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Gromov–Witten Invariants of Stable Maps with Fields

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We construct the Gromov–Witten invariants of moduli of stable maps to $\mathbb{P}^4$ with fields. This is the all genus mathematical theory of the Guffin–Sharpe–Witten model, and is a modified twisted Gromov–Witten invariant of $\mathbb{P}^4$. These invariants are constructed using the cosection localization of Kiem–Li, an algebro-geometric analog of Witten's perturbed equations in Landau–Ginzburg theory. We prove that these invariants coincide, up to sign, with the Gromov–Witten invariants of quintics.

1 Introduction

The Candelas et al. genus zero generating function [3] of the Gromov–Witten invariants of quintic Calabi–Yau three-folds was proved by Givental [12] and Lian et al. [21]; the genus 1 generating function of Bershadsky et al. [2] was proved by Zinger [26]. Both proofs rely on the "hyperplane property" of the Gromov–Witten invariants of quintics, which expresses the invariants in terms of "Euler class of bundles" over the moduli of stable maps to $\mathbb{P}^4$. The hyperplane property for genus 0 was derived by Kontseviech [17]; the case of genus 1 was proved by Li and Zinger [20]. This paper is our first step to build such a theory for all genus Gromov–Witten invariants of quintics [19], and beyond.

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In this paper, we introduce a new class of moduli spaces: the moduli of stable maps to $\mathbb{P}^4$ with fields. These moduli spaces are cones over the usual moduli of stable maps to $\mathbb{P}^4$; they are not proper for positive genus. We use Kiem–Li's cosection localized virtual cycle to construct their localized virtual cycles, and thus their Gromov–Witten invariants. Applying degeneration, we prove that these invariants coincide (up to signs) with the Gromov–Witten invariants of the quintics.

We briefly outline our construction and the main theorem. Given nonnegative integers g and d, we form the moduli $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ of genus g degree d stable maps to $\mathbb{P}^4$ with p-fields:

$$\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p = \{[u, C, p] | [u, C] \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d),\ p \in \Gamma(C, u^*\mathscr{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)\}/\sim .$$

Here $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ is the moduli of degree d genus g stable maps to $\mathbb{P}^4$.

It is a Deligne–Mumford stack; forgetting the fields, the induced morphism

$$\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p \to \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$$

has fiber $H^0(u^*\mathscr{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)$ over $[u, C] \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$. When g is positive, it is not proper.

The moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ has a perfect obstruction theory, and thus has a virtual class. To overcome its nonproperness in order to define its Gromov–Witten invariant, we construct a cosection (homomorphism) of its obstruction sheaf. The choice of the cosection depends on the choice of a degree 5 homogeneous polynomial, like $\mathbf{w} = x_1^5 + \cdots + x_5^5$. The nonsurjective locus (called the degeneracy locus) of the cosection associated to $\mathbf{w}$

$$\sigma : \mathcal{O}b_{\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p} \longrightarrow \mathscr{O}_{\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p}$$

is

$$\overline{\mathcal{M}}_g(Q, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p, \quad Q = (x_1^5 + \cdots + x_5^5 = 0) \subset \mathbb{P}^4,$$

which is proper. Applying Kiem–Li cosection localized virtual class construction, we obtain a localized virtual cycle

$$[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_\sigma \in A_0 \overline{\mathcal{M}}_g(Q, d).$$

We define the Gromov–Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ to be

$$N_g(d)^p_{\mathbb{P}^4} = \deg[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_\sigma.$$

(We also call them the Gromov–Witten invariants of the space $(K_{\mathbb{P}^4}, \mathbf{w})$.)

This relates to the Gromov–Witten invariants the quintic Q:

Theorem 1.1. For $g \geq 0$ and $d > 0$, the Gromov–Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ (or $(K_{\mathbb{P}^4}, \mathbf{w})$) coincides with the Gromov–Witten invariant $N_g(d)_Q$ of the quintic Q up to a sign:

$$N_g(d)^p_{\mathbb{P}^4} = (-1)^{5d+1-g} N_g(d)_Q. \qquad \square$$

When $g = 0$, this is derived in Guffin and Sharpe [13] using path integral. This identity also is the same as Kontsevich's formula for $g = 0$ Gromov–Witten invariants of quintics. If one views the localized virtual cycle of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ as "Euler class of bundles", this theorem is a substitute of the "hyperplane property" of the Gromov–Witten invariants of quintics in high genus.

We believe that this construction will lead to a mathematical approach to Witten's Gauged-Linear-Sigma model for all genus. In [24], Witten constructed a (gauged) topological field theory (for $g = 0$) whose target is the stacky quotient $[\mathbb{C}^6/\mathbb{C}^*]$ (of weights $(1, 1, 1, 1, 1, -5)$) with a superpotential, say $\mathbf{w}$. This theory has two GIT quotients: one is $(K_{\mathbb{P}^4}, \mathbf{w})$, called the massive theory; the other is $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ called the linear Landau–Ginzburg model. (Linear Landau–Ginzburg model means the space is the orbifold quotient of an affine space.) Witten proposed to A-twist both models: the A-twist of $(K_{\mathbb{P}^4}, \mathbf{w})$ likely is a theory of moduli of stable quotients, and the resulting theory is of Landau–Ginzburg type. The A-twist of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ is related to the generalized Witten conjecture [25] for $A_4 = (\mathbb{C}, x^5)$.

The program proposed in [24] provides a possible road map toward an all genus mathematical theory linking the Gromov–Witten theory of quintic to the Landau–Ginzburg model of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$. A bolder speculation is that there is a geometric mirror construction identifying the A-twisted topological string theory of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ with the B-side invariants of its Landau–Ginzburg Mirror.

In [10], Fan et al. constructed the virtual cycle of the A-twisted topological string theories of the linear Landau–Ginzburg model of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$; their construction is via analytic perturbation of Witten's equation. Later, Ruan and Chiodo proved [6] the genus zero mirror symmetry for $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ and its mirror.

For massive theory of $(K_{\mathbb{P}^4}, \mathbf{w})$, Marian, Oprea and Pandharipande constructed the moduli of stable quotients [23], which is believed to be an example of massive instantons. It is interesting to see how the invariants of A-twisting the construction in [23] relate to the invariants of the massive instantons in $(K_{\mathbb{P}^4}, \mathbf{w})$ in Witten's program.

Using Super-String theories, Guffin and Sharpe constructed a special type of genus 0 Landau–Ginzburg model for $(K_{\mathbb{P}^4}, \mathbf{w})$, and equated it with the genus 0 Gromov–Witten invariants of the quintic Q [13]. The notion of p-fields was introduced in this work. Using nonperturbative localization of path-integral, they reduced this theory to the genus 0 Gromov–Witten invariants of quintics. Since this follows Witten's Gauged-Linear-Sigma model program, we call this construction the Guffin–Sharpe–Witten (GSW) model.

Our work is an algebro-geometric construction of the GSW model for all genus. The moduli of stable maps with p-fields is the algebro-geometric substitute of the phase space of all smooth maps with smooth fields. The cosection localized virtual cycle is the analog of Witten's perturbed equation. Theorem 1.1 shows that the Gromov–Witten invariants of the algebro-geometric GSW model of all genus coincide up to signs with the Gromov–Witten invariants of quintic three-folds.

Our construction applies to global complete intersection Calabi–Yau three-folds of toric varieties. In the subsequent papers, we apply the techniques developed to the moduli of stable quotients (cf. [23]) to obtain all genus invariants of massive theory of $(K_{\mathbb{P}^4}, \mathbf{w})$ [4]; we shall also apply it to the linear Landau–Gingzberg model to obtain an alternative algebro-geometric construction of Fan–Jarvis–Ruan–Witten invariants [5]. In the later case, the resulting invariants are equal to those defined using perturbed Witten equations [10].

We believe the new invariants and their equivalence with the Gromov–Witten invariants of quintics provide the first step toward building a geometric bridge establishing the conjectural equivalence of Gromov–Witten invariants of quintics and the Fan–Jarvis–Ruan–Witten invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$. Constructing such a bridge will be the long-term goal of this project.

Conventions. In this paper, the primary focus is on moduli of stable maps with fields to $\mathbb{P}^4$, on a smooth quintic Calabi–Yau $Q \subset \mathbb{P}^4$ defined by $\sum x_i^5 = 0$, and on a deformation of $\mathbb{P}^4$ to the normal cone to $Q \subset \mathbb{P}^4$.

Throughout the paper, we fix homogeneous coordinates $[x_1, \ldots, x_5]$ of $\mathbb{P}^4$, with $x_i \in H^0(\mathbb{P}^4, \mathscr{O}(1))$ and $\mathscr{O}(1) := \mathscr{O}_{\mathbb{P}^4}(1)$. We denote by N the normal bundle to Q in $\mathbb{P}^4$. Using the defining section $\sum x_i^5 = 0$, we obtain a canonical isomorphism $N \cong \mathscr{O}_Q(5)$.

In this paper we fix positive integers g and d throughout. We use $(f, \mathcal{C})$ with subscripts to denote the universal families of various moduli spaces. For instance, after abbreviating $\mathcal{P} = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$, the universal curve and map of $\mathcal{P}$ is denoted by

$$(f_{\mathcal{P}}, \pi_{\mathcal{P}}) : \mathcal{C}_{\mathcal{P}} \longrightarrow \mathbb{P}^4 \times \mathcal{P}.$$

For any locally free sheaf $\mathscr{L}$ on $\mathcal{C}$, we denote by $\mathrm{Vb}(\mathscr{L})$ the underlying vector bundle of $\mathscr{L}$; namely, the sheaf of sections of $\mathrm{Vb}(\mathscr{L})$ is $\mathscr{L}$.

In this paper we use fonts $\mathbb{E}$, etc. to denote derived objects (of complexes). We reserve $\mathbb{L}_{X/Y}$ to denote the cotangent complex of $X \to Y$; we denote by $\mathbb{T}_{X/Y}$ its derived dual $\mathbb{T}_{X/Y} = \mathbb{L}_{X/Y}^{\vee}$, called the tangent complex of $X \to Y$. We use $\phi_{X/Y} : \mathbb{T}_{X/Y} \to \mathbb{E}_{X/Y}$ to denote a relative obstruction theory of $X \to Y$, following Behrend and Fantechi [1].

Without causing confusion, all pull-backs of derived objects (respectively sheaves) are derived pull back (respectively sheaves pull back) unless otherwise stated.

2 Direct Image Cones and Moduli of Sections

In this section, to a locally free sheaf $\mathcal{L}$ over a family of nodal curves $\pi : \mathcal{C} \to \mathfrak{A}$ over an Artin stack $\mathfrak{A}$, we construct its direct image cone $C(\pi_* \mathcal{L})$, and its relative obstruction theory.

2.1 Direct image cones

Let $\mathfrak{A}$ be an Artin stack, $\pi : \mathcal{C} \to \mathfrak{A}$ be a flat family of connected, nodal, arithmetic genus g curves, and $\mathscr{L}$ be a locally free sheaf on $\mathcal{C}$.

Definition 2.1. For any scheme S, we define $C(\pi_* \mathscr{L})(S)$ to be the collection of (ρ, p) so that $\rho : S \to \mathfrak{A}$ is a morphism and $p \in H^0(\mathcal{C}_S, \rho^* \mathscr{L})$, where $\mathcal{C}_S = S \times_{\mathfrak{A}} \mathcal{C}$ and $\rho^* \mathscr{L} = \mathscr{L} \times_{\mathscr{O}_{\mathcal{C}}} \mathscr{O}_{\mathcal{C}_S}$.

An arrow from (ρ, p) to (ρ', p') in $C(\pi_* \mathscr{L})(S)$ consists of an arrow $\tau : \mathcal{C}_S \to \mathcal{C}_S$ in $\mathfrak{A}(S)$ such that under the induced isomorphism $\tau^* \rho'^* \mathscr{L} \cong \rho^* \mathscr{L}$, $p = \tau^* p'$. Given $S \to S'$, we define $C(\pi_* \mathscr{L})(S') \to C(\pi_* \mathscr{L})(S)$ by pull-back. □

We show that $C(\pi_* \mathscr{L})$ is a stack over $\mathfrak{A}$. Given a module $\mathcal{F}$, we denote by $\mathrm{Sym} \mathcal{F}$ the symmetric algebra of $\mathcal{F}$.

Proposition 2.2. Let the notation be as in Definition 2.1. We have a canonical $\mathfrak{A}$-isomorphism

$$C(\pi_*\mathscr{L}) \cong \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym} R^1\pi_*(\mathscr{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}}). \qquad \square$$

Proof. For any scheme S and a morphism $\rho : S \to \mathfrak{A}$, we let

$$C(\pi_*\mathscr{L})(\rho) = \{(\rho, p) \mid p \in H^0(\mathcal{C}_S, \rho^*\mathscr{L})\} \cong \Gamma(\mathcal{C}_S, \rho^*\mathscr{L}).$$

We define a transformation

$$C(\pi_*\mathscr{L})(\rho) \longrightarrow \mathrm{Hom}_S(S, \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym} R^1\pi_*(\mathscr{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}}) \times_{\mathfrak{A}} S) \tag{2.1}$$

as follows. We let $\mathscr{F} = R^1\pi_*(\mathscr{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}})$. Given a $\rho : S \to \mathfrak{A}$, an S-morphism $S \to \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym}\mathscr{F} \times_{\mathfrak{A}} S$ is given by a morphism of sheaves of $\mathscr{O}_{\mathfrak{A}}$-algebras

$$\mathrm{Sym}\mathscr{F} \longrightarrow \mathscr{O}_S,$$

which is equivalent to a morphism of sheaves of $\mathscr{O}_{\mathfrak{A}}$-modules

$$R^1\pi_{S*}(\mathscr{L}_S^\vee \otimes \omega_{\mathcal{C}_S/S}) = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{A}}} \mathscr{O}_S \longrightarrow \mathscr{O}_S.$$

Here we have used the base change property of $R^1\pi_*$.

Applying Serre duality [9] to the complete intersection morphism $\pi_S : \mathcal{C}_S \to S$, we obtain

$$\mathrm{Hom}_S(R^1\pi_{S*}(\mathscr{L}_S^\vee \otimes \omega_{\mathcal{C}_S/S}), \mathscr{O}_S) = \Gamma(\mathcal{C}_S, \mathscr{L}_S).$$

This defines the transformation (2.1). It is direct to check that this is an isomorphism, and satisfies the base change property. This proves the Proposition. ∎

2.2 Moduli of sections

One can also construct the direct image cone via the moduli of sections. Let $\mathcal{C} \to \mathfrak{A}$ be as in Definition 2.1; let $\mathcal{Z} \to \mathcal{C}$ be an Artin stack such that the arrow $\mathcal{Z} \to \mathcal{C}$ is representable and quasi-projective. We define a groupoid $\mathfrak{S}$ (with dependence on $\mathcal{Z}$ implicitly understood) as follows.

For any scheme $S \to \mathfrak{A}$, we define $\mathcal{C}_S = \mathcal{C} \times_{\mathfrak{A}} S$ and $\mathcal{Z}_S = \mathcal{Z} \times_{\mathcal{C}} \mathcal{C}_S$; we view $\mathcal{Z}_S$ as a scheme over $\mathcal{C}_S$ via the projection $\pi_S : \mathcal{Z}_S \to \mathcal{C}_S$. We define

$$\mathfrak{S}(S) = \{s : \mathcal{C}_S \to \mathcal{Z}_S \mid s \text{ are } \mathcal{C}_S\text{-morphisms}\}.$$

The arrows are defined by pull-backs.

Proposition 2.3. The groupoid $\mathfrak{S}$ is an Artin stack with a natural projection to $\mathfrak{A}$. The morphism $\mathfrak{S} \to \mathfrak{A}$ is representable and quasi-projective. □

Proof. This follows from the functorial construction of Hilbert scheme and that $\mathcal{Z} \to \mathcal{C}$ is representable and quasi-projective. ■

Corollary 2.4. Let $\pi : \mathcal{C} \to \mathfrak{A}$ be as in Definition 2.1, and let $\mathcal{Z} = \mathrm{Vb}(\mathscr{L})$, which is the underlying vector bundle of the locally free sheaf $\mathscr{L}$. Then canonically $C(\pi_* \mathscr{L}) \cong \mathfrak{S}$ as stacks over $\mathfrak{A}$. □

2.3 The obstruction theory

We give the perfect obstruction theory of $\mathfrak{S}$. Let $\mathcal{Z} \to \mathcal{C} \to \mathfrak{A}$ be as in Proposition 2.3. Let $\pi_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \to \mathfrak{S}$ be the universal family of $\mathfrak{S}$ and $\mathfrak{e} : \mathcal{C}_{\mathfrak{S}} \to \mathcal{Z}$ be the tautological evaluation map; namely, $(\pi_{\mathfrak{S}}, \mathfrak{e}) : \mathcal{C}_{\mathfrak{S}} \to \mathfrak{S} \times \mathcal{Z}$ is the universal family of $\mathfrak{S}$.

As mentioned at the end of the introduction, we let $\mathbb{T}_{\mathfrak{S}/\mathfrak{A}}$ be the tangent complex of $\mathfrak{S} \to \mathfrak{A}$, which is the dual of the cotangent complex $\mathbb{L}_{\mathfrak{S}/\mathfrak{A}}$.

Proposition 2.5. Let the situation be as stated. Suppose $\mathcal{Z} \to \mathcal{C}$ is smooth; then $\mathfrak{S} \to \mathfrak{A}$ has a perfect relative obstruction theory:

$$\phi_{\mathfrak{S}/\mathfrak{A}} : \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} := R^{\bullet} \pi_{\mathfrak{S}*} \mathfrak{e}^* \Omega^{\vee}_{\mathcal{Z}/\mathcal{C}}.$$ □

Proof. By our construction we have the commutative diagrams

$$\begin{tikzcd} \mathfrak{S} & \mathcal{C}_{\mathfrak{S}} \arrow[l] \arrow[r, "\mathfrak{e}"] \arrow[d] & \mathcal{Z} \arrow[d] \\ \mathfrak{A} \arrow[from=u] & \mathcal{C} \arrow[l] \arrow[r, "="] & \mathcal{C} \end{tikzcd} \tag{2.2}$$

where the left one is Cartesian. Applying the projection formula to

$$\pi_{\mathfrak{S}}^{*}\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \cong \mathbb{T}_{\mathcal{C}_{\mathfrak{S}}/\mathcal{C}} \longrightarrow \mathfrak{e}^{*}\mathbb{T}_{\mathcal{Z}/\mathcal{C}} = \mathfrak{e}^{*}\Omega^{\vee}_{\mathcal{Z}/\mathcal{C}}, \tag{2.3}$$

and using

$$\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow R^{\bullet}\pi_{\mathfrak{S}*}\pi_{\mathfrak{S}}^{*}\mathbb{T}_{\mathfrak{S}/\mathfrak{A}},$$

we obtain

$$\phi_{\mathfrak{S}/\mathfrak{A}} : \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} := R^{\bullet}\pi_{\mathfrak{S}*}\mathfrak{e}^{*}\Omega^{\vee}_{\mathcal{Z}/\mathcal{C}}. \tag{2.4}$$

We claim that $\phi_{\mathfrak{S}/\mathfrak{A}}$ is a perfect obstruction theory.

We prove this by applying the criterion in [1, Theorem 4.5]. Given an extension $T \subset T'$ by ideal J with $J^2 = 0$, and a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\mathfrak{m}} & \mathfrak{S} \\ \big\downarrow & & \big\downarrow \\ T' & \xrightarrow{\mathfrak{n}} & \mathfrak{A} \end{array} \tag{2.5}$$

we say that $\mathfrak{m}$ lifts to an $\mathfrak{m}' : T' \to \mathfrak{S}$ if $\mathfrak{m}'$ fits into (2.5) to form two commuting triangles.

By standard deformation theory, the diagram (2.5) provides a morphism

$$\mathfrak{m}^{*}\mathbb{L}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{L}_{T/T'} \longrightarrow \mathbb{L}^{\geq -1}_{T/T'} = J[1],$$

which gives an element

$$\varpi(\mathfrak{m}) \in \operatorname{Ext}^1_T(\mathfrak{m}^{*}\mathbb{L}_{\mathfrak{S}/\mathfrak{A}}, J) = H^1(T, \mathfrak{m}^{*}\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J). \tag{2.6}$$

Using the morphism $\phi_{\mathfrak{S}/\mathfrak{A}}$ in (2.4), we obtain the homomorphism

$$\phi' : H^1(T, \mathfrak{m}^{*}\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J) \longrightarrow H^1(T, \mathfrak{m}^{*}\mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J).$$

We define

$$\mathrm{ob}(T, T', \mathfrak{m}) := \phi'(\varpi(\mathfrak{m})) \in H^1(T, \mathfrak{m}^{*}\mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J).$$

To prove that $\phi_{\mathfrak{S}/\mathfrak{A}}$ is a perfect relative obstruction theory, by the criterion in [1, Theorem 4.5 (3)], we need to show

(1) $\mathrm{ob}(T, T', \mathfrak{m}) = 0$ if and only if $\mathfrak{m}$ in (2.5) can be lifted to $\mathfrak{m}' : T' \to \mathfrak{C}$;

(2) when $\mathrm{ob}(T, T', \mathfrak{m}) = 0$, the set of liftings $\mathfrak{m}' : T' \to \mathfrak{C}$ forms a torsor under $H^0(T, \mathfrak{m}^*\mathbb{E}_{\mathfrak{C}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J)$.

We now verify (1) and (2). Pulling back $\mathcal{C}$ to T and T' via $\mathfrak{m}$ and $\mathfrak{n}$, respectively, we obtain two families $\pi_T : \mathcal{C}_T \to T$ and $\pi_{T'} : \mathcal{C}_{T'} \to T'$; pulling back $\mathfrak{e}$ to T, we have the evaluation map $\mathfrak{e}_T : \mathcal{C}_T \to \mathcal{Z}$. Let

$$\kappa : H^1(T, \mathfrak{m}^*\mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathscr{O}_T} J) \xrightarrow{\cong} H^1(T, R^\bullet \pi_{T*}(\mathfrak{e}_T^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}} \otimes \pi_T^* J))$$

be the canonical isomorphism defined by the definition of $\mathbb{E}_{\mathfrak{S}/\mathfrak{A}}$ (cf. (2.4)).

Using the standard property of cotangent complex, the commuting square

$$\begin{array}{ccc} \mathcal{C}_T & \xrightarrow{\mathfrak{e}_T} & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{C}_{T'} & \xrightarrow{\tilde{\mathfrak{n}}} & \mathcal{C} \end{array} \tag{2.7}$$

where $\tilde{\mathfrak{n}}$ is the lift of $\mathfrak{n}$ in (2.5), induces homomorphisms

$$\mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}} \cong \mathfrak{e}_T^* \mathbb{L}_{\mathcal{Z}/\mathcal{C}} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_T^* \mathbb{L}_{T/T'} \longrightarrow \mathbb{L}^{\geq -1}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_T^* J[1].$$

Their composite associates to an element

$$\varpi(\mathbf{e}_T, \mathcal{Z}, \mathcal{C}) \in H^1(\mathcal{C}_T, \mathfrak{e}_T^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}} \otimes \pi_T^* J) \cong H^1(T, R^\bullet \pi_{T*}(\mathfrak{e}_T^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}} \otimes \pi_T^* J)).$$

By Lemma A.5, $\varpi(\mathbf{e}_T, \mathcal{Z}, \mathcal{C}) = 0$ if and only if (2.7) admits a lifting $\mathcal{C}_{T'} \to \mathcal{Z}$.

As (2.7) is the composition of (2.5) with (2.2), $\varpi(\mathbf{e}_T, \mathcal{Z}, \mathcal{C}) = \kappa(\phi'(\varpi(\mathfrak{m})))$. Thus $\mathrm{ob}(T, T', m) = 0$ if and only if (2.7) has a lifting, which is equivalent to the fact that $\mathfrak{m}$ lifts to an $\mathfrak{m}' : T' \to \mathfrak{S}$ in (2.5). This verifies criterion (1).

Finally, when $\mathrm{ob}(T, T', \mathfrak{m}) = 0$, any two liftings $\mathcal{C}_{T'} \to \mathcal{Z}$ differ by a section in $H^0(\mathcal{C}_T, \mathfrak{e}_T^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}} \otimes \pi_T^* J)$, and vice versa [14, Theorem 2.1.7]. This proves the criterion (2). These complete the proof of the Proposition. ■

2.4 Moduli of stable maps

Using the stack $\mathfrak{D}_g$ of curves with line bundles, this construction provides a different perspective of the moduli of stable maps to a projective scheme.

Definition 2.6. We define $\mathfrak{D}_g$ to be the groupoid associating to each scheme S the set $\mathfrak{D}_g(S)$ of pairs $(\mathcal{C}_S, \mathscr{L}_S)$, where $\mathcal{C}_S \to S$ is a flat family of connected nodal curves and $\mathscr{L}_S$ is a line bundle on $\mathcal{C}_S$ of degree d along fibers of $\mathcal{C}_S/S$. An arrow from $(\mathcal{C}_S, \mathscr{L}_S)$ to $(\mathcal{C}'_S, \mathscr{L}'_S)$ consists of a pair (ρ, τ), where $\rho : \mathcal{C}_S \to \mathcal{C}'_S$ and $\tau : \rho^*\mathscr{L}'_S \to \mathscr{L}$ are S isomorphisms. □

It is easy to show that $\mathfrak{D}_g$ is a smooth Artin stack. By forgetting the line bundles, one obtains an induced morphism $\mathfrak{D}_g \to \mathfrak{M}_g$, where $\mathfrak{M}_g$ is the Artin stack of all connected nodal curves of genus g. For any $\xi = (C, L) \in \mathfrak{D}_g$ the automorphism group of ξ relative to $\mathfrak{M}_g$, (i.e., automorphisms of L that fix C) is $\mathbb{C}^*$. We denote by $(\mathcal{C}_{\mathfrak{D}_g}, \mathcal{L}_{\mathfrak{D}_g})$, with $\pi_{\mathfrak{D}_g} : \mathcal{C}_{\mathfrak{D}_g} \to \mathfrak{D}_g$ implicitly understood, the universal family of $\mathfrak{D}_g$.

We now let $X \subset \mathbb{P}^n$ be a projective scheme. For the integer d given (the integer d is fixed throughout this paper), we have the moduli of genus g and degree d stable maps to X: $\overline{\mathcal{M}}_g(X, d)$. We now present it as the moduli of sections. We keep the homogeneous coordinates $[x_1, \dots, x_{n+1}]$ of $\mathbb{P}^n$ mentioned in the introduction. The choice of $[x_i]$ provides a presentation

$$\mathbb{P}^n = \mathbb{A}^{n+1*}/\mathbb{C}^*, \quad \mathbb{A}^{n+1*} := \mathbb{A}^{n+1} - 0. \tag{2.8}$$

We form the bundle

$$\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)})^* = \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)}) - 0_{\mathcal{C}_{\mathfrak{D}_g}},$$

where $0_{\mathcal{C}_{\mathfrak{D}_g}}$ is the zero section. Using the $\mathbb{C}^*$-equivariance of the projection $\mathbb{A}^{n+1*} \to \mathbb{P}^n$ induced by (2.8), we obtain a canonical morphism

$$\Psi : \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)})^* \longrightarrow \mathbb{P}^n.$$

We let

$$\mathcal{Z}_X = \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)})^* \times_{\mathbb{P}^n} X \subset \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)})^*.$$

We let $\mathfrak{S}_X$ be the stack of sections constructed in Section 2.2 with $\mathcal{Z}$ replaced by $\mathcal{Z}_X$.

Proposition 2.7. There is a canonical open immersion of stacks $\overline{\mathcal{M}}_g(X, d) \to \mathfrak{S}_X$, as stacks over $\mathfrak{M}_g$. □

Proof. For notational simplicity, in the remainder of this Section, we abbreviate $Y = \overline{\mathcal{M}}_g(X, d)$, and denote by $(f_Y, \pi_Y) : \mathcal{C}_Y \to X \times Y$ the universal family. Pulling back $\mathscr{O}(1)$, we obtain $\mathscr{L}_Y = f_Y^*\mathscr{O}(1)$; pulling back the homogeneous coordinates x_i (viewing

$x_i \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)))$, we obtain $u_i = f_Y^* x_i$. Since f_Y has degree d along fibers of $\mathcal{C}_Y/Y$, $(\mathcal{C}_Y, \mathscr{L}_Y)$ defines a morphism

$$\lambda : Y = \overline{\mathcal{M}}_g(X, d) \longrightarrow \mathfrak{D}_g; \tag{2.9}$$

since $f_Y(\mathcal{C}_Y) \subset X$, $(u_1, \ldots, u_{n+1})$ defines a section $Y \to \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus(n+1)}) \times_{\mathfrak{D}_g} Y$ (of Y) that factors through a section

$$\xi : Y \to \mathcal{Z}_X \times_{\mathfrak{D}_g} Y.$$

This defines a morphism $Y \to \mathfrak{S}_X$.

It is direct to check that this is an open immersion, and is a morphism over $\mathfrak{M}_g$. This proves the Proposition. ■

It is worth comparing the relative obstruction theory $\phi_{Y/\mathfrak{D}_g}$ of $\overline{\mathcal{M}}_g(X, d) \to \mathfrak{D}_g$ constructed using Section 2.3 with the relative obstruction theory $\phi_{Y/\mathfrak{M}_g}$ of $\overline{\mathcal{M}}_g(X, d) \to \mathfrak{M}_g$ given in [1].

Following the notation before Proposition 2.5, we have an evaluation map $\mathfrak{e}_Y : \mathcal{C}_Y \to \mathcal{Z}_X$. The induced morphism $\pi_Y^* \mathbb{T}_{Y/\mathfrak{D}_g} \cong \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{D}_g}} \to \mathfrak{e}_Y^* \mathbb{T}_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}}$ induces

$$\phi_{Y/\mathfrak{D}_g} : \mathbb{T}_{Y/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{Y/\mathfrak{D}_g} := R^\bullet \pi_* \mathfrak{e}_Y^* \mathbb{T}_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}}.$$

Applying Proposition 2.5, $\phi_{Y/\mathfrak{D}_g}$ is a perfect relative obstruction theory of $Y \to \mathfrak{D}_g$.

Lemma 2.8. Suppose that X is smooth. The relative obstruction theories $\phi_{Y/\mathfrak{D}_g}$ and $\phi_{Y/\mathfrak{M}_g}$ are related by a morphism of distinguished triangles:

$$\begin{tikzcd}
R^\bullet \pi_{Y*} \mathscr{O}_{\mathcal{C}_Y} \arrow[r] & \mathbb{E}_{Y/\mathfrak{D}_g} \arrow[r] & \mathbb{E}_{Y/\mathfrak{M}_g} \arrow[r, "+1"] & {} \\
\lambda^* \mathbb{T}_{\mathfrak{D}_g/\mathfrak{M}_g}[-1] \arrow[r] \arrow[u, "||"'] & \mathbb{T}_{Y/\mathfrak{D}_g} \arrow[r] \arrow[u, "\phi_{Y/\mathfrak{M}_g}"'] & \mathbb{T}_{Y/\mathfrak{M}_g} \arrow[r, "+1"] \arrow[u, "\phi_{Y/\mathfrak{D}_g}"'] & {}
\end{tikzcd}$$ □

Proof. Let $\mathcal{C}_{\mathfrak{M}_g}$ be the universal curve on $\mathfrak{M}_g$; let

$$\chi_M : \mathcal{Z}_X \longrightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$$

be the morphism so that its first factor is the composite $\mathcal{Z}_X \to \mathcal{C}_{\mathfrak{D}_g} \to \mathcal{C}_{\mathfrak{M}_g}$, and the second factor is the natural projection. Let

$$f : \mathcal{C}_Y \longrightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$$

be the compost of $\mathfrak{e}_Y : \mathcal{C}_Y \to \mathcal{Z}_X$ with $\chi_M : \mathcal{Z}_X \to \mathcal{C}_{\mathfrak{M}_g} \times X$. Note that the first factor of f is the canonical projection induced by $Y \to \mathfrak{D}_g \to \mathfrak{M}_g$; its the second factor is f_Y.

Taking the tangent complex relative to $\mathfrak{M}_g$, we obtain

$$\pi_Y^* \mathbb{T}_{Y/\mathfrak{M}_g} \cong \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{M}_g}} \longrightarrow f^* \mathbb{T}_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}} \cong f_Y^* T_X.$$

This induces

$$\phi_{Y/\mathfrak{M}_g} : \mathbb{T}_{Y/\mathfrak{M}_g} \longrightarrow \mathbb{E}_{Y/\mathfrak{M}_g} := R^\bullet \pi_* f_Y^* T_X,$$

which is the perfect relative obstruction theory of $Y \to \mathfrak{M}_g$ defined in [1].

We let $\chi_D : \mathcal{Z}_X \to \mathcal{C}_{\mathfrak{D}_g} \times X$ be defined similarly to χ_M, and let $g : \mathcal{C}_{\mathfrak{D}_g} \times X \to \mathcal{C}_{\mathfrak{M}_g} \times X$ be the projection. Note that $g \circ \chi_D = \chi_M$. By the construction, we have the commutative diagrams

$$\begin{tikzcd}
\mathcal{Z}_X \arrow[r, "\chi_D"] \arrow[d, "\rho_0"] & \mathcal{C}_{\mathfrak{D}_g} \times X \arrow[r, "g"] \arrow[d, "\pi_1"] & \mathcal{C}_{\mathfrak{M}_g} \times X \arrow[d] \\
\mathcal{C}_{\mathfrak{D}_g} \arrow[r, equal] & \mathcal{C}_{\mathfrak{D}_g} \arrow[r] & \mathcal{C}_{\mathfrak{M}_g}
\end{tikzcd} \tag{2.10}$$

It induces an exact sequence of locally free sheaves

$$0 \longrightarrow T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g} \times X} \longrightarrow T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow \chi_D^* T_{\mathcal{C}_{\mathfrak{D}_g} \times X/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow 0.$$

Since χ_D is a $\mathbb{C}^*$-principal bundle, $\mathscr{O}_{\mathcal{Z}_X} \cong T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g} \times X}$. Also we have canonical isomorphism $\chi_D^* T_{\mathcal{C}_{\mathfrak{D}_g} \times X/\mathcal{C}_{\mathfrak{D}_g}} \cong \chi_M^* T_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}}$. Let $\lambda_C : \mathcal{C}_Y \to \mathcal{C}_{\mathfrak{D}_g}$ be induced by λ. The above sequence fits into a morphism of distinguished triangles,

$$\begin{tikzcd}
\mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g} \times X} \arrow[r] & \mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] & \mathfrak{e}_Y^* \chi_M^* T_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}} \cong f_X^* T_X \arrow[r, "+1"] & {} \\
\lambda_C^* \mathbb{T}_{\mathcal{C}_{\mathfrak{D}_g}/\mathcal{C}_{\mathfrak{M}_g}}[-1] \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{M}_g}} \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

where the left vertical arrow is the composition

$$\lambda^*\mathbb{T}_{\mathcal{C}_{\mathfrak{D}_g}/\mathcal{C}_{\mathfrak{M}_g}} \cong \mathfrak{e}_Y^*\chi_D^*\pi_1^* T_{\mathcal{C}_{\mathfrak{D}_g}/\mathcal{C}_{\mathfrak{M}_g}} \cong \mathfrak{e}_Y^*\chi_D^* T_{\mathcal{C}_{\mathfrak{D}_g}\times X/\mathcal{C}_{\mathfrak{M}_g}\times X} \longrightarrow \mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}\times X}[1],$$

where the last arrow is given by the distinguished triangle of cotangent complexes associated to the top row of (2.10). Here the commutativity of squares in the above diagram can be checked by diagram chasing using (2.10).

Therefore we have a homomorphism of distinguished triangles

$$\begin{tikzcd}
R^\bullet\pi_{Y*}\mathscr{O}_{\mathcal{C}_Y} \arrow[r] & \mathbb{E}_{Y/\mathfrak{D}_g} \arrow[r] & \mathbb{E}_{Y/\mathfrak{M}_g} \arrow[r, "+1"] & {} \\
\lambda^*\mathbb{T}_{\mathfrak{D}_g/\mathfrak{M}_g}[-1] \arrow[r] \arrow[u] & \mathbb{T}_{Y/\mathfrak{D}_g} \arrow[r] \arrow[u, "\phi_{Y/\mathfrak{D}_g}"'] & \mathbb{T}_{Y/\mathfrak{M}_g} \arrow[r, "+1"] \arrow[u, "\phi_{Y/\mathfrak{M}_g}"'] & {}
\end{tikzcd}$$

By the property of cotangent complex of Picard stacks the left vertical arrow of the above diagram is an isomorphism. ■

Let $[Y/\mathfrak{D}_g]^{\mathrm{vir}}$ and $[Y/\mathfrak{M}_g]^{\mathrm{vir}} \in A_*Y$ be the virtual cycles using the respective perfect relative obstruction theories.

Corollary 2.9. We have an identity

$$[Y/\mathfrak{D}_g]^{\mathrm{vir}} = [Y/\mathfrak{M}_g]^{\mathrm{vir}} \in A_*Y. \qquad \square$$

Proof. Applying [1, Proposition 2.7] to Lemma 2.8, we obtain a diagram of cone stacks

$$\begin{tikzcd}
h^1/h^0(R^\bullet\pi_{Y*}\mathscr{O}_{\mathcal{C}_Y}) \arrow[r] & h^1/h^0(\mathbb{E}_{Y/\mathfrak{D}_g}) \arrow[r, "\theta"] & h^1/h^0(\mathbb{E}_{Y/\mathfrak{M}_g}) \\
h^1/h^0(\lambda^*\mathbb{T}_{\mathfrak{D}_g/\mathfrak{M}_g}[-1]) \arrow[r] \arrow[u, equal] & h^1/h^0(\mathbb{T}_{Y/\mathfrak{D}_g}) \arrow[r, "\theta_{\mathrm{int}}"] \arrow[u, "(\phi_{Y/\mathfrak{D}_g})_*"'] & h^1/h^0(\mathbb{T}_{Y/\mathfrak{M}_g}) \arrow[u, "(\phi_{Y/\mathfrak{M}_g})_*"']
\end{tikzcd}$$

of which the two rows are an exact sequence of abelian cone stacks. Applying an argument analogs to the second line in the proof of [16, Proposition 3], one checks $(\theta_{\mathrm{int}})^*(C_{Y/\mathfrak{M}_g}) = C_{\mathfrak{D}_g/\mathfrak{M}_f}$. Hence θ is a quotient of bundle stacks such that $\theta^*(C_{Y/\mathfrak{M}_g}) = C_{Y/\mathfrak{D}_g}$. By the projection formula

$$[Y/\mathfrak{D}_g]^{\mathrm{vir}} = [Y/\mathfrak{M}_g]^{\mathrm{vir}} \in A_*Y.$$

This proves the Corollary. ∎

We remark that this construction is parallel to the construction of Ciocan-Fontanine and Kim [7, 8] on the obstruction theories of the moduli spaces of maps to toric varieties [8, Theorem 3.2.1]; and to the comparison result with the perfect obstruction theory relative to $\mathfrak{M}_g$ in the same paper [8, Secttion 5].

3 Gromov–Witten Invariant of the GSW Model

In this section, we construct the moduli of stable maps to $\mathbb{P}^4$ coupled with p-fields. We construct its localized virtual cycle, using Kiem–Li's cosection localized virtual cycles. We define its degree to be the virtual counting of stable maps to $\mathbb{P}^4$ with p-field. This class of invariants is a generalization of the genus 0 GSW model $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ [13].

3.1 Moduli of stable maps with p-fields

Let $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ be the moduli of genus g degree d stable maps to $\mathbb{P}^4$. For the moment, we denote by $(f_M, \mathcal{C}_M, \pi_M)$ the universal family of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$, and by $\mathscr{L}_M = f_M^*\mathscr{O}(1)$ the tautological line bundle. We form

$$\mathscr{P}_M := \mathscr{L}_M^{-\otimes 5} \otimes \omega_{\mathcal{C}_M/M},$$

and call it the auxiliary invertible sheaf on $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$.

We define the moduli of genus g degree d stable maps with p-fields to be the direct image cone

$$\mathcal{P} := \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p := C(\pi_{M*}\mathscr{P}_M). \tag{3.1}$$

(We abbreviate it to $\mathcal{P}$, as indicated above.)

Like before, we can embed $\mathcal{P}$ into the moduli of sections for a choice of $\mathcal{Z} \to \mathfrak{D}_g$. Let $[x_1, \ldots, x_5]$ be the homogeneous coordinates of $\mathbb{P}^4$ specified in the Introduction. Let

$$(f_{\mathcal{P}}, \pi_{\mathcal{P}}) : \mathcal{C}_{\mathcal{P}} \to \mathbb{P}^4 \times \mathcal{P}$$

be the universal map of $\mathcal{P}$. We let $\mathscr{L}_{\mathcal{P}} = f_{\mathcal{P}}^*\mathscr{O}(1)$ be the tautological invertible sheaf; $\mathscr{P}_{\mathcal{P}} = \mathscr{L}_{\mathcal{P}}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}$ be the auxiliary invertible sheaf; and

$$\mathfrak{p} \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{P}_{\mathcal{P}}) \quad \text{and} \quad \mathfrak{u}_i = f_{\mathcal{P}}^* x_i \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{L}_{\mathcal{P}}) \tag{3.2}$$

be the universal p-field and the tautological coordinate functions, respectively. Note that $(\mathcal{C}_{\mathcal{P}}, \mathscr{L}_{\mathcal{P}})$ induces a morphism $\mathcal{P} \to \mathfrak{D}_g$ so that $(\mathcal{C}_{\mathcal{P}}, \mathscr{L}_{\mathcal{P}})$ is isomorphic to the pull-back of $(\mathcal{C}_{\mathfrak{D}_g}, \mathscr{L}_{\mathfrak{D}_g})$.

Using the line bundle $\mathscr{L}_{\mathfrak{D}_g}$ on $\mathcal{C}_{\mathfrak{D}_g}$ and its auxiliary invertible sheaf

$$\mathscr{P}_{\mathfrak{D}_g} = \mathscr{L}_{\mathfrak{D}_g}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g},$$

we form the bundle

$$\mathcal{Z} := \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g})$$

over $\mathcal{C}_{\mathfrak{D}_g}$. Then the section $((\mathfrak{u}_i)_{i=1}^5, \mathfrak{p})$ defines a section of

$$\mathcal{Z} \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathcal{C}_{\mathcal{P}} \longrightarrow \mathcal{C}_{\mathcal{P}}.$$

This section induces a $\mathcal{C}_{\mathfrak{D}_g}$-morphism $\mathcal{C}_{\mathcal{P}} \to \mathcal{Z} \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathcal{C}_{\mathcal{P}}$. Composed with the projection $\mathcal{Z} \times_{\mathfrak{D}_g} \mathcal{P} \to \mathcal{Z}$, we obtain the evaluation morphism over $\mathcal{C}_{\mathfrak{D}_g}$:

$$\tilde{\mathfrak{e}} : \mathcal{C}_{\mathcal{P}} \longrightarrow \mathcal{Z}. \tag{3.3}$$

Proposition 3.1. The pair $\mathcal{P} \to \mathfrak{D}_g$ admits a perfect relative obstruction theory

$$\phi_{\mathcal{P}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} := R^{\bullet}\pi_{\mathcal{P}*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathscr{P}_{\mathcal{P}}).$$ □

Proof. The proof follows from Proposition 2.5 applied to the (evaluation) morphism $\tilde{\mathfrak{e}}$, using that $\Omega^{\vee}_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}} = \mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g}$. ■

3.2 Constructing a cosection

We define a multilinear bundle morphism

$$h_1 : \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g}) \longrightarrow \mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}), \quad h_1(z, p) = p \cdot \sum_{i=1}^{5} z_i^5, \tag{3.4}$$

where $(z, p) = ((z_i)_{i=1}^5, p) \in \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g})$. This map is based on the dualpairing $\mathscr{L}_{\mathfrak{D}_g}^{\otimes 5} \otimes \mathscr{P}_{\mathfrak{D}_g} \to \omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}$.

The morphism h_1 induces a homomorphism of tangent complexes

$$dh_1 : \mathbb{T}_{\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5}\oplus\mathscr{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow h_1^*\mathbb{T}_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} = h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}.$$

In explicit form, for any closed $\xi\in\mathcal{C}_{\mathcal{P}}$ and $(z,p)\in\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5}\oplus\mathscr{P}_{\mathfrak{D}_g})|_\xi$, $dh_1|_{(z,p)}$ sends

$$((\mathring{z}_i),\mathring{p})\in\Omega^\vee_{\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5}\oplus\mathscr{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}\Big|_{(z,p)} = (\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5}\oplus\mathscr{P}_{\mathfrak{D}_g})\otimes_{\mathscr{O}_{\mathcal{C}_{\mathfrak{D}_g}}}\mathbf{k}(\xi)$$

to

$$dh_1|_{(z,p)}(\mathring{z},\mathring{p}) = \left(\sum_{i=1}^5 z_i^5\right)\cdot\mathring{p} + p\cdot\sum_{i=1}^5 5z_i^4\cdot\mathring{z}_i. \tag{3.5}$$

On the other hand, by pulling back dh_1 to $\mathcal{C}_{\mathcal{P}}$ via the evaluation morphism $\tilde{\mathfrak{e}}$ (cf. (3.3)), one has (homomorphism and canonical isomorphisms)

$$\tilde{\mathfrak{e}}^*(dh_1) : \tilde{\mathfrak{e}}^*\Omega^\vee_{\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5}\oplus\mathscr{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow \tilde{\mathfrak{e}}^*h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}.$$

Because the right-hand side is canonically isomorphic to $\omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}$, applying $R^\bullet\pi_{\mathcal{P}*}$, we obtain

$$\sigma_1^\bullet : \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \longrightarrow R^\bullet\pi_{\mathcal{P}*}(\mathfrak{e}^*h_1^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}) \cong R^\bullet\pi_{\mathcal{P}*}(\omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}). \tag{3.6}$$

We define

$$\sigma_1 := H^1(\sigma_1^\bullet) : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} = H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1\pi_{\mathcal{P}*}(\omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}) \cong \mathscr{O}_{\mathcal{P}}. \tag{3.7}$$

By Proposition 3.1, σ_1 is in the form (of homomorphism of sheaves)

$$\sigma_1 : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} = R^1\pi_{\mathcal{P}*}\mathscr{L}_{\mathcal{P}}^{\oplus 5}\oplus R^1\pi_{\mathcal{P}*}\mathscr{P}_{\mathcal{P}} \longrightarrow \mathscr{O}_{\mathcal{P}}.$$

3.3 Degeneracy locus of the cosection

We give a coordinate expression of the cosection σ_1. We define by $\mathfrak{u}_i = f_{\mathcal{P}}^* x_i$ and let $\mathfrak{p}\in\Gamma(\mathcal{C}_{\mathcal{P}},\mathscr{P}_{\mathcal{P}})$ be the tautological section of $\mathcal{P}$. Take any étale chart $T\to\mathcal{P}$, and let $\mathcal{C}_T = \mathcal{C}_{\mathcal{P}}\times_{\mathcal{P}} T$. For

$$\mathring{p}\in H^1(\mathcal{C}_T,\mathscr{P}_{\mathcal{P}}) \quad\text{and}\quad \mathring{u} = (\mathring{u}_i)_{i=1}^5 \in H^1(\mathcal{C}_T,\mathscr{L}_{\mathcal{P}}^{\oplus 5}),$$

we define

$$\zeta(\mathring{p},\mathring{u}) := 5p\cdot\sum_{i=1}^{5} u_i^4\cdot\mathring{u}_i+\left(\sum_{i=1}^{5} u_i^5\right)\cdot\mathring{p}, \tag{3.8}$$

where p and u_i are the pull-back of $\mathfrak{p}$ and $\mathfrak{u}_i$ to $\mathcal{C}_T$, respectively. The expression (3.8) is an element in $R^1\pi_{\mathcal{P}*}(\omega_{\mathcal{C}_\mathcal{P}/\mathcal{P}})\otimes_{\mathscr{O}_\mathcal{P}}\mathscr{O}_T\cong\mathscr{O}_T$.

One checks that this defines a homomorphism

$$\zeta : R^1\pi_{\mathcal{P}*}\mathscr{L}_\mathcal{P}^{\oplus 5}\oplus R^1\pi_{\mathcal{P}*}\mathscr{P}_\mathcal{P}\longrightarrow\mathscr{O}_\mathcal{P}.$$

Lemma 3.2. The two homomorphisms ζ and σ_1 coincide. □

Proof. This follows from the explicit expression of dh_1 in affine coordinates generalizing the expression (3.5). It is straightforward. ■

Definition 3.3. We define the degeneracy locus of σ_1 to be

$$D(\sigma_1)=\{\xi\in\mathcal{P}\,|\,\sigma_1|_\xi : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g}\otimes_{\mathscr{O}_\mathcal{P}}\mathbf{k}(\xi)\longrightarrow\mathbf{k}(\xi)\text{ vanishes}\}.$$ □

Following our convention, we denote by $Q\subset\mathbb{P}^4$ the quintic three-fold defined by $\sum x_i^5=0$. We let $\overline{\mathcal{M}}_g(Q,d)$ be the moduli of genus g degree d stable maps to Q. Using $\overline{\mathcal{M}}_g(Q,d)\subset\overline{\mathcal{M}}_g(\mathbb{P}^4,d)$, we obtain an embedding

$$\overline{\mathcal{M}}_g(Q,d)\subset\overline{\mathcal{M}}_g(\mathbb{P}^4,d)\subset\mathcal{P},$$

where the second inclusion is by assigning zero p-fields.

Proposition 3.4. The degeneracy locus of σ_1 is $\overline{\mathcal{M}}_g(Q,d)\subset\mathcal{P}$; it is proper. □

Proof. Let $\xi=(C,L,\phi,p)\in\mathcal{P}$, where $\phi=(\phi_i)_{i=1}^5\in H^0(C,L^{\oplus 5})$. The restriction of $\sigma_1=\zeta$ to ξ takes the form $\sigma_1|_\xi(\mathring{p},\mathring{\phi})=5p\sum\phi_i^4\mathring{\phi}_i+\sum\phi_i^5\mathring{p}$.

Suppose $\sum\phi_i^5\neq 0$; then, by Serre duality, we can find $\mathring{\mathfrak{p}}\in H^1(C,L^{-\otimes 5}\otimes\omega_C)$ so that $\mathring{p}\cdot\sum\phi_i^5\neq 0\in H^1(C,\omega_C)$. Letting $\mathring{\phi}_i=0$, we obtain $\sigma_1|_\xi\neq 0$.

Suppose $\sum\phi_i^5=0$ and $p\neq 0$. Then, since ϕ_i have no common vanishing locus, for some k, $p\cdot\phi_k^4\neq 0$. By Serre duality, we can find a $\mathring{\phi}_k$ so that $p\cdot\phi_k^4\cdot\mathring{\phi}_k\neq 0\in H^1(C,\omega_C)$. By choosing other $\mathring{\phi}_i=0$, we obtain the surjectivity of $\sigma_1|_\xi$. This proves that the degeneracy locus (i.e., the nonsurjective locus) of σ_1 is the collection of (C, L, ϕ, p) such that $\sum \phi_i^5 = 0$ and $p = 0$. This set is $\overline{\mathcal{M}}_g(Q, d) \subset \mathcal{P}$.

We comment that though this argument is set-theoretic, it is easy to see that the identification given is as closed substacks of $\mathcal{P}$. ■

3.4 The cosection factorizes

Let $q : \mathcal{P} \to \mathfrak{D}_g$ be the tautological morphism. We form the distinguished triangle

$$q^*\mathbb{L}_{\mathfrak{D}_g} \longrightarrow \mathbb{L}_{\mathcal{P}} \longrightarrow \mathbb{L}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\delta} q^*\mathbb{L}_{\mathfrak{D}_g}[1]. \tag{3.9}$$

Composing $\phi_{\mathcal{P}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \to \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$ with the dual of δ in the above distinguished triangle, we obtain the morphism

$$\phi_{\mathcal{P}/\mathfrak{D}_g} \circ \delta^\vee : q^*\mathbb{T}_{\mathfrak{D}_g} \longrightarrow \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}[1] \longrightarrow \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}[1].$$

Defining $\eta = H^0(\phi_{\mathcal{P}/\mathfrak{D}_g} \circ \delta^\vee)$, we obtain the composite

$$\eta : q^*T_{\mathfrak{D}_g} \longrightarrow H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) = \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g}. \tag{3.10}$$

Following the construction in [15, (4.3)], the cokernel of (3.10) is the absolute obstruction sheaf of $\mathcal{P}$, which we denote by $\mathcal{O}b_{\mathcal{P}}$.

In this subsection, we show the following proposition.

Proposition 3.5. The cosection $\sigma_1 : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} \to \mathscr{O}_{\mathcal{P}}$ lifts to a $\bar{\sigma}_1 : \mathcal{O}b_{\mathcal{P}} \to \mathscr{O}_{\mathcal{P}}$. □

We continue to use the notation developed in the proof of Proposition 3.1.

Lemma 3.6. The following composition is trivial:

$$0 = H^1(\sigma_1^\bullet \circ \phi_{\mathcal{P}/\mathfrak{D}_g}) : H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1\pi_{\mathcal{P}*}\omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}.$$ □

Proof. Using the universal curve $\pi_{\mathfrak{D}_g} : \mathcal{C}_{\mathfrak{D}_g} \to \mathfrak{D}_g$ of $\mathfrak{D}_g$, we introduce the direct image cone $\mathfrak{C}_\omega = C(\pi_*\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})$; we denote by $\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})$ the underlying bundle of $\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}$. Let $\mathcal{C}_{\mathfrak{C}_\omega} = \mathcal{C}_{\mathfrak{D}_g} \times_{\mathfrak{D}_g} \mathfrak{C}_\omega$ be the universal curve over $\mathfrak{C}_\omega$, and $\pi_{\mathfrak{C}_\omega} : \mathcal{C}_{\mathfrak{C}_\omega} \to \mathfrak{C}_\omega$ be the projection.

Continue to denote by $(f_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}}, \mathscr{L}_{\mathcal{P}})$ the universal family of $\mathcal{P}$, and using $\mathfrak{u}_i = f_{\mathcal{P}}^* x_i \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{L}_{\mathcal{P}})$ and $\mathfrak{p} \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathscr{P}_{\mathcal{P}})$, the universal coordinate functions and p-field, respectively, (cf. (3.2)), we form

$$\epsilon := \mathfrak{p} \cdot (\mathfrak{u}_1^5 + \cdots + \mathfrak{u}_5^5) \in \Gamma(\mathcal{C}_{\mathcal{P}}, \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}).$$

It defines a morphism $\Phi_\epsilon : \mathcal{P} \to \mathfrak{C}_\omega$ so that if we denote by $\tilde{\Phi}_\epsilon : \mathcal{C}_{\mathcal{P}} \to \mathcal{C}_{\mathfrak{C}_\omega}$ the tautological lift of Φ_ϵ using that both $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathfrak{C}_\omega}$ are pull-backs of $\mathcal{C}_{\mathfrak{D}_g}$, and denote by $\mathfrak{e}$ and $\mathfrak{e}'$ the evaluation morphisms as shown, we have a commutative diagram of morphisms of stacks over $\mathcal{C}_{\mathfrak{D}_g}$:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\mathfrak{e}} & \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g}) \\ \downarrow{\scriptstyle \tilde{\Phi}_\epsilon} & & \downarrow{\scriptstyle h_1} \\ \mathcal{C}_{\mathfrak{C}_\omega} & \xrightarrow{\mathfrak{e}'} & \mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}) \end{array} \tag{3.11}$$

Here h_1 is defined in (3.4). This shows that the square below is commutative:

$$\begin{array}{ccccc} \pi_{\mathcal{P}}^* \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} & = & \mathbb{T}_{\mathcal{C}_{\mathcal{P}}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & \mathfrak{e}^* \Omega^\vee_{\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle dh_1} \\ \pi_{\mathcal{P}}^* \tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g} & = & \tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathcal{C}_{\mathfrak{C}_\omega}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & \tilde{\Phi}_\epsilon^* \mathfrak{e}'^* \Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}}/\mathfrak{D}_g)/\mathcal{C}_{\mathfrak{D}_g}} \end{array} \tag{3.12}$$

Applying $R^1\pi_{\mathcal{P}*}$ to the lower horizontal arrow, we obtain the obstruction assignment homomorphism

$$(0 =)\ H^1(\Phi_\epsilon^* \phi_{\mathfrak{C}_\omega/\mathfrak{D}_g}) : H^1(\Phi_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g}) \longrightarrow \Phi_\epsilon^* R^1\pi_{\mathfrak{C}_\omega *} \omega_{\mathcal{C}_{\mathfrak{C}_\omega}/\mathfrak{C}_\omega}, \tag{3.13}$$

which is trivial since $\mathfrak{C}_\omega$ is a vector bundle over $\mathfrak{D}_g$ and $\mathcal{C}_{\mathfrak{C}_\omega} \to \mathcal{C}_{\mathfrak{D}_g}$ is smooth.

Therefore, using the Cartesian squares

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\tilde{\Phi}_\epsilon} & \mathcal{C}_{\mathfrak{C}_\omega} \\ \downarrow{\scriptstyle \pi_{\mathcal{P}}} & & \downarrow{\scriptstyle \pi_{\mathfrak{C}_\omega}} \\ \mathcal{P} & \xrightarrow{\Phi_\epsilon} & \mathfrak{C}_\omega \end{array} \tag{3.14}$$

and the commutativity of (3.12), applying $R^1\pi_{\mathcal{P}*}$, we see that the composite

$$H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1\pi_{\mathcal{P}*}\mathfrak{e}^* \Omega^\vee_{\mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathscr{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow R^1\pi_{\mathcal{P}*}\mathfrak{e}^* h_1^* \Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}$$

coincides with the composite

$$H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\Phi_\epsilon^*\mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g}) \xrightarrow{0} \Phi_\epsilon^* R^1\pi_{\mathfrak{C}_\omega *}\Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}.$$

Since the composite in the second line is trivial (cf. (3.13)), the composite in the first line is trivial. Using

$$\mathfrak{e}^* h_1^* \Omega^\vee_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \cong \omega_{\mathcal{C}_\mathcal{P}/\mathcal{P}},$$

this is exactly the vanishing desired by the Lemma. ■

Proof of Proposition 3.5. The composition of σ with (3.10) is the H^1 of the composition

$$\mathbb{T}_{\mathfrak{D}_g}[-1] \longrightarrow \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\phi_{\mathcal{P}/\mathfrak{D}_g}} \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\sigma_1^\bullet} R^\bullet\pi_{\mathcal{P}*}\omega_{\mathcal{C}_\mathcal{P}/\mathcal{P}},$$

where the first arrow is the $\delta^\vee$ in (3.9). Lemma 3.6 implies the H^1 of the above composition is trivial. ■

Here we comment the background of this construction in Super-String Theories. Let $K_{\mathbb{P}^4}$ be the total space of the canonical line bundle $\mathbb{P}^4$. The quintic polynomial $\sum x_i^5$ defines a regular map $\mathbf{w}_{\mathbb{P}^4} \in \Gamma(\mathscr{O}_{K_{\mathbb{P}^4}})$. Its critical locus is the quintic three-fold $Q \subset \mathbb{P}^4$. In physics literature, the pair $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ is called a Landau–Ginzburg model (nonlinear). In [13], Guffin and Sharpe constructed a path integral for genus 0 A-twisted theory of the Landau–Ginzburg space $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ [13]. In this paper, we have constructed a mathematical theory generalizing it to all genus.

3.5 The virtual dimension

We calculate the virtual dimension of $\mathcal{P}$. Let $\xi = (f, C, L, p) \in \mathcal{P}$ be any closed point. The virtual dimension of $\mathcal{P}/\mathfrak{D}_g$ at ξ is

$$\dim H^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \otimes_{\mathscr{O}_\mathcal{P}} \mathbf{k}(\xi)) - \dim H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \otimes_{\mathscr{O}_\mathcal{P}} \mathbf{k}(\xi)).$$

By the expression of $\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$, the above term equals to

$$h^0(L^{\oplus 5}) + h^0(L^{-\otimes 5} \otimes \omega_C) - h^1(L^{\oplus 5}) - h^1(L^{-\otimes 5} \otimes \omega_C) = 4 - 4g.$$

Because

$$\dim \mathfrak{D}_g = \dim \mathfrak{D}_g/\mathfrak{M}_g + \dim \mathfrak{M}_g = (h^0(\mathscr{O}_C) - 1) + 3g - 3 = 4g - 4,$$

the virtual dimension of $\mathcal{P}$ at ξ is zero.

3.6 Localized virtual cycle

We recall the notion of kernel-stack of a cosection. Let $E = [E^1 \to E^1]$ be a two-term complex of locally free sheaves on a Deligne–Mumford stack X; let $f: H^1(E) \to \mathscr{O}_X$ be a cosection of $H^1(E)$. We denote by $D(f)$ the closed subset of $x \in X$ such that $f|_x = 0 : H^1(E)|_x \to \mathbb{C}_x$. Let $U = X - D(f)$.

Definition 3.7. Let the notation be as stated. We define the kernel stack $h^1/h^0(E)_f$ be

$$h^1/h^0(E)_f := (h^1/h^0(E) \times_X D(f)) \cup \ker\{h^1/h^0(E)|_U \to H^1(E)|_U \to \mathbb{C}_U\}. \qquad \square$$

Here $h^1/h^0(E)|_U \to H^1(E)|_U$ is the tautological projection and $H^1(E)|_U \to \mathbb{C}_U$ is $f|_U$. Since f is surjective over U, the composite in the bracket is surjective, thus the kernel is a bundle-stack. Clearly, the union is closed in $h^1/h^0(E)$; we endow it the reduced structure, and call it the kernel-stack of f.

We apply the theory developed in [15]. As σ_1 is a cosection of $H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})$, we can apply the preceding construction to σ_1 to form the kernel subcone-stack

$$h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1} \subset h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}). \tag{3.15}$$

Proposition 3.8. The virtual normal cone cycle $[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}] \in Z_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})$ lies inside $Z_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1}$. □

Proof. The smoothness of the morphism from $\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}$ to $\mathbf{C}_{\mathcal{P}}$(the intrinsic normal cone of $\mathcal{P}$) and Propostion 3.5 reduces the claim to the absolute case, which is proved in [15, Proposition 4.3]. ■

In [15], Kiem and the second-named author constructed a localized Gysin map

$$0^!_{\sigma_1,\mathrm{loc}} : A_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1} \longrightarrow A_{*-n} D(\sigma_1),$$

where $-n$ is the rank of $\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$.

Definition-Proposition 3.9. We define the localized virtual cycle of $(\mathcal{P}, \sigma_1)$ to be

$$[\mathcal{P}]^{\mathrm{vir}}_{\sigma_1} = [\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\sigma_1} := 0^!_{\sigma_1,\mathrm{loc}}([\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}]) \in A_0\overline{\mathcal{M}}_g(Q, d).$$

We define the virtual enumeration $N_g(d)^p_{\mathbb{P}^4} := \deg [\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\sigma_1}$. □

The number $N_g(d)^p_{\mathbb{P}^4}$ is the virtual counting of the GSW model $(\mathcal{P}, \sigma_1)$. We call it the Gromov–Witten invariants of the moduli of stable maps to $\mathbb{P}^4$ with p-fields, or of the Landau–Ginzburg space $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$.

4 Degeneration of Moduli of Stable Maps with p-Fields

In the second part, we use degeneration to prove that $N_g(d)^p_{\mathbb{P}^4}$ coincides up to a sign with the Gromov–Witten invariants $N_g(d)_Q$ of the quintic three-fold Q.

The degeneration we use is to degenerate the moduli $\mathcal{P}$ to the moduli of stable maps to the normal bundle to $Q \subset \mathbb{P}^4$ coupled with p-field. After constructing a cosection of its obstruction sheaf, the degeneration admits a localized virtual cycle that provides the proof of the equivalence of two classes of invariants.

4.1 The degeneration

We let V be the total space of the deformation of $\mathbb{P}^4$ to the normal bundle of $Q \subset \mathbb{P}^4$; it is the blow-up of $\mathbb{P}^4 \times \mathbb{A}^1$ along $Q \times 0$, after taking out the proper transform of $\mathbb{P}^4 \times 0$. Let

$$q_{\mathbb{A}^1} : V \longrightarrow \mathbb{A}^1 \quad \text{and} \quad q_{\mathbb{P}^4} : V \longrightarrow \mathbb{P}^4 \tag{4.1}$$

be the two projections. Then the fiber of $q^1_{\mathbb{A}}$ over $c \neq 0$ is $\mathbb{P}^4$, and the central fiber (over $0 \in \mathbb{A}^1$) is the normal bundle N to $Q \subset \mathbb{P}^4$. We define the degree of a morphism $u: C \to V$ to be $\deg u = \deg(q_{\mathbb{P}^4} \circ u)^* \mathscr{O}(1)$.

We form the moduli of genus g and degree d stable maps $\overline{M}_g(V, d)$. For the moment, we denote by

$$(\tilde{f}, \tilde{\pi}) : \tilde{\mathcal{C}} \longrightarrow V \times \overline{\mathcal{M}}_g(V, d)$$

the universal family of $\overline{\mathcal{M}}_g(V, d)$. Since $q_{\mathbb{A}^1}$ is proper away from the central fiber $N = V \times_{\mathbb{A}^1} 0$, and since $\mathbb{A}^1$ is affine, the composite $q_{\mathbb{A}^1} \circ \tilde{f} : \tilde{\mathcal{C}} \to \mathbb{A}^1$ factors through a map $\overline{\mathcal{M}}_g(V, d) \to \mathbb{A}^1$. Its fibers over $c \neq 0 \in \mathbb{A}^1$ are $\overline{M}_g(\mathbb{P}^4, d)$; its central fiber is $\overline{M}_g(N, d)$.

We now couple the stable maps with p-field. Let $\tilde{\mathscr{L}} = \tilde{f}^*\mathscr{O}(1)$ and $\tilde{\mathscr{P}} = \tilde{\mathscr{L}}^{-\otimes 5} \otimes \omega_{\tilde{\mathcal{C}}/\overline{\mathcal{M}}_g(V,d)}$ be the tautological and auxiliary invertible sheaves, respectively. Like before, we define the moduli of stable maps coupled with p-fields the be

$$\mathcal{V} := \overline{\mathcal{M}}_g(V,d)^p := C(\tilde{\pi}_*\tilde{\mathscr{P}}),$$

the direct image cone. It is over $\mathbb{A}^1$, and its fibers over $c \neq 0 \in \mathbb{A}^1$ and $0 \in \mathbb{A}^1$ are, respectively,

$$\mathcal{V} \times_{\mathbb{A}^1} c \cong \mathcal{P}, \quad \mathcal{V} \times_{\mathbb{A}^1} 0 := \overline{\mathcal{M}}_g(N,d)^p.$$

Here $\overline{\mathcal{M}}_g(N,d)^p$ is the moduli of stable maps to N coupled with p-fields.

Following our convention, we denote by

$$(f_{\mathcal{V}}, \pi_{\mathcal{V}}) : \mathcal{C}_{\mathcal{V}} \longrightarrow V \times \mathcal{V} \tag{4.2}$$

the universal map of $\mathcal{V}$.

4.2 The cone over V

We construct the tautological cone $C(V)$ over V that will be used to construct the evaluation morphism $\mathfrak{e}_{\mathfrak{v}}$ of $\mathcal{C}_{\mathcal{V}}$. The evaluation map will be used to construct the obstruction theory of $\mathcal{V}$.

We let $B = \mathrm{Vb}(\mathscr{O}(5))$ be the underlying line bundle of $\mathscr{O}(5)$ over $\mathbb{P}^4$; let

$$\mathbf{q}_{\mathbb{P}^4} : B \times \mathbb{A}^1 \to B \to \mathbb{P}^4 \quad \text{and} \quad \mathbf{q}_{\mathbb{A}^1} : B \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$

be the (composite of) projection(s). We let $t \in \Gamma(\mathscr{O}_{\mathbb{A}^1})$ be the standard coordinate function of $\mathbb{A}^1$. We introduce tautological sections over $B \times \mathbb{A}^1$:

$$\tilde{x}_i = \mathbf{q}_{\mathbb{P}^4}^* x_i \in \Gamma(\mathbf{q}_{\mathbb{P}^4}^*\mathscr{O}(1)), \quad \tilde{t} = \mathbf{q}_{\mathbb{A}^1}^* t \in \Gamma(\mathscr{O}_{B\times\mathbb{A}^1}), \quad \text{and} \quad \tilde{y} \in \Gamma(\mathbf{q}_{\mathbb{P}^4}^*\mathscr{O}(5)), \tag{4.3}$$

where $\tilde{y}$ is the section so that the morphism $B \times \mathbb{A}^1 \to \mathrm{Vb}(\mathscr{O}(5))$ induced by $\tilde{y}$ is the projection $B \times \mathbb{A}^1 \to B = \mathrm{Vb}(\mathscr{O}(5))$. (i.e., $\tilde{y}$ is the pull-back of the identity map $B \to B$.)

Lemma 4.1. We have a closed immersion

$$V \cong (\tilde{s} = 0\} \subset B \times \mathbb{A}^1, \quad \tilde{s} = \tilde{x}_1^5 + \cdots + \tilde{x}_5^5 - \tilde{t} \cdot \tilde{y}. \qquad \square$$

Proof. This is proved in [11, Remark 5.1.1]. We only add that the isomorphism is given by extending

$$\Phi : V - V\times_{\mathbb{A}^1} 0 \longrightarrow B\times\mathbb{A}^1$$

to V, which is defined via $\Phi^*(\tilde{x}_i) = q_{\mathbb{P}^4}^*(x_i)$, $\Phi^*(\tilde{t}) = q_{\mathbb{A}^1}^* t$, and $\Phi^*\tilde{y} = t^{-1}\cdot(x_1^5+\cdots+x_5^5)$, where $q_{\mathbb{P}^4} : V\to\mathbb{P}^4$ is the projection, etc. (cf. (4.1)). ∎

In the following, we view $V\subset B\times\mathbb{A}^1$ using this isomorphism. We next construct the desired cone $C(V)$. We let $W_5=\mathbb{C}_{\mathbb{A}^1}$ (respectively $W_1=\mathbb{C}_{\mathbb{A}^1}^{\oplus 5}$) be the trivial line bundle (respectively rank 5 trivial vector bundle) over $\mathbb{A}^1$. We consider the rank 6 bundle

$$\mathrm{pr}_{\mathbb{A}^1} : W_1\times_{\mathbb{A}^1} W_5 \longrightarrow \mathbb{A}^1$$

with the $\mathbb{C}^*$-action: $\mathbb{C}^*$ acts on the base $\mathbb{A}^1$ trivially and acts on fibers of W_1 (respectively W_5) of weight 1 (respectively weight 5); namely, for $z\in W_1$ and $y\in W_5$, $z^\sigma=\sigma z$ and $y^\sigma=\sigma^5 y$.

We let $W_1^* = W_1 - 0_{W_1}$, where 0_{W_1} is the zero section of W_1. We introduce

$$C(V) = (\epsilon=0)\subset W_1^*\times_{\mathbb{A}^1} W_5,\quad \epsilon = z_1^5+\cdots+z_5^5 - t\cdot y.$$

It is smooth and is $\mathbb{C}^*$-invariant.

Sine the $\mathbb{C}^*$ acts trivially on the base $\mathbb{A}^1$, and acts on the fibers of $W_1^*\times_{\mathbb{A}^1} W_5\to\mathbb{A}^1$ with weights $(1,1,1,1,1,5)$, $(W_1^*\times_{\mathbb{A}^1} W_5)/\mathbb{C}^*$ is isomorphic to $B=\mathrm{Vb}(\mathscr{O}(5))$, and under this isomorphism we have the following commuting (horizontal) quotient morphisms:

$$\begin{tikzcd} W_1^*\times_{\mathbb{A}^1} W_5 \arrow[r, "\Psi"] & B\times\mathbb{A}^1 \\ C(V) \arrow[u, "\cup"'] \arrow[r, "/\mathbb{C}^*"] & V \arrow[u, "\cup"'] \end{tikzcd} \tag{4.4}$$

which is also a fiber diagram.

For a later purpose, we describe the tangent bundles $T_{C(V)/\mathbb{A}^1}$ and $T_{C(V)}$. Using the defining equation of $C(V)$, they fit into the exact sequences

$$0\longrightarrow T_{C(V)/\mathbb{A}^1}\longrightarrow \mathscr{O}_{C(V)}^{\oplus 5}\oplus\mathscr{O}_{C(V)} \xrightarrow{\mathrm{d}'\epsilon} \mathscr{O}_{C(V)}\longrightarrow 0,\quad (\mathrm{d}'t=0), \tag{4.5}$$

where d' is the relative differential and $d'\epsilon|_{((z_i),y,t)}$ sends $((\mathring{z}_i), \mathring{y})$ to $\sum 5z_i^4\mathring{z}_i - t\mathring{y}$;

$$0 \longrightarrow T_{C(V)} \longrightarrow \mathscr{O}_{C(V)}^{\oplus 5} \oplus \mathscr{O}_{C(V)} \oplus \mathscr{O}_{C(V)} \xrightarrow{d\epsilon} \mathscr{O}_{C(V)} \longrightarrow 0, \tag{4.6}$$

where $d\epsilon|_{((z_i),y,t)}$ sends $((\mathring{z}_i), \mathring{y}, \mathring{t})$ to $\sum 5z_i^4\mathring{z}_i - t\mathring{y} - y\mathring{t}$.

Together they fit into the exact sequence

$$0 \longrightarrow T_{C(V)/\mathbb{A}^1} \longrightarrow T_{C(V)} \longrightarrow \mathscr{O}_{C(V)} \longrightarrow 0. \tag{4.7}$$

4.3 The evaluation maps

We now construct the evaluation morphism of $\mathcal{C}_{\mathcal{V}}$. Since V is a family over $\mathbb{A}^1$, it is natural to construct the obstruction theory of $\mathcal{V}$ relative to $\mathfrak{D}_g \times \mathbb{A}^1$.

To this purpose, we introduce $\widetilde{\mathfrak{D}}_g = \mathfrak{D}_g \times \mathbb{A}^1$, viewed as a stack over $\mathbb{A}^1$; denote by

$$\mathcal{C}_{\widetilde{\mathfrak{D}}_g} := \mathcal{C}_{\mathfrak{D}_g} \times \mathbb{A}^1 \longrightarrow \mathfrak{D}_g \times \mathbb{A}^1 = \widetilde{\mathfrak{D}}_g$$

the universal curve, and denote by $\mathscr{L}_{\widetilde{\mathfrak{D}}_g}$ the pull-back of $\mathscr{L}_{\mathfrak{D}_g}$ via $\mathcal{C}_{\widetilde{\mathfrak{D}}_g} \to \mathcal{C}_{\mathfrak{D}_g}$.

We form $\mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5})^* = \mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5}) - 0_{\widetilde{\mathfrak{D}}_g}$, and consider the bundle over $\mathcal{C}_{\mathfrak{D}_g}$:

$$\mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5}) \longrightarrow \mathcal{C}_{\widetilde{\mathfrak{D}}_g}. \tag{4.8}$$

Note that, for each $\xi \in \mathcal{C}_{\mathfrak{D}_g}$, the fiber of (4.8) over $\xi \times \mathbb{A}^1 \subset \widetilde{\mathfrak{D}}_g$ is isomorphic to

$$(L^{\oplus 5} - 0) \times L^{\otimes 5} \times \mathbb{A}^1 \cong W_1^* \times_{\mathbb{A}^1} W_5, \quad L := \mathscr{L}_{\mathfrak{D}_g}^{\oplus 5} \otimes_{\mathcal{C}_{\mathfrak{D}_g}} \mathbf{k}(\xi),$$

where the isomorphism is uniquely determined by an isomorphism $L \cong \mathbb{C}$, and two different isomorphisms are equivalent under a scaling of $(\mathbb{C}^5 - 0) \times \mathbb{C}$ by a $c \in \mathbb{C}^*$ with weights $(1, \ldots, 1, 5)$ on the factors of $(\mathbb{C}^5 - 0) \times \mathbb{C}$.

We let $\mathbb{C}^*$ act on the bundle (4.8) fiberwise with these weights. We obtain the quotient $\mathbb{A}^1$-morphisms (the $\mathbb{A}^1$ is the base of $W \to \mathbb{A}^1$ and of $\widetilde{\mathfrak{D}}_g = \mathfrak{D}_g \times \mathbb{A}^1 \to \mathbb{A}^1$)

$$\mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5}) \longrightarrow (\mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5}))/\mathbb{C}^* \longrightarrow (W_1^* \times_{\mathbb{A}^1} W_5)/\mathbb{C}^*.$$

We define

$$\mathcal{Z}' = (\mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5})) \times_{(W_1^* \times_{\mathbb{A}^1} W_5)/\mathbb{C}^*} V; \tag{4.9}$$

it is the preimage of $V\subset (W_1^*\times_{\mathbb{A}^1} W_5)/\mathbb{C}^*$ of the morphism above (4.9). We define

$$\mathcal{Z}=\mathcal{Z}'\times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}\mathrm{Vb}(\mathscr{P}_{\widetilde{\mathfrak{D}}_g}). \tag{4.10}$$

We now construct the evaluation morphism

$$\mathfrak{e}_{\mathfrak{v}}:\mathcal{C}_{\mathcal{V}}\longrightarrow\mathcal{Z}. \tag{4.11}$$

We let $\mathscr{L}_{\mathcal{V}}=f_{\mathcal{V}}^*\mathscr{O}(1)$, where $(f_{\mathcal{V}},\mathcal{C}_{\mathcal{V}})$ is the universal family of $\mathcal{V}$ (cf. (4.2)), let $\mathscr{P}_{\mathcal{V}}=\mathscr{L}_{\mathcal{V}}^{-\otimes 5}\otimes\omega_{\mathcal{C}_{\mathcal{V}}/\mathcal{V}}$ be the auxiliary invertible sheaf, and let

$$\mathfrak{p}\in\Gamma(\mathcal{C}_{\mathcal{V}},\mathscr{P}_{\mathcal{V}}),\ \mathfrak{u}_i=f_{\mathcal{V}}^*\tilde{x}_i\in\Gamma(\mathcal{C}_{\mathcal{V}},\mathscr{L}_{\mathcal{V}})\quad\text{and}\quad\mathfrak{y}=f_{\mathcal{V}}^*\tilde{y}\in\Gamma(\mathcal{C}_{\mathcal{V}},\mathscr{L}_{\mathcal{V}}^{\otimes 5}) \tag{4.12}$$

(cf. (4.3)) be the universal $\mathfrak{p}$-field and the tautological coordinate functions, respectively. Note that $(\mathcal{C}_{\mathcal{V}},\mathscr{L}_{\mathcal{V}})$ induces an $\mathbb{A}^1$-morphism $\mathcal{V}\to\widetilde{\mathfrak{D}}_g$ so that $(\mathcal{C}_{\mathcal{V}},\mathscr{L}_{\mathcal{V}})$ is isomorphic to the pull-back of $(\mathcal{C}_{\widetilde{\mathfrak{D}}_g},\mathscr{L}_{\widetilde{\mathfrak{D}}_g})$.

Then the definition of $V\subset B\times\mathbb{A}^1$ implies that the sections in (4.12) satisfy

$$\mathfrak{u}_1^5+\mathfrak{u}_2^5+\mathfrak{u}_3^5+\mathfrak{u}_4^5+\mathfrak{u}_5^5-t\cdot\mathfrak{y}=0,$$

where t is the coordinate function of $\mathbb{A}^1$ mentioned before. Therefore the section $((\mathfrak{u}_i)_{i=1}^5,\mathfrak{y},\mathfrak{p})$ defines a section of

$$\mathcal{Z}\times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}\mathcal{C}_{\mathcal{V}}\longrightarrow\mathcal{C}_{\mathcal{V}}.$$

This section induces a $\mathcal{C}_{\mathcal{V}}$-morphism $\mathcal{C}_{\mathcal{V}}\to\mathcal{Z}\times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}\mathcal{C}_{\mathcal{V}}$. Composed with the projection $\mathcal{Z}\times_{\widetilde{\mathfrak{D}}_g}\mathcal{V}\to\mathcal{Z}$, we obtain the evaluation morphism over $\mathcal{C}_{\widetilde{\mathfrak{D}}_g}$ in (4.11).

4.4 The obstruction theory of $\mathcal{V}/\widetilde{\mathfrak{D}}_g$

We will build the obstruction theories to carry out the degeneration for virtual cycles. We first construct the relative obstruction theory of $\mathcal{V}\to\widetilde{\mathfrak{D}}_g$. The restriction of this obstruction theory to fibers over $c\in\mathbb{A}^1$ will give the relative obstruction theories of $\mathcal{V}_c=\mathcal{V}\times_{\mathbb{A}^1}c\to\mathfrak{D}_g$.

We begin with a description of the tangent bundle $T_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g}$. Let $\varrho:\mathcal{Z}'\to\widetilde{\mathfrak{D}}_g$ be the tautological projection. Using the explicit description of $T_{C(V)/\mathbb{A}^1}$ given in (4.5), and the construction of $\mathcal{Z}'$ in (4.9), we see that $\Omega^{\vee}_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g}$ fits into the exact sequence

$$0 \longrightarrow \Omega^{\vee}_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g} \longrightarrow \varrho^*\mathscr{L}^{\oplus 5}_{\widetilde{\mathfrak{D}}_g} \oplus \varrho^*\mathscr{L}^{\otimes 5}_{\widetilde{\mathfrak{D}}_g} \xrightarrow{\mathrm{d}\mathcal{E}} \varrho^*\mathscr{L}^{\otimes 5}_{\widetilde{\mathfrak{D}}_g} \longrightarrow 0,$$

where $\mathrm{d}\mathcal{E}$ restricted to $((z_i), y, t) \in \mathcal{Z}'$ sends $((\mathring{z}_i), \mathring{y})$ to $\sum 5z_i^4\mathring{z}_i - t\mathring{y}$. (cf. (4.5).) Using that $\mathscr{L}_{\mathcal{V}} = f_{\mathcal{V}}^*\mathscr{O}(1)$, we obtain

$$\mathfrak{e}_{\mathfrak{v}}^*\Omega^{\vee}_{\mathcal{Z}'/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \cong f_{\mathcal{V}}^*\mathscr{H} \quad \text{and} \quad \mathfrak{e}_{\mathfrak{v}}^*\Omega^{\vee}_{\mathcal{Z}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \cong f_{\mathcal{V}}^*\mathscr{H} \oplus \mathscr{P}_{\mathcal{V}}, \tag{4.13}$$

where $\mathscr{H}$ on $B \times \mathbb{A}^1$ is defined by the exact sequence

$$0 \longrightarrow \mathscr{H} \longrightarrow q_{\mathbb{P}^4}^*\mathscr{O}(1)^{\oplus 5} \oplus q_{\mathbb{P}^4}^*\mathscr{O}(5) \xrightarrow{\mathrm{d}'\tilde{s}} q_{\mathbb{P}^4}^*\mathscr{O}(5) \longrightarrow 0,$$

where $\mathrm{d}'\tilde{s}$ is the differential of $\tilde{s}$ in Lemma 4.1, after setting $\mathrm{d}'t = 0$. (Recall that $V \subset B \times \mathbb{A}^1$ by Lemma 4.1.) Namely, for $\xi \in B \times \mathbb{A}^1$ with $(\mathring{x}_i) \in q_{\mathbb{P}^4}^*\mathscr{O}(1)^{\oplus 5}|_{\xi}$ and $\mathring{y} \in q_{\mathbb{P}^4}^*\mathscr{O}(5)|_{\xi}$, we set

$$\mathrm{d}'\tilde{s}|_{\xi}((\mathring{x}_i), \mathring{y}) = 5\mathrm{u}_1(\xi)^5\mathring{x}_1 + \cdots + 5\mathrm{u}_5(\xi)^5\mathring{x}_5 - \tilde{t}(\xi)\cdot\mathring{y} \in q_{\mathbb{P}^4}^*\mathscr{O}(5)|_{\xi}.$$

We have a similar description

$$\mathfrak{e}_{\mathfrak{v}}^*\Omega^{\vee}_{\mathcal{Z}'/\mathcal{C}_{\mathfrak{D}_g}} \cong f_{\mathcal{V}}^*\mathscr{K} \quad \text{and} \quad \mathfrak{e}_{\mathfrak{v}}^*\Omega^{\vee}_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}} \cong f_{\mathcal{V}}^*\mathscr{K} \oplus \mathscr{P}_{\mathcal{V}}, \tag{4.14}$$

where $\mathscr{K}$ is defined by the exact sequence

$$0 \longrightarrow \mathscr{K} \xrightarrow{i} q_{\mathbb{P}^4}^*\mathscr{O}(1)^{\oplus 5} \oplus q_{\mathbb{P}^4}^*\mathscr{O}(5) \oplus q_{\mathbb{P}^4}^*\mathscr{O} \xrightarrow{\mathrm{d}\tilde{s}} q_{\mathbb{P}^4}^*\mathscr{O}(5) \longrightarrow 0, \tag{4.15}$$

where $\mathrm{d}\tilde{s}$ is the differential of $\tilde{s}$ in Lemma (4.1).

Proposition 4.2. The pair $\mathcal{V} \to \widetilde{\mathfrak{D}}_g$ admits a perfect relative obstruction theory

$$\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} : \mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} := R^{\bullet}\pi_{\mathcal{V}*}(f_{\mathcal{V}}^*\mathscr{H} \oplus \mathscr{P}_{\mathcal{V}}).$$

Its specialization at $c \neq 0 \in \mathbb{A}^1$ (respectively $0 \in \mathbb{A}^1$) gives the perfect relative obstruction theory $\phi_{\mathcal{P}/\mathfrak{D}_g}$ (respectively $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$). □

Proof. We fit $\mathfrak{e}_\mathfrak{v}:\mathcal{C}_\mathcal{V}\to\mathcal{Z}$ (cf. (4.11)) into the commutative diagrams

$$\begin{tikzcd} \mathcal{V} \arrow[d] & \mathcal{C}_\mathcal{V} \arrow[l, "\pi_\mathcal{V}"'] \arrow[r, "\mathfrak{e}_\mathfrak{v}"] \arrow[d] & \mathcal{Z} \arrow[d, "\mathrm{pr}"] \\ \widetilde{\mathfrak{D}}_g & \mathcal{C}_{\widetilde{\mathfrak{D}}_g} \arrow[l, "\pi_{\widetilde{\mathfrak{D}}_g}"'] \arrow[r, equal] & \mathcal{C}_{\widetilde{\mathfrak{D}}_g} \end{tikzcd} \tag{4.16}$$

where the left one is Cartesian. Using

$$\pi_\mathcal{V}^*\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\cong\mathbb{T}_{\mathcal{C}_\mathcal{V}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}\longrightarrow\mathfrak{e}_\mathcal{V}^*\mathbb{T}_{\mathcal{Z}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}=\mathfrak{e}_\mathcal{V}^*T_{\mathcal{Z}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \tag{4.17}$$

and applying the projection formula, we obtain

$$\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}:\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\longrightarrow R^\bullet\pi_{\mathcal{V}*}\mathfrak{e}_\mathcal{V}^*T_{\mathcal{Z}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}. \tag{4.18}$$

Let $\mathfrak{S}$ be the moduli of section of $\mathcal{Z}\to\widetilde{\mathfrak{D}}_g$ constructed in Section 2.2. Because the evaluation morphism $\mathfrak{e}_\mathfrak{v}$ induces an open immersion $\mathcal{V}\to\mathfrak{S}$, using Proposition 2.5 implies that $\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ is a perfect relative obstruction theory.

Finally, the fiber product of every stack in (4.16) with $c\neq 0\in\mathbb{A}^1$ gives the diagram used to construct $\phi_{\mathcal{P}/\mathfrak{D}_g}$. Using $\iota_c:\mathcal{V}\times_{\mathbb{A}^1}c\to\mathcal{V}$, the functoriality of the construction ensures that $\phi_{\mathcal{P}/\mathfrak{D}_g}$ is the composition of $\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}\to\iota_c^*\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ with

$$\iota_c^*(\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}):\iota_c^*\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\longrightarrow\iota_c^*\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\cong\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}.$$

In case $c=0$, we define $\mathbb{E}_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}:=\iota_0^*\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ and let $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$ be the composition of $\mathbb{T}_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}\to\iota_0^*\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ with $\iota_0^*(\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g})$. This defines a perfect obstruction theory for $\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g$ by [1, Proposition 7.2]. This proves the Proposition. ■

4.5 The obstruction theory of $\mathcal{V}/\mathfrak{D}_g$

To compare the virtual cycle of $\mathcal{V}_0$ with $\mathcal{V}_{c\neq 0}$, we need the relative obstruction theory of $\mathcal{V}\to\mathfrak{D}_g$.

We use the $\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ just constructed. We let

$$\mathscr{K}\longrightarrow q_{\mathbb{P}^4}^*\mathscr{O}_{\mathbb{P}^4}\cong\mathscr{O}_{B\times\mathbb{A}^1} \tag{4.19}$$

be the composition of i in (4.15) with the projection to the last factor. We form

$$\mu: R^\bullet\pi_{\mathcal{V}*} f_{\mathcal{V}}^*\mathscr{K} \longrightarrow R^\bullet\pi_{\mathcal{V}*} f_{\mathcal{V}}^*\mathscr{O}_V \longrightarrow R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-1],$$

where the first arrow is $R^\bullet\pi_{\mathcal{V}*}$ of (4.19), and the second arrow is the tautological homomorphism from a two-term complex to its H^1.

We let $C(\mu^\vee)$ be the mapping cone of $\mu^\vee$, and let $C(\mu^\vee)^\vee$ be its dual. It fits into the distinguished triangle

$$R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-2] \longrightarrow C(\mu^\vee)^\vee \longrightarrow R^\bullet\pi_{\mathcal{V}*} f_{\mathcal{V}}^*\mathscr{K} \xrightarrow{+1} R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-1]. \tag{4.20}$$

We define

$$\mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} := R^\bullet\pi_{\mathcal{V}*}(f_{\mathcal{V}}^*\mathscr{K} \oplus \mathscr{P}_{\mathcal{V}}) \quad \text{and} \quad \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} := C(\mu^\vee)^\vee \oplus R^\bullet\pi_{\mathcal{V}*}\mathscr{P}_{\mathcal{V}}.$$

Then one has

$$R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-2] \longrightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} \xrightarrow{\eta} \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \xrightarrow[+1]{\mu} R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-1]. \tag{4.21}$$

By construction $\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}$ is a derived object representable by a two-term complex of locally free sheaves; its H^1 is

$$H^1(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}) = \ker\{H^1(\mu): R^1\pi_{\mathcal{V}*}(f_{\mathcal{V}}^*\mathscr{K} \oplus \mathscr{P}_{\mathcal{V}}) \longrightarrow R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}\}.$$

Since (4.19) is surjective, $H^1(\mu)$ is also surjective.

We now derive the perfect relative obstruction theory of $\mathcal{V} \to \mathfrak{D}_g$. Substituting $\tilde{\mathfrak{D}}_g$ and $\tilde{\mathcal{C}}_{\mathfrak{D}_g}$ in Proposition 4.2 by $\mathfrak{D}_g$ and $\mathcal{C}_{\mathfrak{D}_g}$, respectively, and following the recipe in the proof of Proposition 4.2, we obtain a morphism

$$\phi'_{\mathcal{V}/\mathfrak{D}_g}: \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \longrightarrow R^\bullet\pi_{\mathcal{V}*}\mathfrak{e}_{\mathfrak{v}}^*\mathbb{T}_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}} \cong R^\bullet\pi_{\mathcal{V}*}(f_{\mathcal{V}}^*\mathscr{K} \oplus \mathscr{P}_{\mathcal{V}}) := \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}. \tag{4.22}$$

Since moduli of sections of $\mathcal{Z} \to \widetilde{\mathfrak{D}}_g$ is isomorphic to the moduli of sections of $\mathcal{Z} \to \mathfrak{D}_g$, where $\mathcal{Z} \to \mathfrak{D}_g$ is via the composite $\mathcal{Z} \to \widetilde{\mathfrak{D}}_g \to \mathfrak{D}_g$, both are $\mathcal{V}$, thus Proposition 2.5 implies that $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ is a perfect relative obstruction theory.

According to Proposition 4.2, the obstruction sheaf of $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ has an extra factor $R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_\mathcal{V}}$ compared with that of $\phi_{\mathcal{P}/\mathfrak{D}_g}$ and of $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$. Our solution is to lift it to a new obstruction theory (cf. (4.21))

$$\phi_{\mathcal{V}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}, \tag{4.23}$$

whose obstruction sheaf is parallel to that of $\phi_{\mathcal{P}/\mathfrak{D}_g}$ and of $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$.

We denote by $\tilde{\mathfrak{r}} : \mathcal{C}_\mathcal{V} \to \mathcal{C}_{\widetilde{\mathfrak{D}}_g}$ the tautological morphism covering the tautological projection $\mathfrak{r}$ shown in the Cartesian square

$$\begin{tikzcd} \mathcal{C}_\mathcal{V} \arrow[r, "\tilde{\mathfrak{r}}"] \arrow[d, "\pi_\mathcal{V}"] & \mathcal{C}_{\widetilde{\mathfrak{D}}_g} \arrow[d] \\ \mathcal{V} \arrow[r, "\mathfrak{r}"] & \widetilde{\mathfrak{D}}_g \end{tikzcd}$$

Applying $\mathbb{T}_{\cdot/\mathcal{C}_{\mathfrak{D}_g}}$ to the evaluation $\mathcal{C}_{\mathfrak{D}_g}$-morphism $\mathfrak{e}_\mathfrak{v} : \mathcal{C}_\mathcal{V} \to \mathcal{Z}$ (in (4.16)), the identity

$$\tilde{\mathfrak{r}} = \mathrm{pr} \circ \mathfrak{e}_\mathfrak{v} : \mathcal{C}_\mathcal{V} \xrightarrow{\mathfrak{e}_\mathfrak{v}} \mathcal{Z} \xrightarrow{\mathrm{pr}} \mathcal{C}_{\widetilde{\mathfrak{D}}_g}$$

provides us a commutative square

$$\begin{tikzcd} \mathfrak{e}_\mathfrak{v}^* \Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] & (\mathrm{pr}\circ\mathfrak{e}_\mathfrak{v})^*\Omega^\vee_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}/\mathcal{C}_{\mathfrak{D}_g}} \cong \mathscr{O}_{\mathcal{C}_\mathcal{V}} \\ \mathbb{T}_{\mathcal{C}_\mathcal{V}/\mathcal{C}_{\mathfrak{D}_g}} \cong \pi_\mathcal{V}^*\mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \arrow[u] \arrow[r] & \tilde{\mathfrak{r}}^*\Omega^\vee_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}/\mathcal{C}_{\mathfrak{D}_g}} \cong \pi_\mathcal{V}^*\mathfrak{r}^*\Omega^\vee_{\widetilde{\mathfrak{D}}_g/\mathfrak{D}_g} \arrow[u] \end{tikzcd}$$

applying projection formula to both vertical arrows, we further obtain the commutative diagrams

$$\begin{tikzcd} \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \arrow[r] & R^\bullet\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_\mathcal{V}} \arrow[r] & R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_\mathcal{V}}[-1] \\ \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \arrow[u, "\phi'_{\mathcal{V}/\mathfrak{D}_g}"] \arrow[r] & \mathfrak{r}^*\Omega^\vee_{\widetilde{\mathfrak{D}}_g/\mathfrak{D}_g} = \mathscr{O}_\mathcal{V} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{4.24}$$

This shows that $\mu \circ \phi'_{\mathcal{V}/\mathfrak{D}_g} = 0$ (cf. μ is in (4.21)). Applying $\mathrm{Hom}(\mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}, \cdot)$ to (4.21), we conclude that the morphism $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ (in (4.21)) lifts (nonuniquely) as stated in (4.23) such that

$$\eta \circ \phi_{\mathcal{V}/\mathfrak{D}_g} = \phi'_{\mathcal{V}/\mathfrak{D}_g}. \tag{4.25}$$

Proposition 4.3. The homomorphism $\phi_{\mathcal{V}/\mathfrak{D}_g}$ is a perfect relative obstruction theory of $\mathcal{V} \to \mathfrak{D}_g$. □

Proof. We only need to check the criterion of perfect obstruction theory stated in the proof of Proposition 2.5; namely, we need to show that to any square zero extension $T \subset T'$ of affine schemes by J, and a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\mathfrak{m}} & \mathcal{V} \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\mathfrak{n}} & \mathfrak{D}_g \end{array}$$

the arrow $\phi_{\mathcal{V}/\mathfrak{D}_g}$ assigns an element $\varpi(\mathfrak{m}) \in H^1(T, \mathfrak{m}^*\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} \otimes J)$ (cf. (2.6)) such that there is a lifting $\mathfrak{m}' : T' \to \mathcal{V}$ of the square above if and only if $\varpi(\mathfrak{m}) = 0$.

Recall that $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ is also a perfect relative obstruction theory. We let $\varpi(\mathfrak{m})' \in H^1(T, \mathfrak{m}^*\mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \otimes J)$ be the associated obstruction class. Since $\phi_{\mathcal{V}/\mathfrak{D}_g}$ is a lift of $\phi'_{\mathcal{V}/\mathfrak{D}_g}$, $\varpi(\mathfrak{m})'$ is the image of $\varpi(\mathfrak{m})$ under the homomorphism

$$H^1(\eta) : H^1(T, \mathfrak{m}^*\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} \otimes J) \longrightarrow H^1(T, \mathfrak{m}^*(R^\bullet \pi_{\mathcal{V}*}\mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \otimes J)$$

induced by the η in (4.21). Because of the distinguished triangle (4.21), $H^1(\eta)$ is injective. This proves that $\varpi(\mathfrak{m}) = 0$ if and only if $\varpi(\mathfrak{m})' = 0$. Since the later is the obstruction class, the former is too.

The other part of the criterion follows from the same reason. This proves the Proposition. ■

4.6 Comparison of obstruction theories

Let $c \in \mathbb{A}^1$ be any closed point. We denote the restrictions to fibers over c by

$$\iota_c : \mathcal{V}_c = \mathcal{V} \times_{\mathbb{A}^1} c \xrightarrow{\subset} \mathcal{V} \quad \text{and} \quad \mathfrak{e}_{\mathfrak{v}_c} = \mathfrak{e}_{\mathfrak{v}}|_{\mathcal{C}_{\mathcal{V}_c}} : \mathcal{C}_{\mathcal{V}_c} = \mathcal{C}_{\mathcal{V}} \times_{\mathcal{V}} \mathcal{V}_c \longrightarrow \mathcal{Z}_c = \mathcal{Z} \times_{\mathbb{A}^1} c.$$

Recall by Proposition 4.2 that composing the tautological $\mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \to \iota_c^*\mathbb{T}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}$ with $\iota_c^*\phi_{\mathcal{V}/\tilde{\mathfrak{D}}_g}$ gives the perfect relative obstruction theory

$$\phi_{\mathcal{V}_c/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} := \iota_c^*\mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}.$$

(Note that for $c \neq 0$, $\mathcal{V}_c = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$, and this obstruction theory coincides with the one constructed in Section 3.1.)

We now compare the obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$ with $\phi_{\mathcal{V}_c/\mathfrak{D}_g}$. Using the tautological exact sequence

$$0 \longrightarrow T_{\mathcal{Z}_c/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow T_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}|_{\mathcal{Z}_c} \longrightarrow \mathscr{O}_{\mathcal{Z}_c} \longrightarrow 0, \tag{4.26}$$

we obtain a morphism of distinguished triangles (the top line is an exact sequence of sheaves):

$$\begin{tikzcd}
\mathfrak{e}^*_{\mathfrak{v}_c}\Omega^\vee_{\mathcal{Z}_c/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] & \mathfrak{e}^*_{\mathfrak{v}_c}(\Omega^\vee_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}|_{\mathcal{Z}_c}) \arrow[r] & \mathfrak{e}^*_{\mathfrak{v}_c}\mathscr{O}_{\mathcal{Z}_c} \cong \mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}} \arrow[r, "+1"] & 0 \\
\mathbb{T}_{\mathcal{C}_{\mathcal{V}_c}/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{C}_{\mathcal{V}}/\mathcal{C}_{\mathfrak{D}_g}}|_{\mathcal{C}_{\mathcal{V}_c}} \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{C}_c/\mathcal{C}_{\mathcal{V}}}[1] \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd} \tag{4.27}$$

By projection formula, we have a morphism of distinguished triangles

$$\begin{tikzcd}
R^\bullet\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{V}_c}[-1] \arrow[r] & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta'"] & \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[r, "+1"] & {} \\
\mathbb{T}_{\mathcal{V}_c/\mathcal{V}} \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\gamma_0"] \arrow[u, "\phi_{\mathcal{V}_c/\mathfrak{D}_g}"'] & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[r, "+1"] \arrow[u, "\phi'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}"'] & {}
\end{tikzcd} \tag{4.28}$$

(Here and later, we use $(\cdot)|_{\mathcal{V}} := \iota_c^*(\cdot)$ to denote the derived restriction.) Applying the mapping cone construction (4.21) to the top row of (4.28), and using the octahedral axiom, we obtain a compatible diagram of mapping cones

$$\begin{tikzcd}
 & & R^1\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \arrow[r, "="] & R^1\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \\
R^\bullet\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \arrow[r] & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta'"] & \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[r] \arrow[u, "\mu|_{\mathcal{V}_c}"'] & R^\bullet\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{V}_c} \arrow[u] \\
\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \arrow[r] \arrow[u] & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta_0"] \arrow[u] & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[r, "\beta_1"] \arrow[u, "\eta|_{\mathcal{V}_c}"'] & \pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{V}_c} \arrow[u]
\end{tikzcd} \tag{4.29}$$

Restricting the perfect obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$ (cf. Proposition 4.3) to $\mathcal{V}_c$, we obtain the following (not necessarily commuting) homomorphisms:

$$\begin{tikzcd}
\mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta_0"] & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\
\mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[u, "\phi_{\mathcal{V}_c/\mathfrak{D}_g}"] \arrow[r, "\gamma_0"] & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[u, "\phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}"']
\end{tikzcd} \tag{4.30}$$

We consider

$$\delta = \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \beta_0 \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}.$$

Applying the commutative diagrams (4.25), (4.29), and (4.28), we conclude that

$$\eta|_{\mathcal{V}_c} \circ \delta = \eta|_{\mathcal{V}_c} \circ \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \eta|_{\mathcal{V}_c} \circ \beta_0 \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} = \phi'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \beta' \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} = 0.$$

Therefore, δ factors through $R^1\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{V}_c}[-2] \to \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}$.

Because of this, after applying the truncation functor $\tau_{\leq 1}$ to (4.30), we obtain a commutative square

$$\begin{tikzcd}
\mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta_0"] & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\
\mathbb{T}^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[u, "\phi^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g}"] \arrow[r, "\gamma_0"] & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|^{\leq 1}_{\mathcal{V}_c} \arrow[u, "\phi^{\leq 1}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}"']
\end{tikzcd} \tag{4.31}$$

On the other hand, applying the truncation functor $\tau_{\leq 1}$ to the left square in (4.28), we obtain another commutative square

$$\begin{tikzcd}
\pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \arrow[r] & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \\
\mathbb{T}^{\leq 1}_{\mathcal{V}_c/\mathcal{V}} \arrow[u] \arrow[r] & \mathbb{T}^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[u, "\phi^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g}"']
\end{tikzcd} \tag{4.32}$$

Combined, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{V}_c}[-1] \cong \pi_{\mathcal{V}_c*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \arrow[r] & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[r, "\beta_0"] & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \arrow[r, "+1"] & {} \\
\mathbb{T}^{\leq 1}_{\mathcal{V}_c/\mathcal{V}} \arrow[u] \arrow[r] & \mathbb{T}^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g} \arrow[u, "\phi^{\leq 1}_{\mathcal{V}_c/\mathfrak{D}_g}"'] \arrow[r, "\gamma_0"] & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|^{\leq 1}_{\mathcal{V}_c} \arrow[u, "\phi^{\leq 1}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}"'] &
\end{tikzcd} \tag{4.33}$$

By (4.29) the top row is a distinguished triangle (but not the lower one).

We comment that applying results in [16], this diagram implies that the virtual cycle of $\mathcal{V}_c$ is the pull-back via $\iota_c: \mathcal{V}_c \to \mathcal{V}$ of the virtual cycle of $\mathcal{V}$. In our case, we are using localized virtual cycles via cosections of the obstruction sheaves, thus we need to construct a cosection of the obstruction sheaf

$$\mathcal{O}b_{\mathcal{V}} = \operatorname{coker}\{T_{\mathfrak{D}_g} \otimes_{\mathscr{O}_{\mathfrak{D}_g}} \mathscr{O}_{\mathcal{V}} \longrightarrow H^1(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})\}.$$

4.7 Family cosection

We first construct a cosection of the obstruction sheaf $\mathcal{O}b_{\mathcal{V}/\tilde{\mathfrak{D}}_g}$. The construction is parallel to the case $\mathcal{P} = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$.

First, we define a bi-linear morphism of bundles

$$h: \mathrm{Vb}(\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5} \oplus \mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \oplus \mathscr{P}_{\tilde{\mathfrak{D}}_g}) \longrightarrow \mathrm{Vb}(\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{P}_{\tilde{\mathfrak{D}}_g}) \longrightarrow \mathrm{Vb}(\omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}).$$

Here the first arrow is $(\mathrm{pr}_2, \mathrm{pr}_3)$, where pr_i is the ith projection; the second arrow is induced by tensoring of sheaves of $\mathscr{O}_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}}$-modules $\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \otimes \mathscr{P}_{\tilde{\mathfrak{D}}_g} \to \omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}$. Using that the family $\mathcal{Z} \to \mathcal{C}_{\tilde{\mathfrak{D}}_g}$ in (4.10) is a subfamily

$$\mathcal{Z} \subset \mathrm{Vb}(\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \mathrm{Vb}(\mathscr{P}_{\tilde{\mathfrak{D}}_g}) \subset \mathrm{Vb}(\mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5} \oplus \mathscr{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \oplus \mathscr{P}_{\tilde{\mathfrak{D}}_g}),$$

composing with h, we obtain a $\mathcal{C}_{\tilde{\mathfrak{D}}_g}$-morphism

$$\mathcal{Z} \longrightarrow \mathrm{Vb}(\omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}). \tag{4.34}$$

Lemma 4.4. The homomorphism (4.34) induces a homomorphism

$$\sigma^\bullet : \mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g} \longrightarrow R^1\pi_{\mathcal{V}*}\mathscr{O}_{\mathcal{C}_{\mathcal{V}}}[-1]$$

whose restriction to $\mathcal{V} \times_{\mathbb{A}^1} c \cong \mathcal{P}$, $c \neq 0$, is proportional (by an element in $\mathbb{C}^*$) to $\sigma_1^\bullet$ in (3.7). □

Proof. The proof is exactly as in Section 3.2. We omit it here. ■

We define

$$\sigma = H^1(\sigma^\bullet) : \mathcal{O}b_{\mathcal{V}/\tilde{\mathfrak{D}}_g} := H^1(\mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}) \longrightarrow R^1\pi_{\mathcal{V}*}\omega_{\mathcal{C}_{\mathcal{V}}/\mathcal{V}} \cong \mathscr{O}_{\mathcal{V}}. \tag{4.35}$$

Let $\tilde{\mathfrak{q}}:\mathcal{V}\to\widetilde{\mathfrak{D}}_g$ be the projection. The distinguished triangle $\tilde{\mathfrak{q}}^*\mathbb{L}_{\widetilde{\mathfrak{D}}_g}\to\mathbb{L}_{\mathcal{V}}\to\mathbb{L}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\to\tilde{\mathfrak{q}}^*\mathbb{T}_{\widetilde{\mathfrak{D}}_g}[1]$ gives a morphism $\tilde{\mathfrak{q}}^*\mathbb{T}_{\widetilde{\mathfrak{D}}_g}\to\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}[1]$, which, composed with $\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}:\mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\to\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$, gives

$$\eta:\tilde{\mathfrak{q}}^*\mathbb{T}_{\widetilde{\mathfrak{D}}_g}\longrightarrow\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}[1].$$

Taking the cokernel of the H^0 of this arrow, we obtain the absolute obstruction sheaf

$$\mathcal{O}b_{\mathcal{V}}:=\operatorname{coker}\{H^0(\eta):\tilde{\mathfrak{q}}^*\Omega^\vee_{\widetilde{\mathfrak{D}}_g}\longrightarrow H^1(\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g})\}. \tag{4.36}$$

Lemma 4.5. The following composite vanishes:

$$\tilde{\mathfrak{q}}^*\Omega^\vee_{\widetilde{\mathfrak{D}}_g}\xrightarrow{H^0(\eta)}H^1(\mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g})\xrightarrow{\sigma}R^1\pi_{\mathcal{V}*}\omega_{\mathcal{C}_{\mathcal{V}}/\mathcal{V}}. \tag{4.37}$$

□

Proof. The proof is exactly the same as that of Proposition 3.5, and will be omitted. ■

This immediately gives the following corollary.

Corollary 4.6. The cosection $\sigma:\mathcal{O}b_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}\to\mathscr{O}_{\mathcal{V}}$ lifts to a cosection $\bar{\sigma}:\mathcal{O}b_{\mathcal{V}}\to\mathscr{O}_{\mathcal{V}}$. □

Lastly, we describe the degeneracy (nonsurjective) locus of σ. As before, we say σ is degenerate at $\xi\in\mathcal{V}$ if $\sigma|_\xi$ is not surjective (i.e., is trivial). Let $\xi\in\mathcal{V}$ be any closed point; ξ is represented by

$$((\phi_i),b,p)\in H^0(L^{\oplus 5})\times H^0(L^{\otimes 5})\times H^0(L^{-\otimes 5}\otimes\omega_C)$$

for $(C,L)\in\mathfrak{D}_g$ the point under ξ. Then $\sigma|_\xi:\mathcal{O}b_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}|_\xi\to\mathbb{C}$ is identical to the composite of the inclusion

$$\mathcal{O}b_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}|_\xi\subset H^1(L^{\oplus 5})\oplus H^1(L^{\otimes 5})\oplus H^1(L^{-\otimes 5}\otimes\omega_C)$$

with the pairing

$$H^1(L^{\oplus 5})\oplus H^1(L^{\otimes 5})\oplus H^1(L^{-\otimes 5}\otimes\omega_C)\longrightarrow H^1(\omega_C)$$

defined via $((\mathring{\phi}_i),\mathring{b},\mathring{p})\mapsto\mathring{b}\cdot p+b\cdot\mathring{p}$. Like the proof of Proposition 3.4, this description shows that the degeneracy locus of σ is $\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1\subset\mathcal{V}$, where the inclusion is via vanishing p-fields and the inclusion $\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1\subset\overline{\mathcal{M}}_g(V,d)$ induced by the tautological inclusion $Q\times\mathbb{A}^1\subset V$.

Lemma 4.7. The degeneracy locus of the cosection $\bar\sigma$ is $\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1\subset\mathcal{V}$; it is proper over $\mathbb{A}^1$. □

Proof. We first need to verify that σ is as given. The proof of this is exactly the same as that of Lemma 3.2. Using this description, we argue that the degeneracy locus of the cosection $\sigma:\mathcal{O}b_{\mathcal{V}/\mathfrak{D}_g}\to\mathscr{O}_{\mathcal{V}}$ is $\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1\subset\mathcal{V}$, and thus is proper over $\mathbb{A}^1$. Since $\bar\sigma$ is a lift of σ, the degeneracy locus of $\bar\sigma$ coincides with that of σ. This proves the Lemma. ■

4.8 The constancy of the invariants

By direct verification, the virtual dimension of $\mathcal{V}$ is 1. Using Lemma 4.7 and Corollary 4.6, following the convention introduced in Section 3.6, we denote by

$$h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar\sigma}\subset h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})$$

the kernel of a cone-stack morphism $h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})\to\mathbb{C}_{\mathcal{V}}$ induced by $\bar\sigma$ defined as in Definition 3.7.

Let

$$[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}]\in Z_*h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})$$

be the intrinsic normal cone embedded using the obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$. Because of Lemma 4.7 and Corollary 4.6, applying [15, Theorem 5.1] we conclude that

$$[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}]\in Z_*h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar\sigma}.$$

We then apply the localized Gysin map [15]

$$0^!_{\bar\sigma,\mathrm{loc}}:A_*h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar\sigma}\longrightarrow A_*(\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1).$$

Definition 4.8. We define the localized virtual cycle of $(\mathcal{V},\bar\sigma)$ to be

$$[\mathcal{V}]^{\mathrm{vir}}_{\bar\sigma}:=0^!_{\bar\sigma,\mathrm{loc}}([\mathbf{C}_{\mathcal{V}/\mathfrak{D}_g}])\in A_1(\overline{\mathcal{M}}_g(Q,d)\times\mathbb{A}^1).$$ □

Now let $c\in\mathbb{A}^1$ be any closed point and let $\jmath_c : c \to \mathbb{A}^1$ be the closed inclusion. We define $\mathcal{N} := \mathcal{V}\times_{\mathbb{A}^1} 0$. By the compatibility stated in diagram (4.33) and Corollary 4.6, we apply [15, Theorem 5.2] to obtain the following proposition.

Proposition 4.9. Under the shriek operation of cycles ($c\neq 0$),

$$\jmath_c^!([\mathcal{V}]^{\mathrm{vir}}_{\bar\sigma}) = [\mathcal{P}]^{\mathrm{vir}}_{\bar\sigma_1}\in A_0\overline{\mathcal{M}}_g(Q,d), \quad \jmath_0^!([\mathcal{V}]^{\mathrm{vir}}_{\bar\sigma}) = [\mathcal{N}]^{\mathrm{vir}}_{\bar\sigma_0}\in A_0\overline{\mathcal{M}}_g(Q,d). \qquad \square$$

Here $[\mathcal{N}]^{\mathrm{vir}}_{\bar\sigma_0}$ is the localized virtual cycle using the obstruction theory of $\mathcal{N}$ induced by the restriction of $\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$(Proposition 4.2) and the restriction of the cosection $\bar\sigma_0 = \bar\sigma|_{\mathcal{N}}$.

5 Gromov–Witten Invariant of $(K_N, \mathbf{w}_N)$

We continue to denote by

$$r : N \longrightarrow Q$$

the normal bundle to Q in $\mathbb{P}^4$. Let K_N be the total space of the canonical line bundle of N, which is isomorphic to the underlying line bundle of the pull-back $r^*\mathscr{O}(-5)$. The duality paring $\mathscr{O}_Q(5)\otimes_{\mathscr{O}_Q}\mathscr{O}_Q(-5)\to\mathscr{O}_Q$ defines a regular function $\mathbf{w}_N\in\Gamma(\mathscr{O}_{K_N})$. The degree $\deg[\mathcal{N}]^{\mathrm{vir}}_{\bar\sigma_0}$ are the Gromov–Witten invariants of the Landau–Ginzburg space (K_N, w_N).

We denote by $\overline{\mathcal{M}}_g(N,d)$ the moduli space of genus g degree d stable maps to N, where the degree is measured by their images in $\mathbb{P}^4$ via $N\to Q\subset\mathbb{P}^4$. Because $N = V\times_{\mathbb{A}^1} 0$, canonically $\overline{\mathcal{M}}_g(N,d) = \overline{\mathcal{M}}_g(\mathcal{V},d)\times_{\mathbb{A}^1} 0$. The moduli of stable maps coupled with p-fields is identical to $\mathcal{N}$

$$\mathcal{N} := \mathcal{V}\times_{\mathbb{A}^1} 0 = \overline{\mathcal{M}}_g(V,d)^p\times_{\mathbb{A}^1} 0 \cong \overline{\mathcal{M}}_g(N,d)^p.$$

We let

$$(f_{\mathcal{N}},\pi_{\mathcal{N}}) : \mathcal{C}_{\mathcal{N}} \longrightarrow N\times\mathcal{N}$$

be the universal map of $\mathcal{N}$. By definition, it is the restriction of $(f_{\mathcal{V}},\pi_{\mathcal{V}},\mathcal{C}_{\mathcal{V}})$ to the fiber over $0\in\mathbb{A}^1$.

5.1 The invariants and the equivalence

As indicated in the beginning of Section 4.6, we have evaluation morphism

$$\mathfrak{e}_{\mathcal{N}} : \mathcal{C}_{\mathcal{N}} \longrightarrow \mathcal{Z}_0 = \mathcal{Z}\times_{\mathbb{A}^1} 0.$$

By construction in Proposition 4.2,

$$\phi_{\mathcal{N}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{N}/\mathfrak{D}_g} \longrightarrow R^\bullet \pi_{\mathcal{N}*} \mathfrak{e}_{\mathcal{N}}^* \mathbb{T}_{\mathcal{Z}_0/\mathcal{C}_{\mathfrak{D}_g}} := \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g} \tag{5.1}$$

is a perfect relative obstruction theory of $\mathcal{N}/\mathfrak{D}_g$, which is identical to the restriction of $\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g}$ to the fiber over $0 \in \mathbb{A}^1$.

We let σ_0 be the restriction of σ to $\mathcal{N}$:

$$\sigma_0 = \sigma|_{\mathcal{N}} : \mathcal{O}b_{\mathcal{N}/\mathfrak{D}_g} \longrightarrow \mathscr{O}_{\mathcal{N}}. \tag{5.2}$$

Proposition 5.1. The cosection σ_0 lifts to a cosection $\bar{\sigma}_0 : \mathcal{O}b_{\mathcal{N}} \to \mathscr{O}_{\mathcal{N}}$. The degeneracy (nonsurjective) locus of the cosection $\bar{\sigma}_0$ is $\overline{M}_g(Q, d) \subset \overline{M}_g(N, d)^P$, and thus is proper. □

Proof. This follows directly from Lemma 4.7. ■

Because the virtual dimension of $\mathcal{V}$ is 1, the virtual dimension of $\mathcal{N}$ is 0. By Proposition 3.4, applying a cosection localization Gysin map [15, Theorem 5.1], we obtain the following proposition.

Definition-Proposition 5.2. We define the localized virtual cycle of $\overline{M}_g(N, d)^P$ to be

$$[\overline{M}_g(N, d)^P]^{\mathrm{vir}}_\sigma := 0^!_{\sigma,\mathrm{loc}}([\mathbf{C}_{\overline{M}_g(N,d)^P/\mathfrak{D}_g}]) \in A_0 \overline{M}_g(Q, d);$$

we define $N_g(d)_{K_{N_Q}} = \deg [\overline{M}_g(N, d)^P]^{\mathrm{vir}}_\sigma$. □

We call $N_g(d)_{K_{N_Q}}$ the formal Landau–Ginzburg model.

Theorem 5.3. For any positive d, the invariants coincide: $N_g(d)^p_{\mathbb{P}^4} = N_g(d)_{K_{N_Q}}$. □

Proof. The proof follows directly from Proposition 4.9. ■

5.2 Comparing with the GW invariant of Quintics

We now show that the formal Landau–Ginzburg model gives the same invariants as the Gromov–Witten invariants of Q up to signs.

We first construct a perfect relative obstruction theory of $\mathcal{N} \to \mathcal{Q}$. In the fiber product over $\mathfrak{D}_g$

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\gamma} & \mathfrak{C} := C(\pi_{\mathfrak{D}_g *}(\mathscr{L}_{\mathfrak{D}_g}^{\otimes 5} \oplus \mathscr{P}_{\mathfrak{D}_g})) \\ \downarrow{\scriptstyle \upsilon} & & \downarrow \\ \mathcal{Q} := \overline{\mathcal{M}}_g(Q, d) & \longrightarrow & \mathfrak{D}_g \end{array} \tag{5.3}$$

where $C(\pi_{\mathfrak{D}_g *}(\mathscr{L}_{\mathfrak{D}_g}^{\otimes 5} \oplus \mathscr{P}_{\mathfrak{D}_g}))$ is the direct image cone constructed in Section 2.1, the morphism γ pulls back the relative perfect obstruction theory

$$\mathbb{T}_{\mathfrak{C}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g} \tag{5.4}$$

to the morphism

$$\phi_{\mathcal{N}/\mathcal{Q}} : \mathbb{T}_{\mathcal{N}/\mathcal{Q}} \longrightarrow \mathbb{E}_{\mathcal{N}/\mathcal{Q}} := \gamma^* \mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g}.$$

If one uses the usual convention to define $\mathscr{L}_{\mathcal{N}} = f_{\mathcal{N}}^* \mathscr{O}(1)$, where

$$(f_{\mathcal{N}}, \pi_{\mathcal{N}}) : \mathcal{C}_{\mathcal{N}} \longrightarrow N \times \mathcal{N} \tag{5.5}$$

is the universal family of $\mathcal{N}$ and let $\mathscr{P}_{\mathcal{N}} = \mathscr{L}_{\mathcal{N}}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{N}}/\mathcal{N}}$, then one has

$$\mathbb{E}_{\mathcal{N}/\mathcal{Q}} \cong R^{\bullet} \pi_{\mathcal{N}*}(\mathscr{L}_{\mathcal{N}}^{\otimes 5} \oplus \mathscr{P}_{\mathcal{N}}).$$

By Proposition 2.5, $\phi_{\mathcal{N}/\mathcal{Q}}$ is the perfect relative obstruction theory associated with the direct image cone stack $\mathcal{N} \cong C(\pi_{\mathcal{Q}*}(\mathscr{L}_{\mathcal{Q}}^{\otimes 5} \oplus \mathscr{P}_{\mathcal{Q}}))$ relative to $\mathcal{Q} = \overline{\mathcal{M}}_g(Q, d)$.

We define

$$\mathfrak{Q} = \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\oplus 5})^* \times_{\mathbb{P}^4} Q. \tag{5.6}$$

The evaluation maps of $\mathcal{N}$ and $\mathcal{Q}$ fit into the diagram

$$\begin{array}{ccccc} \mathcal{C}_{\mathcal{N}} & \xrightarrow{\mathbf{e}_{\mathcal{N}}} & \mathcal{Z}_0 & \longrightarrow & \mathrm{Vb}(\mathscr{L}_{\mathfrak{D}_g}^{\otimes 5}) \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathrm{Vb}(\mathscr{P}_{\mathfrak{D}_g}) \\ \downarrow{\scriptstyle \upsilon_{\mathcal{C}}} & & \downarrow & & \downarrow \\ \mathcal{C}_{\mathcal{Q}} & \xrightarrow{\mathbf{e}_{\mathcal{Q}}} & \mathfrak{Q} & \longrightarrow & \mathcal{C}_{\mathfrak{D}_g} \end{array}$$

where the right square is a fiber product of smooth morphisms, and $\upsilon_{\mathcal{C}}$ is induced by the vertical arrow υ in diagram (5.3).

The diagram associates a morphism between distinguished triangles

$$\begin{tikzcd}
\mathbf{e}_{\mathcal{N}}^* T_{\mathcal{Z}_0/\mathfrak{Q}} \arrow[r] & \mathbf{e}_{\mathcal{N}}^* T_{\mathcal{Z}_0/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] & \upsilon_{\mathcal{C}}^*\mathbf{e}_{\mathcal{Q}}^* T_{\mathfrak{Q}/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r, "+1"] & {} \\
\mathbb{T}_{\mathcal{C}_{\mathcal{N}}/\mathcal{C}_{\mathcal{Q}}} \arrow[r] \arrow[u] & \mathbb{T}_{\mathcal{C}_{\mathcal{N}}/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r] \arrow[u] & \upsilon_{\mathcal{C}}^*\mathbb{T}_{\mathcal{C}_{\mathcal{Q}}/\mathcal{C}_{\mathfrak{D}_g}} \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

Identifying $\mathbb{E}_{\mathcal{N}/\mathcal{Q}}$ with $R^\bullet\pi_{\mathcal{N}*}\mathbf{e}_{\mathcal{N}}^* T_{\mathcal{Z}_0/\mathfrak{Q}}$, by the projection formula we have

$$\begin{tikzcd}
\mathbb{E}_{\mathcal{N}/\mathcal{Q}} \arrow[r] & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g} \arrow[r, "h"] & \upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \arrow[r, "+1"] & {} \\
\mathbb{T}_{\mathcal{N}/\mathcal{Q}} \arrow[r] \arrow[u, "\phi_{\mathcal{N}/\mathcal{Q}}"'] & \mathbb{T}_{\mathcal{N}/\mathfrak{D}_g} \arrow[r, "n"] \arrow[u, "\phi_{\mathcal{N}/\mathfrak{D}_g}"'] & \upsilon^*\mathbb{T}_{\mathcal{Q}/\mathfrak{D}_g} \arrow[r, "+1"] \arrow[u, "\upsilon^*\phi_{\mathcal{Q}/\mathfrak{D}_g}"'] & {}
\end{tikzcd} \tag{5.7}$$

Now let $U := \mathcal{N} - \mathcal{Q}$; it is open in $\mathcal{N}$, and has the property that both σ_0 and σ_0' are surjective on U. By the octahedral axiom, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_U[-1] \arrow[r, "="] & \mathscr{O}_U[-1] & & \\
\mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U \arrow[r] \arrow[u] & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U \arrow[r, "h|_U"] \arrow[u] & \upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}|_U \arrow[r, "+1"] & {} \\
\mathbb{E}'_{U/\mathcal{Q}} \arrow[r] \arrow[u, "\chi_Q"'] & \mathbb{E}'_{U/\mathfrak{D}_g} \arrow[r, "h_\chi := h|_U \circ \chi"] \arrow[u, "\chi"'] & \upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}|_U \arrow[r, "+1"] \arrow[u, "||"'] & {}
\end{tikzcd} \tag{5.8}$$

where all rows and columns are distinguished triangles, and the two vertical arrows to $\mathscr{O}_U[-1]$ are induced by σ_0 and σ_0', respectively. We show that the perfect obstruction theories of $\mathcal{N} \to \mathcal{Q}$ and $\mathcal{N} \to \mathfrak{D}_g$ restricted to U can be lifted to $\mathbb{E}'_{U/\mathcal{Q}}$ and $\mathbb{E}'_{U/\mathfrak{D}_g}$, respectively.

Lemma 5.4. There are perfect relative obstruction theories $\phi'_{U/\mathcal{Q}}$ of $U/\mathcal{Q}$ and $\phi'_{U/\mathfrak{D}_g}$ of $U/\mathfrak{D}_g$ that fit into a compatible diagram

$$\begin{tikzcd}
\mathbb{E}'_{U/\mathcal{Q}} \arrow[r, "\theta_E"] & \mathbb{E}'_{U/\mathfrak{D}_g} \arrow[r, "h_\chi"] & (\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g})|_U \\
\mathbb{T}^{\le 1}_{U/\mathcal{Q}} \arrow[r, "\theta"] \arrow[u, "\phi'_{U/\mathcal{Q}}"'] & \mathbb{T}^{\le 1}_{U/\mathfrak{D}_g} \arrow[r, "n' := n^{\le 1}|_U"] \arrow[u, "\phi'_{U/\mathfrak{D}_g}"'] & (\upsilon^*\mathbb{T}_{\mathcal{Q}/\mathfrak{D}_g})|_U^{\le 1} \arrow[u, "\upsilon^*\phi^{\le 1}_{\mathcal{Q}/\mathfrak{D}_g}|_U"']
\end{tikzcd} \tag{5.9}$$

and whose composition with the lower row of arrows in (5.8) is the $\tau^{[0,1]}$ truncation of (5.7) restricted over U. □

Proof. Applying the truncation functor $\tau_{\leq 1}$ to $\phi_{\mathcal{N}/\mathfrak{D}_g}|_U$, we obtain

$$\phi^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U : \mathbb{T}^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U \longrightarrow \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U.$$

Then arguing similarly to Lemma 3.6, we conclude that in the commutative diagram

$$\begin{tikzcd} \mathbb{T}^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U \arrow[r, "\phi^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U"] \arrow[d] & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U \arrow[d] \\ H^1(\mathbb{T}_{\mathcal{N}/\mathfrak{D}_g}|_U) \arrow[r] & \mathcal{O}b_{\mathcal{N}/\mathfrak{D}_g}|_U \end{tikzcd}$$

the composition of $\phi^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U$ with $\mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U \to \mathscr{O}_U[-1]$ vanishes. Hence $\phi^{\leq 1}_{\mathcal{N}/\mathfrak{D}_g}|_U = \chi \circ \phi'_{U/\mathfrak{D}_g}$ for some

$$\phi'_{U/\mathfrak{D}_g} : \mathbb{T}^{\leq 1}_{U/\mathfrak{D}_g} \longrightarrow \mathbb{E}'_{U/\mathfrak{D}_g}.$$

It is direct to check that $\phi'_{U/\mathfrak{D}_g}$ is a perfect obstruction theory, and the middle square of the diagram (5.9) commutes.

By similar reason the $\tau_{\leq 1}$ truncation of $\phi_{\mathcal{N}/\mathcal{Q}}|_U$,

$$\phi_{\mathcal{N}/\mathcal{Q}}|_U : \mathbb{T}^{\leq 1}_{U/\mathcal{Q}} \longrightarrow \mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U,$$

has the property that its composition with $\mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U \to \mathscr{O}_U[-1]$ vanishes and lifts to a

$$\phi'_{U/\mathcal{Q}} : \mathbb{T}^{\leq 1}_{U/\mathcal{Q}} \longrightarrow \mathbb{E}_{U/\mathfrak{D}_g},$$

such that $\phi_{\mathcal{N}/\mathcal{Q}}|_U = \chi_{\mathcal{Q}} \circ \phi'_{U/\mathcal{Q}}$. The map $\Delta := \theta_E \circ \phi'_{U/\mathcal{Q}} - \phi'_{U/\mathfrak{D}_g} \circ \theta$ in (5.9) thus satisfies $\chi \circ \Delta = 0$, hence Δ factors through a morphism $\mathscr{O}_U[-2] \longrightarrow \mathbb{E}'_{U/\mathfrak{D}_g}$; because after applying the truncation functor $\tau_{\leq 1}$, we have $\tau_{\leq 1}(\Delta) = \Delta$ and $\tau_{\leq 1}(\mathscr{O}_U[-2]) = 0$, and we conclude $\Delta = 0$. This proves that the left square is commutative. ■

To proceed, we apply the work of Kim–Kresch–Pantev on deformation of intrinsic normal cone [16]. (It is recalled in the Appendix.) Let $h^1/h^0(c_0(\tilde{h}))$ be the bundle stack over $\mathcal{N} \times \mathbb{P}^1$ introduced in Lemma A.3. Following [16], it is a deformation of $h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})$ to

$h^1/h^0(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}) \times_{\mathcal{N}} h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})$ (cf. Definition A.1). By (A.5) and Lemma A.3, the diagram (5.7) induces an inclusion of cone-stacks

$$\mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}} \subset h^1/h^0(c_0(\tilde{n})) \subset h^1/h^0(c_0(\tilde{h})), \tag{5.10}$$

thus giving a cycle

$$[\mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}}] \in Z_* h^1/h^0(c_0(\tilde{h})). \tag{5.11}$$

We next introduce a cosection of $H^1(c_0(\tilde{h}))$. Composing the cosection $\sigma_0 : \mathcal{O}b_{\mathcal{N}/\mathfrak{D}_g} \to \mathscr{O}_{\mathcal{N}}$ (cf. (5.2)) with $H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \to H^1(\mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})$, we obtain

$$\sigma_0' : \mathcal{O}b_{\mathcal{N}/\mathcal{Q}} := H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \longrightarrow \mathscr{O}_{\mathcal{N}}. \tag{5.12}$$

Arguing similar to Proposition 5.1, one sees that the degeneracy locus of σ_0' equals $\mathcal{Q} \subset \mathcal{N}$, and the same argument as in Lemma 3.6 shows that the composite of σ_0' with $H^1(\mathbb{T}_{\mathcal{N}/\mathcal{Q}}) \to H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})$ is trivial. Thus σ_0' lifts to a cosection of $\mathcal{O}b_{\mathcal{N}}$.

We remark that the σ_0 defined earlier is the extension of σ_0' by sending elements in $\upsilon^*\mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g}$ to zero:

$$\sigma_0 : H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \cong \upsilon^*\mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathcal{O}b_{\mathcal{N}/\mathcal{Q}} \to \mathscr{O}_{\mathcal{N}}. \tag{5.13}$$

Since $\upsilon^*\mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathcal{O}b_{\mathcal{N}/\mathcal{Q}} = H^1(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})$, by Definition 3.7, we have the kernel cone-stack

$$h^1/h^0(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})_{\sigma_0} \subset h^1/h^0(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}). \tag{5.14}$$

Using the construction after (A.3), we have the distinguished triangle

$$c_0(\tilde{h}) \longrightarrow \upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g} \xrightarrow{\tilde{h}} \upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}(1) \xrightarrow{+1}. \tag{5.15}$$

We define

$$\sigma_h : H^1(c_0(\tilde{h})) \longrightarrow \mathscr{O}_{\mathcal{N}\times\mathbb{P}^1}, \tag{5.16}$$

to be the composite of $H^1(c_0(\tilde{h})) \to H^1(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})$ induced by the first arrow in (5.15) with σ_0. By

$$H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \cong \upsilon^*\mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathcal{O}b_{\mathcal{N}/\mathcal{Q}},$$

we have a canonical isomorphism

$$H^1(c_0(\tilde{h})) \cong p_{\mathcal{N}}^*(\upsilon^* \mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathcal{O}b_{\mathcal{N}/\mathcal{Q}})$$

of $\mathscr{O}_{\mathcal{N}\times\mathbb{P}^1}$-modules, where $p_{\mathcal{N}}: \mathcal{N}\times\mathbb{P}^1 \to \mathcal{N}$ is the projection, and under this isomorphism σ_h is the pull-back of σ_0.

Using that σ_h is a cosection of $H^1(c_0(\tilde{h}))$, we form the kernel cone-stack

$$h^1/h^0(c_0(\tilde{h}))_{\sigma_h} \subset h^1/h^0(c_0(\tilde{h}))$$

as defined in Definition 3.7.

Lemma 5.5. The cycle $[\mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}}]$ in (5.11) is a cycle in $Z_* h^1/h^0(c_0(\tilde{h}))_{\sigma_h}$. □

Proof. Applying Lemma A.3 to the n' in (5.9), and to the restriction $n|_{\mathcal{U}}$ in (5.7), we have $h^1/h^0(c_0(\tilde{n}))|_{\mathcal{U}} \cong h^1/h^0(c_0(\tilde{n}'))$. Let

$$Z_*(h^1/h^0(c_0(\tilde{n}'))) \longrightarrow Z_*(h^1/h^0(c_0(\tilde{h}))|_{\mathcal{U}}) \tag{5.17}$$

be the composite of $Z_*(h^1/h^0(c_0(\tilde{n}'))) \to Z_*(h^1/h^0(c_0(\tilde{h}_\chi)))$ induced by (5.9) with the tautological embedding $Z_*(h^1/h^0(c_0(\tilde{h}_\chi))) \to Z_*(h^1/h^0(c_0(\tilde{h}))|_{\mathcal{U}})$ induced by the lower two rows of (5.8).

By Lemma 5.4, the morphism

$$Z_*(h^1/h^0(c_0(\tilde{n}))|_{\mathcal{U}}) \longrightarrow Z_*(h^1/h^0(c_0(\tilde{h}))|_{\mathcal{U}})$$

induced by (5.7), factors through (5.17). (The induced homomorphism between the cycle groups $Z_*h^1/h^0(c_0(\tilde{g}))$ and $Z_*h^1/h^0(c_0(\tilde{k}))$ does not depend on the choice of morphisms between mapping cones $c_0(\tilde{g}) \to c_0(\tilde{k})$.) Since we have

$$h^1/h^0(c_0(\tilde{h}_\chi)) \cong h^1/h^0(c_o(\tilde{h}))_{\sigma_h}|_{\mathcal{U}},$$

we conclude that $[\mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}}|_{\mathcal{U}}]$ lies inside $(h^1/h^0(c_o(\tilde{h}))_{\sigma_h})|_{\mathcal{U}}$. This implies that $[\mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}}] \in Z_*h^1/h^0(c_0(\tilde{h}))_{\sigma_h}$. ■

We now quote the virtual pull-back construction of Manolache in [22, Section 2.1, Construction 1]. First the compatibility diagram (5.7) fits into the condition 2 in the construction of Manolache [22]. Let $\mathbf{C}_{\mathcal{N}/\mathcal{Q}}$ be the intrinsic normal cone of $\mathcal{N}$ relative to $\mathcal{Q}$ and let

$$i : \mathbf{C}_{\mathcal{N}/\mathcal{Q}} \to h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})$$

be the inclusion by the relative perfect obstruction theory $\phi_{\mathcal{N}/\mathcal{Q}}$. Using the cosection σ_0' (cf. (5.12)) and following Definition 3.7, we form the kernel cone-stack

$$h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{\sigma_0'} \subset h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}).$$

It is direct to check that in diagram (5.3) the cosection σ_0' is the composition of a cosection $\sigma_{\mathfrak{C}}$ of $H^1(\mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g})$ with $H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \cong \gamma^* H^1(\mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g})$. By the same argument as in the proof of Proposition 3.8, the cycle $[\mathbf{C}_{\mathfrak{C}/\mathfrak{D}_g}]$ lies inside $h^1/h^0(\mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g})_{\sigma_{\mathfrak{C}}}$ (cf. [15, Proposition 4.3]). Since the diagram (5.3) implies that the support of $\mathbf{C}_{\mathcal{N}/\mathcal{Q}}$ lies inside the support of $\gamma^*\mathbf{C}_{\mathfrak{C}/\mathfrak{D}_g}$, we have

$$i_*[\mathbf{C}_{\mathcal{N}/\mathcal{Q}}] \in Z_* h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{\sigma_0'}. \tag{5.18}$$

In the following, we define $G = h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{\sigma_0'}$. Generalizing the intrinsic pull-back of [22], we construct a virtual pull-back morphism of cosection localized classes

$$\upsilon_G^! : A_*\mathcal{Q} \longrightarrow A_*\mathcal{Q}$$

defined as follows. We let ζ be the map that sends a cycle $\sum n_i[V_i] \in Z_*\mathcal{Q}$ to the cycle class $\sum n_i[\mathbf{C}_{V_i\times_{\mathcal{Q}}\mathcal{N}/V_i}]$. It preserves rational equivalence, and defines a map $\zeta : A_*\mathcal{Q} \to A_*\mathbf{C}_{\mathcal{N}/\mathcal{Q}}$. Because of (5.18), the push-forward i_* factors through $A_*G \subset h^1/h^0((\mathbb{E}_{\mathcal{N}/\mathcal{Q}})$. We then let $0^!_{\sigma_0',\mathrm{loc}}$ be the localized Gysin map defined in [15]. The $\upsilon_G^!$ is the composite

$$\upsilon_G^! : A_*\mathcal{Q} \xrightarrow{\zeta} A_*\mathbf{C}_{\mathcal{N}/\mathcal{Q}} \xrightarrow{i_*} A_*G' \xrightarrow{0^!_{\sigma_0',\mathrm{loc}}} A_*\mathcal{Q}. \tag{5.19}$$

We prove an analog result to [22, Theorem 4 and Corollary 4].

Theorem 5.6. We have $\upsilon_{G'}^!([\mathcal{Q}]^{\mathrm{vir}}) = [\overline{M}_g(N,d)^p]^{\mathrm{vir}}_{\sigma_0} \in A_0\mathcal{Q}$. □

For convenience, we define $\mathbf{C} := \mathbf{C}_{\mathcal{N}\times\mathbb{P}^1/M^0_{\mathcal{Q}/\mathfrak{D}_g}}$. The proof of [22, Theorem 4] can be applied here without change except that we need to use the cosection localized Gysin map $0^!_{\sigma_h}$, applied to the cone cycle $[\mathbf{C}]$. In order to prove that the identities in [22] hold after replacing the ordinary Gysin map by the localized Gysin map, we need to check that the rational equivalences used all lie in the kernel of the (family) cosection σ_h over $\mathcal{N}\times\mathbb{P}^1$. For $x\in\mathbb{P}^1$ we define $\mathbf{C}_x = \mathbf{C}\times_{\mathbb{P}^1} x$.

Proof. Lemma 5.5 shows the first rational equivalence $\mathbf{C}$ used in the proof of [22, Theorem 4] lies in the kernel of σ_h. We analyze other rational equivalences now.

Let $p_{\mathbb{P}^1} : \mathcal{N}\times\mathbb{P}^1 \to \mathbb{P}^1$ be the projection. We first extend σ_h to

$$\tilde{\sigma}_h : H^1(c_0(\tilde{h})) \oplus p^*_{\mathbb{P}^1} T_{\mathbb{P}^1} \longrightarrow \mathscr{O}_{\mathcal{N}\times\mathbb{P}^1}$$

by setting $\tilde{\sigma}_h|_{p^*_{\mathbb{P}^1}T_{\mathbb{P}^1}} = 0$. The degeneracy locus $D(\tilde{\sigma}_h) = \mathcal{Q}\times\mathbb{P}^1$. Restricting $\tilde{\sigma}_h$ to $\mathcal{N}\times x \subset \mathcal{N}\times\mathbb{P}^1$, we obtain

$$\tilde{\sigma}_h|_x : H^1(c_0(\tilde{h})|_{\mathcal{N}\times x}) \oplus L_x \longrightarrow \mathscr{O}_{\mathcal{N}\times x},$$

where $L_x = p^*_{\mathbb{P}^1} T_{\mathbb{P}^1}|_{\mathcal{N}\times x}$. Over $a = [0,1]$, $\tilde{\sigma}_h|_a$ is the extension of σ_0 to

$$\tilde{\sigma}_0 := \tilde{\sigma}_h|_a : \upsilon^*\mathcal{O}b_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathcal{O}b_{\mathcal{N}/\mathcal{Q}} \oplus L_a \longrightarrow \mathscr{O}_{\mathcal{N}}$$

by setting $\tilde{\sigma}_0|_{L_a} = 0$. The kernel cone-stack of $\tilde{\sigma}_0$ (as defined in (3.15)) takes the form (cf. (5.13) and (5.14))

$$h^1/h^0(c_0(\tilde{h})|_{\mathcal{N}\times a} \oplus L_a[-1])_{\tilde{\sigma}_0} = \upsilon^* h^1/h^0(\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}) \times_{\mathcal{N}} h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{\sigma_0'} \times_{\mathcal{N}} L_a.$$

For convenience, in the following we call a cycle in $h^1/h^0(c_0(\tilde{h})|_{\mathcal{N}\times x} \oplus L_x[-1])$ annihilated by $\tilde{\sigma}_h|_x$ if it lies in $h^1/h^0(c_0(\tilde{h})|_{\mathcal{N}\times x} \oplus L_x[-1])_{\tilde{\sigma}_h|_x}$. Over $\mathcal{N}\times x$, the normal cone $\mathbf{C}_{\mathbf{C}_x/\mathbf{C}} \subset \mathbf{C}_x \times_{\mathcal{N}} L_x$ to $\mathbf{C}_x := \mathbf{C}\times_{\mathcal{N}\times\mathbb{P}^1} (\mathcal{N}\times x)$ in $\mathbf{C}$) is annihilated by $\tilde{\sigma}_h|_x$; applying the localized Gysin map

$$0^!_{\tilde{\sigma}_h|_x,\mathrm{loc}} : Z_* h^1/h^0(c_0(\tilde{h})|_{\mathcal{N}\times x} \oplus L_x[-1]) \longrightarrow A_*\mathcal{Q},$$

we obtain the cosection localized class

$$0^!_{\tilde{\sigma}_h|_x,\mathrm{loc}}([\mathbf{C}_{\mathbf{C}_x/\mathbf{C}}]) \in A_*(\mathcal{Q}). \tag{5.20}$$

For $x \neq a \in \mathbb{P}^1$, $\mathbf{C}$ is flat over $\mathbb{P}^1$ near $x\in\mathbb{P}^1$, $\mathbf{C}_{\mathbf{C}_x/\mathbf{C}} = \mathbf{C}_x \times_{\mathcal{N}} L_x$, which implies that the class (5.20) is $[\overline{M}_g(N,d)^p)]^{\mathrm{vir}}_{\sigma_0}$. For $x = a$ the class (5.20) is $0^!_{\tilde{\sigma}_0,\mathrm{loc}}([\mathbf{C}_{\mathbf{C}_a/\mathbf{C}}])$. Applying

[15, Theorem 4.5], the classes (5.20) are independent of $x \in \mathbb{P}^1$. Hence we have

$$[\overline{M}_g(N,d)^p)]^{\mathrm{vir}}_{\sigma_0} = 0^!_{\tilde{\sigma}_h|_x,\mathrm{loc}}([\mathbf{C}_{\mathbf{C}_x/\mathbf{C}}]) = 0^!_{\tilde{\sigma}_0,\mathrm{loc}}([\mathbf{C}_{\mathbf{C}_a/\mathbf{C}}]) \in A_*(\mathcal{Q}).$$

By [15, 18] we have a rational equivalence $R \in W_*(\mathbf{C}_a \times_{\mathcal{N}} L_a)$ that gives

$$\partial R = [\mathbf{C}_{\mathbf{C}_a/\mathbf{C}}] - [\mathbf{C}_{\mathcal{N}/\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}} \times_{\mathcal{N}} L_a] \in Z_*(\mathbf{C}_a \times_{\mathcal{N}} L_a). \tag{5.21}$$

Lemma 5.5 implies that $\mathbf{C}_a$ is annihilated by σ_0. Hence $\mathbf{C}_a \times_{\mathcal{N}} L_a$ is annihilated by $\tilde{\sigma}_0$, which implies that

$$R \in W_* h^1/h^0(c_0(\tilde{h})|_{\mathcal{N}\times a} \oplus L_a[-1])_{\tilde{\sigma}_0}. \tag{5.22}$$

Since the localized Gysin map preserves rational equivalences in (5.22), (5.21) implies

$$0^!_{\tilde{\sigma}_0,\mathrm{loc}}([\mathbf{C}_{\mathbf{C}_a/\mathbf{C}}]) = 0^!_{\tilde{\sigma}_0,\mathrm{loc}}([\mathbf{C}_{\mathcal{N}/\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}} \times_{\mathcal{N}} L_a]) = 0^!_{\sigma_0,\mathrm{loc}}([\mathbf{C}_{\mathcal{N}/\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}}]).$$

It remains to show that

$$0^!_{\sigma_0,\mathrm{loc}}([\mathbf{C}_{\mathcal{N}/\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}}]) = \upsilon^!_{G'}([\mathcal{Q}]^{\mathrm{vir}}). \tag{5.23}$$

For brevity we define $K = h^1/h^0(\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g})$. We let $s: \mathcal{Q} \to K$ be the zero section. We have morphisms

$$\mathcal{N} \xrightarrow{\upsilon} \mathcal{Q} \xrightarrow{s} K, \tag{5.24}$$

and, by $\mathbf{C}_{\mathcal{Q}/K} \cong K$, we obtain a canonical embedding

$$i'': \mathbf{C}_{\mathcal{N}/K} \subset \mathbf{C}_{\mathcal{N}/\mathcal{Q}} \times_{\mathcal{N}} \upsilon^* K \subset h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \times_{\mathcal{Q}} K \tag{5.25}$$

(denoted by i'').

We define (cf. 5.14)

$$G'' := h^1/h^0(\upsilon^*\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \oplus \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})_{\sigma_0''}.$$

By (5.18) we have

$$(i'')_*[\mathbf{C}_{\mathcal{N}/K}] \in Z_* G''.$$

Mimicking the definition of $\upsilon^!_G$, we define

$$(s\circ\upsilon)^!_{G''}: A_*K \longrightarrow A_*\mathcal{Q}$$

to be the composite

$$A_*H_{\mathcal{Q}} \xrightarrow{\zeta''} A_*\mathbf{C}_{\mathcal{N}/H_{\mathcal{Q}}} \xrightarrow{i''_*} A_*G'' \xrightarrow{0^!_{\sigma_0,\mathrm{loc}}} A_*\mathcal{Q}, \tag{5.26}$$

where $\mathcal{N}\to K$ is via (5.24); $0^!_{\sigma_0,\mathrm{loc}}$ is the localized Gysin map; and ζ'' sends cycles $\sum n_i[V_i]$ to the cycle classes $\sum n_i[\mathbf{C}_{V_i\times_K\mathcal{N}/V_i}]$. In this way, the left-hand side of (5.23) is

$$0^!_{\sigma_0,\mathrm{loc}}([\mathbf{C}_{\mathcal{N}/\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}}]) = (s\circ\upsilon)^!_{G''}(\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}).$$

Since $[\mathcal{Q}]^{\mathrm{vir}}$ is zero-dimensional, we can write $[\mathcal{Q}]^{\mathrm{vir}} = \sum r_i[p_i] \in A_0(\mathcal{Q})$, $r_i\in\mathbb{Q}$, and p_i are closed points in $h^1/h^0(\mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g})$. Let $m: K\to\mathcal{Q}$ be the projection to the base, and denote by $m^{-1}p_i = K\times_{\mathcal{Q}} p_i$. Since $[\mathcal{Q}]^{\mathrm{vir}} = 0^![\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}]$, we have a rational equivalence $R''\in W_*(K)$ such that

$$\partial R'' = [\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}] - \sum r_i[m^{-1}p_i] \in Z_*(K). \tag{5.27}$$

Hence

$$\begin{aligned}(s\circ\upsilon)^!_{G''}([\mathbf{C}_{\mathcal{Q}/\mathfrak{D}_g}]) &= (s\circ\upsilon)^!_{G''}\left(\sum r_i[m^{-1}p_i]\right) = \sum r_i 0^!_{\sigma_0,\mathrm{loc}}([\mathbf{C}_{m^{-1}p_i\times_K\mathcal{N}/m^{-1}p_i}])\\ &= 0^!_{\sigma'_0,\mathrm{loc}}\left(\sum r_i[\mathbf{C}_{p_i\times_{\mathcal{Q}}\mathcal{N}/p_i}]\right) = \nu^!_G([\mathcal{Q}]^{\mathrm{vir}}),\end{aligned}$$

where $\mathcal{N}\to K$ is via (5.24), and the third identity follows from

$$\mathbf{C}_{m^{-1}p_i\times_K\mathcal{N}/m^{-1}p_i} = \mathbf{C}_{p_i\times_{\mathcal{Q}}\mathcal{N}/p_i}\times m^{-1}p_i \subset \mathbf{C}_{\mathcal{N}/\mathfrak{D}_g}\times_{\mathcal{N}}\upsilon^*K.$$

This proves the theorem. ∎

Remark: Our proof can be generalized to show

$$(s\circ\upsilon)^!_{G''}(\alpha) = \upsilon^!_G\circ s^!(\alpha), \quad \alpha\in A_*H_{\mathcal{Q}}, \tag{5.28}$$

where $s^!: A_*(K)\to A_*(\mathcal{Q})$ is the shriek operation by the regular embedding s in (5.24). The above formula (5.28) is the analog of [22, Theorem 4] for cosection localized virtual pull-back.

Theorem 5.7. We have

$$N_g(d)^p_{\mathbb{P}^4} = N_g(d)_{K_{N_Q}} = (-1)^{5d+1-g} \cdot N_g(d)_Q. \qquad \square$$

Proof. We first compute the degree of the zero cycle $\upsilon^!_G([\xi]) \in A_0\mathcal{Q}$, where ξ is any closed point in $\mathcal{Q}$. Let ξ be $[u, C] \in \mathcal{Q}$. We define

$$V_1 = H^0(C, u^*\mathscr{O}(5)), \quad V_2 = H^0(C, u^*\mathscr{O}(-5) \otimes \omega_C), \quad V = V_1 \oplus V_2.$$

Note that when ξ varies, the dimension of V_i, V also varies. The fiber of $\upsilon : \mathcal{N} \to \mathcal{Q}$ can be described as

$$\upsilon^{-1}\xi := \mathcal{N} \times_{\mathcal{Q}} \xi \cong V, \quad G|_{\upsilon^{-1}\xi} \cong [V \times V^\vee / V],$$

where the action of V on $V \times V^\vee$ is via the zero homomorphism $0 : V \to V \times V^\vee$. One also checks that the cosection σ'_0 restricted to $\upsilon^{-1}\xi$ is induced by

$$\sigma_\xi : V \times V^\vee = (V_1 \oplus V_2) \times (V_1^\vee \oplus V_2^\vee) \longrightarrow \mathbb{C},$$

given by dual parings $V_i \times V_i^\vee \to \mathbb{C}$.

Applying the composition (5.26) step by step, from

$$\zeta([\xi]) = [\mathbf{C}_{V/\xi}] \in A_*(G'|_\xi),$$

we have

$$\upsilon^!_G([\xi]) = 0^!_{\sigma_\xi,\mathrm{loc}}(\mathbf{C}_{V/\xi}) = (-1)^{\mathrm{rank} V}[\xi] = (-1)^{5d+1-g}[\xi] \in A_0\mathcal{Q}.$$

Here the second equality follows from

$$\mathbf{C}_{V/\xi} = [V \times 0/V] \subset [V \times V^\vee / V] = G|_{\upsilon^{-1}\xi},$$

and [15, Example 2.9]. The third equality follows from

$$\mathrm{rank} V \equiv \chi(u^*\mathscr{O}(5)) + 2h^1(u^*\mathscr{O}(5)) \equiv 5d + 1 - g \mod 2.$$

Taking degrees,

$$\deg\ \upsilon^!_G([\xi]) = (-1)^{5d+1-g}.$$

Since both $[Q]^{\mathrm{vir}}$ and $[\overline{M}_g(N,d)^p]^{\mathrm{vir}}_{\sigma_0}$ in Lemma 5.6 are zero-dimensional, taking degrees, we obtain

$$\deg\,[\overline{M}_g(N,d)^p]^{\mathrm{vir}}_{\sigma_0} = \deg\ \upsilon^!_G([\xi])\cdot \deg\,[Q]^{\mathrm{vir}} = (-1)^{5d+1-g}N_g(d)_Q.$$

This proves the second identity in the statement of the theorem. The first identity is Theorem 5.3. ■

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Appendix A

We recall some useful facts known to the experts.

A.1 Kresch–Kim–Pantev's construction

Let S be a stack.

Convention. For a complex (derived object) $\mathbb{G}$ on S, we define $\mathbb{G}(k)$ without further commenting to be

$$\mathbb{G}(k) := p_S^*\mathbb{G}\otimes p_{\mathbb{P}^1}^*\mathscr{O}(k);$$

further, whenever we see a complex over S appearing in a sequence involving complexes over $S\times\mathbb{P}^1$, we understand the complex as its pull-back from S.

Definition A.1. Let $\mathbb{E}_1 \xrightarrow{b} \mathbb{E}_2 \longrightarrow \mathbb{E}_3 \xrightarrow{+1}$ be a distinguished triangle of objects in $\mathbf{D}(S)$ with cohomologies concentrated at nonpositive degrees. Assume that $\mathbb{E}_1$ is of amplitude in $[-1, \infty]$. Let $[x, y]$ be the homogeneous coordinates of $\mathbb{P}^1$, and let

$$\bar{b} : \mathbb{E}_1(-1) \to \mathbb{E}_1 \oplus \mathbb{E}_2$$

be defined by $(x \cdot 1, y \cdot b)$. We form the mapping cone $c(\bar{b})$ of $\bar{b}$, which fits into the distinguished triangle

$$\mathbb{E}_1(-1) \xrightarrow{\bar{b}} \mathbb{E}_1 \oplus \mathbb{E}_2 \longrightarrow c(\bar{b}) \xrightarrow{+1}.$$

Applying the h^1/h^0 construction to $c(\bar{b})^\vee$, we obtain $h^1/h^0(c(\bar{b})^\vee)$, which is a cone-stack over $S \times \mathbb{P}^1$ [1]. Following [16], we call it the deformation of $h^1/h^0(\mathbb{E}_2^\vee)$ to $h^1/h^0(\mathbb{E}_1^\vee) \times_S h^1/h^0(\mathbb{E}_3^\vee)$. □

Let $i : X \to Y$ and $j : Y \to Z$ be morphisms of relative Deligne–Mumford type, between stacks. Let

$$i^*\mathbb{L}_{Y/Z} \xrightarrow{\beta} \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \xrightarrow{+1} \qquad \text{(A.1)}$$

be the induced distinguished triangle of cotangent complexes. We quote the main theorem of [16].

Proposition A.2. [16] We have a natural isomorphism

$$N_{X\times\mathbb{P}^1/M^\circ_{Y/Z}} \cong h^1/h^0(c(\bar{\beta})^\vee). \qquad \square$$

Now we state a truncated version which is dual to Definition A.1.

Lemma A.3. Let

$$\mathbb{T}^{\leq 1}_{X/Y} \longrightarrow \mathbb{T}^{\leq 1}_{X/Z} \xrightarrow{k} i^*\mathbb{T}^{\leq 1}_{Y/Z}$$

be the truncation by $\tau_{\leq 1}$ of the dual of the distinguished triangle (A.1). (It is not a distinguished triangle.) Let $c_0(\tilde{k})$ be defined by making

$$c_0(\tilde{k}) \longrightarrow i^*\mathbb{T}^{\leq 1}_{Y/Z} \oplus \mathbb{T}^{\leq 1}_{X/Z} \xrightarrow{\tilde{k}} i^*\mathbb{T}^{\leq 1}_{Y/Z} \otimes \mathscr{O}_{\mathbb{P}^1}(1)$$

a distinguished triangle, where $\tilde{k} = (x, y \cdot k)$ as in Definition A.1. Then there is a natural isomorphism

$$h^1/h^0(c(\bar{\beta})^\vee) \cong h^1/h^0(c_0(\tilde{k})). \qquad \square$$

Proof. Using the simplicial resolution of Illusie, we can represent $\imath^*\mathbb{L}_{Y/Z}$ and $\mathbb{L}_{X/Z}$ by perfect complex (over X globally) of amplitude $[-\infty, 0]$ such that $\beta : i^*\mathbb{L}_{Y/Z} \to \mathbb{L}_{X/Z}$ is given by a homomorphism between these two complexes. From this it is direct to show that the canonical morphism

$$c_0(\tilde{k}) \longrightarrow c(\bar{\beta})^\vee \tag{A.2}$$

induces isomorphisms on H^1 and H^0 of the two complexes in (A.2). Hence their truncations by $\tau_{\leq 1}$ are isomorphic under this arrow, which shows that the cone-stacks of the h^1/h^0 constructions of the two complexes in (A.2) are isomorphic under the arrow induced by (A.2). ■

A.2 Application

We recall the rational equivalence inside the deformations of ambient cone-stacks constructed by Kim et al. [16].

Let Z be an Artin stack, locally of finite type and of pure dimension. Let Y be a stack and $Y \to Z$ be a morphism of relative Deligne–Mumford type in the derived category of coherent sheaves on X. Let $\mathbb{E}^\vee$ (respectively $\mathbb{F}^\vee$, $\mathbb{V}^\vee$) be a perfect relative obstruction theory of X/Z (respectively Y/Z, X/Y).

Definition A.4. We say $\mathbb{F}$ and $\mathbb{E}$ are *truncated-compatible* (verses (V, s)) if there exists a commutative diagram

$$\begin{tikzcd} \mathbb{V} \arrow[r] & \mathbb{E} \arrow[r, "g"] & \mathbb{F}|_X \arrow[r, "+1"] & {} \\ \mathbb{T}^{\leq 1}_{X/Y} \arrow[r] \arrow[u] & \mathbb{T}^{\leq 1}_{X/Z} \arrow[r, "k"] \arrow[u] & \mathbb{T}_{Y/Z}|^{\leq -1}_X \arrow[u] \end{tikzcd} \tag{A.3}$$

such that its top row is a distinguished triangle, and its bottom row is the first line in Lemma A.3. □

Accordingly, the morphisms g and k in (A.3) induce homomorphisms $\tilde{g}$ and $\tilde{k}$, respectively, that fit into a homomorphism of distinguished triangle's

$$\begin{tikzcd} c_0(\tilde{g}) \arrow[r] & \mathbb{F}|_X \oplus \mathbb{E} \arrow[r, "\tilde{g}"] & \mathbb{F}|_X(1) \arrow[r, "+1"] & {} \\ c_0(\tilde{k}) \arrow[r] \arrow[u] & \mathbb{T}^{\leq 1}_{Y/Z}|_X \oplus \mathbb{T}^{\leq 1}_{X/Z} \arrow[r, "\tilde{k}"] \arrow[u] & \mathbb{T}^{\leq 1}_{Y/Z}|_X \otimes \mathscr{O}_{\mathbb{P}^1}(1) \arrow[r, "+1"] \arrow[u] & {} \end{tikzcd}$$

where $c_0(\tilde{g})$ is to make the first row a distinguished triangle as $c_0(\tilde{k})$ did in Lemma A.3. We let $M^0_{Y/Z}$ be the deformation of Z to the normal cone $\mathbf{C}_{Y/Z}$, let $\mathbf{C}_{X\times\mathbb{P}^1/M^0_{Y/Z}}$ be the normal cone to $X\times\mathbb{P}^1$ in $M^0_{Y/Z}$, and let $N_{X\times\mathbb{P}^1/M^0_{Y/Z}}$ be the normal sheaf of $X\times\mathbb{P}^1$ in $M^0_{Y/Z}$. By the functoriality of the h^1/h^0 construction, we have

$$\mathcal{D} := \mathbf{C}_{X\times\mathbb{P}^1/M^0_{Y/Z}} \subset N_{X\times\mathbb{P}^1/M^0_{Y/Z}} \cong h^1/h^0(c_0(\tilde{k})), \tag{A.4}$$

where the isomorphism is proved in [16] and Lemma A.3. We also have the inclusion

$$h^1/h^0(c_0(\tilde{k})) \subset h^1/h^0(c_0(\tilde{g})) \cong h^1/h^0(c(\overline{g^\vee})^\vee), \tag{A.5}$$

where $h^1/h^0(c(\overline{g^\vee})^\vee)$ is the deformation of $h^1/h^0(\mathbb{E})$ to $h^1/h^0(\mathbb{F}|_X)\times_X h^1/h^0(\mathbb{V})$ as in Definition A.1. This shows that the truncated compatibility (A.3) is sufficient to apply the Kresch–Kim–Pantev construction of rational equivalence.

A.3 Obstruction class assignments

Assume that there is a smooth morphism of Artin stacks $H\to W$. Suppose $T\subset T'$ is a pair of affine schemes such that $J := I_{T/T'}$ and $J^2=0$. Fix a morphism $T'\to\mathfrak{D}_g$, which pulls back $\pi_{\mathfrak{D}_g}:\mathcal{C}_{\mathfrak{D}_g}\to\mathfrak{D}_g$ to $\pi_T:\mathcal{C}_T\to T$ and $\pi_{T'}:\mathcal{C}_{T'}\to T'$. Assume that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_T & \xrightarrow{\mathfrak{e}} & H \\ \downarrow & & \downarrow \\ \mathcal{C}_{T'} & \longrightarrow & W \end{array} \tag{A.6}$$

Since the ideal sheaf of $\mathcal{C}_T\subset\mathcal{C}_{T'}$ is $\pi_{T'}^*J$, it is a square zero extension. We define $V_T := \mathfrak{e}^*\Omega^\vee_{H/W}$; then V_T is a locally free sheaf over $\mathcal{C}_T$. The diagram (A.6) provides a morphism

$$V_T^\vee \cong \mathfrak{e}^*\mathbb{L}_{H/W} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_T^*\mathbb{L}_{T/T'} \longrightarrow \mathbb{L}^{\geq -1}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_T^*J[1], \tag{A.7}$$

(here $\mathfrak{e}^*$ denotes a derived pull-back) which defines an element

$$\omega(\mathfrak{e}, H, W) \in \operatorname{Ext}^1_{\mathcal{C}_T}(V_T^\vee, \pi_T^*J) \cong H^1(\mathcal{C}_T, V_T\otimes\pi_T^*J).$$

Lemma A.5. $\omega(\mathfrak{e}, H, W)=0$ if and only if the diagram (A.6) admits a lifting $\mathcal{C}_{T'}\to H$ that commutes with the diagram. □

Proof. We form the diagram

$$\begin{tikzcd} X_0 := \mathcal{C}_T \arrow[r, "\mathfrak{i}"] \arrow[d, "\bar{\mathfrak{e}}"] & X := \mathcal{C}_{T'} \\ Y_0 := H \times_W \mathcal{C}_T \arrow[r, "\mathfrak{j}"] \arrow[d, "\Delta"] & Y := H \times_W \mathcal{C}_{T'} \arrow[d] \\ \mathcal{C}_T \arrow[r, "\subset"] & S := \mathcal{C}_{T'} \end{tikzcd} \tag{A.8}$$

where $\mathfrak{i}$ and $\mathfrak{j}$ are extensions over S. By construction the associated homomorphism of sheaves

$$v : \bar{\mathfrak{e}}^* I_{Y_0/Y} \longrightarrow I_{X_0/X} = \pi_T^* J$$

is an isomorphism. If $\bar{\mathfrak{e}}$ lifts to a $\mathcal{C}_{T'}$-morphism $\bar{\mathfrak{e}}' : X \to Y$, then the $X \to \mathcal{C}_{T'}$ is an isomorphism.

Following the steps in the proof of [14, Theorem 2.1.7], the obstruction to the existence of such $\bar{\mathfrak{e}}'$ (in the notation of [14]) are constructed as follows. First one has a sequence of cotangent complexes

$$\mathbb{L}_{X_0/Y_0}[-1] \longrightarrow \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/S} \longrightarrow \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/Y} \longrightarrow \bar{\mathfrak{e}}^*\mathbb{L}^{\geq -1}_{Y_0/Y} \longrightarrow \pi_T^* J[1], \tag{A.9}$$

where the first (left) morphism comes from the triple $X_0 \to Y_0 \to S$; the middle morphism is induced by $\mathbb{L}_{Y_0/S} \to \mathbb{L}_{Y_0/Y}$.

Using $\mathbb{L}_{X_0/Y_0} = V_T^\vee[1]$, this sequence associates an element

$$\omega(\bar{\mathfrak{e}}, \mathfrak{j}) \in \mathrm{Ext}^2_{X_0}(\mathbb{L}_{X_0/Y_0}, \pi_T^* J) = \mathrm{Ext}^1_{X_0}(V_T^\vee, \pi_T^* J) = H^1(\mathcal{C}_T, V_T \otimes \pi_T^* J).$$

The argument in [14, Theorem 2.1.7] shows that $\omega(\bar{\mathfrak{e}}, \mathfrak{j}) = 0$ if and only if a lift $\bar{\mathfrak{e}}' : X \to Y$ exists in the diagram (A.8). Such a lift exists if and only if a lift $\mathfrak{e}' : \mathcal{C}_{T'} \to H$ exists in the diagram (A.6). Hence we only need to verify that $\omega(\bar{\mathfrak{e}}, \mathfrak{j}) = \omega(\mathfrak{e}, H, W)$.

To this end, we verify the commutativity of the following diagram:

$$\begin{tikzcd} \mathbb{L}_{X_0/Y_0}[-1] \arrow[r] \arrow[d, "\cong"] & \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/S} \arrow[r] & \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/Y} \\ \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/X_0} & \bar{\mathfrak{e}}^*\mathfrak{j}^*\mathbb{L}_{Y/S} \arrow[l, "\cong"'] \arrow[u] \arrow[r] & \mathbb{L}_{X_0/S} \arrow[u, "\cong"'] \end{tikzcd} \tag{A.10}$$

where the first vertical arrow is an isomorphism because $\mathbb{L}_{X_0} = 0$; the left square is commutative because the canonical $\mathbb{L}_{Y_0/S} \to \mathbb{L}_{Y_0/X_0}$ induces a $\bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/S} \to \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/X_0}$ that splits the left square into two commutative triangles of cotangent complexes; the third vertical arrow is composing $\mathbb{L}_{X_0/S} \cong \bar{\mathfrak{e}}^*\Delta^*\mathbb{L}_{X_0/S}$ with the isomorphism $\Delta^*\mathbb{L}_{X_0/S} \xrightarrow{\cong} \mathbb{L}_{Y_0/Y}$. The right square is commutative because one has a canonical pull-back $\bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/S} \to \mathbb{L}_{X_0/S}$ and a commutative diagram

$$\begin{tikzcd} \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/S} \arrow[r] \arrow[d] & \bar{\mathfrak{e}}^*\mathbb{L}_{Y_0/Y} \\ \mathbb{L}_{X_0/S} & \bar{\mathfrak{e}}^*\Delta^*\mathbb{L}_{X_0/S} \arrow[l, "\cong"'] \arrow[u, "\cong"'] \end{tikzcd}$$

The upper and lower rows of the diagram (A.10) are, respectively, sequence (A.7) and (A.9). Thus the commutative diagram (A.10) implies $\omega(\bar{\mathfrak{e}}, \mathfrak{j}) = \omega(\mathfrak{e}, H, W)$. ■

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