\Lambda} m_i D_i\Big), \tag{3.3}$$

*where*

(1) *$A = \sum_{i=1}^{k_1} a_i A_i$, where $A_1, \cdots, A_{k_1}$ are disjoint sections of $C \to S$ such that for any pair $(i, j)$, either $A_i \cap \Sigma_j = \emptyset$ or $A_i = \Sigma_j$, $a_i \in \frac{1}{r}\mathbb{Z}_{>0}$ when $A_i = \Sigma_j$ for some pre-stacky $\Sigma_j$, otherwise $a_i \in \mathbb{Z}_{>0}$;*

(2) *$B=\sum_{i=1}^{k_2} b_iB_i$, where $b_i\in\mathbb{Z}_{>0}$, $B_1,\cdots,B_{k_2}$ are disjoint sections of $C\to S$ such that for any pair $(i,j)$, either $B_i\cap\Sigma_j=\emptyset$ or $B_i=\Sigma_j$, and when $B_i=\Sigma_j$, $\Sigma_j$ must be assigned regular;*

(3) *$A_1,\cdots,A_{k_1},B_1,\cdots,B_{k_2}$ are mutually disjoint and intersect $C_0$ transversally;*

(4) *$rm_i\in\mathbb{Z}$ and $l_i\in\mathbb{Z}$;*

*such that*

$$\mathcal{O}_C(B+\sum l_iD_i)\cong\mathcal{O}_C(rA+\sum rm_iD_i)\otimes\omega_{C/S}^{\log}. \tag{3.4}$$

*We call $\mathcal{B}$ a basket if in addition it satisfies $l_i\ge 0$, $m_i\ge 0$ and $l_im_i=0$ for all $i$.*

**Definition 3.8.** *We say a basket $\mathcal{B}$ final if it satisfies*
*(i). for every $i\in\Lambda$, $B\cap D_i=\emptyset$ if $m_i\neq 0$, and $A\cap D_i=\emptyset$ if $l_i\neq 0$;*
*(ii). for distinct $i\neq j\in\Lambda$ such that $l_im_j\neq 0$, $D_i\cap D_j=\emptyset$.*

Let $(\mathcal{C},\mathcal{L},\mathcal{N},\rho,\nu)$ and $\mathcal{D}$ be given by Proposition 3.5. Let $\{\mathcal{D}_i\mid i\in\Lambda\}$ be the set of irreducible components of $\mathcal{C}_0$. We form

$$\mathcal{A}=\overline{(\nu_{2*}=0)},\ \mathcal{B}=\overline{(\rho_*=0)},\ (\nu_2=0)=\mathcal{A}+\sum m_i\mathcal{D}_i,\ \mathcal{D}=-\sum l_i\mathcal{D}_i, \tag{3.5}$$

where the summations run over all $i\in\Lambda$. By the construction, $\nu_2$ and $\rho$ induce isomorphisms $\mathcal{N}\cong\mathcal{O}_{\mathcal{C}}(\mathcal{A}+\sum m_i\mathcal{D}_i)$ and $\mathcal{O}_{\mathcal{C}}\cong\mathcal{L}^{-r}\otimes\omega_{\mathcal{C}/S}^{\log}(\mathcal{D}-\mathcal{B})$. Using $\mathcal{L}^\vee\cong\mathcal{N}$, we obtain an isomorphism

$$\mathcal{O}_{\mathcal{C}}(\mathcal{B}+\sum l_i\mathcal{D}_i)\cong\mathcal{O}_{\mathcal{C}}(r\mathcal{A}+\sum rm_i\mathcal{D}_i)\otimes\omega_{\mathcal{C}/S}^{\log}. \tag{3.6}$$

Let $(\Sigma,C)$ be the good $S$-family of pre-stacky pointed nodal curves that is the un-stacking of $(\Sigma^{\mathcal{C}},\mathcal{C})$ as explained before Definition 3.7. Let $D_i\subset C_0$ be the image of $\mathcal{D}_i$. Since $\mathcal{C}$ away from $(\nu_2=0)$ is a scheme, and by the construction carried out in the proof of Proposition 3.5, we have $\mathcal{B}=\sum_{i=1}^{k_2}b_i\mathcal{B}_i$, where $\mathcal{B}_i$ are sections of $\mathcal{C}\to S$ and $b_i\in\mathbb{Z}_{>0}$, and for any $(i,j)$ either $\mathcal{B}_i\cap\Sigma_j^{\mathcal{C}}=\emptyset$ or $\mathcal{B}_i=\Sigma_j^{\mathcal{C}}$, and in the later case $\Sigma_j^{\mathcal{C}}$ is a scheme. We let $B_i\subset C$ be the image of $\mathcal{B}_i$. For $\mathcal{A}$, it can also be written as $\mathcal{A}=\sum_{i=1}^{k_1}a_i\mathcal{A}_i$, where $\mathcal{A}_i$ are sections of $\mathcal{C}\to S$. Let $A_i$ be the image of $\mathcal{A}_i$. We form

$$A=\sum_{A_i\notin\{\text{pre-stacky }\Sigma_j\}}a_iA_i+\sum_{A_i\in\{\text{pre-stacky }\Sigma_j\}}\frac{a_i}{r}A_i,\quad\text{and}\quad B=\sum_{i=1}^{k_2}b_iB_i.$$

**Lemma 3.9.** *Let $(\Sigma,C)$ be as before, let $\mathcal{B}$ in (3.3) be such that the coefficients $l_i$ and $m_i$ are given in (3.5), and let $A$ and $B$ be given in the identities above. Then $\mathcal{B}$ is a pre-basket.*

*Proof.* That $\mathcal{B}$ satisfies (1)-(4) in Definition 3.7 follows from the proof of Proposition 3.5. For the isomorphism (3.4), we notice that by our choice of $A$ and $B$, we have

$$\mathcal{O}_{\mathcal{C}}(\mathcal{B}+\sum l_i\mathcal{D}_i)\cong\mathcal{O}_C(B+\sum l_iD_i)\otimes_{\mathcal{O}_C}\mathcal{O}_{\mathcal{C}}$$

and

$$\mathcal{O}_{\mathcal{C}}(r\mathcal{A}+\sum rm_i\mathcal{D}_i)\otimes\omega_{\mathcal{C}/S}^{\log}\cong\big(\mathcal{O}_C(rA+\sum rm_iD_i)\otimes\omega_{C/S}^{\log}\big)\otimes_{\mathcal{O}_C}\mathcal{O}_{\mathcal{C}}.$$

Therefore, (3.4) follows from (3.6). This proves the Lemma. □

3.3. **Restacking.** In this subsection, we fix an $S$-family of pre-stacky $n$-pointed nodal curves $(\Sigma, C)$ and a final basket $\mathcal{B}$ on it in the notations of Definition 3.7 and 3.8. Let $t$ be a uniformizing parameter of $R$, where $S = \operatorname{Spec} R$, that is the pullback of the standard coordinate variable of $\mathbb{A}^1$ via the map $S \to \mathbb{A}^1$ specified at the beginning of §3.1. Let $R_a = R[z]/(z^a - t)$, and let $S_a = \operatorname{Spec} R_a$.

**Lemma 3.10.** *Let $C$ be a flat $S$-family of nodal curves, let $N$ be the singular points of the central fiber $C_0$, and let $M$ be an (integral) effective Cartier divisor on $C$ such that $M = rM_h + M_0$, where $M_0$ (resp. $M_h$) is an integral Weil divisor contained in $C_0$ (resp. none of its irreducible components lie in $C_0$). Then there is a unique $S_r$-family of twisted curves $\tilde{\mathcal{C}}$ such that*

1. *let $\tilde N \subset \tilde{\mathcal{C}}_0$ be the singular points of $\tilde{\mathcal{C}}_0$, then $\tilde{\mathcal{C}} - \tilde N \cong (C - N) \times_S S_r$;*
2. *let $\tilde\phi : \tilde{\mathcal{C}} - \tilde N \to C$ be the morphism induced by (1), then $\tilde\phi^*(M)$ is divisible by $r$, and $\frac{1}{r}\tilde\phi^*(M)$ extends to a Cartier divisor on $\tilde{\mathcal{C}}$, denoted by $\tilde M_{\frac{1}{r}}$;*
3. *each $\zeta \in \tilde N$ is either a scheme point or a $\boldsymbol{\mu}_a$-stacky point of $\tilde{\mathcal{C}}$, where $a|r$, such that the tautological map $\operatorname{Aut}(\zeta) \to \operatorname{Aut}(\mathcal{O}_{\tilde{\mathcal{C}}}(\tilde M_{\frac{1}{r}})|_\zeta)$ is injective.*

*Proof.* Let $p \in N$ be a singular point of $C_0$. Pick an étale open neighborhood $q\colon V \to C$ of $p \in C$ so that $V$ is an open subscheme of $(xy = t^k) \subset \operatorname{Spec}(R[x, y])$, as $S$-schemes. Let $D_1$ and $D_2$ in $V$ be $(x = t = 0)$ and $(y = t = 0)$, respectively (Example 6.5.2 in [Har]). When $k \neq 1$, $D_1$ and $D_2$ are Weil but not Cartier divisors. Write $q^{-1}M = rA + n_1D_1 + n_2D_2$, where $A = q^{-1}M_h$ is an integral Weil divisor with no irreducible components contained in $D_1 \cup D_2$. Let $\mathrm{CL}(V)$ (resp. $\mathrm{Car}(V)$) be the Weil (resp. Cartier) divisor class groups of $V$ respectively. It is known that $\mathrm{CL}(V)/\mathrm{Car}(V) = \mathbb{Z}/k\mathbb{Z}$, generated by $D_1$ (or $D_2$). Thus $A$ is linearly equivalent to $lD_1 + B$ for an integer $l$ and a Cartier divisor $B$. Since $M$ is a Cartier divisor, we have $rl + n_1 - n_2 \equiv 0(k)$ (i.e. $\equiv 0 \mod (k)$). Here we used the fact that $D_1 + D_2 = 0 \in \mathrm{CL}(V)/\mathrm{Car}(V)$.

Consider the base change $\tilde C := C \times_S S_r \to C$. Let $\tilde p$ be the node in the central fiber of $\tilde C$ corresponding to $p$. Let $\tilde V = V \times_S S_r$ and $\tilde q : \tilde V \to V$ be the projection. It is an étale neighborhood of $\tilde p \in \tilde C$, and is an open subsheme of $(xy = z^{rk}) \subset \operatorname{Spec}(R_r[x, y])$.

Let $\tilde A$ be $\tilde q^{-1}(A)$, $\tilde D_1 = (x = z = 0)$, and $\tilde D_2 = (y = z = 0)$. Since $\tilde q^{-1}(D_i) = r\tilde D_i$, the pullback $\tilde q^{-1}(M) = r\tilde A + rn_1\tilde D_1 + rn_2\tilde D_2$, and $\frac{1}{r}\tilde q^{-1}(M)$ away from $\tilde p$ is Cartier. Since $A = lD_1 \in \mathrm{CL}(V)/\mathrm{Car}(V)$, $\tilde A = l(r\tilde D_1) \in \mathrm{CL}(\tilde V)/\mathrm{Car}(\tilde V)$. Thus

$$\tilde q^{-1}(M) = r\tilde A + rn_1\tilde D_1 + rn_2\tilde D_2 \equiv (2rl + rn_1 - rn_2)\tilde D_1 \in \mathrm{CL}(\tilde V)/\mathrm{Car}(\tilde V).$$

Let $a'$ be defined by

$$(r\ell + n_1 - n_2, rk) = a'k, \quad a' \in [1, r], \tag{3.7}$$

and let $a = r/a'$. Since $k|(r\ell + n_1 - n_2)$, $a \in \mathbb{Z}$ and $a|r$. Thus $\frac{1}{r}\tilde q^{-1}(M)$ is Cartier only when $a = 1$.

To make it Cartier when $a > 1$, we introduce $\boldsymbol{\mu}_a$-stacky structure at $\tilde p$ as follows. Consider $(uv = z^k) \subset \operatorname{Spec}(R_a[u, v])$ and the morphism $(uv = z^k) \to (xy = z^{ak})$

via $x \mapsto u^a$ and $y \mapsto v^a$. Let $U \subset (uv = z^k)$ be the open subscheme mapped onto $\tilde{V}$, and let $\zeta \in \boldsymbol{\mu}_a$ act on $U$ via $\zeta \cdot (u, v) = (\zeta u, \zeta^{-1} v)$. Then the quotient $U/\boldsymbol{\mu}_a \cong \tilde{V}$. Let $\phi : U \to V$ be the induced projection, let $\bar{D}_1 = (u = z = 0)$ and $\bar{D}_2 = (v = z = 0)$. Then $\phi^{-1} D_i = a\bar{D}_i$ and hence $\frac{1}{a}\phi^{-1}(M) = \bar{A} + n_1\bar{D}_1 + n_2\bar{D}_2$, where $\bar{A} = \phi^{-1}(M_h)$, and is $\boldsymbol{\mu}_a$-invariant. Since $A \equiv lD_1 \in \mathrm{CL}(V)/\mathrm{Car}(V)$, $\bar{A} = l\phi^{-1}(D_1) = al\bar{D}_1 \in \mathrm{CL}(U)/\mathrm{Car}(U)$, and is $\boldsymbol{\mu}_a$-invariant. Hence

$$\frac{1}{r}\phi^{-1}(M) \equiv \frac{(rl + n_1 - n_2)}{a'}\bar{D}_1 \equiv 0 \in \mathrm{CL}(U)/\mathrm{Car}(U),$$

because of (3.7). Thus is Cartier.

Therefore, if $a = 1$, we do nothing at $\tilde{p}$; otherwise, we introduce a stacky structure at $\tilde{p}$ by replacing a neighborhood of $\tilde{p} \in \tilde{C}$ by the quotient stack $[U/\boldsymbol{\mu}_a]$ (cf. [AV]) and denote the resulting stack by $\tilde{\mathcal{C}}$. By repeating this over all $p \in N$, we obtain $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ such that $\tilde{M}_{\frac{1}{r}} = \frac{1}{r}\tilde{\phi}^{-1}(M)$ is an integral Cartier divisor satisfying the requirements of the Lemma. ☐

**Corollary 3.11.** *Let $C$ be a flat $S$-family of nodal curves, let $N$ be the singular points of $C_0$, and $v \in H^0(C - N, \mathcal{M})$ be a section of an invertible sheaf $\mathcal{M}$ on $C - N$ so that $\mathcal{M}^{\otimes r}$ extends to an invertible sheaf on $C$. Then there is a unique $S_r$-family of twisted nodal curves $\tilde{\mathcal{C}}$ such that*

1. *let $\tilde{N} \subset \tilde{\mathcal{C}}_0$ be the singular points of $\tilde{\mathcal{C}}_0$, then $\tilde{\mathcal{C}} - \tilde{N} \cong (C - N) \times_S S_r$;*
2. *there is an invertible sheaf $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{C}}$ and a section $\tilde{v} \in H^0(\tilde{\mathcal{M}})$ so that, letting $\tilde{\phi} : \tilde{\mathcal{C}} - \tilde{N} \to C$ be the morphism induced by (1), then $\tilde{\mathcal{M}}|_{\tilde{\mathcal{C}} - \tilde{N}} \cong \tilde{\phi}^*\mathcal{M}$, and $\tilde{v}|_{\tilde{\mathcal{C}} - \tilde{N}} = \tilde{\phi}^* v$;*
3. *each $p \in \tilde{N}$ is either a scheme point or a $\boldsymbol{\mu}_a$-stacky point, $a|r$, of $\tilde{\mathcal{C}}$, and the tautological map $\mathrm{Aut}(\tilde{p}) \to \mathrm{Aut}(\tilde{\mathcal{M}}|_{\tilde{p}})$ is injective.*

*Proof.* Since $C$ is normal, and $\mathcal{M}^{\otimes r}$ extends to an invertible sheaf on $C$, $v^r$ extends to a regular section over $C$, thus $M = \overline{(v^r = 0)}$ is a Cartier divisor on $C$. As $M|_{C-N} = r(v = 0)$, we can write $M = rM_h + M_0$, where $M_0$ is supported on $C_0$ and no irreducible components of $M_h$ lie in $C_0$.

Let $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ be the $S_r$-family of twisted curves constructed in the previous Lemma for the Cartier divisor $M = rM_h + M_0$, and $\tilde{M}_{\frac{1}{r}}$ be the Cartier divisor so that $\tilde{\phi}^{-1}(M) = r\tilde{M}_{\frac{1}{r}}$. Let $\tilde{\mathcal{M}} = \mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{M}_{\frac{1}{r}})$. Then $\tilde{\mathcal{M}}$ is invertible, with a tautological section $\tilde{v} \in H^0(\tilde{\mathcal{M}})$ so that $(\tilde{v} = 0) = \tilde{M}_{\frac{1}{r}}$. Because

$$(\tilde{v} = 0) \cap (\tilde{\mathcal{C}} - \tilde{N}) = \big((v = 0) \times_S S_r\big) \cap (\tilde{\mathcal{C}} - \tilde{N}),$$

we conclude that we have isomorphism $\tilde{\mathcal{M}}|_{\tilde{\mathcal{C}} - \tilde{N}} \cong \tilde{\phi}^*\mathcal{M}|_{\tilde{\mathcal{C}} - \tilde{N}}$ so that $\tilde{v}|_{\tilde{\mathcal{C}} - \tilde{N}} = \tilde{\phi}^* v|_{\tilde{\mathcal{C}} - \tilde{N}}$. This proves the corollary. ☐

Let $(\Sigma, C)$ be a good $S$-family of pre-stacky pointed curves, and let $\mathcal{B}$ be a final basket, in the notation of Definition 3.4 and 3.6. We shall provide a procedure to construct a family in $\mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$.

We first restacking the pre-stacky pointed curve $(\Sigma \times_S S_r, C \times_S S_r)$ to obtain an $S$-family of pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$. Let $C$ be the coarse moduli of $\mathcal{C}$; let

$q : \mathcal{C} - (\mathcal{C}_0)_{\rm sing} \to C$ be the projection. Let $M = rA + \sum rm_iD_i$, which is a Cartier divisor on $C$ so that $D_i$ are supported along $C_0$ and no irreducible component of $A$ lies in $C_0$. Thus $\frac{1}{r}q^*M$ is a Cartier divisor. We then apply Lemma 3.10 and Corollary 3.11 to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ to obtain an $S_r$-family of pointed twisted curve $(\Sigma^{\tilde{\mathcal{C}}}, \tilde{\mathcal{C}})$ such that it is isomorphic to $(\Sigma^{\mathcal{C}}, \mathcal{C})$ away from the singular points of the central fiber, and there is an invertible sheaf $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{C}}$ with a section $\tilde{v}$ so that $\tilde{\mathcal{M}}$ is the extension of $\mathcal{O}_{\tilde{\mathcal{C}}-(\tilde{\mathcal{C}}_0)_{\rm sing}}(\frac{1}{r}q^*M)$ and $\tilde{v}$ is the extension of the tautological section of the latter.

Let $\tilde{\phi} : \tilde{\mathcal{C}} \to C$ be the tautological morphism, let $\tilde{\mathcal{N}} = \tilde{\mathcal{M}}$, and let $\tilde{\nu}_2 = \tilde{v} \in H^0(\tilde{\mathcal{N}}) = H^0(\tilde{\mathcal{M}})$. Let $\tilde{\nu}_1 : \tilde{\mathcal{L}} \cong \tilde{\mathcal{N}}^\vee$. By (3.4), we conclude that

$$\mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{\phi}^*(B + \sum l_iD_i)) \cong \tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}.$$

Let $\tilde{\rho} \in \Gamma(\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S})$ be induced by the above isomorphism and the tautological inclusion $\mathcal{O}_{\tilde{\mathcal{C}}} \subset \mathcal{O}_{\tilde{\mathcal{C}}}(\tilde{\phi}^*(B + \sum l_iD_i))$. Because $\rho_*$ vanishes along $\Sigma^{\mathcal{C}_*}_{(1,\rho)}$, $\tilde{\rho}$ lifts to a section in $\Gamma(\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}(-\Sigma^{\tilde{\mathcal{C}}}_{(1,\rho)}))$. This proves

**Lemma 3.12.** *Let notations be as stated. Then* $\tilde{\xi} = (\Sigma^{\tilde{\mathcal{C}}}, \tilde{\mathcal{C}}, \tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi} = 0, \tilde{\rho}, \tilde{\nu})$ *constructed based on a final basket* $\mathcal{B}$ *belongs to* $\mathcal{W}^{pre-}_{g,\gamma,\mathbf{d}}(S)$ *for a choice of* $(g, \gamma, \mathbf{d})$.

3.4. **Modifying brackets.** In the following subsections, we will perform a series of blowups, base changes and stablizations to obtain an extension in $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$. Let $(\Sigma, C)$ be a good $S$-family of pre-stacky pointed nodal curves.

**Definition 3.13.** *Let* $\mathcal{B}$ *be a pre-basket of* $(\Sigma, C)$. *We say* $\mathcal{B}'$ *is a modification of* $\mathcal{B}$ *if there is a finite base change* $S' \to S$, *a good* $S'$*-family* $(\Sigma', C')$ *of pre-stacky pointed curves so that* $\mathcal{B}'$ *is a basket of* $(\Sigma', C')$, $(\Sigma', C') \times_{S'} S'_* \cong (\Sigma, C) \times_S S'_*$ *as pre-stacky pointed nodal curves, and under this isomorphism* $\mathcal{B}' \times_{S'} S'_* = \mathcal{B} \times_S S'_*$.

We first show that we can find a basket $\mathcal{B}'$ that is a modification of $\mathcal{B}$ on $(\Sigma, C)$. Indeed, let $\bar{r}_i = \min(rm_i, l_i)$, and let

$$\mathcal{B}' = (B + \sum(l_i - \bar{r}_i)D_i, A + \sum(m_i - \bar{r}_i/r)D_i). \tag{3.8}$$

It is easy to see that $\mathcal{B}'$ is a modification of $\mathcal{B}$.

In the following, we assume $\mathcal{B}$ is a basket as in Definition 3.13. We will construct modifications of the basket $\mathcal{B}$ that will reduce the $\mathbb{Z}_{\geq 0}$-valued quantities

$$V_1(\mathcal{B}) = \sum_{B\cap D_j \neq \emptyset} rm_j, \quad V_2(\mathcal{B}) = \sum_{A\cap D_i \neq \emptyset} l_i, \quad \text{and} \quad V_3(\mathcal{B}) = \sum_{D_i\cap D_j \neq \emptyset} rl_im_j.$$

**Lemma 3.14.** *Let* $(\Sigma, C)$ *and* $\mathcal{B}$ *be as stated. Then there is a modification* $\mathcal{B}'$ *of* $\mathcal{B}$ *such that* $V_1(\mathcal{B}') = 0$.

*Proof.* Let $\bar{j}$ be such that $m_{\bar{j}} > 0$ and $p \in B \cap D_{\bar{j}} \neq \emptyset$. Since $\mathcal{B}$ is a basket, $l_{\bar{j}} = 0$. By the definition of basket, $C_0$ is smooth at $p$ and $p \notin A$. We let $\tau \colon \tilde{C} \to C$ be the blowup of $C$ at $p$, let $E$ be the exceptional divisor, and let $\tilde{\pi} \colon \tilde{C} \to S$ be the induced projection. In the following, for any Cartier divisor $G \subset C$, we denote by

$\tilde{G}$ its strict transform in $\tilde{C}$. Because $B$ is an integral divisor, $\tau^*B = \tilde{B} + lE$ with $1 \le l \in \mathbb{Z}$. By the blowing up formula, we have $\omega^{\log}_{\tilde{C}/S} = \tau^*\omega^{\log}_{C/S}(\epsilon E)$, where

$$\epsilon = 0 \text{ when } p \in \Sigma; \quad \epsilon = 1 \text{ when } p \notin \Sigma. \tag{3.9}$$

We give $\tilde{\Sigma}$ the pre-stacky assignments according to that of $\Sigma$. Form

$$\tilde{\mathcal{B}} = (\tilde{B} + \sum l_i\tilde{D}_i + (l+\epsilon)E, \tilde{A} + \sum m_j\tilde{D}_j + m_{\bar{j}}E).$$

We claim that it is a pre-basket of $(\tilde{\Sigma}, \tilde{C})$. Indeed, the conditions (1)-(4) in the Definition 3.7 can be easily verified. It remains to verify the isomorphism (3.4). Obviously we have

$$\tau^*\mathcal{O}_C(B + \sum l_iD_i) \cong \mathcal{O}_{\tilde{C}}(\tilde{B} + \sum lE + \sum l_iD_i),$$

$$\tau^*\big((\mathcal{O}_C(rA + \sum rm_iD_i) \otimes \omega^{\log}_{C/S}\big) \cong \mathcal{O}_{\tilde{C}}(r\tilde{A} + \sum rm_iD_i + rm_{\bar{j}}E) \otimes \tau^*\omega^{\log}_{C/S},$$

and $\tau^*\omega^{\log}_{C/S} \cong \omega^{\log}_{\tilde{C}/S}(-\epsilon E)$. Combined, we get

$$\mathcal{O}_{\tilde{C}}(\tilde{B} + \sum(l+\epsilon)E + \sum l_iD_i) \cong \mathcal{O}_{\tilde{C}}(r\tilde{A} + \sum rm_iD_i + rm_{\bar{j}}E) \otimes \omega^{\log}_{\tilde{C}/S}.$$

Thus $\tilde{\mathcal{B}}$ is a pre-basket. Then $(\tilde{\mathcal{B}})'$ given in (3.8) is a basket. By construction, it is a modification of $\mathcal{B}$.

We check that $V_1((\tilde{\mathcal{B}})') < V_1(\mathcal{B})$. In fact,

$$(\tilde{\mathcal{B}})' = \big(\tilde{B} + \sum l_i\tilde{D}_i + (l+\epsilon-\bar{r})E, \tilde{A} + \sum m_j\tilde{D}_j + (m_{\bar{j}} - \frac{\bar{r}}{r})E\big),$$

where $1 \le \bar{r} = \min\{rm_{\bar{j}}, l+\epsilon\} \in \mathbb{Z}$ since $0 \ne rm_{\bar{j}} \in \mathbb{Z}$. Since $\tilde{B} \cap \tilde{D}_{\bar{j}} = \emptyset$ and $0 \le m_{\bar{j}} - \bar{r}/r < m_{\bar{j}}$, we have

$$V_1((\tilde{\mathcal{B}})') = \sum_{\tilde{B}\cap\tilde{D}_j\ne\emptyset, j\ne\bar{j}} rm_j + r(m_{\bar{j}} - \frac{\bar{r}}{r}) = \sum_{B\cap D_j\ne\emptyset} rm_j - \bar{r} = V_1(\mathcal{B}) - \bar{r} < V_1(\mathcal{B}).$$

The lemma is proved by induction. □

**Lemma 3.15.** *Let* $(\Sigma, C)$ *and* $\mathcal{B}$ *be as stated with* $V_1(\mathcal{B}) = 0$. *Then there is a modification* $\mathcal{B}'$ *of* $\mathcal{B}$ *such that* $V_1(\mathcal{B}') = V_2(\mathcal{B}') = 0$.

*Proof.* Suppose there is an $l_{\bar{i}} > 0$ such that $A \cap D_{\bar{i}} \ne \emptyset$. Since $\mathcal{B}$ is a basket, $m_{\bar{i}} = 0$. Pick $p \in A \cap D_{\bar{i}}$. Let $\tau\colon \tilde{C} \to C$ be the blowup of $C$ at $p$. If $p$ lies on a marking $\Sigma_i$, let $\tilde{\Sigma}_i \subset \tilde{C}$ be the strict transform of $\Sigma_i$. By transversality, $\tau^*A = \tilde{A} + mE$ where $m \in \frac{1}{r}\mathbb{Z}_{>0}$. Consider

$$\tilde{\mathcal{B}} = (\tilde{B} + \sum l_i\tilde{D}_i + (l_{\bar{i}}+\epsilon)E, \tilde{A} + \sum m_j\tilde{D}_j + mE),$$

where $\epsilon$ is as in (3.9). We have $\tilde{B} \cap E = \emptyset$, $\tilde{A} \cap \tilde{D}_{\bar{i}} = \emptyset$, and $\tilde{A}$ intersects $E$ transversally. Like in the proof of the previous Lemma, it is direct to verify that $\tilde{\mathcal{B}}$ is a pre-basket. Like in (3.8), let

$$\mathcal{B}' = \big(\tilde{B} + \sum l_i\tilde{D}_i + (l_{\bar{i}}+\epsilon-\bar{r})E, \tilde{A} + \sum m_j\tilde{D}_j + (m - \frac{\bar{r}}{r})E\big),$$

where $\bar r = \min\{rm, l_{\bar i} + \epsilon\}$. We claim that $\bar r \geq 1 + \epsilon$. Indeed, when $p \notin \Sigma$, $m \geq 1$; when $p \in \Sigma$, then $rm \geq 1$ but $\epsilon = 0$. Thus $0 \leq l_{\bar i} + \epsilon - \bar r < l_{\bar i}$, and $V_2(\mathcal{B}') < V_2(\mathcal{B})$. Repeating this construction, we prove the Lemma. □

**Lemma 3.16.** *Let $(\Sigma, C)$ and $\mathcal{B}$ be as stated with $V_1(\mathcal{B}) = V_2(\mathcal{B}) = 0$. Then there is a final basket $\mathcal{B}'$ which is a modification of $\mathcal{B}$.*

*Proof.* Suppose there are pairs $D_{\bar i} \neq D_{\bar j}$ such that $p \in D_{\bar i} \cap D_{\bar j}$, and $\ell_{\bar i} > 0$ and $m_{\bar j} > 0$. Take a base change $S_2 \to S$, and let $C' = S_2 \times_S C$. Then near every node of the central fiber of $C'$, $C'$ is locally of the form $xy = t^2$. Minimally resolve $C'$ to get a smooth $\tilde C$ with a $(-2)$-curve corresponding to each node. Let $E$ be the $(-2)$-curve corresponding to the point $p$ mentioned earlier. Then $\omega^{\log}_{\tilde C/S_2} = \tilde\tau^*\omega^{\log}_{C'/S_2} = \tau^*\omega^{\log}_{C/S}$, where $\tilde\tau$ is the minimal resolution morphism $\tilde C \to C'$ and $\tau$ is the composition of $\tilde\tau$ with the base change map $\tau' \colon C' \to C$.

Let $\mathcal{B} = (B + \sum \ell_i D_i, A + \sum m_j D_j)$. Since $A$ and $B$ do not intersect $D_{\bar i} \cap D_{\bar j}$, we have $\tau^* A = \tilde A$ and $\tau^* B = \tilde B$. From the base-change and the minimal resolution, we get a new pre-basket on $\tilde C$: $(\tilde B + \sum \ell_i \tilde D_i + \ell_{\bar i} E, \tilde A + \sum m_j \tilde D_j + m_{\bar j} E)$. Then let

$$\tilde{\mathcal{B}} = (\tilde B + \sum \ell_i \tilde D_i + (\ell_{\bar i} - \bar r)E, \tilde A + \sum m_j \tilde D_j + (m_{\bar j} - \bar r/r)E),$$

where $\bar r = \min\{rm_{\bar j}, \ell_{\bar i}\}$ as in (3.8). It is a basket. Furthermore, we have $m_{\bar j} - \bar r/r < m_{\bar j}$ and $\ell_{\bar i} - \bar r < \ell_{\bar i}$, $E$ intersects $\tilde D_{\bar i}$ and $\tilde D_{\bar j}$ at the nodes, and $\tilde D_{\bar i} \cap \tilde D_{\bar j} = \emptyset$. It is clear that $V_3(\tilde{\mathcal{B}}) < V_3(\mathcal{B})$. Also note that the central fiber of $\tilde C$ is reduced. Repeating this procedure, we prove the Lemma. □

3.5. **Existence of extensions.** In this subsection, we prove

**Proposition 3.17.** *Let $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$ be such that $\mathcal{C}_*$ is smooth. Then possibly after a finite base change of $S$, $\xi_*$ extends to a $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$.*

*Proof.* Let $\xi_* = (\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \cdots)$. We distinguish several cases. The case $\rho_* = 0$ is proved in Proposition 3.1. The next case is when $\varphi_* = \nu_{2*} = 0$. In this case $(\rho_* = 0) = \emptyset$ and $\mathcal{L}_*^\vee \cong \mathcal{N}_*$. So we get a $d$-spin twisted curve $(\mathcal{C}_*, \mathcal{L}_*)$. By [AJ], we can extend $(\mathcal{C}_*, \mathcal{L}_*)$ to a $d$-spin twisted curve $(\mathcal{C}, \mathcal{L})$ over possibly a finite base change of $S$. Here the extension $\xi$ is in $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$.

The last case is when $\varphi_* = 0$, $\rho_* \neq 0$ and $\nu_{2*} \neq 0$. This case is proved by the combination of Proposition 3.5, Lemma 3.9, the finalization of baskets in §3.4 and the restacking Lemma 3.12. □

3.6. **Stabilization.** Let $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* = \xi \times_S S_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$. We will show how to modify $\xi$ along $\mathcal{C}_0$ to obtain a new family $\xi' \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S)$, possibly after a finite base change, such that $\xi_* \cong \xi'_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$.

**Lemma 3.18.** *Let $\xi \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$. Suppose the central fiber $\mathcal{C}_0$ is irreducible, then $\xi_0 \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\eta_0)$.*

*Proof.* Suppose $\xi_0$ is unstable. Since $\mathcal{C}_0$ is irreducible, by Lemma 3.3, either $\mathcal{C}_0$ is a smooth rational curve satisfying one of (1)-(3) in Lemma 3.3, or $\mathcal{C}_0$ satisfies (4) of the same Lemma.

In case $\mathcal{C}_0$ is a smooth rational curve and satisfies one of (1)-(3) mentioned, since the properties in (1)-(3) are deformation invariant, for general $s \in S_*$, $\mathcal{C}_s$ satisfies the same property, forcing $\xi_s$ unstable, contradicting to that $\xi_*$ is a family of stable objects.

Therefore, $\mathcal{C}_0$ must be the Case (4) in Lemma 3.3. Therefore, for a closed point $s \in S_*$, $\Sigma^{\mathcal{C}_s} = \emptyset$, and $\deg \mathcal{L}_s = \deg \mathcal{N}_s = 0$. Here $\Sigma^{\mathcal{C}_s} = \Sigma^{\mathcal{C}} \cap \mathcal{C}_s$, ect. By the non-vanishing assumption on $(\rho, \nu_2)$, $(\varphi, \nu_1)$ and $(\nu_1, \nu_2)$, as in the proof of Lemma 3.3, when $\rho|_{\mathcal{C}_s} \neq 0$, we conclude that $\mathcal{L}^{\otimes r}|_{\mathcal{C}_s} \cong \mathcal{O}_{\mathcal{C}_s}$ and $\mathcal{L}|_{\mathcal{C}_s}^{\vee} \cong \mathcal{N}|_{\mathcal{C}_s}$; when $\varphi|_{\mathcal{C}_s} \neq 0$, we conclude that $\mathcal{L}|_{\mathcal{C}_s} \cong \mathcal{N}|_{\mathcal{C}_s} \cong \mathcal{O}_{\mathcal{C}_s}$. Therefore, $\xi_s$ for general $s \in S_*$ belongs to Case (4) of Lemma 3.3, thus must be unstable. This proves the Lemma. □

We prove the desired existence for the general case.

**Proposition 3.19.** *Let $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S_*)$ and $\mathcal{C}_*$ is smooth. Then possibly after a finite base change, we can find a $\xi' \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S)$ such that $\xi_* \cong \xi'_*$.*

*Proof.* Suppose $\xi_0$ is not stable, and $\mathcal{C}_0$ is irreducible, by Lemma 3.18, we are done. In case $\mathcal{C}_0$ is reducible, then by Lemma 3.3, we can find a rational curve $\mathcal{E} \subset \mathcal{C}$ so that either (1) or (2) of Lemma 3.3 holds.

Let $\mathcal{E} \subset \mathcal{C}$ be of Case (1). We divide it into two subcases: Case-(1a) when both $\mathcal{E} \cap \Sigma^{\mathcal{C}}$ and $\mathcal{E} \cap \mathcal{C}_{0,\mathrm{sing}}$ consist of one point, and Case-(1b) when $\mathcal{E} \cap \mathcal{C}_{0,\mathrm{sing}}$ consists of two points. We look at Case-(1a). Let $p \in \mathcal{E}$ be the node of $\mathcal{C}_0$ that lies in $\mathcal{E}$. Let $(E \subset C, \Sigma)$ be the coarse moduli of $(\mathcal{E} \subset \mathcal{C}, \Sigma^{\mathcal{C}})$. Then $E$ is a (-1)-curve of $C$. Let $q : C \to C'$ be the contraction of $E$, let $p' = q(E) \in C'$ and $\Sigma' = q(\Sigma)$. We then introduce the stacky structure along $\Sigma'$ and $(C'_0)_{\mathrm{sing}}$ to obtain a family of twisted curves $\mathcal{C}' \to S$ so that $q$ introduces an isomorphism $\phi : \mathcal{C}' - p' \cong \mathcal{C} - \mathcal{E}$.

We claim that $\phi^*(\mathcal{L}, \mathcal{N}, \rho, \varphi, \nu)$ extends to $(\mathcal{L}', \mathcal{N}', \rho', \varphi', \nu')$ on $\mathcal{C}'$. Indeed, since $\mathcal{C}'$ is smooth at $p'$, $\phi^*\mathcal{L}$ and $\phi^*\mathcal{N}$ extends to invertible sheaves $\mathcal{L}'$ and $\mathcal{N}'$ on $\mathcal{C}'$, respectively, and the sections $\phi^*\rho$, $\phi^*\varphi$ and $\phi^*\nu$ extends to regular sections $\rho'$, $\varphi'$ and $\nu'$.

We now check that $(\rho', \nu'_2)$ is nonzero at $p'$. Indeed, since $\deg \mathcal{L}|_{\mathcal{E}} = 0$ and $\omega^{\log}_{\mathcal{C}/S}|_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$, if $\rho|_{\mathcal{E}} \neq 0$ then it is nowhere vanishing. By (4) of Definition 2.3, at least one of $\rho|_{\mathcal{E}}$ and $\nu_2|_{\mathcal{E}}$ is nontrivial, thus at least one of $\rho|_{\mathcal{E}}$ and $\nu_2|_{\mathcal{E}}$ is nowhere vanishing. Consequently, at least one of $\rho'(p')$ or $\nu'_2(p')$ is nonzero. By the same reason, we conclude that both $(\varphi', \nu'_1)|_{p'}$ and $(\nu'_1, \nu'_2)|_{p'}$ are nonzero. This concludes that $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots) \in \mathcal{W}^{\mathrm{pre}}_{g,\gamma,\mathbf{d}}(S)$. Finally, since $\xi \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$, we have $\xi' \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$.

We next consider then $\mathcal{E} \subset \mathcal{C}$ is of Case (2). As in Case-(1a), we form the coarse moduli $(E \subset C, \Sigma)$ of $(\mathcal{E} \subset \mathcal{C}, \Sigma^{\mathcal{C}})$, contract $E$ to obtain $C \to C'$, and then reintroduce the stacky structure on $C'$ to get $\mathcal{C}'$ so that $\Sigma^{\mathcal{C}'} \subset \mathcal{C}' - p'$ is isomorphic to $\Sigma^{\mathcal{C}} \subset \mathcal{C} - \mathcal{E}$. Next, we extend the pullback of $\mathcal{L}$, etc. from $\mathcal{C}' - p'$ to $\mathcal{C}'$, and show that the extensions give a new family $\xi' \in \mathcal{W}^{\mathrm{pre}-}_{g,\gamma,\mathbf{d}}(S)$.

The new family $\xi'$ has the property that $\xi_* \cong \xi'_*$, $\mathcal{C}'$ is smooth, and the number of rational curves $\mathcal{E}'$ in $\mathcal{C}'$ listed as Case-(1a) or (2) is one less than that in $\mathcal{C}$. Therefore, after iteration, we can find a $\xi' \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(S)$ so that $\xi_* = \xi'_*$, $\mathcal{C}'$ is smooth, and no rational curve $\mathcal{E}'$ in $\mathcal{C}'$ belongs to Case-(1a) or (2) in Lemma 3.3.

Therefore, to prove the Lemma, we only need to consider the case where the rational $\mathcal{E} \subset \mathcal{C}$ that makes $\xi_0$ unstable characterized by Lemma 3.3 are all in Case-(1b). Let $\mathcal{D} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k \subset \mathcal{C}$ be a maximal connected chain of Case-(1b) rational curves for the family $\xi$. As before, let $(D \subset C, \Sigma)$ be the coarse moduli of $(\mathcal{D} \subset \mathcal{C}, \Sigma^{\mathcal{C}})$. Then $D \subset C$ is a connected chain of (-2)-curves. Let $q : C \to C'$ be the contraction of $D$, $p' = q(D) \in C'$ and $\Sigma' = q(\Sigma)$. Note that $p' \cap \Sigma' = \emptyset$.

We distinguish two cases. The first is when $\mathcal{L}|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$. We claim that then $\mathcal{N}|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$ as well. Indeed, since $\xi_0 \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(\eta_0)$, when $\rho|_{\mathcal{D}} \neq 0$, then $\varphi|_{\mathcal{D}} = 0$, which forces $\nu_1|_{\mathcal{D}}$ nowhere vanishing. Thus $\mathcal{N}|_{\mathcal{D}} \cong \mathcal{L}^\vee|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$. When $\rho|_{\mathcal{D}} = 0$, then $\nu_2|_{\mathcal{D}}$ is nowhere vanishing, implying that $\mathcal{N}|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$. Note that by (2) of Definition 2.3, $\mathcal{C}$ is a scheme along $\mathcal{D}$.

We then introduce stacky structure on $C'$ along $\Sigma'$ and $(C'_0)_{\text{sing}}$ to obtain a family of twisted curves $\mathcal{C}'$ so that the contraction morphism $q : C \to C'$ induces a contraction morphism $\psi : \mathcal{C} \to \mathcal{C}'$ so that $\psi|_{\mathcal{C}-\mathcal{D}} : \mathcal{C} - \mathcal{D} \to \mathcal{C}' - p'$ is an isomorphism of pointed twisted curves, and that $p'$ is a scheme point of $\mathcal{C}'$. Let $(\mathcal{L}', \mathcal{N}', \varphi', \rho', \nu') = \psi_*(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$. It is direct to check that $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots) \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(S)$ and satisfies $\xi_* \cong \xi'_*$.

The other case is when $\mathcal{L}|_{\mathcal{C}} \ncong \mathcal{O}_{\mathcal{D}}$. In this case, since $\deg \mathcal{L}|_{\mathcal{E}_i} = 0$ for all $\mathcal{E}_i \subset \mathcal{D}$, we have $\varphi|_{\mathcal{D}} = 0$. Thus $\rho|_{\mathcal{D}}$ and $\nu_1|_{\mathcal{D}}$ are nowhere vanishing, implying that $\mathcal{N}|_{\mathcal{D}} \cong \mathcal{L}^\vee|_{\mathcal{D}}$.

To proceed, we introduce stacky structures on $C'$ along $\Sigma'$ and $(C'_0)_{\text{sing}} - p'$ to obtain a family of twisted curves $\mathcal{C}'$ so that the contraction morphism $q : C \to C'$ induces an isomorphism $\phi : \mathcal{C}' - p' \cong \mathcal{C} - \mathcal{D}$. (For the moment we keep $p'$ a scheme point of $\mathcal{C}'$.)

Let $(\bar{\mathcal{L}}, \bar{\mathcal{N}}, \bar\rho, \bar\varphi, \bar\nu)$ be the pullback of $(\mathcal{L}, \mathcal{N}, \rho, \varphi, \nu)$ via $\phi$. Since $\mathcal{L}^r|_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}$, $\bar{\mathcal{L}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$. As $\bar{\mathcal{L}}$ is isomorphic to $\bar{\mathcal{N}}^\vee$ near $p'$, $\bar{\mathcal{N}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$.

We now consider $\bar\nu_2 \in H^2(\mathcal{C}' - p', \bar{\mathcal{N}})$. Since $\bar{\mathcal{N}}^{\otimes r}$ extends to an invertible sheaf on $\mathcal{C}'$, we can apply Corollary 3.11 to $(\bar\nu_2, \bar{\mathcal{N}})$ to introduce a $\boldsymbol{\mu}_r$-stacky structure at $p' \in \mathcal{C}'$ if necessary. After a finite base change, we continue to denote by $\mathcal{C}'$ the resulting family of twisted curves. Then $\bar{\mathcal{N}}$ extends to an invertible sheaf $\mathcal{N}'$ on $\mathcal{C}'$ so that $\bar\nu_2$ extends to a regular section $\nu'_2$ of $\mathcal{N}'$. Since $\bar{\mathcal{L}}^\vee$ is isomorphic to $\bar{\mathcal{N}}$ near $p'$, we extend $\bar{\mathcal{L}}$ to an invertible sheaf $\mathcal{L}'$ on $\mathcal{C}'$ so that $\mathcal{L}'^\vee$ is isomorphic to $\mathcal{N}'$ near $p'$, and extend the known isomorphism between $\bar{\mathcal{L}}^\vee$ and $\bar{\mathcal{N}}$. Because $\mathcal{C}'$ is normal near $p'$, we can extend $\bar\varphi$, $\bar\rho$ and $\bar\nu_1$ to regular sections $\varphi'$, $\rho'$ and $\nu'_1$ over $\mathcal{C}'$.

We claim that $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots) \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(S)$. For this, we only need to check that $(\varphi', \nu'_1)|_{p'}$, $(\rho', \nu'_2)|_{p'}$ and $(\nu'_1, \nu'_2)|_{p'}$ are nonzero. Indeed, since $(\rho = 0)$ and $(\nu_1 = 0)$ are disjoint from $\mathcal{D}$, the closures of $(\rho' = 0) - p'$ and $(\nu'_1 = 0) - p'$ do not contains $p'$. Since $(\rho' = 0)$ and $(\nu'_1 = 0)$ are pure codimension one closed subsets of $\mathcal{C}'$, we conclude that $\rho'$ and $\nu'_1$ are nonzero at $p'$. This proves that

$(\varphi', \nu_1')|_{p'}$, $(\rho', \nu_2')|_{p'}$ and $(\nu_1', \nu_2')|_{p'}$ are nonzero. Therefore, $\xi' \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(S)$ and satisfies $\xi_* = \xi'_*$.

We repeat this contraction and introducing stacky structures at the contracted point if necessary to all connected chains of (-2) curves $\mathcal{E} \subset \mathcal{C}_0$ as argued, possibly after a finite base change, we obtain a family $\xi' \in \mathcal{W}^{\text{pre}-}_{g,\gamma,\mathbf{d}}(S)$ so that $\xi_* \cong \xi'_*$, and that there are no rational curves $\mathcal{E} \subset \mathcal{C}'_0$ belongs to the list in Lemma 3.3. This shows that $\xi' \in \mathcal{W}^{-}_{g,\gamma,\mathbf{d}}(S)$ is a desired extension. □

3.7. **Proof of properness.** We prove the properness by gluing the extensions constructed in the previous subsections, using the construction in [AGV, Appendix] (cf. [AF, Def. 1.4.1]).

Let $\mathcal{X}$ be an $S$-family of not necessary connected twisted nodal curves with two markings $\Gamma_1$ and $\Gamma_2$ which are $\boldsymbol{\mu}_r$-gerbes over $S$, $X$ be the moduli of $\mathcal{X}$ with the natural projection $\pi\colon \mathcal{X} \to X$, $s_1, s_2\colon S \to X$ be two sections such that $s_i(S) = \pi(\Gamma_i)$. The line bundle $N^{\otimes 5}_{\Gamma_i/\mathcal{X}}$ descends to the normal bundle of $\pi(\Gamma_i)$ in $X$.

**Lemma 3.20** ([AF, Def. 1.4.1])**.** *With notations and assumptions as above. Assume $s_1^* N^r_{\Gamma_1/\mathcal{X}} \cong s_2^* N^{-r}_{\Gamma_2/\mathcal{X}}$. Then possibly after a finite base change, we can find an $S$-family of not necessary connected twisted nodal curves $\mathcal{X}'$ together with an $S$-morphism $\alpha : \mathcal{X} \to \mathcal{X}'$ that is the gluing of $\mathcal{X}$ via (an appropriate $S$-isomorphism) $\Gamma_1 \cong \Gamma_2$.*

*Proof.* Possibly after a finite base change, we can find an $S$-isomorphism $\gamma : \Gamma_1 \to \Gamma_2$ and $N_{\Gamma_1/\mathcal{X}} \otimes \gamma^* N_{\Gamma_2/\mathcal{X}} \cong \mathcal{O}_{\Gamma_1}$ that induces the given isomorphism $s_1^* N^r_{\Gamma_1/\mathcal{X}} \cong s_2^* N^{-r}_{\Gamma_2/\mathcal{X}}$. In case $\Gamma_1$ and $\Gamma_2$ lie in different connected components of $\mathcal{X}$, the gluing is given in [AF, Definition 1.4.1]. The case $\Gamma_1$ and $\Gamma_2$ lie in the same connected component of $\mathcal{X}$ can be deduced by adopting the construction in the Appendix of [AGV]. □

We can also glue the sheaves and sections. Let the situation be as in Lemma 3.20, and let $\gamma : \Gamma_1 \to \Gamma_2$ be the isomorphism given in its proof.

**Corollary 3.21.** *Suppose we have an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ and an isomorphism $\gamma^*(\mathcal{L}|_{\Gamma_2}) \cong \mathcal{L}|_{\Gamma_1}$. Then the sheaf $\mathcal{L}$ glues to get an invertible sheaf $\mathcal{L}'$ on $\mathcal{X}'$ via the exact sequence*

$$0 \longrightarrow \mathcal{L}' \longrightarrow \alpha_* \mathcal{L} \stackrel{\epsilon}{\longrightarrow} \alpha_*(\mathcal{L}|_{\Gamma_1}) \longrightarrow 0,$$

*where the arrow $\epsilon$ takes the form*

$$\big(\alpha_* \mathcal{L}\big)|_{\Gamma} \cong \alpha_*(\mathcal{L}|_{\Gamma_1}) \oplus \alpha_*(\mathcal{L}|_{\Gamma_2}) \stackrel{(\epsilon_1, -\epsilon_2)}{\longrightarrow} \alpha_*(\mathcal{L}|_{\Gamma_1}),$$

*where $\epsilon_1$ is the identity $\alpha_*(\mathcal{L}|_{\Gamma_1}) = \alpha_*(\mathcal{L}|_{\Gamma_1})$, and $\epsilon_2$ is the isomorphism $\alpha_*(\mathcal{L}|_{\Gamma_2}) \cong \alpha_*(\mathcal{L}|_{\Gamma_1})$ induced by the isomorphism $\gamma^*(\mathcal{L}|_{\Gamma_2}) \cong \mathcal{L}|_{\Gamma_1}$ given. Furthermore, suppose $s \in H^0(\alpha_* \mathcal{L})$ is a section so that $\epsilon(s) = 0$, then $s$ lifts to a section $s' \in H^0(\mathcal{L}')$.*

**Proposition 3.22.** *The degeneracy locus $\mathcal{W}^{-}_{g,\gamma,\mathbf{d}}$ is a proper, closed substack of $\mathcal{W}_{g,\gamma,\mathbf{d}}$.*

*Proof.* The closedness follows from the criterion (2.10). We now prove the properness.

Let $\xi_* = (\Sigma^{\mathcal{C}_*}, \mathcal{C}_*, \cdots) \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S_*)$. Possibly after a finite base change, we can assume that every connected component of the singular locus $(\mathcal{C}_*)_{\rm sing}$ is the image of a section of $\mathcal{C}_* \to S_*$. Let

$$\pi : \mathcal{C}^{\rm nor}_* = \coprod \mathcal{C}_{\alpha *} \to \mathcal{C}_*$$

be the normalization of $\mathcal{C}_*$, with every $\mathcal{C}_{\alpha *}$ connected. After a finite base change, we can assume that $\mathcal{C}_{\alpha *} \to S_*$ have connected fibers. Let $\tau_{\alpha *} : \mathcal{C}_{\alpha *} \to \mathcal{C}_*$ be the tautological morphism. For each $\mathcal{C}_{\alpha *}$, we endow it with the markings the (disjoint) union of $\tau^{-1}_{\alpha *}(\Sigma^{\mathcal{C}_*})$ and $\tau^{-1}_{\alpha *}((\mathcal{C}_*)_{\rm sing})$. Let $\mathcal{L}_{\alpha *}$, etc., be the pullbacks of $\mathcal{L}_*$, etc., via $\tau_{\alpha *}$. By Corollary 3.4,

$$\xi_{\alpha *} = (\Sigma^{\mathcal{C}_{\alpha *}}, \mathcal{C}_{\alpha *}, \mathcal{L}_{\alpha *}, \mathcal{N}_{\alpha *}, \varphi_{\alpha *}, \rho_{\alpha *}, \nu_{\alpha *}) \in \mathcal{W}^-_{g_\alpha, n_\alpha, \mathbf{d}_\alpha}(S_*),$$

for a choice of $(g_\alpha, n_\alpha, \mathbf{d}_\alpha)$.

Applying Proposition 3.19, after a finite base change $S_\alpha \to S$, we can extend $\xi_{\alpha *}$ to a $\xi'_\alpha \in \mathcal{W}^-_{g_\alpha, n_\alpha, \mathbf{d}_\alpha}(S_\alpha)$. We then pick a finite base change $\tilde{S} \to S$, factoring through all $S_\alpha \to S$, and form $\xi_\alpha = \xi'_\alpha \times_{S_\alpha} \tilde{S}$. Therefore, after denoting $\tilde{S}$ by $S$, we conclude that possibly after a finite base change, every $\xi_{\alpha *}$ extends to a $\xi_\alpha \in \mathcal{W}^-_{g_\alpha, n_\alpha, \mathbf{d}_\alpha}(S)$.

We now glue $\xi_\alpha$'s to a $\xi$ that is a stable extension of $\xi_*$. Let $\tilde{\mathcal{C}} = \coprod \mathcal{C}_\alpha$, $\Upsilon_* \subset \mathcal{C}_*$ be a section of $(\mathcal{C}_*)_{\rm sing}$, and $\Upsilon_{1_*} \coprod \Upsilon_{2_*} \subset \mathcal{C}^{\rm nor}_*$ be the preimage of $\Upsilon_*$. Using $(\mathcal{C}_*)^{\rm nor} \to \mathcal{C}_*$, they are markings in $\tilde{\mathcal{C}}_*$. Since markings in $\mathcal{C}_{\alpha *}$ extend to markings in $\mathcal{C}_\alpha$, after a finite base change, we can assume that all $\Upsilon_{i*}$ extend to sections $\Upsilon_i$ in $\tilde{\mathcal{C}}$ such that the $S_*$-isomorphisms $\Upsilon_{1_*} \cong \Upsilon_* \cong \Upsilon_{2_*}$ extend to an $S$-isomorphisms $\sigma : \Upsilon_1 \cong \Upsilon_2$.

Then possibly after a finite base change, we can find an isomorphism $\sigma^* N_{\Upsilon_2/\tilde{\mathcal{C}}} \otimes N_{\Upsilon_1/\tilde{\mathcal{C}}} \cong \mathcal{O}_{\Upsilon_1}$ whose restriction to $\Upsilon_{1*}$ is consistent with the isomorphism $\mathcal{E}xt^1(\Omega_{\mathcal{C}_*}, \mathcal{O}_{\mathcal{C}_*}) \cong \mathcal{O}_{\mathcal{C}_*}$. Applying Lemma 3.20, we obtain a gluing $\mathcal{C}$ of $\tilde{\mathcal{C}}$ along $\Upsilon_1 \cong \Upsilon_2$, resulting a family of twisted pointed nodal curves. After performing such gluing to all sections of $(\mathcal{C}_*)_{\rm sing}$, we obtain an $S$-family of twisted nodal curves $\mathcal{C} \to S$ that is an extension of $\mathcal{C}_* \to S_*$. We denote by

$$\beta : \tilde{\mathcal{C}} \longrightarrow \mathcal{C} \tag{3.10}$$

the gluing morphism.

We next glue the sheaves and fields. Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}, \tilde{\varphi}, \tilde{\rho}, \tilde{\nu})$ be the sheaves and sections on $\tilde{\mathcal{C}}$ so that its restriction to $\mathcal{C}_\alpha$ is part of the extension $\xi_\alpha$. We will show that possibly after a finite base change, we can find $(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ over $\mathcal{C}$ together with isomorphisms $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}}) \cong \beta^*(\mathcal{L}, \mathcal{N})$ and $(\tilde{\varphi}, \tilde{\rho}, \tilde{\nu}) = \beta^*(\varphi, \rho, \nu)$

Without loss of generality, we can assume that $(\mathcal{C}_*)_{\rm sing}$ consists of a single $S_*$-section. Thus $\mathcal{C}$ is the gluing of $\tilde{\mathcal{C}}$ along $\Upsilon_1 \cong \Upsilon_2$. Let

$$\iota_i : \Upsilon \longrightarrow \tilde{\mathcal{C}}$$

be the composite $\Upsilon \cong \Upsilon_i \to \tilde{\mathcal{C}}$ of the tautological maps. We first consider the case where $\iota_1^*\tilde{\varphi} \neq 0$. Then necessarily $\iota_2^*\tilde{\varphi} \neq 0$. Since $\xi_\alpha \in \mathcal{W}^-_{g_\alpha,\gamma_\alpha,\mathbf{d}_\alpha}(S)$, $\tilde{\rho} = 0$ in a neighborhood $\tilde{\mathcal{U}}$ of $\Upsilon_1 \cup \Upsilon_2$ in $\tilde{\mathcal{C}}$, thus $\tilde{\nu}_2$, which is nowhere vanishing in $\tilde{\mathcal{U}}$, induces an isomorphism $\tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}} \cong \mathcal{O}_{\tilde{\mathcal{U}}}$, hence inducing $\iota_i^*\tilde{\mathcal{N}} \cong \mathcal{O}_\Upsilon$ so that $\iota_i^*\tilde{\nu}_2 = 1$. Note that in this case, $\tilde{\mathcal{C}}$ is a scheme along $\Upsilon_i$.

For $i = 1$ or 2, we consider $(\iota_i^*\tilde{\varphi}, \iota_i^*\tilde{\nu}_1)$, which is a nowhere vanishing section of $H^0(\iota_i^*\tilde{\mathcal{L}}^{\oplus(n+1)})$. It induces a morphism $\beta_i : \Upsilon \to \mathbb{P}^n$. Because $\beta_1|_{\Upsilon_*} = \beta_2|_{\Upsilon_*}$, we have $\beta_1 = \beta_2$. Consequently, we have a unique isomorphism $\phi : \iota_1^*\tilde{\mathcal{L}} \cong \iota_2^*\tilde{\mathcal{L}}$ so that

$$\phi^*(\iota_2^*\tilde{\varphi}, \iota_2^*\tilde{\nu}_1) = (\iota_1^*\tilde{\varphi}, \iota_1^*\tilde{\nu}_1). \tag{3.11}$$

Using $\iota_i^*\tilde{\mathcal{N}} \cong \mathcal{O}_\Upsilon$ and $\iota_i^*\tilde{\nu}_2 = 1$, we have isomorphism $\phi' : \iota_1^*\tilde{\mathcal{N}} \cong \iota_2^*\tilde{\mathcal{N}}$ so that $\phi'^*\iota_2^*\tilde{\nu}_2 = \iota_1^*\tilde{\nu}_2$.

Applying the scheme version of Corollary 3.21, we obtain invertible sheaves $\mathcal{L}$ and $\mathcal{N}$ on $\mathcal{C}$ with isomorphisms $\beta^*\mathcal{L} \cong \tilde{\mathcal{L}}$ and $\beta^*\mathcal{N} \cong \tilde{\mathcal{N}}$ whose restrictions to $\Upsilon$ are $\phi$ and $\phi'$ respectively. By (3.11) and Corollary 3.21, we also obtain sections $\varphi \in H^0(\mathcal{L})^{\oplus n}$ and $\nu_2 \in H^0(\mathcal{N})$ that are liftings of $\beta_*\tilde{\varphi}$ and $\beta_*\tilde{\nu}_2$, respectively, which thus satisfy $\beta^*\varphi = \tilde{\varphi}$ and $\beta^*\nu_2 = \tilde{\nu}_2$, under the given isomorphisms.

It remains to check that $\tilde{\nu}_1$ and $\tilde{\rho}$ can be glued to sections over $\mathcal{C}$. In this case, since $\mathcal{C}$ is a scheme along $\beta(\Upsilon_1) = \beta(\Upsilon_2)$, using that $\tilde{\mathcal{L}}$ glues to $\mathcal{L}$ we conclude that $\tilde{\mathcal{L}}^{-r} \otimes \omega^{\log}_{\tilde{\mathcal{C}}/S}$ glues to $\mathcal{L}^{-r} \otimes \omega^{\log}_{\mathcal{C}/S}$. Since $\tilde{\rho}$ vanishes along $\Upsilon_1 \cup \Upsilon_2$, $\beta_*\tilde{\rho}$ lifts to $\rho$ so that $\rho|_{\beta(\Upsilon_1)} = 0$. The gluing of $\tilde{\nu}_1$ is similar. This proves the existence of gluing in this case.

The other case is when $\iota_1^*\rho \neq 0$, which implies that $\iota_2^*\tilde{\rho} \neq 0$. In this case, we must have $\tilde{\varphi}|_{\tilde{\mathcal{U}}} = 0$ over a neighborhood $\tilde{\mathcal{U}}$ of $\Upsilon_1 \cup \Upsilon_2$ in $\tilde{\mathcal{C}}$. Consequently, $\tilde{\nu}_1|_{\tilde{\mathcal{U}}}$ is nowhere vanishing, forcing $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$. In particular, we have the induced isomorphism $\iota_i^*(\tilde{\mathcal{L}} \otimes \tilde{\mathcal{N}}) \cong \mathcal{O}_\Upsilon$ and $\iota_i^*\tilde{\nu}_1 = 1$ under this isomorphism.

To proceed, we use the canonical isomorphisms $\iota_i^*\omega^{\log}_{\tilde{\mathcal{C}}/S} \cong \mathcal{O}_\Upsilon$ due to that $\Upsilon_i \subset \tilde{\mathcal{C}}$ is a section along smooth locus of fibers of $\tilde{\mathcal{C}} \to S$. Using this isomorphism, we can view $\iota_i^*\tilde{\rho}$ as a section in $H^0(\iota_i^*\tilde{\mathcal{L}}^{-r})$. Using $\iota_i^*\tilde{\mathcal{L}}^\vee \cong \iota_i^*\tilde{\mathcal{N}}$, $\iota_i^*\tilde{\nu}_2$ is a section in $H^0(\iota_i^*\tilde{\mathcal{L}}^\vee)$. Because $(\iota_i^*\tilde{\rho}, \iota_i^*\tilde{\nu}_2)$ is nowhere vanishing, it defines a morphism $\beta_i : \Upsilon_i \to \mathbb{P}_{(r,1)}$. Because $\beta_1 \times_S S_* = \beta_2 \times_S S_*$, we have $\beta_1 = \beta_2$. Thus there are isomorphisms

$$\phi' : \iota_1^*\tilde{\mathcal{L}} \cong \beta_1^*\mathcal{O}_{\mathbb{P}_{(r,1)}}(1) = \beta_2^*\mathcal{O}_{\mathbb{P}_{(r,1)}}(1) \cong \iota_2^*\tilde{\mathcal{L}}$$

so that $\iota_1^*\tilde{\rho} = \iota_2^*\tilde{\rho}$ and $\iota_1^*\tilde{\nu}_2 = \iota_2^*\tilde{\nu}_2$ (with the known $\iota_i^*\omega^{\log}_{\tilde{\mathcal{C}}/S} \cong \mathcal{O}_\Upsilon$).

Like before, using $\phi'$, and applying Corollary 3.21, we can glue $\tilde{\mathcal{L}}$ to get $\mathcal{L}$ on $\mathcal{C}$ so that, letting $\iota : \Upsilon \cong \alpha(\Upsilon_1) \subset \mathcal{C}$ be the tautological inclusion, the isomorphisms $\iota_1^*\tilde{\mathcal{L}} \cong \iota^*\mathcal{L} \cong \iota_2^*\tilde{\mathcal{L}}$ induce the isomorphism $\phi'$. We next glue $\tilde{\mathcal{N}}$. Let $\mathcal{U}$ be the image of $\tilde{\mathcal{U}}$ under $\tilde{\mathcal{C}} \to \mathcal{C}$, which is a neighborhood of $\Upsilon \subset \mathcal{C}$. Then using $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$, we see that $\tilde{\mathcal{N}}$ glues to get $\mathcal{N}$ on $\mathcal{C}$ so that $\mathcal{N}|_{\mathcal{U}} \cong \mathcal{L}^\vee|_{\mathcal{U}}$, consistent with the isomorphism $\tilde{\mathcal{L}}^\vee|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}}$.

As before, applying Corollary 3.21 and using that $\iota_1^*\tilde{\nu}_2 = \iota_2^*\tilde{\nu}_2$ and $\iota_1^*\tilde{\rho}_1 = \iota_2^*\tilde{\rho}_2$ under $\phi' : \iota_1^*\tilde{\mathcal{L}} \cong \iota_2^*\tilde{\mathcal{L}}$, we conclude that $\tilde{\nu}_2$ and $\tilde{\rho}$ glue to $\nu_2$ and $\rho$ of $\mathcal{N}$ and

$\mathcal{L}^{-r} \otimes \omega_{\mathcal{C}/S}^{\log}$, respectively. Since $\tilde{\varphi}|_{\mathcal{U}} = 0$, it glues to $\varphi$ such that $\varphi|_{\mathcal{U}} = 0$. For $\tilde{\nu}_1$, since it induces isomorphism $\tilde{\mathcal{N}}|_{\tilde{\mathcal{U}}} \cong \tilde{\mathcal{L}}^{\vee}|_{\tilde{\mathcal{U}}}$, and this isomorphism descends to $\mathcal{N}|_{\mathcal{U}} \cong \mathcal{L}'^{\vee}|_{\mathcal{U}}$, we see that $\tilde{\nu}_1$ glues to get $\nu_1$. Finally, we let $\Sigma^{\mathcal{C}}$ be the image of $\Sigma^{\tilde{\mathcal{C}}} - (\Upsilon_1 \cup \Upsilon_2)$, Then

$$\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu) \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(S).$$

This proves the Proposition. ☐

As the proof doesn't use the condition $\varphi_1^r + \ldots + \varphi_n^r = 0$, we have

**Corollary 3.23.** *Let $\mathcal{W}^{\sim}_{g,\gamma,\mathbf{d}} \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$ be the reduced closed substack where close points are $\xi \in \mathcal{W}_{g,\gamma,\mathbf{d}}(\mathbb{C})$ such that $(\varphi = 0) \cup (\rho = 0) = \mathcal{C}$ (c.f. Lemma 2.12). Then $\mathcal{W}^{\sim}_{g,\gamma,\mathbf{d}}$ is proper.*

## 4. Finite presentation of the degeneracy locus

In this section, we prove that $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated and $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is of finite type.

4.1. **Separatedness.** In this subsection, we prove that $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated. As before, $\eta_0 \in S$ is a closed point in a smooth curve over $\mathbb{C}$, and $S_* = S - \eta_0$.

**Lemma 4.1.** *Let $\xi$, $\xi' \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S)$ be such that $\xi_* \cong \xi'_* \in \mathcal{W}_{g,\gamma,\mathbf{d}}(S_*)$. Suppose $\mathcal{C}_*$ is smooth. Then $\xi \cong \xi'$.*

*Proof.* Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \cdots)$ and $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \mathcal{L}', \cdots)$, let $C$ (resp. $C'$) be the coarse moduli of $\mathcal{C}$ (resp. $\mathcal{C}'$), and let $\pi : X \to C$ (resp. $\pi' : X' \to C'$) be the minimal desingularization. Thus $\pi$ and $\pi'$ are contractions of chains of $(-2)$-curves.

Let $f : X \dashrightarrow X'$ be the birational map induced by $\xi_* \cong \xi'_*$, let $U_0 \subset X$ be the largest open subset over which $f$ is well-defined. Suppose $U_0 \subsetneq X$, then $X - U_0$ is discrete. Let $X_1$ be the blowing up of $X$ at $X - U_0$. Inductively, suppose $X_k \to X$ is a successive blowing up along points, let $U_k \subset X_k$ be the largest open subset over which the birational $f : X_k \dashrightarrow X'$ is well-defined, then let $X_{k+1}$ be the blowing up of $X_k$ along $X_k - U_k$. After finite steps, we have $X_{\bar{k}} = U_{\bar{k}}$, thus $X_{\bar{k}} = X_{\bar{k}+1}$ for large $\bar{k}$. Denote $Y = X_{\bar{k}}$ for large $\bar{k}$ with

$$\bar{\pi} : Y \longrightarrow X \quad \text{and} \quad \bar{f} : Y \longrightarrow X'$$

the induced projection and birational morphism.

Let $E \subset Y$ be the exceptional divisor of $\bar{\pi}$. Write $E = \sum_{k \geq 1} E_k$, where $E_k \subset E$ is the proper transform of the exceptional divisor of $X_k \to X_{k-1}$ when $k \geq 2$ and the exceptional divisor of the map $X_1 \to X$ when $k = 1$. Let $E' \subset Y$ be the exceptional divisor of $\bar{f}$. By our construction, $E$ and $E'$ share no common irreducible curves. Let $Y_0 = \cup_j^n D_j$ be the irreducible component decomposition of the central fiber $Y_0$.

Furthermore, by our construction for $V = Y - E'$, $\bar{f}|_V : V \to \bar{f}(V)$ is an isomorphism, and by the blowing up formula, we have

$$\omega_{Y/S}^{\log} \cong \bar{\pi}^* \omega_{X/S}^{\log}(\sum_i iE_i) \quad \text{and} \quad \omega_{Y/S}^{\log}|_V \cong \bar{f}^* \omega_{X'/S}^{\log}|_V. \tag{4.1}$$

Let $L$ and $N$ (resp. $L'$ and $N'$) be the line bundles on $X$ (resp. $X'$) that are the pullbacks of the descents of $\mathcal{L}^{\otimes r}$ and $\mathcal{N}^{\otimes r}$ (resp. $\mathcal{L}'^{\otimes r}$ and $\mathcal{N}'^{\otimes r}$) to $C$ (resp. $C'$). Using $\xi_* \cong \xi'_*$, we can find integers $a_i$ and $b_i$ so that

$$\bar{f}^* L' \cong \bar{\pi}^* L(\sum a_i D_i) \quad \text{and} \quad \bar{f}^* N' \cong \bar{\pi}^* N(\sum b_i D_i). \tag{4.2}$$

Let $u_1$, $u_2$ and $h_j$ be the pullbacks of the descents of $\nu_1^r$, $\nu_2^r$ and $\varphi_j^r$, which are sections of $L\otimes N$, $N$ and $L$, respectively. Denote by $u_1'$, $u_2'$ and $h_j'$ be the pullbacks of the descents of $\nu_1'^r$, $\nu_2'^r$ and $\varphi_j'^r$ similarly. By the same reason, we will view $\rho$ and $\rho'$ as sections in $H^0(X, L^{-1}\otimes\omega_{X/S}^{\log}) = H^0(\mathcal{C}, \mathcal{L}^{-r}\otimes\omega_{\mathcal{C}/S}^{\log})$, and in $H^0(X', L'^{-1}\otimes\omega_{X'/S}^{\log})$ respectively.

We now show that $E=\emptyset$, namely, $f$ is a morphism. Suppose not, let $D_j\subset E$ be an irreducible component with $x=\bar{\pi}(D_j)\in X$. We first remark that $x$ is a smooth point of $X_0$. Indeed, that $x$ is a singular point of $X_0$ implies that $V_0 = V\times_S \eta_0$ is not reduced. On the other hand, since $\bar{f}|_V : V\to X'$ is an $S$-isomorphism onto its image and since $X_0'$ is reduced, we conclude that $V_0$ is reduced too. This proves that $x$ is a smooth point of $X_0$.

Let $x=\bar{\pi}(D_j)$ be as before. Then by the construction of $\bar{\pi}$, $\bar{\pi}^{-1}(x)$ is a tree of rational curves. By reindexing $Y_0=\cup_{j=1}^n D_j$, we can assume that $D_1+\cdots+D_k$ form a maximal chain of rational curves in $\bar{\pi}^{-1}(x)$, namely, $D_i\subset E_i$ for $i\le k$, and $D_i\cap D_{i+1}\neq\emptyset$ for $i<k$, thus $D_k\subset Y$ is a $(-1)$-curve. Therefore by our construction of $(\bar{\pi},\bar{f})$, $\bar{f}(D_1),\cdots,\bar{f}(D_k)$ is a chain of rational curves in $X'$, and $\bar{f}(D_k)$ is a $(-1)$-curve in $X'$ since $x$ is a smooth point of $X_0$. In particular, the image of $\bar{f}(D_k)$ in $\mathcal{C}_0'$, denoted by $\mathcal{D}'$, is a rational curve. Let $z\in D_k$ be a general point and let $y=\bar{f}(z)$. Note $x=\bar{\pi}(z)$.

**Sublemma 4.2.** *Let the situation be as stated. Then we have $u_2(x)=0$.*

*Proof.* We prove by contradiction. Suppose $u_2(x)\neq 0$. We claim that $a_k=b_k=0$.

We divide it into two cases. The first is when $u_1(x)\neq 0$. Since $\bar{\pi}^*u_2\in H^0(V,\bar{\pi}^*N)$ and non-trivial along $V\cap D_k$, and since $\bar{f}^*u_2'\in H^0(V,\bar{f}^*N')$, using (4.2) and that $\bar{\pi}^*u_2|_{V\times_S S_*} = \bar{f}^*u_2'|_{V\times_S S_*}$, we conclude that $b_k\ge 0$. Similarly, using that $u_1(x)\neq 0$, we conclude that $a_k+b_k\ge 0$.

We claim that $a_k+b_k=0$. Suppose not, then by (4.2), we have $u_1'(y)=0$, thus $u_2'(y)\neq 0$ and $(h_k'(y))\neq 0$, which forces $b_k\le 0$ and $a_k\le 0$, contradicting to $a_k+b_k>0$. This proves $a_k+b_k=0$.

We now prove $a_k=b_k=0$. Suppose not. Since $a_k+b_k=0$ and $b_k\ge 0$, we must have $b_k>0$. Then $a_k<0$ and $u_2'(y)=0$. Note that because of (4.1) and (4.2), for any $i\le k$ and a dense open $U\subset D_i$ that is disjoint from the nodes of $Y_0$,

$$\bar{f}^*(L'^{-1}\otimes\omega_{X'/S}^{\log})|_U \cong \bar{\pi}^*(L^{-1}\otimes\omega_{X/S}^{\log})((i-a_i)D_i)|_U. \tag{4.3}$$

Applying to $i=k$, we conclude that $\rho'(y)=0$, contradicting to $u_2'(y)=0$. Thus $a_k=b_k=0$.

The other case is when $u_1(x)=0$. Since $u_1(x)=0$, we have $(h_i(x))\neq 0$. Similar to the argument above, using $h_i(x))\neq 0$ (resp. $u_2(x)\neq 0$), we conclude that $a_k\ge 0$ (resp. $b_k\ge 0$).

We now show that $b_k = 0$. Suppose not, that is $b_k > 0$, then we must have $\bar{f}^*u_2'|_{D_k} = 0$, which forces $\bar{f}^*\rho'|_{D_k} \neq 0$. Applying (4.1) and (4.2), we must have $k - a_k \leq 0$, thus $a_k \geq k \geq 1$, which forces $\bar{f}^*u_1'|_{D_k} = 0$, violating that $(u_1', u_2')$ is nowhere vanishing. This proves $b_k = 0$.

A similar argument shows that $a_k > 0$ would lead to $\bar{f}^*h_k'|_{D_k} = \bar{f}^*u_1'|_{D_k} = 0$, a contradiction. Therefore, $a_k = b_k = 0$ in this case, too.

Let $\mathcal{D}' \subset \mathcal{C}'$ be the irreducible component whose image in $C'$ is the same as the image of $D_k$ under $Y \to X' \to C'$. Since $\bar{f}(D_k)$ is a (-1)-curve, $\mathcal{D}'$ is a smooth rational curve in $\mathcal{C}'$ and contains exactly one node of $\mathcal{C}_0'$ and at most one marking of $\Sigma^{\mathcal{C}'}$.

Next, we use (4.1) and $a_k = b_k = 0$ to conclude that $\rho'|_{\mathcal{D}'} = 0$. Therefore, $\mathcal{N}'|_{\mathcal{D}'} \cong \mathcal{O}_{\mathcal{D}'}$, and $(\varphi_1', \cdots, \varphi_n', \nu_1')|_{\mathcal{D}'}$ defines a morphism $\beta' : \mathcal{D}' \to \mathbb{P}^n$. Let $q' \in \mathcal{D}'$ be the node of $\mathcal{C}_0'$. By (4.2) and that $a_k = b_k = 0$, we conclude that the pullback of $(\varphi_1', \cdots, \varphi_n', \nu_1')|_{\mathcal{D}'-q'}$ to $Y$ equals to the pullback of $(\varphi_1, \cdots, \varphi_n, \nu_2)|_x$, thus $\beta' : \mathcal{D}' \to \mathbb{P}^n$ is a constant map. Since $\mathcal{D}'$ contains one node of $\mathcal{C}_0'$ and at most one marking in $\Sigma^{\mathcal{C}'}$, adding that $\rho'|_{\mathcal{D}'} = 0$, by Lemma 3.3 we conclude that $\xi_0'$ is unstable, a contradiction. This proves the Sublemma. □

We continue to denote by $D_1 + \cdots + D_k$ a maximal chain of rational curves in $\bar{\pi}^{-1}(x)$ with $D_i \in E_i$.

**Sublemma 4.3.** *We have $a_k = k$, $a_i \leq a_{i+1} - 2$ for $i < k$, and $a_i + b_i = 0$ for all $i \leq k$.*

*Proof.* First, we have (4.3). We claim that for $i \leq k$,

$$i - a_i \geq 0 \quad \text{and} \quad a_i + b_i = 0. \tag{4.4}$$

In fact, by the previous Sublemma, we know $u_2(x) = 0$, thus $\rho(x) \neq 0$ and $\bar{\pi}^*\rho|_{D_i} \neq 0$. Adding that $\bar{f}^*\rho'$ is regular along $V$ and coincides with $\bar{\pi}^*\rho$ over $V \times_S S_*$, we obtain the first inequality in (4.4). Similarly, using $u_1(x) \neq 0$, we obtain $a_i + b_i \geq 0$.

We now show $a_i + b_i = 0$. Suppose not, say $a_i + b_i > 0$, then $\bar{f}^*u_1'|_{D_i} = 0$, which forces $\bar{f}^*h_j'|_{D_i} \neq 0$ for some $j$, and by (4.2), we obtain $a_i \leq 0$. As $a_i + b_i > 0$, we obtain $b_i > 0$, and hence $\bar{f}^*u_2'|_{D_i} = 0$, contradicting to $(u_1'(y), u_2'(y)) \neq 0$. This proves (4.4).

We next prove $a_k = k$. Suppose not, by (4.4) we have $k - a_k \geq 1$. Then $\bar{f}^*\rho'|_{D_k} = 0$, which forces $\bar{f}^*u_2'|_{D_k}$ nowhere vanishing. By (4.2), we have $b_k \leq 0$; adding $a_k + b_k = 0$, we have $a_k \geq 0$.

We claim $a_k > 0$. Suppose not, i.e., $a_k = 0$. Let $\mathcal{D}' \subset \mathcal{C}'$, as before, be the irreducible component whose image in $C'$ is the same as the image of $D_k$ under $Y \to X' \to C'$. As argued in the proof of the previous Sublemma, $\mathcal{D}'$ is a smooth rational curve in $\mathcal{C}'$ and contains one node $q'$ of $\mathcal{C}_0'$. As $a_k = 0$, $\rho'|_{\mathcal{D}'} = 0$, $\mathcal{N}'|_{\mathcal{D}'} \cong \mathcal{O}_{\mathcal{D}'}$, and $(\varphi_1', \cdots, \varphi_n', \nu_1')|_{\mathcal{D}'}$ defines a morphism $\beta' : \mathcal{D}' \to \mathbb{P}^n$, which turns out to be constant, as argued before. Therefore, as $\mathcal{D}'$ contains at most one marking of $\Sigma^{\mathcal{C}'}$, $\xi_0'$ becomes unstable, a contradiction. This proves that $a_k$ is positive.

Therefore, by the property $\bar{f}^*\rho'|_{D_k} = 0$, etc., we conclude that $\rho'|_{\mathcal{D}'} = \varphi'|_{\mathcal{D}'} = 0$, and both $\nu_1'|_{\mathcal{D}'}$ and $\nu_2'|_{\mathcal{D}'}$ are nowhere vanishing. Because $\mathcal{D}'$ contains one node and at most one marking of $\mathcal{C}_0'$, $\xi_0'$ becomes unstable, a contradiction. This proves that $a_k = k$.

Finally, we prove $a_i \le a_{i+1} - 2$. Let $\Lambda = \{1 \le i < k \mid a_i \not\le a_{i+1} - 2\}$. The intended inequality is equivalent to $\Lambda = \emptyset$. Suppose $\Lambda \ne \emptyset$, and let $i$ be the largest element in $\Lambda$. Suppose $i = k-1$. Since $a_k = k$, we have $a_{k-1} \ge a_k - 1 = k-1$. By (4.4), we conclude that $a_{k-1} = k-1$, which implies $\deg \bar f^* L'|_{D_k} = -1$.

Consequently, using (4.3), we conclude that $\bar f^* \rho'|_{D_k}$ is nowhere vanishing. Like before, let $\mathcal{D}' \subset \mathcal{C}_0'$ be the irreducible component associated to $D_k \subset E$, via $Y \to C'$ and $\mathcal{C}' \to C'$. Then $\mathcal{D}'$ is a rational curve, contains one node of $\mathcal{C}_0'$, and with $\varphi'|_{\mathcal{D}'} = 0$, $\rho'|_{\mathcal{D}'}$ nowhere vanishing and $\deg \mathcal{L}'|_{\mathcal{D}'} = -\frac{1}{5}$. By Lemma 3.3, this makes $\xi_0'$ unstable, a contradiction. Therefore, $i < k-1$.

Since $i+1 \notin \Lambda$, $(i+1) - a_{i+1} \ge 1$, which forces $\bar f^* \rho'|_{D_{i+1}} = 0$, and then $\bar f^* u_2'|_{D_{i+1}}$ is nowhere vanishing and $\bar f^* N'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$. We claim that $\bar f^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$. SInce $\bar f^* N'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$, $(\bar f^* h_1', \cdots, \bar f^* h_n', \bar f^* u_1')|_{D_{i+1}}$ defines a morphism $\beta : D_{i+1} \to \mathbb{P}^n$. By the second inequality in (4.4) and (4.2), it is a constant map. Thus $\bar f^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$.

Now let $D_i, D_{i+2}, D_{k_2}, \cdots, D_{k_l}$ be the irreducible components in $E$ that interest with $D_{i+1}$. Since $i+1 \notin \Lambda$, $a_{i+1} \le a_{i+2} - 2$. Possibly by changing to a different maximal chain of rational curves in $\bar\pi^{-1}(x)$, we can assume without loss of generality that $a_{i+1} \le a_{k_s} - 2$ for all $2 \le s \le l$. Because $\bar f^* L'|_{D_{i+1}} \cong \mathcal{O}_{D_{i+1}}$, we must have $(a_i - a_{i+1}) + \sum_s (a_{k_s} - a_{i+1}) = 0$. Therefore, $a_i - a_{i+1} \le -2$, a contraction. This proves $\Lambda = \emptyset$, and the Sublemma follows. □

We continue our proof of Lemma 4.1. We keep the maximal chain of rational curves $D_1, \cdots, D_k \subset E$. We claim $k = 1$. Otherwise, $k \ge 2$ and $a_1 \le a_2 - 2 \le 0$. By (4.3) we obtain $\bar f^* \rho'|_{D_1} = 0$ and thus $\bar f^* u_2'|_{D_1}$ is nowhere vanishing. Since $u_2(x) = 0$ by Sublemma 4.2, we get $\bar\pi^* u_2|_{D_1} = 0$. Thus from (4.2), we get $b_1 < 0$ and hence $a_1 = -b_1 > 0$, a contradiction to $a_1 \le 0$. Thus $k = 1$.

Next we prove that every $D_\ell \subset Y_0$ not in $E$ that intersects one of $D_i \subset E$ must be contracted by $\bar f$. Indeed, if $\bar f(D_\ell)$ is not a point, since $\bar\pi(D_\ell)$ is not a point as well, we must have $a_\ell = b_\ell = 0$ and $D_\ell \cap V$ is dense in $D_\ell$. Since $D_i$ is a $(-1)$-curve, the combination of (4.1), (4.2) and $\rho(x) \ne 0$ (since $u_2(x) = 0$) implies $\bar f^* \rho'|_{D_i}$ is nowhere vanishing. From (4.2), we also have $\deg(\bar f^* L'|_{D_i}) = -1$ since $a_i = 1$ and $a_\ell = 0$. Thus $\xi_0'$ satisfies the condition (2) in the Lemma 3.3 and hence is not stable, a contradiction.

In conclusion, we have proved that $E$ is a disjoint union of (-1)-curves, likewise for $E'$, and that every irreducible component in $Y_0$ not in $E$ must be in $E'$. Since $Y \to X$ is by first blowing up smooth points of $X_0$, and since $Y_0$ is connected, this is possible only if both $E \cong \mathbb{P}^1$, and then $Y$ is the blowing up of $X = S \times \mathbb{P}^1$ at a single point in $X_0$.

Then a direct analysis shows that this is impossible, assuming both $\xi_0$ and $\xi_0'$ are stable. (As this analysis is straightforward, we omit the details here.) This proves that $E = \emptyset$ and $f : X \to X'$ is a birational morphism. By symmetry, $f^{-1} : X' \to X$ is also a birational morphism. Therefore, $f : X \cong X'$ is an isomorphism.

Knowing that $f$ is an isomorphism, a parallel argument shows that

$$f^*L' \cong L, \quad f^*N' \cong N, \quad \text{and} \quad f^*(\rho', h_k', u_i') = (\rho, h_k, u_i). \tag{4.5}$$

We prove that this implies $C \cong C'$. Indeed, it is easy to show that a $D_i \subset X$ is contracted by $\mathrm{pr} : X \to C$ if and only if $D_i \subset X$ is a (-2)-curve and $L|_{D_i} \cong N|_{D_i} \cong \mathcal{O}_{D_i}$. Therefore, $D_i$ is contracted by the map $X \to C$ if and only if $f(D_i)$ is contracted by the map $X' \to C'$. This proves that $f : X \cong X'$ induces an isomorphism $\bar\phi : C \cong C'$.

Let $\Delta$ (resp. $\Delta'$) be the set of singular points of $\mathcal{C}_0$ (resp. $\mathcal{C}_0'$). Let $p : \mathcal{C} \to C$ and $p' : \mathcal{C}' \to C'$ be the coarse moduli morphisms. Then the isomorphisms $\bar\phi$ and (4.5) (with $a_i = b_i = 0$) induce isomorphisms

$$\bar\phi^* p'_*(\Sigma^{\mathcal{C}'}, \mathcal{L}'^{\otimes 5}, \mathcal{N}'^{\otimes 5}, \varphi_k'^5, \rho', \nu_i'^5) \cong p_*(\Sigma^{\mathcal{C}}, \mathcal{L}^{\otimes 5}, \mathcal{N}^{\otimes 5}, \varphi_k^5, \rho, \nu_i^5); \tag{4.6}$$

and isomorphisms

$$\phi : \mathcal{C} - \Delta \xrightarrow{\cong} \mathcal{C}' - \Delta', \quad \phi^*\mathcal{L}' \cong \mathcal{L}, \quad \phi^*\mathcal{N}' \cong \mathcal{N}, \quad \phi^*(\varphi', \rho', \nu_i') = (\varphi, \rho, \nu_i), \tag{4.7}$$

extending $\xi_* \cong \xi_*'$.

Now let $p \in \Delta$ be a point and let $p' \in \Delta'$ be the corresponding point. Pick an open subset $\mathcal{U} \subset \mathcal{C}$ of $p \in \mathcal{C}$ so that $\mathcal{U} \cap \Delta = p$. Let $\mathcal{U}' \subset \mathcal{C}'$ be the open subset of $p' \in \mathcal{C}'$ so that $\mathcal{U}' \cap \Delta' = p'$ and $\phi(\mathcal{U} - p) = \mathcal{U}' - p'$. If $p$ is a scheme point, then $\phi$ extends to a morphism $\tilde\phi : (\mathcal{C} - \Delta) \cup \mathcal{U} \to (\mathcal{C}' - \Delta') \cup \mathcal{U}'$ so that (4.7) extends to

$$\tilde\phi^*\mathcal{L}' \cong \mathcal{L}, \quad \tilde\phi^*\mathcal{N}' \cong \mathcal{N}, \quad \tilde\phi^*(\varphi', \rho', \nu_i') = (\varphi, \rho, \nu_i). \tag{4.8}$$

However, by (2) in Definition 2.3, this implies $p'$ is also a scheme point, and $\tilde\phi$ is an isomorphism. If $p'$ is a scheme point, the same conclusion holds by switching the role of $\mathcal{C}$ and $\mathcal{C}'$. Finally, when both $p$ and $p'$ are stacky points, then both are $\boldsymbol{\mu}_a$-srtacky points for an $a|r$. Thus a local argument shows that $\phi$ extends to an isomorphism $\tilde\phi : (\mathcal{C} - \Delta) \cup \mathcal{U} \to (\mathcal{C}' - \Delta') \cup \mathcal{U}'$ so that (4.7) extends to (4.8). By going through this local extension throughout all points in $\Delta$, we conclude that $\phi$ extends to an isomorphism $\tilde\phi : \mathcal{C} \to \mathcal{C}'$ so that (4.7) extends to (4.8). This proves that $\xi \cong \xi'$. □

**Proposition 4.4.** *Lemma 4.1 holds without assuming that $\mathcal{C}_*$ is smooth.*

*Proof.* Let $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$ be as before, and let $\xi_\alpha$, $\alpha \in \Xi$, be families constructed as in Corollary 3.4. Let $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \cdots)$ and likewise $\xi_\alpha'$, $\alpha \in \Xi$, be the similar decomposition. Here both $\xi_\alpha$ and $\xi_\alpha'$ are indexed by the same set $\Xi$ because $\xi_* \cong \xi_*'$. By Corollary 3.4, all $\xi_\alpha$ and $\xi_\alpha'$ are stable families of MSP-fields.

Because $\xi_* \cong \xi_*'$, we have $\xi_{\alpha *} \cong \xi_{\alpha *}'$. By Lemma 4.1, $\xi_{\alpha *} \cong \xi_{\alpha *}'$ extends to $\xi_\alpha \cong \xi_\alpha'$. Then a direct argument shows that as the isomorphisms $\xi_\alpha \cong \xi_\alpha'$ are consistent with the isomorphism $\xi_* \cong \xi_*'$, they induce an isomorphism $\xi \cong \xi'$, extending $\xi_* \cong \xi_*'$. As the argument is straightforward, we omit the details here. □

Applying valuative criterion of separateness of DM stacks, Proposition 4.4 proves

**Proposition 4.5.** *The stack $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is separated.*

4.2. **$\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ is of finite type.** Let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ be of the presentation $(\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots)$, let $\mathcal{C}_0 = \mathcal{C} \cap (\nu_1 = 0)_{\rm red}$, $\mathcal{C}_\infty = \mathcal{C} \cap (\nu_2 = 0)_{\rm red}$, and $\mathcal{C}_1 = \mathcal{C} \cap (\rho = \varphi = 0)_{\rm red}$ be as before . Let $\mathcal{C}_{01}$ (resp. $\mathcal{C}_{1\infty}$) be the union of irreducible components of $\mathcal{C}$ in $(\rho = 0)$ (resp. $(\varphi = 0)$) but not in $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\infty$.

**Lemma 4.6.** *We have*

1. *The curves* $\mathcal{C}_0$, $\mathcal{C}_1$ *and* $\mathcal{C}_\infty$ *are mutually disjoint;*
2. *no two of* $\mathcal{C}_0, \mathcal{C}_{01}, \mathcal{C}_1, \mathcal{C}_{1\infty}, \mathcal{C}_\infty$ *share common irreducible components;*
3. $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty$.

*Proof.* For $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$, (1) follows from their definitions. Furthermore, any irreducible component $\mathcal{A} \subset \mathcal{C}_{01}$ has $\varphi|_{\mathcal{A}} \neq 0$. Likewise, any irreducible $\mathcal{A} \subset \mathcal{C}_{1\infty}$ has $\rho|_{\mathcal{A}} \neq 0$. Since $(\varphi = 0) \cup (\rho = 0) = \mathcal{C}$, we conclude that $\mathcal{C}_{01}$ and $\mathcal{C}_{1\infty}$ share no common irreducible components. The other parts of (2) are similar. Finally, by the same reasoning $(\varphi = 0) \cup (\rho = 0) = \mathcal{C}$, we have (3). □

We derive some information of the degrees of $\mathcal{L}$ and $\mathcal{N}$ along $\mathcal{C}_a$. First, since $\nu_2|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$ is nowhere vanishing, $\mathcal{N}|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1} \cong \mathcal{O}_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$. Thus

$$d_\infty = \deg \mathcal{N} = \deg \mathcal{N}|_{\mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}. \tag{4.9}$$

Similarly, since $\nu_1$ restricted to $\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty$ is nowhere vanishing,

$$\mathcal{L} \otimes \mathcal{N}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty} \cong \mathcal{O}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}.$$

Combined, we get

$$\deg \mathcal{L}|_{\mathcal{C}_0 \cup \mathcal{C}_{01}} = d_0, \quad \deg \mathcal{L}|_{\mathcal{C}_1} = 0, \quad \deg \mathcal{L}|_{\mathcal{C}_{1\infty} \cup \mathcal{C}_\infty} = -d_\infty. \tag{4.10}$$

For $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$, let $\Upsilon_\xi$ be the dual graph of $\Sigma^{\mathcal{C}} \subset \mathcal{C}$. Namely, vertices of $\Upsilon_\xi$ are associated to irreducible components of $\mathcal{C}$, edges of $\Upsilon_\xi$ are associated to nodes connecting two different irreducible components of $\mathcal{C}$, and legs of $\Upsilon_\xi$ are associated to markings $\Sigma^{\mathcal{C}}$. Let $V(\Upsilon_\xi)$ be the set of vertices, and $E(\Upsilon_\xi)$ be the set of edges. Furthermore, each vertex $v \in V(\Upsilon_\xi)$ is decorated by $g_v := g(\mathcal{C}_v)$, the arithmetic genus of the irreducible component $\mathcal{C}_v$ associated to $v$.

Given a decorated graph $\Upsilon$ as above (i.e. a connected graph with legs, vertices $v$ decorated by $g_v \in \mathbb{Z}_{\geq 0}$, and without circular-edges), we say $v \in V(\Upsilon)$ is stable (resp. semistable) if $2g_v - 2 + |E_v| \geq 1$ (resp. $\geq 0$), where $E_v$ is the set of legs and edges in $\Upsilon$ attached to $v$.

**Proposition 4.7.** *The set* $\Theta := \{\Upsilon_\xi \mid \xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})\}$ *is a finite set.*

*Proof.* Let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$. As the curve $(\Sigma^{\mathcal{C}} \subset \mathcal{C})$ may not stable, knowing the total genus and the number of legs of $\Upsilon_\xi$ is not sufficient to bound the geometry of $\Upsilon_\xi$. Our approach is to use the information of line bundles $\mathcal{L}$ and $\mathcal{N}$ on $\mathcal{C}$ given by $\xi$ to add legs to $\Upsilon_\xi$ to form a semistable $\tilde{\Upsilon}_\xi$

First, let $\mathbb{I} = \{0, 01, 1, 1\infty, \infty\}$. Using (3) of Lemma 4.6, for $a \in \mathbb{I}$ we define $V(\Upsilon_\xi)_a = \{v \in V(\Upsilon_\xi) \mid \mathcal{C}_v \subset \mathcal{C}_a\}$. By the the stability criterion Lemma 3.3, we know that all $v \in V(\Upsilon_\xi)_1 \cup V(\Upsilon_\xi)_\infty$ are stable.

We now add new legs to vertices in $V(\Upsilon_\xi)$, called auxiliary legs. For every $v\in V(\Upsilon_\xi)_0\cup V(\Upsilon_\xi)_{01}$, we add $3\deg\mathcal{L}|_{\mathcal{C}_v}$ auxiliary legs to $v$. By (4.10), the total new legs added to all vertices in $V(\Upsilon_\xi)_0\cup V(\Upsilon_\xi)_{01}$ is $3d_0$.

We next treat the vertices in $V(\Upsilon_\xi)_{1\infty}$. We first introduce

$$E_{V(\Upsilon_\xi)_\infty}=\{e\in E(\Upsilon_\xi)\mid e\in E_v \text{ for some } v\in V(\Upsilon_\xi)_\infty\}.$$

For $v\in V(\Upsilon_\xi)_{1\infty}$, we define

$$\delta(v)=r\deg\mathcal{N}|_{\mathcal{C}_v}-|E_v\cap E_{V(\Upsilon_\xi)_\infty}|\in\mathbb{Z}_{\geq 0}. \tag{4.11}$$

Here it takes value in $\mathbb{Z}_{\geq 0}$ because $\nu_2$ is non-vanishing along $\mathcal{C}_v$ and vanishes at $\mathcal{C}_v\cap\mathcal{C}_\infty$. To every $v\in V(\Upsilon_\xi)_{1\infty}$, we add $2\delta(v)$ number of auxiliary legs to $v$.

We now show that the number of new legs added to $V(\Upsilon_\xi)_{1\infty}$ is bounded by $2rd_\infty+4g+2\ell$. Let $\mathcal{C}^1_\infty,\ldots,\mathcal{C}^s_\infty$ be the connected components of $\mathcal{C}_\infty$, $m_i$ be the number of markings on $\mathcal{C}^i_\infty$. Because $\rho|_{\mathcal{C}^i_\infty}$ is nowhere vanishing, using the discussion leading to (4.10), we have

$$0=-r\deg\mathcal{L}|_{\mathcal{C}^i_\infty}+\deg\omega^{\log}_{\mathcal{C}}|_{\mathcal{C}^i_\infty}=r\deg\mathcal{N}|_{\mathcal{C}^i_\infty}+\big(2g(\mathcal{C}^i_\infty)-2+|\mathcal{C}^i_\infty\cap\mathcal{C}_{1\infty}|+m_i\big).$$

Therefore, using that $\deg\mathcal{N}|_{\mathcal{C}_v}=0$ unless $v\in V(\Upsilon_\xi)_{1\infty}\cup V(\Upsilon_\xi)_\infty$, and using that $\sum_{v\in V(\Upsilon_\xi)_{1\infty}}|E_v\cap E_{V(\Upsilon_\xi)_\infty}|=\sum_{i=1}^r|\mathcal{C}^i_\infty\cap\mathcal{C}_{1\infty}|$, we obtain

$$\begin{aligned}d_\infty=\deg\mathcal{N}&=\sum_{i=1}^s\frac{1}{r}\big(2-2g(\mathcal{C}^i_\infty)-|\mathcal{C}^i_\infty\cap\mathcal{C}_{1\infty}|-m_i\big)+\sum_{v\in V(\Upsilon_\xi)_{1\infty}}\deg\mathcal{N}|_{\mathcal{C}_v}\\&=\frac{2s}{r}-\sum_{i=1}^s\frac{1}{r}\big(m_i+2g(\mathcal{C}^i_\infty)\big)+\frac{1}{r}\sum_{v\in V(\Upsilon_\xi)_{1\infty}}\delta(v).\end{aligned}$$

Thus the total number of auxiliary legs added to vertices in $V(\Upsilon_\xi)_{1\infty}$ is bounded by $2rd_\infty+4g+2\ell$; the number $s$ of connected components of $\mathcal{C}_\infty$ is bounded by the same number too.

Let $\tilde\Upsilon_\xi$ be the resulting graph after adding auxiliary legs to $v\in V(\Upsilon_\xi)$ according to the rules specified above. We now study the stability of vertices of $\tilde\Upsilon_\xi$. Let $v\in V(\Upsilon_\xi)$ be a not-stable vertex. Then $v\in V(\Upsilon_\xi)_{1\infty}$ and $|E_v|=1$ or 2. In case $|E_v|=1$, by (4.11) we have $\deg\mathcal{N}|_{\mathcal{C}_v}=\frac{1}{r}|E_v\cap E_{V(\Upsilon_\xi)_\infty}|=\frac{1}{r}$. By Lemma 3.3, $\xi$ is not stable, impossible. Thus $|E_v|=2$, and $v$ is a strictly semistable vertex of $\tilde\Upsilon_\xi$.

We now show that $\tilde\Upsilon_\xi$ contains no chain of strictly semistable vertices of length more than two. Indeed, suppose $v_1,v_2,v_3\in V(\tilde\Upsilon_\xi)$ with edges $e_1$ and $e_2$ forming a chain of unstable vertices in $\tilde\Upsilon_\xi$, where $e_1$ connects $v_1$ and $v_2$, and $e_2$ connects $v_2$ and $v_3$. By our construction, all $v_i\in V(\Upsilon_\xi)_{1\infty}$. Since $\xi$ is stable, by Lemma 3.3, $\deg\mathcal{N}|_{\mathcal{C}_{v_2}}\geq\frac{1}{r}$. Since both $v_1$ and $v_3$ are not in $V(\Upsilon_\xi)_\infty$, $E_{v_2}\cap E_{V(\Upsilon_\xi)_\infty}=\emptyset$, a contradiction.

Let $\Xi$ be the set of connected graphs with legs and whose vertices $v$ are decorated by non-negative integers $g_v$. For a $\Upsilon\in\Xi$, we define its genus $g(\Upsilon)$ to be $\frac{1}{2}\dim H^1(\Upsilon,\mathbb{Q})+\sum_{v\in V(\Upsilon)}g_v$. For a pair of integers $(g,l)$, let $\Xi_{g,l}$ be the set of genus $g$ graphs in $\Xi$ having exactly $l$ edges. We say $\Upsilon\in\Xi$ quasi-stable if all its vertices are semistable and that there is no chain of strictly semistable vertices in

$\Upsilon$ of length more than two. Let $\Xi_{g,l}^{q.s.}$ be the set of quasi-stable graphs in $\Xi_{g,l}$. Since the subset of stable $\Upsilon$ in $\Xi_{g,l}$ is finite, we see easily that $\Xi_{g,l}^{q.s.}$ is also finite.

By our bound on the legs added to $\Upsilon_\xi$ to derive $\tilde{\Upsilon}_\xi$, we conclude that

$$\{\tilde{\Upsilon}_\xi \mid \xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})\} \subset \coprod_{l\le 3\ell+2rd_\infty+4g} \Xi_{g,l}^{q.s.},$$

where the later is finite. Consequently, $\Theta = \{\Upsilon_\xi \mid \xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})\}$ is finite. □

**Proposition 4.8.** *The collection $\mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ is bounded.*

*Proof.* Since $\Theta$ is finite, we only need to show that to any $\Upsilon \in \Theta$, the set $\mathcal{W}_\Upsilon = \{\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid \Upsilon_\xi \cong \Upsilon\}$ is bounded.

Let $\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$ and let $\Upsilon_\xi$ be its decorated dual graph. The information of $\xi$ indeed provides us three more assignments

$$\iota_\xi : V(\Upsilon_\xi) \to \mathbb{I} \quad \text{and} \quad d_{\mathcal{L},\xi},\ d_{\mathcal{N},\xi} : V(\Upsilon_\xi) \longrightarrow \mathbb{Z},$$

where $\iota_\xi(v) = a$ if $\mathcal{C}_v \subset \mathcal{C}_a$, $d_{\mathcal{L},\xi}(v) = \deg \mathcal{L}|_{\mathcal{C}_v}$, and $d_{\mathcal{N},\xi}(v) = \deg \mathcal{N}|_{\mathcal{C}_v}$.

We show that given $\Upsilon \in \Theta$, the set

$$\{(\Upsilon_\xi, \iota_\xi, d_{\mathcal{L},\xi}, d_{\mathcal{M},\xi}) \mid \xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) \text{ and } \Upsilon_\xi \cong \Upsilon\} \tag{4.12}$$

is finite. First, since $\mathbb{I}$ is finite, the possible choices of $\iota_\xi : V(\Upsilon) \to \mathbb{I}$ are finite. Once $\iota_\xi$ is chosen, the map $d_{\mathcal{N},\xi}$ has the following properties: $d_{\mathcal{N},\xi}$ takes value 0 on $\iota_\xi^{-1}\big(\{0, 01, 1\}\big)$, for $v \in \iota_\xi^{-1}(\infty)$, $d_{\mathcal{N},\xi}(v) = \frac{1}{r}(2-2g_v-|E_v|)$, and $d_{\mathcal{N},\xi}$ restricted to $\iota_\xi^{-1}\big(1\infty\big)$ takes values in $\frac{1}{r}\mathbb{Z}_{>0}$. Since $\sum_v d_{\mathcal{N},\xi}(v) = d_\infty$, the choices of $d_{\mathcal{N},\xi}$ are finite.

For the assignment $d_{\mathcal{L},\xi} : V(\Upsilon) \to \frac{1}{r}\mathbb{Z}$, as $d_{\mathcal{L},\xi}$ takes positive values on $\iota_\xi^{-1}(\{0, 01\})$, takes 0 on $\iota_\xi^{-1}(1)$, and that $d_{\mathcal{L},\xi} + d_{\mathcal{N},\xi}$ takes value 0 on $\iota_\xi^{-1}(\{1\infty, \infty\})$, the possible choices of $d_{\mathcal{L},\xi}$ are also finite. Combined, we prove that the set (4.12) is finite.

Finally, fixing a $\xi' \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C})$, it is direct to check that the collection

$$\{\xi \in \mathcal{W}^-_{g,\gamma,\mathbf{d}}(\mathbb{C}) \mid (\Upsilon_\xi, \iota_\xi, d_{\mathcal{L},\xi}, d_{\mathcal{M},\xi}) \cong (\Upsilon_{\xi'}, \iota_{\xi'}, d_{\mathcal{L},\xi'}, d_{\mathcal{N},\xi'})\}$$

is bounded. This proves the proposition. □

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