A VANISHING ASSOCIATED WITH IRREGULAR MSP FIELDS

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ABSTRACT. In [CL^{3}1] and [CL^{3}2], the notion of Mixed-Spin-P fields is introduced and their moduli space $\mathcal{W}_{g,\gamma,\mathbf{d}}$ is constructed together with a $\mathbb{C}^*$ action. By applying virtual localization to their virtual classes $[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}$, polynomial relations among Gromov Witten(GW) and Fan-Jarvis-Ruan-Witten(FJRW) invariants of Fermat quintics are derived. In this paper, we prove a vanishing of a class of terms in $[(\mathcal{W}_{g,\gamma,\mathbf{d}})^{\mathbb{C}^*}]^{\mathrm{vir}}$. This vanishing plays a key role in [CL^{3}2]'s proof that in Witten's gauged linear sigma model for Fermat quintics, the FJRW invariants with insertions 2/5 determine the GW invariants of quintic Calabi-Yau threefolds through CY-LG phase transitions.

1. Introduction

In [CL^{3}1], the authors introduced the notion of Mixed-Spin-P fields (abbr. MSP fields), and constructed the properly supported $\mathbb{G}_m$-equivariant virtual cycles of their moduli spaces. Applying virtual localization [GP], they obtained a vanishing of the sum of the contributions indexed by the set of localization graphs.

The set of localization graphs are divided into regular and irregular graphs. The contributions of *regular* graphs are polynomials of the GW invariants of the quintic CY threefolds, and of the FJRW invariants of the Fermat quintics. Provided that the contributions from the *irregular* graphs all vanish, the sum of these polynomials (of the GW and FJRW invariants mentioned) indexed by the regular graphs will provide relations ([CL^{3}2]) that determine, up to finite ambiguity, the full GW (resp. FJRW) invariants of the quintic threefolds (resp. the quintic singularities) via recursions on genus and degree.

Later in [NMSP1, NMSP2, NMSP3], built up on [CL^{3}1, CL^{3}2] and this irregular vanishing property, it is proved that the GW potential F_g of quintics satisfies (i) the finite generation conjecture of Yamaguchi and Yau [YY], (ii) the Yamaguchi-Yau (functional) equations, (iii) convergence with a positive radius, and (iv) BCOV's Feynman rules [BCOV] which determine every F_g explicitly and recursively, based on $3g-3$ initial data.

This paper sets out to prove such an "irregular vanishing" (Theorem 1.2).

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Recall that an MSP field is a collection of objects

$$\xi=(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,\nu_1,\nu_2), \tag{1.1}$$

consisting of a pointed twisted curve $\Sigma^{\mathcal{C}}\subset\mathcal{C}$, invertible sheaves $\mathcal{L}$ and $\mathcal{N}$, and a collection of fields $(\varphi,\rho,\nu_1,\nu_2)$ (cf. Definition 2.1). The MSP field ξ comes with numerical invariants: the genus g of $\mathcal{C}$, the monodromy γ_i of $\mathcal{L}$ at the i-th marking $\Sigma^{\mathcal{C}}_i$, and the bi-degrees $d_0=\deg\mathcal{L}\otimes\mathcal{N}$ and $d_\infty=\deg\mathcal{N}$.

Given g, $\gamma=(\gamma_1,\cdots,\gamma_\ell)$ and $\mathbf{d}=(d_0,d_\infty)$, we let $\mathcal{W}$ be the moduli of stable MSP fields of numerical data $(g,\gamma,\mathbf{d})$. $\mathcal{W}$ is a separated DM stack, locally of finite type. (The data $(g,\gamma,\mathbf{d})$ will be fixed throughout this paper.)

As shown in [CL^{3}1, CL^{3}2], $\mathcal{W}$ is a $T=\mathbb{G}_m$ DM stack (cf. (2.3)), admits a T-equivariant perfect obstruction theory and an invariant cosection $\sigma_{\mathcal{W}}:\mathcal{O}b_{\mathcal{W}}\to\mathcal{O}_{\mathcal{W}}$, giving rise to a cosection-localized virtual cycle [KL]

$$[\mathcal{W}]^{\mathrm{vir}}_{\mathrm{loc}}\in A^T_*\mathcal{W}^-,$$

where $\mathcal{W}^-$ is the vanishing locus of σ. In [CL^{3}1]), it is proved that $\mathcal{W}^-$ is proper and of finite type.

Following [CL^{3}2], we decompose the fixed locus $\mathcal{W}^T$ into disjoint open and closed substacks

$$\mathcal{W}^T=\coprod_{\Gamma\in\Delta^{\mathrm{fl}}}\mathcal{W}_{(\Gamma)},$$

indexed by a set of (flat) decorated graphs Δ^{fl}. By the virtual localization [GP, CKL], after inverting the generator $\mathfrak{t}\in A^1_T(pt)$,

$$[\mathcal{W}]^{\mathrm{vir}}_{\mathrm{loc}}=\sum_\Gamma\frac{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}})}\in\left(A^T_*\mathcal{W}^-\right)_{\mathfrak{t}}. \tag{1.2}$$

We call a graph a pure loop if it has no legs and no stable vertices, and every vertex has exactly two edges attached to it. In [CL^{3}2], we divided the set Δ^{fl} into regular and irregular graphs (Definition 2.8).

Definition 1.1. *Let $Z\subset\mathcal{W}^T$ be a proper closed substack, viewed as a T-stack with the trivial T action. We say $\alpha\in A^T_*Z$ is weakly trivial, denoted by $\alpha\sim 0$, if there is a closed proper substack $Z'\subset\mathcal{W}^T$ with $Z\subset Z'$ so that α is mapped to zero under the induced homomorphism $A^T_*Z\to A^T_*Z'$.*

We will prove Theorem 1.2 below.

Theorem 1.2. *Let Γ be an irregular graph and not a pure loop. Then $[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}_{\mathrm{loc}}\sim 0$.*

Let $[\cdot]_0:A^T_*\mathcal{W}^-\to A_0(pt)$ be the proper pushforward induced by $\mathcal{W}\to pt$. Then Theorem 1.2 implies that for the Γ as stated in Theorem 1.2, and for any $\beta\in A^*_T\mathcal{W}$,

$$\left[\beta\cap\frac{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}})}\right]_0=0.$$

This vanishing theorem implies that the only quintic FJRW invariants that contribute to the relations derived from the theory of MSP fields are those with pure insertions 2/5 (see [CL^{3}2, Thm. 1.1]).

If one considers the MSP moduli for a general Fermat hypersurface which may not be a CY threefold(quintic), one can determine the GW invariants of the Fermat hypersurface using MSP recursions as in [CL^{3}2, (1.2)]. In these recursions, irregular graphs will possibly have nonzero contributions. Calculating these contributions requires a generalization of Theorem 1.2.

2. Irregular graphs

In this section, we recall the notions of MSP fields and decorated graphs associated with T-invariant MSP fields. These notions were first introduced in [CL^{3}2].

2.1. **MSP fields.** Let $\boldsymbol{\mu}_5 = \langle \zeta_5 \rangle \le \mathbb{G}_m$ be the subgroup of fifth-roots of unity, generated by $\zeta_5 = \exp(\frac{2\pi\sqrt{-1}}{5})$. Let

$$\boldsymbol{\mu}_5^{\rm na} = \{(1,\rho),(1,\varphi),\zeta_5,\cdots,\zeta_5^4\} \quad \text{and} \quad \boldsymbol{\mu}_5^{\rm br} = \{(1,\rho),(1,\varphi)\} \cup \boldsymbol{\mu}_5.$$

Here $(1,\varphi)$ and $(1,\rho)$ are symbols, and they function as the identity elements with special properties; thus the subgroup they generate $\langle (1,\rho)\rangle = \langle (1,\varphi)\rangle = \{1\} \le \mathbb{G}_m$ is the trivial subgroup. Note that $\boldsymbol{\mu}_5^{\rm na}$ obtained by removing 1 from $\boldsymbol{\mu}_5^{\rm br}$. The data in $\boldsymbol{\mu}_5^{\rm na}$ are called narrow, while the 1 in $\boldsymbol{\mu}_5^{\rm br}$ is called broad.

Let

$$g \ge 0, \quad \gamma = (\gamma_1,\cdots,\gamma_\ell) \in (\boldsymbol{\mu}_5^{\rm br})^{\times \ell}, \quad \mathbf{d} = (d_0, d_\infty) \in \mathbb{Q}^{\times 2}.$$

For an ℓ-pointed twisted curve $\Sigma^{\mathcal{C}} \subset \mathcal{C}$, and for $\alpha \in \boldsymbol{\mu}_5^{\rm br}$, we denote

$$\omega_{\mathcal{C}/S}^{\log} = \omega_{\mathcal{C}/S}(\Sigma^{\mathcal{C}}), \quad \text{and} \quad \Sigma_\alpha^{\mathcal{C}} = \coprod_{\gamma_i = \alpha} \Sigma_i^{\mathcal{C}}.$$

Definition 2.1 ([CL^{3}1])**.** *A $(g,\gamma,\mathbf{d})$ MSP field ξ is $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$ (as in* (1.1)*) such that*

(1) $\cup_{i=1}^{\ell}\Sigma_i^{\mathcal{C}} = \Sigma^{\mathcal{C}} \subset \mathcal{C}$ *is an ℓ-pointed, genus g, twisted curve such that the i-th marking $\Sigma_i^{\mathcal{C}}$ is banded by the group $\langle\gamma_i\rangle \le \mathbb{G}_m$;*

(2) $\mathcal{L}$ *and* $\mathcal{N}$ *are invertible sheaves on* $\mathcal{C}$, $\mathcal{L}\oplus\mathcal{N}$ *is representable,* $\deg \mathcal{L}\otimes\mathcal{N} = d_0$, $\deg \mathcal{N} = d_\infty$, *and the monodromy of* $\mathcal{L}$ *along* $\Sigma_i^{\mathcal{C}}$ *is* γ_i *when* $\langle\gamma_i\rangle \ne \langle 1\rangle$;

(3) $\nu = (\nu_1,\nu_2) \in H^0(\mathcal{L}\otimes\mathcal{N}) \oplus H^0(\mathcal{N})$, *and* (ν_1,ν_2) *is nowhere zero;*

(4) $\varphi = (\varphi_1,\ldots,\varphi_5) \in H^0(\mathcal{L})^{\oplus 5}$, (φ,ν_1) *is nowhere zero, and* $\varphi|_{\Sigma^{\mathcal{C}}_{(1,\varphi)}} = 0$;

(5) $\rho \in H^0(\mathcal{L}^{\vee\otimes 5}\otimes\omega_{\mathcal{C}/S}^{\log})$, (ρ,ν_2) *is nowhere zero, and* $\rho|_{\Sigma^{\mathcal{C}}_{(1,\rho)}} = 0$.

We call ξ (or γ) narrow if $\gamma \in (\boldsymbol{\mu}_5^{\rm na})^\ell$. We call ξ stable if $|\operatorname{Aut}(\xi)| < \infty$.

The definition of monodromy can be found, for example, in [FJR, CLL]. A typical example of monodromy is as follows. Consider $\mathcal{C} = [\mathbb{A}^1/\boldsymbol{\mu}_5]$, where $\boldsymbol{\mu}_5$ acts on $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[x]$ via $\zeta_5 \cdot x = \zeta_5^{-1}x$. Then the $\mathcal{O}_{\mathcal{C}}$-module $x^{-2}\mathbb{C}[x]$ has monodromy ζ_5^2 at the stacky point.

Throughout this paper, unless otherwise stated, by an MSP field ξ we mean $\xi=(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\cdots)$ as given in (1.1) with *narrow* γ.

By the main theorem of [CL^{3}1], the category $\mathcal{W}$ of families of MSP fields of data $(g,\gamma,\mathbf{d})$ is a separated DM stack. The group $T=\mathbb{G}_m$ acts on $\mathcal{W}$ via

$$t\cdot(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu_1,\nu_2)=(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,t\nu_1,\nu_2). \tag{2.1}$$

The structure of T-invariant MSP fields can be summarized as follows. Let $\xi\in\mathcal{W}^T$. Then there is a homomorphism h and T-linearizations (τ_t,τ_t')

$$h:T\longrightarrow \mathrm{Aut}(\mathcal{C},\Sigma^{\mathcal{C}}),\quad \tau_t:h_{t*}\mathcal{L}\longrightarrow\mathcal{L}\quad\text{and}\quad \tau_t':h_{t*}\mathcal{N}\longrightarrow\mathcal{N} \tag{2.2}$$

such that

$$t\cdot(\varphi,\rho,\nu_1,\nu_2)=(\tau_t,\tau_t')(h_{t*}\varphi,h_{t*}\rho,t\cdot h_{t*}\nu_1,h_{t*}\nu_2),\quad t\in T. \tag{2.3}$$

(Here we allow fractional weight for the T actions on curves and bundles.) Since $\xi\in\mathcal{W}^T$ is stable, such (h,τ_t,τ_t') is unique. We call h,τ_t,τ_t' the T-actions and linearizations induced from $\xi\in\mathcal{W}^T$.

Let $\mathbf{L}_k$ be the one-dimensional weight k T-representation. Let

$$\mathcal{L}^{\log}=\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\quad\text{and}\quad \mathcal{P}^{\log}=\mathcal{L}^{-5}\otimes\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)}). \tag{2.4}$$

Then (2.3) can be rephrased as

$$(\varphi,\rho,\nu_1,\nu_2)\in H^0((\mathcal{L}^{\log})^{\oplus 5}\oplus\mathcal{P}^{\log}\oplus\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1\oplus\mathcal{N})^T. \tag{2.5}$$

2.2. **Decorated graphs of T-MSP fields.** We describe the structure of $\mathcal{W}^T$, following [CL^{3}2]. Let $\xi\in\mathcal{W}^T$, with domain curve $\mathcal{C}$, etc., as in (1.1). We decompose $\mathcal{C}$ as follows: Let

$$\mathcal{C}_0=(\nu_1=0)_{\mathrm{red}},\quad \mathcal{C}_\infty=(\nu_2=0)_{\mathrm{red}},\quad \mathcal{C}_1=(\rho=\varphi=0)_{\mathrm{red}}\subset\mathcal{C};$$

let $\mathcal{A}$ be the set of irreducible components of $\overline{\mathcal{C}-\mathcal{C}_0\cup\mathcal{C}_1\cup\mathcal{C}_\infty}$. Also let

$$\mathcal{C}_{01}=\bigcup_{\mathcal{C}_a\in\mathcal{A},\ \rho|_{\mathcal{C}_a}=0}\mathcal{C}_a,\quad \mathcal{C}_{1\infty}=\bigcup_{\mathcal{C}_a\in\mathcal{A},\ \varphi|_{\mathcal{C}_a}=0},\quad \mathcal{C}_{0\infty}=\bigcup_{\mathcal{C}_a\in\mathcal{A},\ \rho|_{\mathcal{C}_a}\neq0,\ \varphi|_{\mathcal{C}_a}\neq0}\mathcal{C}_a.$$

We know that $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$ are mutually disjoint, and the action $h:T\to\mathrm{Aut}(\mathcal{C},\Sigma^{\mathcal{C}})$ acts trivially on $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$. We also know that every irreducible component $\mathcal{C}_a\subset\mathcal{C}_{01}$ (resp. $\mathcal{C}_a\subset\mathcal{C}_{1\infty}$; resp. $\mathcal{C}_a\subset\mathcal{C}_{0\infty}$) is a smooth rational twisted curve with two T-fixed points lying on $\mathcal{C}_0$ and $\mathcal{C}_1$ (resp. $\mathcal{C}_1$ and $\mathcal{C}_\infty$; resp. $\mathcal{C}_0$ and $\mathcal{C}_\infty$).

We associate a decorated graph with each $\xi\in\mathcal{W}^T$. For a graph Γ, besides its vertices $V(\Gamma)$, edges $E(\Gamma)$ and legs $L(\Gamma)$, its set of flags is

$$F(\Gamma)=\{(e,v)\in E(\Gamma)\times V(\Gamma): v\in e\}.$$

Given $\xi\in\mathcal{W}^T$, let $\pi:\mathcal{C}^{\mathrm{nor}}\to\mathcal{C}$ be its normalization. For any $y\in\pi^{-1}(\mathcal{C}_{\mathrm{sing}})$, we denote by γ_y the monodromy of $\pi^*\mathcal{L}$ along y.

Definition 2.2. *With $\xi\in\mathcal{W}^T$ we associate a graph Γ_ξ as follows:*

(1) *(vertex) Let* $V_0(\Gamma_\xi)$, $V_1(\Gamma_\xi)$, *and* $V_\infty(\Gamma_\xi)$ *be the set of connected components of* $\mathcal{C}_0$, $\mathcal{C}_1$, *and* $\mathcal{C}_\infty$ *respectively, and let* $V(\Gamma_\xi)$ *be their union;*
(2) *(edge) Let* $E_0(\Gamma_\xi)$, $E_\infty(\Gamma_\xi)$ *and* $E_{0\infty}(\Gamma_\xi)$ *be the set of irreducible components of* $\mathcal{C}_{01}$, $\mathcal{C}_{1\infty}$ *and* $\mathcal{C}_{0\infty}$ *respectively, and let* $E(\Gamma_\xi)$ *be their union;*
(3) *(leg) Let* $L(\Gamma_\xi) \cong \{1, \cdots, \ell\}$ *be the ordered set of markings of* $\Sigma^{\mathcal{C}}$ *where* $i \in L(\Gamma_\xi)$ *is attached to* $v \in V(\Gamma_\xi)$ *if* $\Sigma_i^{\mathcal{C}} \in \mathcal{C}_v$*;*
(4) *(flag)* $(e, v) \in F(\Gamma_\xi)$ *if and only if* $\mathcal{C}_e \cap \mathcal{C}_v \neq \emptyset$.

We call $v \in V(\Gamma_\xi)$ *stable if* $\mathcal{C}_v \subset \mathcal{C}$ *is one-dimensional, otherwise it is unstable.*

We now specify the decorations. In the following, let $V^S(\Gamma_\xi) \subset V(\Gamma_\xi)$ be the set of stable vertices. Given $v \in V(\Gamma_\xi)$, let

$$S_v = \{\Sigma_j^{\mathcal{C}} \in \mathcal{C}_v \mid \Sigma_j^{\mathcal{C}} \in \Sigma^{\mathcal{C}}\} \quad \text{and} \quad E_v = \{e \in E(\Gamma_\xi) : (e, v) \in F(\Gamma_\xi)\},$$

consist of the markings on $\mathcal{C}_v$ and of the edges attached to v, respectively.

For $v \in V^S(\Gamma_\xi)$, we define

$$\Sigma_{\rm inn}^{\mathcal{C}_v} = \Sigma^{\mathcal{C}} \cap \mathcal{C}_v, \quad \Sigma_{\rm out}^{\mathcal{C}_v} = \overline{(\mathcal{C} - \mathcal{C}_v)} \cap \mathcal{C}_v, \quad \text{and} \quad \Sigma^{\mathcal{C}_v} = \Sigma_{\rm inn}^{\mathcal{C}_v} \cup \Sigma_{\rm out}^{\mathcal{C}_v}, \tag{2.6}$$

called the inner, outer, and total markings of $\mathcal{C}_v$, respectively, and indexed by S_v, E_v and $S_v \cup E_v$.

We adopt the following convention: for $a \in V(\Gamma_\xi) \cup E(\Gamma_\xi)$, we define

$$d_{0a} = \deg \mathcal{L} \otimes \mathcal{N}|_{\mathcal{C}_a}, \quad d_{\infty a} = \deg \mathcal{N}|_{\mathcal{C}_a}, \quad \text{and} \quad d_a = \deg \mathcal{L}|_{\mathcal{C}_a} = d_{0a} - d_{\infty a}.$$

(This is consistent with $d_0 = \deg \mathcal{L} \otimes \mathcal{N}$ and $d_\infty = \deg \mathcal{N}$.) For $e \in E_v$, we assign $\gamma_{(e,v)}$ according to the following rule:

(1) when $d_e \notin \mathbb{Z}$, assign $\gamma_{(e,v)} = e^{-2\pi\sqrt{-1}d_e}$;
(2) when $d_e \in \mathbb{Z}$ and $v \in V_\infty(\Gamma_\xi) \cup V_1(\Gamma_\xi)$, assign $\gamma_{(e,v)} = (1, \varphi)$;
(3) when $d_e \in \mathbb{Z}$ and $v \in V_0(\Gamma_\xi)$, assign $\gamma_{(e,v)} = (1, \rho)$.

Definition 2.3. *We endow the graph* Γ_ξ *with the following decorations:*

(a) *(genus) Define* $\vec{g} : V(\Gamma_\xi) \to \mathbb{Z}_{\geq 0}$ *by* $\vec{g}(v) = h^1(\mathcal{O}_{\mathcal{C}_v})$.
(b) *(degree) Define* $\vec{d} : E(\Gamma_\xi) \cup V(\Gamma_\xi) \to \mathbb{Q}^{\oplus 2}$ *by* $\vec{d}(a) = (d_{0a}, d_{\infty a})$.
(c) *(marking) Define* $\vec{S} : V(\Gamma_\xi) \to 2^{L(\Gamma_\xi)}$ *by* $v \mapsto S_v \subset L(\Gamma_\xi)$.
(d) *(monodromy) Define* $\vec{\gamma} : L(\Gamma_\xi) \to \boldsymbol{\mu}_5^{\rm na}$ *by* $\vec{\gamma}(\Sigma_i^{\mathcal{C}}) = \gamma_i$.
(e) *(level) Define lev* $: V(\Gamma_\xi) \to \{0, 1, \infty\}$ *by lev*$(v) = a$ *for* $v \in V_a(\Gamma_\xi)$.

We form

$$V^{a,b}(\Gamma_\xi) = \{v \in V(\Gamma_\xi) - V^S(\Gamma_\xi) : |S_v| = a, \, |E_v| = b\}, \tag{2.7}$$

and adopt the convention $V_j^S(\Gamma_\xi) = V_j(\Gamma_\xi) \cap V^S(\Gamma_\xi)$; we do the same for $V_j^{a,b}(\Gamma_\xi)$.

We say $\Gamma_\xi \sim \Gamma_{\xi'}$ if there is an isomorphism of graphs Γ_ξ and $\Gamma_{\xi'}$ that preserves the decorations (a)-(e). We define

$$\Delta = \{\Gamma_\xi \mid \xi \in \mathcal{W}^T\} / \sim .$$

2.3. **Decomposition along nodes.** We describe the decomposition of a T-MSP field along its T-unbalanced nodes.

Definition 2.4. *Let $\mathcal{C}$ be a T-twisted curve (i.e. a twisted curve with a T-action) and q be a node of $\mathcal{C}$. Let $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ be the two branches of the formal completion of $\mathcal{C}$ along q. We call q T-balanced if $T_q\hat{\mathcal{C}}_1 \otimes T_q\hat{\mathcal{C}}_2 \cong \mathbf{L}_0$ as T-representations.*

For $\Gamma \in \Delta$, we let

$$N(\Gamma) = V^{0,2}(\Gamma) \cup \{(e,v) \in F(\Gamma) \mid v \in V^S(\Gamma)\}. \tag{2.8}$$

(Recall $v \in V^{0,2}(\Gamma)$ when $\mathcal{C}_v$ is a node in $\mathcal{C}$.) Note that every $a \in N(\Gamma_\xi)$ has its associated node q_a of $\mathcal{C}$.

Definition 2.5. *We call $a \in N(\Gamma_\xi)$ T-balanced if the associated node q_a is a T-balanced node in $\mathcal{C}$. Let $N(\Gamma_\xi)^{\rm un} \subset N(\Gamma_\xi)$ be the subset of T-unbalanced nodes.*

Clearly, if $v \in N(\Gamma_\xi)$ is T-balanced, then $v \in V_1^{0,2}(\Gamma_\xi)$. Recall $d_e = \deg \mathcal{L}|_{\mathcal{C}_e}$.

Lemma 2.6 ([CL^{3}2, Lemm. 2.14])**.** *For $v \in V_1^{0,2}(\Gamma_\xi)$ with (distinct) (e,v) and $(e',v) \in F(\Gamma_\xi)$, and letting $q_v = \mathcal{C}_e \cap \mathcal{C}_{e'}$ be the associated node, then q_v is T-balanced if and only if $d_e + d_{e'} = 0$, and $(\mathcal{C}_e \cup \mathcal{C}_{e'}) \cap \mathcal{C}_\infty$ is a node or a marking of $\mathcal{C}$.*

Although a T-balanced $a \in N(\Gamma_\xi)$ is characterized by q_a being T-balanced, the previous reasoning shows that it can also be characterized by the information of the graph Γ_ξ. Thus for any $\Gamma \in \Delta$, we can talk about $N(\Gamma)^{\rm un} \subset N(\Gamma)$ without referring to any ξ.

We now introduce flat graphs and regular graphs. We call a graph $\Gamma \in \Delta$ flat if $N(\Gamma)^{\rm un} = N(\Gamma)$. Let $\Delta^{\rm fl} \subset \Delta$ be the set of flat graphs. In case $N(\Gamma)^{\rm un} \subsetneq N(\Gamma)$, we will associate a unique flat $\Gamma^{\rm fl}$, called the flattening of Γ, as follows. For each T-balanced $v \in N(\Gamma)$, which lies in $V_1^{0,2}(\Gamma)$, we eliminate the vertex v from Γ, replace the two edges $e \in E_\infty(\Gamma)$ and $e' \in E_0(\Gamma)$ incident to v by a single edge $\tilde{e}$ incident to the other two vertices that are incident to e or e', and demand that $\tilde{e}$ lies in $E_{0\infty}$. For the decorations, we set $\vec{g}(\tilde{e}) = 0$ and $(d_{0\tilde{e}}, d_{\infty\tilde{e}}) = (d_{\infty e}, d_{\infty e})$ (since $d_{0e'} = d_{\infty e}$, using $d_{0e} = d_{\infty e'} = 0$), while keeping the rest unchanged. Let $\Gamma^{\rm fl}$ be the resulting decorated graph after applying this procedure to all T-balanced v in $N(\Gamma)$. Then $\Gamma^{\rm fl}$ is flat. We introduce

$$\Delta^{\rm fl} = \{\Gamma^{\rm fl} \mid \Gamma \in \Delta\}/\sim .$$

Indeed, it is easy to check that $\Delta^{\rm fl} = \{\Gamma \in \Delta \mid \Gamma \text{ is flat}\}$.

Given a flat $\Gamma \in \Delta^{\rm fl}$, we define a Γ-framed T-MSP field to be a pair (ξ, ϵ), where $\epsilon : \Gamma_\xi^{\rm fl} \cong \Gamma$ is an isomorphism (of decorated graphs). As in [CL^{3}2], we can make sense of families of Γ-framed T-MSP fields (cf. [CL^{3}2, Section 2.4]). We then form the groupoid $\mathcal{W}_\Gamma$ of Γ-framed T-MSP fields with obviously defined arrows; $\mathcal{W}_\Gamma$ is a DM stack, with a forgetful morphism

$$\iota_\Gamma : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}^T.$$

Let $\mathcal{W}_{(\Gamma)}$ be the image of ι_Γ; it is an open and closed substack of $\mathcal{W}^T$. The factored morphism $\mathcal{W}_\Gamma \to \mathcal{W}_{(\Gamma)}$ is an $\mathrm{Aut}(\Gamma)$-torsor.

The cosection-localized virtual cycles $[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}_{\mathrm{loc}}$ are the terms appearing in the localization formula (1.2). Because $\mathcal{W}_\Gamma \to \mathcal{W}_{(\Gamma)}$ is an $\mathrm{Aut}(\Gamma)$-torsor, the similarly defined virtual cycle $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ has ([CL^{3}2, Coro. 3.8])

$$[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = |\,\mathrm{Aut}(\Gamma)| \cdot [\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

For a vertex $v \in V_\infty(\Gamma)$ with $\gamma_v = \{\zeta_5^{a_1}, \cdots, \zeta_5^{a_c}\}$, we abbreviate $\gamma_v = (0^{e_0} \cdots 4^{e_4})$, where e_i is the number of times i appear in $\{a_1, \cdots, a_c\}$. (We require $a_j \in [0,4]$.)

Definition 2.7. *We call a vertex $v \in V^S_\infty(\Gamma)$ exceptional if $g_v = 0$ and $\gamma_v = (1^{2+k}4)$ or $(1^{1+k}23)$, for some $k \geq 0$.*

Definition 2.8. *We call a vertex $v \in V_\infty(\Gamma)$ regular if the followings hold:*

(1) *In case v is stable, then either v is exceptional, or for every $a \in S_v$ and $e \in E_v$, we have γ_a and $\gamma_{(e,v)} \in \{\zeta_5, \zeta_5^2\}$.*
(2) *In case v is unstable and $\mathcal{C}_v$ is a scheme point, then $\mathcal{C}_v$ is a non-marking smooth point of $\mathcal{C}$.*

We call Γ regular if it is flat, and all its vertices $v \in V_\infty(\Gamma)$ are regular. We call Γ irregular if it is not regular.

Remark 2.9. *Suppose $v \in V_\infty(\Gamma)$ is an unstable scheme point, and assume v is exactly the intersection $q := \mathcal{C}_{e_1} \cap \mathcal{C}_{e_2}$ for $e_1, e_2 \in E_\infty(\Gamma)$. Then $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$ by the following reason. We may assume $\mathcal{C} = \mathcal{C}_{e_1} \cup \mathcal{C}_{e_2}$ by our factorization of virtual cycles. Then $\deg \mathcal{L}|_{\mathcal{C}_{e_i}} < 0$ for $i = 1, 2$ implies $H^0(\mathcal{C}_{e_i}, \mathcal{L}^{\oplus 5}) = 0$, and thus $\mathcal{W}_\Gamma$ is a gerbe and hence of dimension zero. Then we have an exact sequence*

$$H^0(\mathcal{C}_{e_1}, \mathcal{L}^{\oplus 5}) \oplus H^0(\mathcal{C}_{e_2}, \mathcal{L}^{\oplus 5}) \xrightarrow{r} \mathcal{L}^{\oplus 5}|_q \longrightarrow H^1(\mathcal{C}, \mathcal{L}^{\oplus 5}).$$

Since $H^0(\mathcal{C}_{e_i}, \mathcal{L}^{\oplus 5}) = 0$, $r = 0$. The sequence, after taking the invariant part, becomes an inclusion

$$\mathcal{L}^{\oplus 5}|_q \longrightarrow H^1(\mathcal{C}, \mathcal{L}^{\oplus 5})^T = \mathcal{O}b^T_\phi,$$

whose family version exhibits a rank five subbundle and thus $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$.

Theorem 1.2 states that *for a non-pure loop irregular* Γ, $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \sim 0$. We prove an easy and useful corollary of Lemma 2.6.

Corollary 2.10. *Let $\Gamma \in \Delta^{\mathrm{fl}}$ be a flat graph that contains an $e \in E_{0\infty}(\Gamma)$. Suppose $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \neq 0$, then $d_e = 0$ and $\mathcal{C}_e \cap \mathcal{C}_\infty$ is a node or a marking of $\mathcal{C}$.*

Proof. Since $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_*(\mathcal{W}_\Gamma \cap \mathcal{W}^-)$, $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \neq 0$ implies that $\mathcal{W}_\Gamma \cap \mathcal{W}^- \neq \emptyset$. Let $\xi \in \mathcal{W}_\Gamma \cap \mathcal{W}^-$, then $E_{0\infty}(\Gamma_\xi) = \emptyset$. Thus the $e \in E_{0\infty}(\Gamma)$ must come from flattening a pair of edges in $E_0(\Gamma_\xi)$ and $E_\infty(\Gamma_\xi)$. By applying Lemma 2.6, the corollary is proved. $\square$

3. The Virtual cycle $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc}$

We begin by recalling the construction of the cosection-localized virtual cycle $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc}$. Let $\mathcal{D}$ be the stack of flat families of $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N})$, where $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ are pointed twisted curves, and $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$. The stack $\mathcal{D}$ is a smooth Artin stack, with a forgetful morphism $\mathcal{W} \to \mathcal{D}$. By [CL^{3}1], we have a perfect relative obstruction theory $\mathbb{T}_{\mathcal{W}/\mathcal{D}} \to \mathbb{E}_{\mathcal{W}/\mathcal{D}}$ and a cosection $\sigma : \mathcal{O}b_{\mathcal{W}/\mathcal{D}} \to \mathcal{O}_{\mathcal{W}}$. Let $\mathbb{E}_{\mathcal{W}} = \text{cone}(\mathbb{T}_{\mathcal{D}}[-1] \to \mathbb{E}_{\mathcal{W}/\mathcal{D}})$ be the mapping cone, and let $\bar\sigma$ be the lift of σ. This way, we obtain a perfect obstruction theory and cosection

$$\phi^\vee_{\mathcal{W}} : \mathbb{T}_{\mathcal{W}} \longrightarrow \mathbb{E}_{\mathcal{W}} \quad \text{and} \quad \bar\sigma : \mathcal{O}b_{\mathcal{W}} = H^1(\mathbb{E}_{\mathcal{W}}) \longrightarrow \mathcal{O}_{\mathcal{W}}.^1$$

Let $\iota_\Gamma : \mathcal{W}_\Gamma \to \mathcal{W}^T$ be the tautological finite étale morphism, which factor through an $\mathrm{Aut}(\Gamma)$-torsor $\mathcal{W}_\Gamma \to \mathcal{W}_{(\Gamma)}$, with $\mathcal{W}_{(\Gamma)} \subset \mathcal{W}^T \subset \mathcal{W}$ open and closed. Taking the T-fixed part of the obstruction theory of $\mathcal{W}$, and using the tautological $\mathbb{T}_{\mathcal{W}_\Gamma} \to \mathbb{T}_{\mathcal{W}}$, we obtain another obstruction theory (c.f. [GP, Prop. 1])

$$\phi^\vee_{\mathcal{W}_\Gamma} : \mathbb{T}_{\mathcal{W}_\Gamma} \longrightarrow \mathbb{E}_{\mathcal{W}_\Gamma}. \tag{3.1}$$

We then restrict $\iota_\Gamma^*\bar\sigma$ to the T-fixed part of $\iota_\Gamma^*\mathcal{O}b_{\mathcal{W}}$ to obtain a cosection

$$\iota_\Gamma^*\bar\sigma^T : \mathcal{O}b_{\mathcal{W}_\Gamma} = (\iota_\Gamma^*\mathcal{O}b_{\mathcal{W}})^T \longrightarrow \mathcal{O}_{\mathcal{W}_\Gamma}.$$

Let $\mathcal{W}_\Gamma^- = \mathcal{W}_\Gamma \cap \mathcal{W}^-$ be the degeneracy locus of $\iota_\Gamma^*\bar\sigma^T$.

Applying the cosection-localized Gysin map in [KL], we obtain

$$[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc} = 0^!_{\rm loc}[\mathfrak{C}_{\mathcal{W}_\Gamma}] \in A^T_*(\mathcal{W}_\Gamma^-), \tag{3.2}$$

where $\mathfrak{C}_{\mathcal{W}_\Gamma} \in h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma})$ is the intrinsic normal cone.

In the remainder of this section, we assume that Γ is an irregular graph with $V_1(\Gamma) = \emptyset$. To prove the desired vanishing $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc} \sim 0$, we will work with a construction of $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc}$ via the obstruction theories of $\mathcal{W}_\Gamma$ relative to the auxiliary stack of Γ-framed curves $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$ (in $\mathcal{D}$).

As we will be working with T-curves extensively, we set the following convention. Let $(\Sigma^{\mathcal{C}}, \mathcal{C})$ be a pointed T-curve, meaning that T acts on the pointed twisted curve $(\Sigma^{\mathcal{C}}, \mathcal{C})$. Denote by $\mathcal{C}^{T,\rm dec}$ the curve after decomposing $\mathcal{C}$ along all its T-unbalanced nodes. Recall that given a flat Γ and a $(\xi, \epsilon) \in \mathcal{W}_\Gamma$, where $\xi = (\mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \cdots)$, etc., we not only have an identification of the T-unbalanced nodes of $\mathcal{C}$ with $N(\Gamma)$, but also an identification of the connected components of $\mathcal{C}^{T,\rm dec}$ with $V^S(\Gamma) \cup E(\Gamma)$. Further the T-linearizations of $\mathcal{C}$ and of $(\mathcal{L}, \mathcal{N})$ restricted to each component $\mathcal{C}_a$ in $\mathcal{C}^{T,\rm dec}$ are specified by the data in Γ.

Definition 3.1. *A Γ-framed (twisted) curve is a T-equivariant $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$ (in $\mathcal{D}$), together with an identification ϵ identifying*

(1) *the marking $\Sigma^{\mathcal{C}}$ with the legs of Γ,*

[1] As argued in [CL^{3}2, Section 3.1], $\phi^\vee_{\mathcal{W}}$ is an arrow in $D^+_{\rm qcoh}(\mathcal{O}_{[\mathcal{W}/T]})$; and $\bar\sigma$ is T-equivariant. (See [CL^{3}2, Section 3.1] for notation.)

(2) *the T-unbalanced nodes of $\mathcal{C}$ with $N(\Gamma)$, and*
(3) *the connected components of $\mathcal{C}^{T,\mathrm{dec}}$ with $V^S(\Gamma)\cup E(\Gamma)$,*

so that these are consistent with the geometry of $(\mathcal{C},\Sigma^{\mathcal{C}})$, and the T-linearization of $\mathcal{L}$ and $\mathcal{N}$ restricted to each component $\mathcal{C}_a$ in $\mathcal{C}^{T,\mathrm{dec}}$, as specified by the data in Γ,

(4) *when $e\in E_{0\infty}(\Gamma)$, either $\mathcal{C}_e$ is irreducible and then $\mathcal{C}_e\cong\mathbb{P}^1$, or $\mathcal{C}_e$ is reducible and then $\mathcal{C}_e=\mathcal{C}_{e-}\cup\mathcal{C}_{e+}$ is a union of two $\mathbb{P}^1$'s so that, $\mathcal{C}_{e-}\cap\mathcal{C}_0\neq\emptyset$, $\mathcal{C}_{e+}\cap\mathcal{C}_\infty\neq\emptyset$, and $\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1|_{\mathcal{C}_{e+}}\cong\mathcal{O}_{\mathcal{C}_{e+}}$ and $\mathcal{N}|_{\mathcal{C}_{e_-}}\cong\mathcal{O}_{\mathcal{C}_{e-}}$.*

Because the conditions in these definitions are open, we can speak of flat families of Γ-framed curves. Let $\mathcal{D}_\Gamma$ be the stack of flat families of Γ-framed curves, where arrows are T-equivariant arrows in $\mathcal{D}$ that preserve the data of Γ-framings. Clearly, $\mathcal{D}_\Gamma$ is a smooth Artin stack, with a forgetful morphism $\mathcal{W}_\Gamma\to\mathcal{D}_\Gamma$.

Definition 3.2. *A Γ-framed gauged twisted curve is a collection of T-equivariant objects $\eta=(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\nu_1,\nu_2)$ with an identification ϵ such that*

(1) *$((\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N}),\epsilon)\in\mathcal{D}_\Gamma$;*
(2) *$(\nu_1,\nu_2)\in H^0(\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1)^T\oplus H^0(\mathcal{N})^T$, such that $\nu_1|_{\mathcal{C}_0}=\nu_2|_{\mathcal{C}_\infty}=0$, and $\nu_1|_{\mathcal{C}_\infty}$ and $\nu_2|_{\mathcal{C}_0}$ are nowhere vanishing;*
(3) *in case of (4) in Definition 3.1, $\nu_1|_{\mathcal{C}_{e+}}$ and $\nu_2|_{\mathcal{C}_{e_-}}$ are nowhere vanishing.*

Note that the conditions (2) and (3) are dictated by (3)-(5) of Definition 2.1 in the presence of the fields (φ,ρ). Because of (3), The T-action on the domain curve of any $\xi\in\mathcal{D}_\Gamma$ or $\mathcal{D}_{\Gamma,\nu}$ are completely determined by Γ.

Similarly we can speak of flat families of Γ-framed gauged curves. Let $\mathcal{D}_{\Gamma,\nu}$ be the stack of flat families of Γ-framed gauged twisted curves as in Definition 3.2. This stack is a smooth Artin stack. By forgetting the ν fields, the φ fields and the ρ fields, we obtain the forgetful morphisms $\mathcal{D}_{\Gamma,\nu}\to\mathcal{D}_\Gamma$.

Let $\mathcal{D}_{\Gamma,[\nu]}\subset\mathcal{D}_\Gamma$ be the image stack of the forgetful $\mathcal{D}_{\Gamma,\nu}\to\mathcal{D}_T$. Let

$$\mathcal{D}_{\Gamma,\nu}\xrightarrow{p_1}\mathcal{D}_{\Gamma,[\nu]}\xrightarrow{p_2}\mathcal{D}_\Gamma \tag{3.3}$$

be the induced morphisms.

Lemma 3.3. *All stacks in (3.3) are smooth. The morphism p_1 is smooth of DM type and the morphism p_2 is a closed embedding. Assuming $V_1(\Gamma)=\emptyset$, then the fiber dimension of p_1 is $|V(\Gamma)|$, and the codimension of image(p_2) is $\sum_{v\in V^S(\Gamma)}g_v$.*

Proof. The proof that all stacks in (3.3) are smooth, p_1 is smooth and p_2 is a closed embedding is straightforward and hence omitted. Let $\xi=(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\nu_1,\nu_2)$ be a closed point in $\mathcal{D}_{\Gamma,\nu}$. In case $V_1(\Gamma)=\emptyset$, then the fiber dimension of p_1 at ξ is the dimension of choices of locally constant sections $\nu_1|_{\mathcal{C}_0}$ and $\nu_2|_{\mathcal{C}_\infty}$, namely the number of connected components of $\mathcal{C}_0\cup\mathcal{C}_\infty$, which is $|V_0(\Gamma)|+|V_\infty(\Gamma)|=|V(\Gamma)|$.

The proof of the codimension is similar, and hence omitted. □

Let $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$ with $\pi:\mathcal{C}\to\mathcal{W}_\Gamma$ be the universal family of $\mathcal{W}_\Gamma$. Let $\mathcal{L}^{\log}=\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})$, $\mathcal{P}^{\log}=\mathcal{L}^{-5}\otimes\omega^{\log}_{\mathcal{C}/\mathcal{W}_\Gamma}(-\Sigma^{\mathcal{C}}_{(1,\rho)})$, and

$$\mathcal{U}=(\mathcal{L}^{\log})^{\oplus 5}\oplus\mathcal{P}^{\log}\quad\text{and}\quad \mathcal{V}=\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1\oplus\mathcal{N}. \tag{3.4}$$

Using the T-invariant version of [CL2, Prop. 2.5], the standard relative obstruction theory of $\mathcal{W}_\Gamma\to\mathcal{D}_\Gamma$ is given by

$$\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}:\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}\longrightarrow\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}:=R\pi^T_*(\mathcal{U}\oplus\mathcal{V});$$

the standard relative obstruction theory of $\mathcal{W}_\Gamma\to\mathcal{D}_{\Gamma,\nu}$ is given by

$$\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}:\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}\longrightarrow\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}:=R\pi^T_*\mathcal{U}.$$

Like the discussion before (3.2), using their respective standard cosections, we obtain their localized virtual cycles $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma}$ of $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ and $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,\nu}$ of $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}$. Let $\mathcal{W}_\Gamma^-$ be the vanishing locus of the cosection of $\phi_{\mathcal{W}_\Gamma}$ mentioned before (3.2). We will show that the vanishing locus of the cosections of $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ and of $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}$ are identical to $\mathcal{W}_\Gamma^-$.

Proposition 3.4. *Let Γ be irregular. Then*

$$[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,\nu}\in A_*\mathcal{W}_\Gamma^-.$$

Proof. We will choose a relative perfect obstruction theory

$$\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}:\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}\longrightarrow\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}},$$

and show that its associated localized virtual cycle $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,[\nu]}$ fits with the identities

$$[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,\nu}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,[\nu]}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}. \tag{3.5}$$

We begin with constructing $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}$. First, because $p_1:\mathcal{D}_{\Gamma,\nu}\to\mathcal{D}_{\Gamma,[\nu]}$ is smooth, $\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}$ is a locally free sheaf. Let

$$\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}\longrightarrow\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}\longrightarrow q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}\xrightarrow{+1} \tag{3.6}$$

be the d.t. associated with $\mathcal{W}_\Gamma\to\mathcal{D}_{\Gamma,\nu}\to\mathcal{D}_{\Gamma,[\nu]}$. Here q is the forgetful morphism from $\mathcal{W}_\Gamma$ to either $\mathcal{D}_{\Gamma,\nu}$ or $\mathcal{D}_{\Gamma,[\nu]}$, whose meaning will be apparent from the context. We claim that this d.t. splits naturally via a

$$\tau:q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}\to\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}. \tag{3.7}$$

Indeed, let $\xi\in\mathcal{W}_\Gamma$ be any closed point, represented by $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,\nu_1,\nu_2)$. Let $\bar\xi=(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\nu_1,\nu_2)$ be its image in $\mathcal{D}_{\Gamma,\nu}$. Then any $x\in\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}|_{\bar\xi}$ is represented by an extension $(\tilde\nu_1,\tilde\nu_2)$ of (ν_1,ν_2) as a section of $(\mathcal{L},\mathcal{N})\times B_2$ over $(\Sigma^{\mathcal{C}},\mathcal{C})\times B_2$, where $B_2=\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$. We define $\tau(\xi)(x)\in\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}|_\xi$ as the family

$$\big((\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho)\times B_2,\tilde\nu_1,\tilde\nu_2\big).$$

This definition extends in the family version and gives a homomorphism τ as in (3.7) that splits (3.6). It follows that $q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} \to \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}[1]$ is zero, and

$$\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} = \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} \oplus q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}. \tag{3.8}$$

By the construction of $\mathcal{D}_{\Gamma,\nu} \to \mathcal{D}_{\Gamma,[\nu]}$, we see that canonically we have

$$q^*\phi^\vee_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} : q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} \xrightarrow{\cong} \pi^T_*\mathcal{V}. \tag{3.9}$$

This together with (3.8) gives us

$$\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} = \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} \oplus q^*\phi_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} : \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} \longrightarrow \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} := R\pi^T_*\mathcal{U} \oplus \pi^T_*\mathcal{V}$$

that fits into the following homomorphism of d.t.s:

$$\begin{array}{ccccccc}
\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \longrightarrow & \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} & \longrightarrow & \pi^T_*\mathcal{V} & \xrightarrow{+1} & \\
\uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}} & & \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}} & & \uparrow{\scriptstyle q^*\phi_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}} & & \\
\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} & \longrightarrow & q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} & \xrightarrow{+1} & .
\end{array} \tag{3.10}$$

Note that by our construction, $\pi^T_*\mathcal{V}$ is a locally free sheaf of rank $|V(\Gamma)|$. By inspection, as $q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}$ is a sheaf, we can easily see that $q^*\phi_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}$ is an isomorphism.

We form the following diagram:

$$\begin{array}{ccccccc}
\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} & \longrightarrow & \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & R^1\pi^T_*\mathcal{V}[-1] & \xrightarrow{+1} & \\
\uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}} & & \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}} & & \uparrow{\scriptstyle \zeta} & & \\
\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}} & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & q^*\mathbb{T}_{\mathcal{D}_{\Gamma,[\nu]}/\mathcal{D}_\Gamma} & \xrightarrow{+1} & ,
\end{array} \tag{3.11}$$

where the top line is induced by $\pi^T_*\mathcal{V} \to R\pi^T_*\mathcal{V} \to R^1\pi^T_*\mathcal{V}$, and the bottom line is induced by $\mathcal{W}_\Gamma \to \mathcal{D}_{\Gamma,[\nu]} \to \mathcal{D}_\Gamma$. The arrow ζ is the one making the above a homomorphism of d.t.s after we have shown that the left square is commutative.

We now show that the left square in (3.11) is commutative. Using the direct sum (3.8), and the definition of $\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}$, we see that the desired commutativity follows from the commutativity of the following two squares:

$$\begin{array}{ccccccc}
\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \longrightarrow & \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \qquad & \pi^T_*\mathcal{V} & \longrightarrow & R\pi^T_*\mathcal{V} \\
\uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}} & & \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}} & & \uparrow{\scriptstyle q^*\phi^\vee_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}} & & \uparrow{\scriptstyle \mathrm{pr}_2\circ\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}} \\
\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & & q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} & \xrightarrow{e} & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma},
\end{array} \tag{3.12}$$

where the horizontal arrow e is defined via the canonical

$$q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}} = H^0(q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}}) \to H^0(q^*\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_\Gamma}) \to H^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) \to \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}.$$

We will prove that the left square is commutative in Proposition 7.2. For the other square, by the construction of the obstruction theory $\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$, and because both $R^i\pi^T_*\mathcal{V}$ are locally free, we conclude that the second square is also commutative.

We also need to show that ζ is an isomorphism. We first check that for any closed $\xi \in \mathcal{W}_\Gamma$, $H^1(\zeta|_\xi)$ is injective. Then because both $H^1(\mathbb{T}_{\mathcal{D}_{\Gamma,\nu}/\mathcal{D}_{\Gamma,[\nu]}})$ and $R^1\pi_*^T\mathcal{V}$ are locally free of identical rank, we conclude that $H^1(\zeta)$ is an isomorphism. Because $H^{i\neq 1}(\zeta)=0$, ζ is proved to be an isomorphism.

Let ξ be any closed point in $\mathcal{W}_\Gamma$, represented by $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N},\varphi,\rho,\nu_1,\nu_2)$. Let $x\neq 0\in H^1(q^*\mathbb{T}_{\mathcal{D}_{\Gamma,[\nu]}/\mathcal{D}_\Gamma}|_\xi)$, which is represented by a first-order deformation of $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N})$ so that $(\Sigma^{\mathcal{C}},\mathcal{C})$ remains constant, and $(\mathcal{L},\mathcal{N})$ is deformed so that (ν_1,ν_2) cannot be extended. Then for the same first-order deformation of $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L},\mathcal{N})$, $(\varphi,\rho,\nu_1,\nu_2)$ does not extend. Then by [BF, Thm. 4.5], $H^1(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}|_\xi)(x)$ is the obstruction to the existence of such an extension, thus

$$H^1(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}|_\xi)(x)\neq 0\in H^1(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}|_\xi).$$

On the other hand, because the existence of the extensions of these four fields are independent of each other, and because extending (ν_1,ν_2) is already obstructed, by the construction of the relative obstruction theory $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$,

$$H^1(\zeta|_\xi)(x)=\mathrm{pr}_2\big(H^1(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}|_\xi)(x)\big)\neq 0\in H^1(R^1\pi_*^T\mathcal{V}[-1]|_\xi).$$

This proves that $H^1(\zeta|_\xi)$ is injective, thus ζ is an isomorphism.

We now show the first identity in (3.5), namely $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,\nu}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc},\Gamma,[\nu]}$. We first apply [BF, Prop. 2.7] to (3.10) to obtain a commutative diagram of cone stacks

$$\begin{array}{ccccc}
h^1/h^0(\pi_*^T\mathcal{V}[-1]) & \longrightarrow & h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}) & \xrightarrow{\lambda} & h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}) \\
\uparrow\cong & & \uparrow(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}})_* & & \uparrow(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}})_* \\
h^1/h^0(\pi_*^T\mathcal{V}[-1]) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}),
\end{array}$$

where both rows are exact sequences of abelian cone stacks. Let

$$\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}\subset h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}})\quad\text{and}\quad \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}\subset h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}})$$

be their respective virtual normal cones [BF, LT]. Applying an argument analogous to [CL2, Coro. 2.9] (see also [KKP, Prop. 3]), we conclude that $\lambda^*(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}})=\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}$. Because the two cosections of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}$ and $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}$ can be lift to the same cosection of the absolute obstruction sheaf $\mathcal{O}b_{\mathcal{W}_\Gamma}$, we conclude that the first identity in (3.5) holds.

We prove the second identity in (3.5). By the same reasoning, from (3.11) we obtain a commutative diagram of cone stacks

$$\begin{array}{ccccc}
h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}) & \xrightarrow{\lambda'} & h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) & \longrightarrow & h^1/h^0(R^1\pi_*^T\mathcal{V}[-1]) \\
\uparrow h^1/h^0(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}) & & \uparrow h^1/h^0(\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) & & \uparrow\cong \\
h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) & \longrightarrow & h^1/h^0(q^*\mathbb{T}_{\mathcal{D}_{\Gamma,[\nu]}/\mathcal{D}_\Gamma})
\end{array}$$

whose first row is exact. Notice that $\mathcal{D}_{\Gamma,[\nu]} \to \mathcal{D}_\Gamma$ is a smooth closed embedding with normal bundle $R^1\pi_*^T\mathcal{V}$, and λ' is a regular embedding whose normal bundle is isomorphic to $R^1\pi_*^T\mathcal{V}$.

By the normal cone construction [Ful], we see that $\lambda'\big(h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}})\big)$ intersects $\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ transversally, and $\lambda'^{-1}(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) = \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}$. Because the two cosections of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,[\nu]}}$ and $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ lift to the same cosection of the absolute obstruction sheaf $\mathcal{O}b_{\mathcal{W}_\Gamma}$, we conclude that the second identity in (3.5) holds.

Finally, we prove the third identity in (3.5). From the canonical diagram

$$\begin{array}{ccc} \mathcal{W}_\Gamma & \xrightarrow{\iota_\Gamma} & \mathcal{W} \\ \big\downarrow q & & \big\downarrow \tilde q \\ \mathcal{D}_\Gamma & \xrightarrow{p} & \mathcal{D}, \end{array} \tag{3.13}$$

we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} = (\iota_\Gamma^*\mathbb{E}_{\mathcal{W}/\mathcal{D}})^T & \xrightarrow{\subset} & \iota_\Gamma^*\mathbb{E}_{\mathcal{W}/\mathcal{D}} \\ \big\uparrow \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & & \big\uparrow \phi^\vee_{\mathcal{W}/\mathcal{D}} \\ \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & \iota_\Gamma^*\mathbb{T}_{\mathcal{W}/\mathcal{D}}. \end{array} \tag{3.14}$$

By the construction of the cosection, $\sigma_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} = (\iota_\Gamma^*\sigma_{\mathcal{W}/\mathcal{D}})^T$. Further (3.13) induces an arrow $q^*\mathbb{T}_{\mathcal{D}_\Gamma}[-1] \to \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$, which when composed with $\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ in (3.14) defines the arrow c below:

$$\begin{array}{ccccccccc} \longrightarrow & q^*\mathbb{T}_{\mathcal{D}_\Gamma}[-1] & \xrightarrow{c} & \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & \mathbb{E}_{\mathcal{W}_\Gamma} & \xrightarrow{+1} \\ & \big\downarrow \epsilon & & \big\| & & \big\downarrow \epsilon' & \\ \longrightarrow & (\iota_\Gamma^*\tilde q^*\mathbb{T}_{\mathcal{D}})^T[-1] & \longrightarrow & (\iota_\Gamma^*\mathbb{E}_{\mathcal{W}/\mathcal{D}})^T & \longrightarrow & (\iota_\Gamma^*\mathbb{E}_{\mathcal{W}})^T & \xrightarrow{+1} & . \end{array} \tag{3.15}$$

Let ϵ be the tautological homomorphism. By the construction of $\mathbb{E}_{\mathcal{W}_\Gamma}$ and $\mathbb{E}_{\mathcal{W}}$, both rows are d.t.s. We chose the third vertical arrow ϵ' to be the one making (3.15) a homomorphism of d.t.s. It is an isomorphism after ϵ is shown to be an isomorphism.

The proof that ϵ is an isomorphism can be achieved with the aid of the stack $\mathcal{M}$, which is the stack of pointed twisted nodal curves. Let $\mathcal{M}_T$ be the stack of pointed twisted nodal curves together with T-actions. As the composites $\mathcal{D}_\Gamma \xrightarrow{p} \mathcal{D} \xrightarrow{f} \mathcal{M}$ and $\mathcal{D}_\Gamma \xrightarrow{h} \mathcal{M}_T \to \mathcal{M}$ are identical, we obtain the following homomorphism of d.t.s:

$$\begin{array}{ccccccc} (p^*\mathbb{T}_{\mathcal{D}/\mathcal{M}})^T & \longrightarrow & (p^*\mathbb{T}_{\mathcal{D}})^T & \longrightarrow & (p^*f^*\mathbb{T}_{\mathcal{M}})^T & \xrightarrow{+1} \\ \big\uparrow \alpha_1 & & \big\uparrow \alpha_2 & & \big\uparrow \alpha_3 & \\ \mathbb{T}_{\mathcal{D}_\Gamma/\mathcal{M}_T} & \longrightarrow & \mathbb{T}_{\mathcal{D}_\Gamma} & \longrightarrow & h^*\mathbb{T}_{\mathcal{M}_T} & \xrightarrow{+1} & . \end{array} \tag{3.16}$$

We can then easily verify that both α_1 and α_3 are isomorphisms. By the five lemma, α_2 is an isomorphism. This proves that ϵ in (3.15) is an isomorphism.

By (3.14) and [GP, Prop. 1], the composite $\mathbb{T}_{\mathcal{W}_\Gamma} \to \mathbb{T}_{\mathcal{W}} \to \iota_\Gamma^* \mathbb{E}_{\mathcal{W}}$ lifts to an obstruction theory $\phi^\vee_{\mathcal{W}_\Gamma}$, making the following square commutative:

$$\begin{array}{ccc} \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & \mathbb{E}_{\mathcal{W}_\Gamma} \\ \uparrow \phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & & \uparrow \phi^\vee_{\mathcal{W}_\Gamma} \\ \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma}. \end{array} \tag{3.17}$$

We then take H^1 of the third column in (3.15) to obtain $\mathcal{O}b_{\mathcal{W}_\Gamma} \cong (\iota_\Gamma^* \mathcal{O}b_{\mathcal{W}})^T$. Further we can easily check that the two cosections coincide, which implies that they have identical vanishing locus $\mathcal{W}_\Gamma^-$. By the same reasoning as before, we conclude that the localized virtual class $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ defined in (3.2) is identical to the class $0^!_{\mathrm{loc}}[\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}]$ (also see [KKP, Prop. 3]). This proves the lemma. □

4. The vanishing in no-string cases

We first prove a special case of Theorem 1.2. Let $\Gamma \in \Delta^{\mathrm{fl}}$. A *string* of Γ is an $e \in E_{0\infty}(\Gamma)$ so that the vertex v of e lying in $V_0(\Gamma)$ is unstable and has no other edge attached to it.

Proposition 4.1. *Let $\Gamma \in \Delta^{\mathrm{fl}}$ be irregular and not a pure loop. Suppose it does not contain strings, then $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$.*

Remark 4.2. *Recall the convention on flat graphs. Let $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots) \in \mathcal{W}_\Gamma$ be any closed point. Note that Γ might be different from Γ_ξ, which happens when Γ_ξ is not flat, while Γ is the flattening of Γ_ξ. In the case $V_1(\Gamma) = \emptyset$, then this happens when every $v \in V_1(\Gamma_\xi)$ has two edges e_{v-} and e_{v+} attached to it, and $\{v, e_{v-}, e_{v+}\}$ in Γ_ξ is replaced by a single edge $e(v) \in E_{0\infty}(\Gamma)$. Our convention is that $\mathcal{C}_{e(v)} = \mathcal{C}_{e_{v-}} \cup \mathcal{C}_{e_{v+}}$.*

We say Γ is *bare* if $V_1(\Gamma) = \emptyset$. We begin with a special case.

Lemma 4.3. *Let the situation be as in Proposition 4.1. Suppose $V_1(\Gamma) = \emptyset$ and $V_0(\Gamma) \neq \emptyset$. Then $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$.*

Proof. Since Γ is fixed throughout this proof, for simplicity we will use V, E, etc., to denote $V(\Gamma)$, $E(\Gamma)$, etc. Recall that for $v \in V^S (= V^S(\Gamma))$, E_v is the set of nodes $\mathcal{C}_v \cap (\cup_{e \in E} \mathcal{C}_e)$, and S_v is the set of legs incident to v (cf. (2.6)).

We introduce more notations. Let $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots) \in \mathcal{W}_\Gamma$ be a closed point; let $v \in V$. For $a \in S_v$, in case $\langle \gamma_a \rangle \neq \{1\}$, let $m_a \in [1, 4]$ so that $\gamma_a = \zeta_5^{m_a}$. Let $S^1_v \subset S_v$ be the subset of legs decorated with $(1, \varphi)$ or $(1, \rho)$. (Since γ is narrow, no legs are decorated by 1.) We denote $S^1 = \cup_{v \in V} S^1_v$, $S^1_\infty = \cup_{v \in V_\infty} S^1_v$, etc. Similarly, we denote $S^{\neq 1}_v = S_v - S^1_v$, and $S^{\neq 1} = \cup_{v \in V} S^{\neq 1}_v$. By the definition of MSP fields, $S^{\neq 1}_v = \emptyset$ when $v \notin V_\infty$, implying $S^{\neq 1} = \cup_{v \in V_\infty} S^{\neq 1}_v$.

We calculate $\operatorname{vir.dim} \mathcal{W}_\Gamma$. Because the perfect obstruction theory of $\mathcal{W}_\Gamma$ is the one relative to $\mathcal{D}_\Gamma$, we have

$$\operatorname{vir.dim} \mathcal{W}_\Gamma = \operatorname{vir.dim} \mathcal{W}_\Gamma/\mathcal{D}_\Gamma + \dim \mathcal{D}_\Gamma \tag{4.1}$$

where

$$\dim \mathcal{D}_\Gamma = \sum_{v\in V^S} (3g_v - 3 + |E_v| + |S_v|) + \sum_{v\in V^S} 2g_v + 2h^1(\Gamma) - |E| - 2. \tag{4.2}$$

Here $3g_v - 3 + |E_v| + |S_v|$ represents deformations of $\Sigma^{\mathcal{C}_v} \subset \mathcal{C}_v$ (where $\Sigma^{\mathcal{C}_v}$ is defined in (2.6)). $\sum_{v\in V^S} 2g_v$ represents deformations of $\mathcal{L}$ and $\mathcal{N}$ restricting to $\mathcal{C}_0$ and $\mathcal{C}_\infty$. The term $2h^1(\Gamma)$ is the deformations of $\mathcal{L}$ and $\mathcal{N}$ attributed to loops in Γ; $|E|$ represents automorphisms of $\mathcal{C}$; and -2 is due to the automorphisms of $\mathcal{L}$ and $\mathcal{N}$.

Next, using the relative perfect obstruction theory of $\mathcal{W}_\Gamma/\mathcal{D}_\Gamma$, we know that $\operatorname{vir.dim} \mathcal{W}_\Gamma/\mathcal{D}_\Gamma$ is the sum of (4.3) and (4.4):

$$\chi_T(\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1) + \chi_T(\mathcal{N}); \tag{4.3}$$

$$\chi_T\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus 5}\big) + \chi_T\big(\mathcal{L}^{\vee\otimes 5}\otimes\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})\big). \tag{4.4}$$

Next, since $\nu_1|_{\mathcal{C}_\infty} = \nu_2|_{\mathcal{C}_0} = 1$, as T sheaves we have $\mathcal{N}|_{\mathcal{C}_0} \cong \mathcal{O}_{\mathcal{C}_0}$ and $\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1|_{\mathcal{C}_\infty} \cong \mathcal{O}_{\mathcal{C}_\infty}$. Let $e \in E_{0\infty}(\Gamma)$ be such that the associated curve $\mathcal{C}_e \cong \mathbb{P}^1$, and let $q_0 = \mathcal{C}_e \cap \mathcal{C}_0$ and $q_\infty = \mathcal{C}_e \cap \mathcal{C}_\infty$. Then using $\nu_1|_{q_0} = 0$, the invariance of ν_1 implies that T acts non-trivially on $\mathcal{C}_e$, thus forcing T to act non-trivially on $\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1|_{q_0}$. Then as $\nu_2|_{q_\infty} = 0$, T also acts non-trivially on $\mathcal{N}|_{q_\infty}$. In case $\mathcal{C}_e$ is a union of two $\mathbb{P}^1$'s, a parallel argument shows that the same conclusion holds. Thus as Γ is connected,

$$(4.3) = \chi(\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1|_{\mathcal{C}_\infty}) + \chi(\mathcal{N}|_{\mathcal{C}_0}) = \sum_{v\in V_0}(1-g_v) + \sum_{v\in V_\infty}(1-g_v).$$

Here when $\mathcal{C}_v$ is a point, we set $g_v = 0$.

To proceed, we let $\xi = (\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots) \in \mathcal{W}_\Gamma$ as before. Let χ_T of a T-sheaf be the T-equivariant χ of the sheaf. We claim

$$\chi_T\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\big) = \chi\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\big). \tag{4.5}$$

Indeed, let $v \in V_0(\Gamma)$, then because $\varphi|_{\mathcal{C}_v} \neq 0$ and since T-acts trivially on $\mathcal{C}_v$, T acts trivially on $\mathcal{L}|_{\mathcal{C}_v}$. For the same reason, for $v \in V_\infty(\Gamma)$, T acts trivially on both $\mathcal{C}_v$ and $\mathcal{L}|_{\mathcal{C}_v}$. On the other hand, suppose $E_{0\infty}(\Gamma) = \{e\}$ has only one element, with the associated curve $\mathcal{C}_e$. In case $\mathcal{C}_e \cong \mathbb{P}^1$, by Lemma 2.6 we have $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}$. Then (4.5) follows. In case $\mathcal{C}_e$ consists of two $\mathbb{P}^1$'s, write $\mathcal{C}_e = \mathcal{C}_{e-} \cup \mathcal{C}_{e+}$, with $q = \mathcal{C}_{e-} \cap \mathcal{C}_{e+}$, $y_- = \mathcal{C}_{e-} \cap \mathcal{C}_0$ and $y_+ = \mathcal{C}_{e+} \cap \mathcal{C}_\infty$. By Lemma 2.6, $\deg \mathcal{L}|_{\mathcal{C}_{e-}} = -\deg \mathcal{L}|_{\mathcal{C}_{e+}} > 0$, thus

$$H^i_T\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\big) = H^i\big(\mathcal{L}|_{\mathcal{C}_\infty}(-y_{e+})\big) \oplus H^i\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})|_{\mathcal{C}_0}\big),$$

and consequently (4.5) follows. The case where $E_{0\infty}(\Gamma)$ contains many edges is similar. This proves (4.5).

We next claim that (4.5) holds with $\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})$ replaced by $\mathcal{L}^{\vee\otimes 5}\otimes\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})$. As before, we first consider the case $E_{0\infty}(\Gamma)=\{e\}$. Because Γ contains no strings, $\mathcal{C}_e\cap\mathcal{C}_0$ is a node of $\mathcal{C}$. By Lemma 2.6, $\mathcal{C}_e\cap\mathcal{C}_\infty$ is also a node of $\mathcal{C}$. Thus $\deg\mathcal{L}^{\vee\otimes 5}\otimes\omega^{\log}_{\mathcal{C}}|_{\mathcal{C}_e}=0$. Then the proof of (4.5) shows that the claim holds in this case. The case $|E_{0\infty}(\Gamma)|>1$ can be treated similarly. This proves the claim.

Consequently,

$$\begin{aligned}(4.4) &= 5\cdot\chi\big(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\big)+\chi\big(\mathcal{L}^{\vee\otimes 5}\otimes\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})\big)\\ &= -5|\Sigma^{\mathcal{C}}_{(1,\varphi)}|+5\big(\deg\mathcal{L}+1-g-\sum_{a\in S^{\neq 1}}\frac{m_a}{5}\big)+\\ &\qquad +\big(2g-2+|S|-5\deg\mathcal{L}-|\Sigma^{\mathcal{C}}_{1,\varrho}|+1-g\big)\\ &= 4(1-g)-4|S^1_\infty|-\sum_{a\in S^{\neq 1}}(m_a-1).\end{aligned}$$

Because Γ is bare, $V^S=V^S_0\cup V^S_\infty$; because Γ has no string, $V_0^{1,1}=\emptyset$. Therefore $\Sigma^{\mathcal{C}}_{(1,\rho)}=\cup_{v\in V_0}S_v=\cup_{v\in V_0^S}S_v$. Similarly, for any $v\in V^U_\infty(=V_\infty-V^S_\infty)$ that has a leg attached to it, v has exactly one edge e attached to it, which must lie in $E_{0\infty}$ as $V_1=\emptyset$. By Corollary 2.10, the leg of v must be a scheme marked point (i.e. in $\Sigma_{(1,\varphi)}$). Thus $S^{\neq 1}=\cup_{v\in V_\infty}S^{\neq 1}_v$ is the same as $\cup_{v\in V^S_\infty}S^{\neq 1}_v$. Putting the above together we obtain

$$\sum_{v\in V^S}|S_v|=|\Sigma^{\mathcal{C}}_{(1,\rho)}|+|S^{\neq 1}|+\sum_{v\in V^S_\infty}|S^1_v|. \tag{4.6}$$

Assumption I. *No leg of Γ is decorated by $(1,\rho)$, and $m_a\neq 1$ for every $a\in S^{\neq 1}$.*

Under this assumption, we have the Euler equation $|E|-|V|=h^1(\Gamma)-1$, $g=\sum_{v\in V^S}g_v+h^1(\Gamma)$, and $\Sigma^{\mathcal{C}}_{(1,\rho)}=\emptyset$. Using (4.1), and adding (4.2), (4.3) and (4.4), we obtain

$$\operatorname{vir.dim}\mathcal{W}_\Gamma=\big(\sum_{v\in V^S}|E_v|-4|S^1_\infty|+\sum_{v\in V^S_\infty}|S^1_v|\big)-3(|E|-|V^U|)-\sum_{a\in S^{\neq 1}}(m_a-2). \tag{4.7}$$

Note that when Γ is a pure loop, it is zero. We now prove that under the assumption of the Proposition 4.1, (4.7) is negative when $[\mathcal{W}_{(\Gamma)}]^{\text{vir}}_{\text{loc}}\neq 0$, which is impossible.

We first consider the case where $V^S=\emptyset$. Since Γ is not a pure loop, $\mathcal{C}$ is a chain of $\mathbb{P}^1$'s connecting two vertices v and v'. Since $V_1=\emptyset$, $E=E_{0\infty}$. Since Γ has no strings, both v and $v'\in V_\infty$. Then by Corollary 2.10, each v and v' each has one leg in $\Sigma_{(1,\varphi)}$ attached to it. Thus $|S^1_\infty|=2$ and $|E|-V^U|=-1$, implying that (4.7) is $-4\cdot 2+3<0$.

We now assume $V^S \neq \emptyset$. Our strategy is to divide the contribution in (4.7) by looking at the maximal simple chains in Γ. Here a simple chain in Γ consists of distinct edges $E_1, \cdots, E_k$ and vertices $v_0, \cdots, v_k$ so that E_i has vertices v_{i-1} and v_i, and $v_{0<i<k}$ are unstable. Since Γ is not a pure loop and $|V^S| > 0$, if $\{E_1, \cdots, E_k\}$ is a maximal simple chain in Γ, then one of $\{v_0, v_k\}$ must be stable.

Clearly maximal simple chains give partitions of E and V^U. Now let $\{E_1, \cdots, E_k\}$ be a maximal simple chain in Γ. Suppose v_0 is stable but v_k is not, then $v_k \in V^U_\infty$ because Γ contains no strings. Thus $|S^1_{v_k}| = 1$ by Corollary 2.10. Therefore the contribution to (4.7) from $\{E_1, \cdots E_k, v_1, \cdots, v_k\}$ is

$$1 - 4|S^1_{v_k}| = -3. \tag{4.8}$$

The other case is when both v_0 and v_k are stable in the maximal chain. Then the contribution to (4.7) from $\{E_1, \cdots E_k, v_1, \cdots, v_{k-1}\}$ is

$$2 - 3 = -1. \tag{4.9}$$

We now show that (4.7) is non-positive. Let Γ' be the graph resulting from removing all edges, all unstable vertices, and all legs attached to unstable vertices. Because every $e \in E$ or $v \in V^U$ is contained in exactly one maximal simple chain, the previous argument shows that

$$\mathrm{vir.}\dim \mathcal{W}_\Gamma \leq \mathrm{vir.}\dim \mathcal{W}_{\Gamma'}.$$

Applying formula (4.7) to $\mathrm{vir.}\dim \mathcal{W}_{\Gamma'}$, we see that it contains terms of the following kind: (i) terms associated with elements in $\cup_{v \in V^S} S^1_v$, each contributing $-4+1=-3$; (ii) terms associated with elements in $\cup_{v \in V^U} S^1_v$, each contributing -4; and (iii) terms associated with elements $a \in S^{\neq 1}$, each contributing $m_a - 2 \leq 0$ since $m_a \geq 2$ by our *simplifying assumption*. This shows that (4.7) is ≤ 0.

When (4.7) is zero, we must have $E = V^U = S^1 = \emptyset$, and $m_a = 2$ for every $a \in S^{\neq 1}$. But this is impossible because Γ is irregular. This proves that under the *simplifying assumption* and when Γ is not a pure loop, $\mathrm{vir.}\dim \mathcal{W}_\Gamma < 0$, implying $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$.

We now prove the proposition without making *Assumption I*. First, suppose Γ has a leg i_0 (i_0-th leg) decorated by $\gamma_{i_0} = \zeta_5$ attached to $v \in V_\infty$. We claim that v is stable. Indeed, if v is not stable, it would have an edge e attached to it. Since Γ is bare $e \in E_{0\infty}(\Gamma)$; but by Lemma 2.6 $d_e \in \mathbb{Z}$, which contradicts to the assumption that i_0 is decorated by ζ_5. Thus v is stable. We let Γ' be the graph obtained by removing the leg i_0 from Γ, as long as Γ is not an one vertex graph with $(g_v, |S_v|) = (0,3)$. (Note that since Γ is irregular, $(g_v, |S_v|) = (1,1)$ is impossible.) Note that if $g_v = 0$, $|S_v| + |E_v| = 3$, and v has at least one edge, then v in Γ' becomes unstable.

Following [CLL, Thm. 4.5], we have a forgetful morphism

$$\mathcal{F} : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_{\Gamma'}$$

that sends $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \cdots) \in \mathcal{W}_\Gamma$ to $\xi' = (\Sigma^{\mathcal{C}'}, \mathcal{C}', \cdots) \in \mathcal{W}_{\Gamma'}$ by *forgetting the marking and stabilizing*.

Marking forgetting and stabilizing. *The curve $\mathcal{C}'$ is obtained from $\mathcal{C}$ by forgetting the marking $\Sigma^{\mathcal{C}}_{i_0}$, making $\mathcal{C}$ scheme along $\Sigma^{\mathcal{C}}_{i_0}$, and stabilizing if necessary, with $\mathcal{C}'$ as the resulting curve; $\Sigma^{\mathcal{C}'}$ is $\Sigma^{\mathcal{C}}$ with $\Sigma^{\mathcal{C}}_{i_0}$ deleted; let $\epsilon:\mathcal{C}\to\mathcal{C}'$ be the resulting morphism, and let $\mathcal{L}'=\epsilon_*\mathcal{L}$ and $\mathcal{L}'\otimes\mathcal{N}'=\epsilon_*(\mathcal{L}\otimes\mathcal{N})$, while φ', etc., is the pushforward of φ, etc., respectively.*

We next compare the virtual cycles $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ and $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}$. First, by *marking forgetting and stabilizing*, we obtain a morphism $f:\mathcal{D}_\Gamma\to\mathcal{D}_{\Gamma'}$, which fits into the following commutative square:

$$\begin{array}{ccc} \mathcal{W}_\Gamma & \xrightarrow{\mathcal{F}} & \mathcal{W}_{\Gamma'} \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}_\Gamma & \xrightarrow{f} & \mathcal{D}_{\Gamma'}. \end{array} \tag{4.10}$$

Let $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\cdots)$ and $(\mathcal{C}',\Sigma^{\mathcal{C}'},\mathcal{L}',\cdots)$ be the universal families of $\mathcal{W}_\Gamma$ and $\mathcal{W}_{\Gamma'}$, respectively, with $\pi:\mathcal{C}\to\mathcal{W}_\Gamma$ and $\pi':\mathcal{C}'\to\mathcal{W}_{\Gamma'}$ their projections. The stabilization defines the Ψ below

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\Psi} & \mathcal{C}'\times_{\mathcal{W}_{\Gamma'}}\mathcal{W}_\Gamma & \xrightarrow{\mathrm{pr}} & \mathcal{C}' \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\tilde\pi} & & \big\downarrow{\scriptstyle\pi'} \\ \mathcal{W}_\Gamma & = & \mathcal{W}_\Gamma & \xrightarrow{\mathcal{F}} & \mathcal{W}_{\Gamma'}. \end{array}$$

It is easy to check that we have canonical isomorphisms $\mathrm{pr}^*\mathcal{L}'\cong\Psi_*\mathcal{L}$, and $R^1\Psi_*\mathcal{L}=0$. This implies

$$R\pi^T_*\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\cong R\tilde\pi^T_*\Psi_*\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})\cong\mathcal{F}^*R\pi'^T_*\mathcal{L}'(-\Sigma^{\mathcal{C}'}_{(1,\varphi)}),$$

and similar isomorphisms with $\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})$ replaced by $\mathcal{L}^{-5}\otimes\omega_{\mathcal{C}/\mathcal{W}_\Gamma}(-\Sigma^{\mathcal{C}}_{(1,\rho)})$, etc.

As in [CLL], the above shows that the relative obstruction theory of $\mathcal{W}_{\Gamma'}\to\mathcal{D}_{\Gamma'}$ is pulled back to that of $\mathcal{W}_\Gamma\to\mathcal{D}_\Gamma$, and the cosection of $\mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}}$ is pulled back to that of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$. Thus, letting $\theta=\mathcal{F}|_{\mathcal{W}^-_\Gamma}:\mathcal{W}^-_\Gamma\to\mathcal{W}^-_{\Gamma'}$, we have

$$\theta^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}=[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}. \tag{4.11}$$

In case Γ has a leg decorated by $(1,\rho)$, we remove this leg from Γ, resulting in a new graph Γ'. (In this case since Γ is irregular, Γ cannot be a single vertex graph.) Then we have a similarly defined forgetful morphism $\mathcal{F}:\mathcal{W}_\Gamma\to\mathcal{W}_{\Gamma'}$ (with stabilization if necessary) and θ as before so that (4.11) holds.

By repeating this procedure (of removing legs labeled by ζ_5 or $(1,\rho)$), we obtain a graph Γ' and morphisms $\mathcal{F}$ and θ as before so that (4.11) holds. As Γ' is bare, not a pure-loop and satisfies *Assumption I*, we have $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}=0$. By (4.11), $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}=0$. This proves the lemma. □

Proof of Proposition 4.1. By a result proved at the end of [CL^{3}2, Section 3], we know that $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}=0$ if there is a $v\in V^{0,2}_\infty(\Gamma)$ so that the two edges e in Γ

incident to v both lie in $E_\infty(\Gamma)$ and have $d_e = \deg \mathcal{L}|_{\mathcal{C}_e} \in \mathbb{Z}$. We now suppose that Γ has no such vertices.

We next trim all edges of Γ in $E_0(\Gamma) \cup E_\infty(\Gamma)$. For $e \in E_0(\Gamma)$, in case e is incident to a stable $v \in V_0^S(\Gamma)$, or in case e is incident to an unstable $v \in V_0^U(\Gamma)$ so that another edge in $E(\Gamma)$ is also incident to v, we remove e and add a new leg decorated by $(1,\rho)$ and attach it to v; otherwise we remove e, v, and any other legs incident to v.

For $e \in E_\infty(\Gamma)$, in case e is incident to a stable $v \in V_\infty^S(\Gamma)$, or in case e is incident to an unstable $v \in V_0^U(\Gamma)$ so that another edge in $E(\Gamma)$ is also incident to v, we remove e and add a new leg decorated by $\gamma_{(e,v)}$[2] and attach it to v; otherwise we remove e, v, and any other legs incident to v. After performing these operations on all e in $E_0(\Gamma)$ and $E_\infty(\Gamma)$, and after discarding all vertices in $V_1(\Gamma)$, we obtain a new graph Γ'. Let $\{\Gamma_i\}$ be the connected components of Γ'.

Applying the discussion [CL^{3}2, Section 3] to this situation, we conclude that if $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} = 0$, then $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$. By our assumption on Γ, we know that all Γ_i in $\{\Gamma_i\}$ are non-loop and bare; and at least one such Γ_i is irregular. Because

$$[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} = \prod [\mathcal{W}_{\Gamma_i}]^{\mathrm{vir}}_{\mathrm{loc}},$$

by applying Lemma 4.3, we have that $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} = 0$. This proves the proposition. □

Corollary 4.4. *In case Γ consists of a single stable vertex $v \in V_\infty(\Gamma)$ such that its legs are decorated by $\gamma_1, \cdots, \gamma_\ell \in \boldsymbol{\mu}_5 - \{1\}$ and that at least one $\gamma_i \in \{\zeta_5^3, \zeta_5^4\}$, then $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = 0$ except when $g_v = 0$ and $\gamma = (1^{1+k}23)$ or $= (1^{2+k}4)$, for a $k \geq 0$.*

5. Reduction to no-string cases

The proof of the general case is by reduction to no-string cases. To this end, we introduce the operation *trimming a leaf edge* from a graph.[3]

Definition 5.1. *Let $\Gamma \in \Delta^{\mathrm{fl}}$ and let $e \in E_{0\infty}(\Gamma)$ be a string (thus a leaf edge). Let $v_- \in V_0(\Gamma)$ and $v_+ \in V_\infty(\Gamma)$ be its vertices. The edge e is trimmed from Γ by first removing e, v_- and all legs attached to v_-, and then attaching a leg, called the distinguished leg, decorated by $(1,\varphi)$ to v_+.*

In the following, we will apply induction on the number of strings to prove Theorem 1.2. We fix a Γ with a string e, and its two associated vertices $v_\pm$, as in Definition 5.1. We assume $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \neq 0$, and shall derive a contradiction toward the end. We denote by Γ' the graph after trimming e from Γ.

As before, let $\mathcal{D}_{\Gamma,\nu}$ be the stack of Γ-framed gauged curves $((\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu), \epsilon)$. For any family $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu)$ (with ϵ implicitly understood) in $\mathcal{D}_{\Gamma,\nu}$, because e is a string of Γ, the correspondence $a = (e, v_+) \in F(\Gamma)$ is associated with a section

[2]We assign $\gamma_{(e,v)} = (1,\varphi)$ in case $d_e \in \mathbb{Z}$, otherwise $\gamma_{(e,v)} = e^{-2\pi\sqrt{-1}d_e}$ (cf. before Defi. 2.3).

[3]A leaf edge is an edge so that one of its vertex is unstable and has only one edge attached to it.

of nodes $\mathcal{R}_a \subset \mathcal{C}$ that splits off a family of rational curves $\mathcal{C}^e \subset \mathcal{C}$ (associated with e), called the e-tail of $\mathcal{C}$.[4] We let

$$\mathcal{C}^\diamond = \overline{\mathcal{C} - \mathcal{C}^e} \subset \mathcal{C}$$

be the complement of $\mathcal{C}^e$ in $\mathcal{C}$.

We consider the family

$$(\mathcal{C}^\diamond, \Sigma^{\mathcal{C}} \cap \mathcal{C}^\diamond + \mathcal{R}_a, \mathcal{L}|_{\mathcal{C}^\diamond}, \mathcal{N}|_{\mathcal{C}^\diamond}, \nu|_{\mathcal{C}^\diamond}). \tag{5.1}$$

Together with the induced framing, (5.1) is a family in $\mathcal{D}_{\Gamma',\nu}$. As this construction is canonical, we obtain a forgetful morphism

$$\mathcal{D}_{\Gamma,\nu} \longrightarrow \mathcal{D}_{\Gamma',\nu}.$$

We need another stack of elements in $\mathcal{D}_{\Gamma,\nu}$ paired with fields on its e-tail. Given $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu) \in \mathcal{D}_{\Gamma,\nu}$, we abbreviate

$$\mathcal{L}^{\log} = \mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)}), \quad \text{and} \quad \mathcal{P}^{\log} = \mathcal{L}^{\vee\otimes 5} \otimes \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)}).$$

Definition 5.2. *Let $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu) \in \mathcal{D}_{\Gamma,\nu}$. A (φ,ρ)-field on its e-tail is*

$$(\varphi^e, \rho^e) = (\varphi^e_1, \cdots, \varphi^e_5, \rho^e) \in H^0(\mathcal{L}^{\log}|_{\mathcal{C}^e})^{\oplus 5} \oplus H^0(\mathcal{P}^{\log}_{\mathcal{C}^e}).$$

A partial e-field on a Γ-framed gauged curve consists of a $\zeta \in \mathcal{D}_{\Gamma,\nu}$ and a (φ,ρ)-field on its e-tail.

We let $\mathcal{Y}_{\Gamma,\nu,e}$ be the groupoid of families of partial e-fields on Γ-framed gauged curves. That is, elements in $\mathcal{Y}_{\Gamma,\nu,e}$ are $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu, \varphi^e, \rho^e)$ (with the Γ-framing implicitly understood) as in Definition 5.2.

Let $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ to be the universal family on $\mathcal{W}_\Gamma$. As before, the flag $a = (e, v_+) \in F(\Gamma)$ is associated with a section of nodes $\mathcal{R}_a \subset \mathcal{C}$ that splits $\mathcal{C}$ into two subfamilies $\mathcal{C}^e$ and $\mathcal{C}^{e\circ}$. The family

$$(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \nu, \varphi|_{\mathcal{C}^e}, \rho|_{\mathcal{C}^e})$$

then is a family in $\mathcal{Y}_{\Gamma,\nu,e}$, which induces a forgetful morphism $\delta : \mathcal{W}_\Gamma \to \mathcal{Y}_{\Gamma,\nu,e}$. Of course, by forgetting the fields on the e-tail, we obtain a forgetful morphism $\zeta : \mathcal{Y}_{\Gamma,\nu,e} \to \mathcal{D}_{\Gamma,\nu}$.

To proceed, let $\bar{e}$ be the graph which is an edge e with two vertices v_- and v_+, together with the decorations on e and the legs on v_- (if any), plus a new leg decorated by $1 = \zeta_5^0$ attached to v_+. Note that because of the decoration 1, $\bar{e}$ is of the broad type (cf. the first paragraph in section 2.1).

Let $\mathcal{W}_{\Gamma'}$ and $\mathcal{W}_{\bar{e}}$ be the moduli stack of stable Γ' and $\bar{e}$-framed MSP fields, respectively. By restricting the universal family of $\mathcal{Y}_{\Gamma,\nu,e}$ to its e-tails, we obtain a family on $\mathcal{C}^e$, which induces a morphism $\mathcal{Y}_{\Gamma,\nu,e} \to \mathcal{W}_{\bar{e}}$. We list these morphisms together:

$$\mathcal{W}_\Gamma \xrightarrow{\delta} \mathcal{Y}_{\Gamma,\nu,e} \longrightarrow \mathcal{W}_{\bar{e}} \quad \text{and} \quad \mathcal{Y}_{\Gamma,\nu,e} \xrightarrow{\zeta} \mathcal{D}_{\Gamma,\nu}. \tag{5.2}$$

[4] Fibers of $\mathcal{C}^e$ can be one $\mathbb{P}^1$, or a union of two $\mathbb{P}^1$'s. See Remark 4.2.

By restricting the universal family of $\mathcal{W}_\Gamma$ to $\mathcal{C}^\diamond$, we would like to obtain a family in $\mathcal{W}_{\Gamma'}$, thereby obtaining a morphism from $\mathcal{W}_\Gamma$ to $\mathcal{W}_{\Gamma'}$. Unfortunately, this in general is not possible because $\varphi|_{\mathcal{R}_a}$ might not vanish identically, thus the restriction does not necessarily induce a morphism $\mathcal{W}_\Gamma \to \mathcal{W}_{\Gamma'}$. (Recall that $\mathcal{R}_a$ is associated with a marking of Γ' labeled by $(1,\varphi)$.)

To remedy this situation, let $\mathcal{W}^\mu_{\bar e} = (\mathcal{W}_{\bar e})_{\rm red}$ be $\mathcal{W}_{\bar e}$ with the reduced stack structure; let

$$\mathcal{Y}^\mu_{\Gamma,\nu,e} = \mathcal{Y}_{\Gamma,\nu,e} \times_{\mathcal{W}_{\bar e}} \mathcal{W}^\mu_{\bar e}, \quad \text{and} \quad \mathcal{W}^\mu_\Gamma = \mathcal{W}_\Gamma \times_{\mathcal{W}_{\bar e}} \mathcal{W}^\mu_{\bar e}. \tag{5.3}$$

Lemma 5.3. *The stack $\mathcal{W}_{\bar e}$ has pure dimension four; it has hypersurface singularities, and is acted on by the group $GL(5,\mathbb{C})$. The coarse moduli of $\mathcal{W}^\mu_{\bar e} = (\mathcal{W}_{\bar e})_{\rm red}$ is isomorphic to $\mathbb{P}^4$, and the induced $GL(5,\mathbb{C})$ action on this coarse moduli is the standard $GL(5,\mathbb{C})$ action on $\mathbb{P}^4$.*

Proof. We begin by classifying the closed points of $\mathcal{W}_{\bar e}$. Let $\xi = (\mathcal{C},\Sigma^{\mathcal{C}},\cdots) \in \mathcal{W}_{\bar e}$ be a closed point, and let Γ_ξ be its associated graph. We claim that $\Gamma_\xi \neq \Gamma^{\rm fl}_\xi$. Indeed, in case $\Gamma_\xi = \Gamma^{\rm fl}_\xi$, then $\mathcal{C} \cong \mathbb{P}^1$ and T acts on $\mathcal{C}$ with two fixed points, p_- and p_+, associated with the vertices $v_- \in V_0(\Gamma_\xi)$ and $v_+ \in V_\infty(\Gamma_\xi)$, respectively. Because we have assumed that $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc} \neq 0$, by Corollary 2.10, we have $\deg \mathcal{L} = 0$. Since p_+ is a marking decorated by 1, and p_- is either a non-marking or a marking decorated by $(1,\rho)$, we have $\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, forcing $\rho = 0$, contradicting $\rho|_{p_+} \neq 0$. This proves $\Gamma_\xi \neq \Gamma^{\rm fl}_\xi$.

When $\Gamma_\xi \neq \Gamma^{\rm fl}_\xi$, Γ_ξ contains two edges: $e_+ \in E_\infty(\Gamma_\xi)$ and $e_- \in E_0(\Gamma_\xi)$. Let $\mathcal{C}_\pm \subset \mathcal{C}$ be the irreducible component associated with $e_\pm$. Then $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$ with one node q associated with the vertex in $V_1(\Gamma_\xi)$. Let $p_\pm \in \mathcal{C}_\pm \subset \mathcal{C}$ be the two T fixed points (other than q) as before. Then by the definition of MSP fields, $\mathcal{N}|_{\mathcal{C}_-}$ and $\mathcal{L}\otimes\mathcal{N}|_{\mathcal{C}_+}$ are trivial. Adding $\deg\mathcal{L} = 0$ and $\deg\mathcal{N} = c$, where $c = d_{\infty e}$, we get $\mathcal{L}|_{\mathcal{C}_-} \cong \mathcal{O}_{\mathcal{C}_-}(c)$, $\mathcal{L}|_{\mathcal{C}_+} \cong \mathcal{O}_{\mathcal{C}_+}(-c)$ and $\mathcal{N}|_{\mathcal{C}_+} \cong \mathcal{O}_{\mathcal{C}_+}(c)$. Consequently,

$$\varphi|_{\mathcal{C}_+} = \rho|_{\mathcal{C}_-} = 0, \tag{5.4}$$

and because $\varphi|_{p_-}$ and $\rho|_{p_+}$ are non-trivial,

$$\varphi|_{\mathcal{C}_-} \in H^0(\mathcal{O}_{\mathcal{C}_-}(c)^{\oplus 5})^T - 0 \cong \mathbb{C}^5 - 0, \quad \rho|_{\mathcal{C}_+} \in H^0(\mathcal{O}_{\mathcal{C}_-})^T - 0 \cong \mathbb{C} - 0.$$

Because ν_1 and ν_2 are non-trivial and unique up to scaling (T-equivariant), we see that ξ is uniquely parameterized by

$$[\varphi_1(p_-),\cdots,\varphi_5(p_-)] \in \mathbb{P}^4.$$

Repeating a family version of this argument, we prove that the coarse moduli of $\mathcal{W}^\mu_{\bar e}$ is isomorphic to $\mathbb{P}^4$.

The mentioned $GL(5,\mathbb{C})$ action on $\mathcal{W}_{\bar e}$ is an obvious one. Given any family in $\mathcal{W}_{\bar e}$, which is given by $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$, we define $\sigma\cdot(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi,\rho,\nu)$ to be $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\sigma\cdot\varphi,\rho,\nu)$, where $\sigma\cdot\varphi$ is the standard matrix multiplication after viewing φ as a column vector with component φ_i, and viewing σ as a 5×5

invertible matrix. This defines a $GL(5,\mathbb{C})$ action on $\mathcal{W}_{\bar e}$, and its action on the coarse moduli of $(\mathcal{W}_{\bar e})_{\rm red}\cong\mathbb{P}^4$ is the standard action of $GL(5,\mathbb{C})$ on $\mathbb{P}^4$.

Finally, we prove that $\mathcal{W}_{\bar e}$ has hypersurface singularity. First, we calculate the tangent space and the obstruction space of $\mathcal{W}_{\bar e}$ at its closed points. Let $\xi=(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\cdots)$ be a closed point of $\mathcal{W}_{\bar e}$. As argued before, $\mathcal{C}=\mathcal{C}_-\cup\mathcal{C}_+$, with $\deg\mathcal{L}|_{\mathcal{C}_\pm}=\mp c$ for a $c\in\mathbb{Z}_+$, $\deg\mathcal{N}|_{\mathcal{C}_-}=0$ and $\deg\mathcal{N}|_{\mathcal{C}_+}=c$. A direct calculation shows that

$$H^1(\mathcal{L}\otimes\mathcal{N}\otimes\mathbf{L}_1)^T=H^1(\mathcal{N})^T=H^1(\mathcal{L}^{\log})^T=0,\quad\text{and}\quad H^1(\mathcal{P}^{\log})=\mathbb{C}.$$

This shows that the obstruction space to deformations of $\xi\in\mathcal{W}_{\bar e}$ is always one dimensional. Because $\mathcal{W}_{\bar e}$ has pure dimension 4, we conclude that $\dim T_\xi\mathcal{W}_{\bar e}=5$ and that $\mathcal{W}_{\bar e}$ is locally defined by one equation in a smooth 5-fold, and thus $\mathcal{W}_{\bar e}$ has hypersurface singularities. □

We now compare the stacks $\mathcal{W}^\mu_\Gamma$, $\mathcal{Y}^\mu_{\Gamma,\nu,e}$, etc. We first show that the family (5.1) together with $(\varphi,\rho)|_{\mathcal{C}^\diamond}$ defines a morphism

$$\mathcal{W}^\mu_\Gamma:=\mathcal{W}_\Gamma\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathcal{Y}^\mu_{\Gamma,\nu,e}\longrightarrow\mathcal{W}_{\Gamma'}. \tag{5.5}$$

Indeed, by the prior discussion, it suffices to show that

$$\varphi|_{\mathcal{R}_a\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathcal{Y}^\mu_{\Gamma,\nu,e}}=0. \tag{5.6}$$

By the vanishing $\varphi|_{\mathcal{C}_+}=0$ in (5.4), the φ-field of any closed $\xi\in\mathcal{W}_{\bar e}$ restricted to $v_+\in\mathcal{C}$ vanishes. This shows that (5.6) holds, and the morphism (5.5) exists.

Next, by definition, the composite morphism $\mathcal{W}^\mu_\Gamma\to\mathcal{W}_\Gamma\to\mathcal{W}_{\bar e}$ (cf. (5.2)) factors through $\mathcal{W}^\mu_\Gamma\to\mathcal{W}^\mu_{\bar e}$. Pairing it with (5.5), we obtain a morphism β shown as follows:

$$\begin{array}{ccc} \mathcal{W}^\mu_\Gamma & \xrightarrow{\beta} & \mathcal{W}_{\Gamma'}\times\mathcal{W}^\mu_{\bar e}\\ \downarrow & & \downarrow\\ \mathcal{Y}^\mu_{\Gamma,\nu,e} & \xrightarrow{\beta'} & \mathcal{D}_{\Gamma',\nu}\times\mathcal{W}^\mu_{\bar e}\end{array}\quad\text{and}\quad \begin{array}{ccc} \mathcal{Y}_{\Gamma,\nu,e} & \xrightarrow{R_e} & \mathcal{W}_{\bar e}\\ \downarrow & & \downarrow\\ \mathcal{D}_{\Gamma,\nu} & \xrightarrow{r_e} & \mathcal{D}_{\bar e,\nu}.\end{array} \tag{5.7}$$

The other arrows in (5.7) are as follows. Let $\mathcal{C}_{\mathcal{Y}_{\Gamma,\nu,e}}$ be the domain curve of the universal family of $\mathcal{Y}_{\Gamma,\nu,e}$. Because curves in the family $\mathcal{C}_{\mathcal{Y}_{\Gamma,\nu,e}}$ are Γ-framed, the family contains a distinguished section of nodes $\mathcal{R}_a\subset\mathcal{C}_{\mathcal{Y}_{\Gamma,\nu,e}}$, where $a=(e,v_+)$, which splits off the e-tails $\mathcal{C}^e_{\mathcal{Y}_{\Gamma,\nu,e}}$ of $\mathcal{C}_{\mathcal{Y}_{\Gamma,\nu,e}}$. The universal family of $\mathcal{Y}_{\Gamma,\nu,e}$ restricted to $\mathcal{C}^e_{\mathcal{Y}_{\Gamma,\nu,e}}$ induces the morphism $R_e:\mathcal{Y}_{\Gamma,\nu,e}\to\mathcal{W}_{\bar e}$. The similar construction gives r_e as shown. Next, by removing the φ^e and ρ^e from the universal family of $\mathcal{Y}_{\Gamma,\nu,e}$ and then restricting the remaining part to $\mathcal{C}^\diamond_{\mathcal{Y}_{\Gamma,\nu,e}}$, we obtain a family in $\mathcal{D}_{\Gamma',\nu}$, which defines a morphism $\mathcal{Y}^\mu_{\Gamma,\nu,e}\to\mathcal{D}_{\Gamma',\nu}$. Paired this morphism with the tautological $\mathcal{Y}^\mu_{\Gamma,\nu,e}\to\mathcal{W}^\mu_{\bar e}$, we obtain the β' in (5.7). By constructions, these two squares are commutative.

Lemma 5.4. *The horizontal arrows in* (5.7) *are smooth. The morphisms β is a $\boldsymbol{\mu}_5$-torsor, and the square involving R_e and r_e is Cartesian.*

Proof. We prove that β is a $\boldsymbol{\mu}_5$-torsor. Following the construction, we see that β is surjective. We now show that it is a $\boldsymbol{\mu}_5$-torsor. Indeed, given any closed point

$$z = ((\mathcal{C}', \Sigma^{\mathcal{C}'}, \mathcal{L}, \cdots), (\mathcal{C}^e, \Sigma^{\mathcal{C}^e}, \mathcal{L}^e, \cdots)) \in \mathcal{W}_{\Gamma'} \times \mathcal{W}^{\mu}_{\bar{e}},$$

any point in $\beta^{-1}(z)$ is obtained by gluing $\mathcal{C}'$ and $\mathcal{C}^e$ along the markings in $\mathcal{C}'$ and $\mathcal{C}^e$ associated with (e_+, v_+), and gluing the $\mathcal{L}$'s and $\mathcal{N}$'s on $\mathcal{C}'$ and $\mathcal{C}^e$. As the markings are scheme points, the gluing of markings is unique. Because the section ν_1 is non-vanishing at the markings, the gluing of $\mathcal{L} \otimes \mathcal{N}$ is also unique. On the other hand, the gluing of $\mathcal{L}$ is constrained by the non-vanishing of ρ's. When one restricts ρ to the marking to be glued, it becomes a section of $\mathcal{L}^{\vee\otimes 5}$ at the marking. Thus the gluing of $\mathcal{L}$ is unique up to $\boldsymbol{\mu}_5$. As this argument works for the family, we have shown that β is a $\boldsymbol{\mu}_5$-torsor.

The other conclusions can be proved similarly and are thus omitted. □

Following [CL2, Prop. 2.5] as before, we endow $\mathcal{W}_{\Gamma'}$ and $\mathcal{W}_{\bar{e}}$ with their tautological perfect relative obstruction theories, relative to $\mathcal{D}_{\Gamma,\nu}$ and $\mathcal{D}_{\bar{e}}$, respectively. For $\mathcal{W}_{\bar{e}}$, as it is proper by Lemma 5.3, we let $[\mathcal{W}_{\bar{e}}]^{\mathrm{vir}} \in A_*\mathcal{W}_{\bar{e}}$ be its virtual class. For $\mathcal{W}_{\Gamma'}$, like $\mathcal{W}_{\Gamma}$, we form its standard cosection $\sigma_{\Gamma',\nu} : \mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} \to \mathcal{O}_{\mathcal{W}_{\Gamma'}}$, which is liftable to a cosection of $\mathcal{O}b_{\mathcal{W}_{\Gamma'}}$. Let $\mathcal{W}^-_{\Gamma'} \subset \mathcal{W}_{\Gamma'}$ be its degeneracy locus (with the reduced structure), and let $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\mathcal{W}^-_{\Gamma'}$ be its associated cosection-localized virtual class.

Let

$$\mathcal{W}^{\sim}_{\Gamma} = \mathcal{W}^{\mu}_{\Gamma} \times_{\kappa, \mathcal{W}_{\Gamma'}} \mathcal{W}^-_{\Gamma'} \subset \mathcal{W}^{\mu}_{\Gamma},$$

where $\kappa : \mathcal{W}^{\mu}_{\Gamma} \xrightarrow{\beta} \mathcal{W}_{\Gamma'} \times \mathcal{W}^{\mu}_{\bar{e}} \xrightarrow{\mathrm{pr}} \mathcal{W}_{\Gamma'}$ is the composite. Let

$$\tilde{\kappa} : \mathcal{W}^{\sim}_{\Gamma} \longrightarrow \mathcal{W}^-_{\Gamma'} \tag{5.8}$$

be induced by κ. Because β is a $\boldsymbol{\mu}_5$-torsor, $\tilde{\kappa}$ is flat. Because $\mathcal{W}^{\nu}_{\bar{e}}$ is proper, κ is a proper morphism.

Proposition 5.5. *The stack $\mathcal{W}^{\sim}_{\Gamma}$ is proper, and contains $\mathcal{W}^-_{\Gamma}$ as its closed substack. Let $\jmath : \mathcal{W}^-_{\Gamma} \to \mathcal{W}^{\sim}_{\Gamma}$ be the inclusion. Then there is a rational $c \in \mathbb{Q}$ such that*

$$\jmath_*[\mathcal{W}_{\Gamma}]^{\mathrm{vir}}_{\mathrm{loc}} = c \cdot \tilde{\kappa}^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(\mathcal{W}^{\sim}_{\Gamma}).$$

We prove Theorem 1.2 by assuming Proposition 5.5 is true.

Proof of Theorem 1.2. Let $\Gamma \in \Delta^{\mathrm{fl}}$ be irregular and not a pure loop. In case Γ has no strings, then the vanishing follows from Proposition 4.1.

Now assume Γ has strings. Let e be a string of Γ, and let Γ' be the result after trimming e from Γ. In case $\Gamma' = \emptyset$, by Lemma 2.6, the marking $\mathcal{C}_{v_+}$ is a scheme marking of type $(1, \phi)$. Thus $\mathrm{vir}\dim \mathcal{W}_{\Gamma} = \mathrm{vir}\dim \mathcal{W}_{\bar{e}} - 5 = 3 - 5 < 0$, implying $[\mathcal{W}_{\Gamma}]^{\mathrm{vir}}_{\mathrm{loc}} = 0$.

Otherwise $\Gamma' \in \Delta^{\mathrm{fl}}$ is irregular, not a pure loop, and has one less string than Γ. Thus by induction, we have $[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}} \sim 0$. By Proposition 5.5, we get $\jmath_*[\mathcal{W}_{\Gamma}]^{\mathrm{vir}}_{\mathrm{loc}} \sim 0$. In other words, there is a proper substack $\mathcal{Z}'$, $\mathcal{W}^-_{\Gamma'} \subset \mathcal{Z}' \subset \mathcal{W}_{\Gamma'}$, so that the cycle

$[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}$ pushed forward to $A_*\mathcal{Z}'$ is zero. Let $\mathcal{Z}=\kappa^{-1}(\mathcal{Z}')$. Since κ is proper, $\mathcal{Z}$ is also proper. Also, $\mathcal{W}_\Gamma^-\subset\mathcal{Z}$. Then Theorem 5.5 implies that the pushforward of $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ to $A_*\mathcal{Z}$ is zero. This proves $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}\sim 0$. □

6. Proof of Proposition 5.5

We continue to denote by $\delta:\mathcal{W}_\Gamma\to\mathcal{Y}_{\Gamma,\nu,e}$ the (representable) morphism induced by restriction. The relative obstruction theory of $\mathcal{Y}_{\Gamma,\nu,e}\to\mathcal{D}_{\Gamma,\nu}$ pullback to $\mathcal{W}_\Gamma$ takes the form

$$\delta^*\phi^\vee_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}:\delta^*\mathbb{T}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}\longrightarrow\delta^*\mathbb{E}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}=R\pi^T_*(\mathcal{U}|_{\mathcal{C}^e}).$$

Here $\mathcal{C}^e$ and $\mathcal{C}^\diamond\subset\mathcal{C}$ are the two families of subcurves (of the universal curve $\mathcal{C}$ of $\mathcal{W}_\Gamma$) after decomposing $\mathcal{C}$ along $\mathcal{R}_a$, where $a=(e,v_+)$; $\mathcal{U}$ is defined in (3.4). Let

$$\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}=R\pi^T_*\big(\mathcal{U}|_{\mathcal{C}^\diamond}(-\mathcal{R}_a)\big).$$

Recall $\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}=R\pi^T_*\mathcal{U}$. Using the exact sequence $\mathcal{U}|_{\mathcal{C}^\diamond}(-\mathcal{R}_a)\to\mathcal{U}\to\mathcal{U}|_{\mathcal{C}^e}$ and the pair $\delta:\mathcal{W}_\Gamma\to\mathcal{Y}_{\Gamma,\nu,e}$, we form the top and the bottom d.t.s

$$\begin{array}{ccccccc}
\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}} & \xrightarrow{\alpha} & \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \xrightarrow{\beta} & \delta^*\mathbb{E}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}} & \xrightarrow{+1} & \\
\uparrow{\scriptstyle\tilde\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}} & & \uparrow{\scriptstyle\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}} & & \uparrow{\scriptstyle\delta^*\phi^\vee_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}} & & \\
\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}} & \xrightarrow{\tilde\alpha} & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \xrightarrow{\tilde\beta} & \delta^*\mathbb{T}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}} & \xrightarrow{+1} &
\end{array}\tag{6.1}$$

where the second and the third vertical arrows are the perfect obstruction theories constructed by direct image cones, and the square is commutative because of Proposition 7.5. Let $\tilde\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}$ be the one making (6.1) a morphism of d.t.s. Applying the five lemma, $\tilde\phi^\vee_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}$ is also a perfect obstruction theory.

Let $\sigma_{\Gamma,\nu}$ be the cosection of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}$ mentioned after Definition 3.2; let

$$\tilde\sigma_{\Gamma,\nu}:=\sigma_{\Gamma,\nu}\circ H^1(\alpha):\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}\longrightarrow\mathcal{O}_{\mathcal{W}_\Gamma}.\tag{6.2}$$

Lemma 6.1. *The degeneracy locus $D(\tilde\sigma_{\Gamma,\nu})=\{\xi\in\mathcal{W}_\Gamma\mid\tilde\sigma_{\Gamma,\nu}|_\xi=0\}$ is proper.*

Proof. The construction of $\sigma_{\Gamma,\nu}$ is as in [CLL], where it is proved that $\sigma_{\Gamma,\nu}$ can be lift to $\bar\sigma_{\Gamma,\nu}:\mathcal{O}b_{\mathcal{W}_\Gamma}\to\mathcal{O}_{\mathcal{W}_\Gamma}$ (cf. [CLL, Prop. 3.4]).

We now show that $D(\tilde\sigma_{\Gamma,\nu})$ is proper. Let $\xi\in\mathcal{W}_\Gamma$ be a closed point, represented by $\xi=(\mathcal{C},\Sigma^{\mathcal{C}},\cdots,\nu)$. Let $\mathcal{R}_a\subset\mathcal{C}$ be the node associated with $a=(e,v^+)\in F(\Gamma)$, which decomposes $\mathcal{C}$ into subcurves $\mathcal{C}^\diamond$ and $\mathcal{C}^e$. By the description of the obstruction theory of $\mathcal{W}_\Gamma\to\mathcal{Y}_{\Gamma,\nu,e}$,

$$\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}|_\xi=H^1\big(\mathcal{L}^{\log}|_{\mathcal{C}^\diamond}(-\mathcal{R}_a)^{\oplus 5}\oplus\mathcal{P}^{\log}|_{\mathcal{C}^\diamond}(-\mathcal{R}_a)\big)^T,$$

where $\mathcal{L}^{\log}$ and $\mathcal{P}^{\log}$ are as defined before (3.4).

Let

$$\xi^\diamond:=(\mathcal{C}^\diamond,\Sigma^{\mathcal{C}^\diamond}=\Sigma^{\mathcal{C}}\cap\mathcal{C}^\diamond+\mathcal{R}_a,\mathcal{L}|_{\mathcal{C}^\diamond},\cdots,\nu_2|_{\mathcal{C}^\diamond}),$$

where the marking $\mathcal{R}_a$ is decorated by $(1,\varphi)$. Then $\xi^\diamond$ is a point in $\mathcal{W}_{\Gamma'}$. Following the construction of the obstruction theory of $\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}$, we see that

$$\mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}|_{\xi^\diamond} = H^1\big(\mathcal{L}|_{\mathcal{C}^\diamond}(-\Sigma^{\mathcal{C}^\diamond}_{(1,\varphi)})^{\oplus 5} \oplus \mathcal{L}^{\vee\otimes 5}|_{\mathcal{C}^\diamond} \otimes \omega^{\log}_{\mathcal{C}^\diamond}(-\Sigma^{\mathcal{C}^\diamond}_{(1,\rho)})\big)^T.$$

Because of the identities

$$\mathcal{P}|_{\mathcal{C}^\diamond} = \mathcal{L}|^{\vee\otimes 5}_{\mathcal{C}^\diamond} \otimes \omega^{\log}_{\mathcal{C}^\diamond}, \quad \Sigma^{\mathcal{C}}_{(1,\rho)}|_{\mathcal{C}^\diamond} = \Sigma^{\mathcal{C}^\diamond}_{(1,\rho)}, \quad \text{and} \quad \Sigma^{\mathcal{C}}_{(1,\varphi)}|_{\mathcal{C}^\diamond} + \mathcal{R}_a = \Sigma^{\mathcal{C}^\diamond}_{(1,\varphi)},$$

we have $\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})|_{\mathcal{C}^\diamond}(-\mathcal{R}_a) = \mathcal{L}|_{\mathcal{C}^\diamond}(-\Sigma^{\mathcal{C}^\diamond}_{(1,\varphi)})$, and the exact sequence

$$0 \longrightarrow \mathcal{P}^{\log}|_{\mathcal{C}^\diamond}(-\mathcal{R}_a) \longrightarrow \mathcal{L}|^{\vee\otimes 5}_{\mathcal{C}^\diamond} \otimes \omega^{\log}_{\mathcal{C}^\diamond}(-\Sigma^{\mathcal{C}^\diamond}_{(1,\rho)}) \longrightarrow \mathcal{P}^{\log}|_{\mathcal{R}_a} \longrightarrow 0. \tag{6.3}$$

Therefore we get the induced surjective

$$r : \mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}|_\xi \longrightarrow \mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}|_{\xi^\diamond}. \tag{6.4}$$

By the definition of the cosections $\sigma_{\Gamma,\nu}|_\xi$ and $\sigma_{\Gamma',\nu}|_{\xi^\diamond}$, we see that (cf. (6.2))

$$\begin{array}{ccc} \mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}|_\xi & \xrightarrow{\ r\ } & \mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}|_{\xi^\diamond} \\ \downarrow{\scriptstyle \tilde\sigma_{\Gamma,\nu}|_\xi} & & \downarrow{\scriptstyle \sigma_{\Gamma',\nu}|_{\xi^\diamond}} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

is commutative. Therefore, $\tilde\sigma_{\Gamma,\nu}|_\xi = 0$ implies that $\kappa(\xi) \in D(\sigma_{\Gamma',\nu})$. (cf. $\kappa : \mathcal{W}^\mu_\Gamma \to \mathcal{W}_{\Gamma'}$ is defined before (5.8).) This proves that

$$D(\tilde\sigma_{\Gamma,\nu}) \subset \kappa^{-1}(D(\sigma_{\Gamma',\nu})).$$

As $D(\sigma_{\Gamma',\nu})$ is proper ([CL^{3}1]) and κ is proper, $D(\tilde\sigma_{\Gamma',\nu})$ is also proper. □

Let

$$\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}} \subset h^1/h^0\big(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}\big) \subset \mathfrak{A}_{\Gamma,e} := h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}})$$

be the virtual normal cone (cf. [BF]). Following [KL], the cosection $\tilde\sigma_{\Gamma,\nu}$ defines a bundle stack homomorphism $\tilde\sigma_{\Gamma,\nu} : \mathfrak{A}_{\Gamma,e} \to \mathcal{O}_{\mathcal{W}_\Gamma}$. Let $\mathfrak{A}_{\Gamma,e}(\tilde\sigma_{\Gamma,\nu}) \subset \mathfrak{A}_{\Gamma,e}$ be the kernel stack of $\tilde\sigma_{\Gamma,\nu}$, which is a closed substack of $\mathfrak{A}_{\Gamma,e}$ defined via

$$\mathfrak{A}_{\Gamma,e}(\tilde\sigma_{\Gamma,\nu}) := \coprod_{\xi\in\mathcal{W}_\Gamma} \ker\{\tilde\sigma_{\Gamma,\nu}|_\xi : \mathfrak{A}_{\Gamma,e}|_\xi \longrightarrow \mathbb{C}\}, \tag{6.5}$$

endowed with the reduced stack structure.

Lemma 6.2. *We have* $(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}})_{\mathrm{red}} \subset \mathfrak{A}_{\Gamma,e}(\tilde\sigma_{\Gamma,\nu})$.

Proof. Let

$$\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} \subset h^1/h^0\big(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}\big) \subset \mathfrak{A}_\Gamma := h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}})$$

be the similarly defined virtual normal cone. By [KL],

$$(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}})_{\mathrm{red}} \subset \mathfrak{A}_\Gamma(\sigma_{\Gamma,\nu}),$$

where $\mathfrak{A}_\Gamma(\sigma_{\Gamma,\nu})\subset\mathfrak{A}_\Gamma$ is the kernel stack of $\sigma_{\Gamma,\nu}$. Applying the functoriality of the h^1/h^0 construction to (6.1), we obtain the commutative diagram

$$\begin{array}{ccccc} \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}} & \xrightarrow{\subset} & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}) & \xrightarrow{\subset} & \mathfrak{A}_{\Gamma,e}=h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle h^1/h^0(\alpha)} \\ \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}} & \xrightarrow{\subset} & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}) & \xrightarrow{\subset} & \mathfrak{A}_\Gamma=h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}}). \end{array} \tag{6.6}$$

Because $(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_{\Gamma,\nu}})_{\rm red}\subset\mathfrak{A}_\Gamma(\sigma_{\Gamma,\nu})$, by the definition of $\tilde\sigma_{\Gamma,\nu}$ (cf. (6.2)) we conclude that $(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}})_{\rm red}\subset\mathfrak{A}_{\Gamma,e}(\tilde\sigma_{\Gamma,\nu})$. □

Our next step is to use the virtual pullback of [CKL, Def. 2.8] (also [Man, Constr. 3.6]) to re-express the cycle $[\mathcal{W}_\Gamma]^{\rm vir}_{\rm loc}$. For this, we need a description of the virtual normal cone of $\mathcal{Y}_{\Gamma,\nu,e}\to\mathcal{D}_{\Gamma,\nu}$:

$$\mathfrak{C}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}\subseteq h^1/h^0\big(\mathbb{T}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}\big)\subseteq\mathfrak{B}:=h^1/h^0(\mathbb{E}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}). \tag{6.7}$$

We show that the identities in (6.7) hold.

Indeed, by Lemma 5.3, $\mathcal{W}_{\bar e}$ has pure dimension 4, equaling the expected dimension of $\mathcal{W}_{\bar e}$, and has local complete intersection singularities. The intrinsic normal cone $\mathfrak{C}_{\mathcal{W}_{\bar e}/\mathcal{D}_{\bar e,\nu}}$ equals the bundle stack $\mathfrak{A}_{\bar e}$ as shown below.

$$\mathfrak{C}_{\mathcal{W}_{\bar e}/\mathcal{D}_{\bar e,\nu}}=\mathfrak{A}_{\bar e}:=h^1/h^0(\mathbb{E}_{\mathcal{W}_{\bar e}/\mathcal{D}_{\bar e,\nu}}). \tag{6.8}$$

Because the second square in (5.7) is a Cartesian square, we have

$$\mathfrak{C}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}=\mathfrak{C}_{\mathcal{W}_{\bar e}/\mathcal{D}_{\bar e,\nu}}\times_{\mathcal{W}_{\bar e}}\mathcal{Y}_{\Gamma,\nu,e}=\mathfrak{A}_{\bar e}\times_{\mathcal{W}_{\bar e}}\mathcal{Y}_{\Gamma,\nu,e}=\mathfrak{B}. \tag{6.9}$$

We form Cartesian products and projections as follows

$$\begin{array}{ccccc} \mathfrak{A}_{\Gamma,e|\mathfrak{B}}:=\mathfrak{A}_{\Gamma,e}\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathfrak{B} & \xrightarrow{\pi_2} & \mathcal{W}_{\Gamma|\mathfrak{B}}:=\mathcal{W}_\Gamma\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathfrak{B} & \longrightarrow & \mathfrak{B} \\ \downarrow & & \downarrow{\scriptstyle\pi_1} & & \downarrow \\ \mathfrak{A}_{\Gamma,e} & \xrightarrow{\beta} & \mathcal{W}_\Gamma & \longrightarrow & \mathcal{Y}_{\Gamma,\nu,e}. \end{array} \tag{6.10}$$

Note that π_2 is the pullback of β via π_1. Viewing $\tilde\sigma_{\Gamma,\nu}:\mathfrak{A}_{\Gamma,e}\to\mathcal{O}_{\mathcal{W}_\Gamma}$ as a bundle stack homomorphism, its pullback

$$\pi_1^*(\tilde\sigma_{\Gamma,\nu}):\mathfrak{A}_{\Gamma,e|\mathfrak{B}}\longrightarrow\mathcal{O}_{\mathcal{W}_{\Gamma|\mathfrak{B}}}$$

is also a bundle stack homomorphism. Its degeneracy locus is then

$$D(\pi_1^*(\tilde\sigma_{\Gamma,\nu}))=D(\tilde\sigma_{\Gamma,\nu})\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathfrak{B}\subset\mathcal{W}_{\Gamma|\mathfrak{B}}, \tag{6.11}$$

and its associated kernel stack $\mathfrak{A}_{\Gamma,e|\mathfrak{B}}(\pi_1^*\tilde\sigma_{\Gamma,\nu})$ (cf. (6.5)) is

$$\mathfrak{A}_{\Gamma,e|\mathfrak{B}}(\pi_1^*\tilde\sigma_{\Gamma,\nu})=\mathfrak{A}(\tilde\sigma_{\Gamma,\nu})\times_{\mathcal{W}_\Gamma}\mathcal{W}_{\Gamma|\mathfrak{B}}\subset\mathfrak{A}_{\Gamma,e|\mathfrak{B}}.$$

We denote the inclusion by ι:

$$\iota:(\mathfrak{C}_{\mathcal{W}_{\Gamma|\mathfrak{B}}/\mathfrak{B}})_{\rm red}\subset(\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}}\times_{\mathcal{Y}_{\Gamma,\nu,e}}\mathfrak{B})_{\rm red}\subset\mathfrak{A}_{\Gamma,e|\mathfrak{B}}(\tilde\sigma_{\Gamma,\nu}), \tag{6.12}$$

where the first inclusion follows from the definition of $\mathcal{W}_{\Gamma|\mathfrak{B}}$ and the second follows from Lemma 6.2.

To proceed, let us recall the virtual pullbacks introduced in [Man]. Following [Man], we form the composite

$$f^! : A_*\mathfrak{B} \xrightarrow{\epsilon} A_*\mathfrak{A}_{\Gamma,e|\mathfrak{B}} \xrightarrow{0^!_{\pi_2}} A_*\mathcal{W}_{\Gamma|\mathfrak{B}} \xrightarrow{0^!_{\pi_1}} A_*\mathcal{W}_\Gamma. \tag{6.13}$$

Here the arrow ϵ is defined as follows. Let $\bar\epsilon' : Z_*\mathfrak{B} \to Z_*(\mathfrak{C}_{\mathcal{W}_{\Gamma|\mathfrak{B}}/\mathfrak{B}})$ be the linear map defined via $\bar\epsilon'([V]) = [\mathfrak{C}_{V\times_{\mathfrak{B}}\mathcal{W}_{\Gamma|\mathfrak{B}}/V}]$. Since $\mathcal{W}_\Gamma$ is a DM stack, both $\mathcal{W}_\Gamma \to \mathcal{Y}_{\Gamma,\nu,e}$ and $\mathcal{W}_{\Gamma|\mathfrak{B}} \to \mathfrak{B}$ are of DM type. Applying the proof of [Man, Thm. 2.31] to [Man, Constr. 3.6], we conclude that $\bar\epsilon'$ descends to the $\bar\epsilon$ in (6.14). Let $\bar\iota_* : A_*\mathfrak{C}_{\mathcal{W}_{\Gamma|\mathfrak{B}}/\mathfrak{B}} \to A_*\mathfrak{A}_{\Gamma,e|\mathfrak{B}}$ be induced by the inclusion (6.12). We define ϵ to be the composite

$$\epsilon : A_*\mathfrak{B} \xrightarrow{\bar\epsilon} A_*(\mathfrak{C}_{\mathcal{W}_{\Gamma|\mathfrak{B}}/\mathfrak{B}}) \xrightarrow{\bar\iota_*} A_*\mathfrak{A}_{\Gamma,e|\mathfrak{B}}. \tag{6.14}$$

The arrows $0^!_{\pi_1}$ and $0^!_{\pi_2}$ in (6.13) are Gysin maps after intersecting with the zero sections of the bundle stacks π_1 and π_2, respectively.

The version we will use is the localized analogue of (6.13):

$$f^!_{\mathrm{loc}} : A_*\mathfrak{B} \xrightarrow{\tilde\epsilon} A_*(\mathfrak{A}_{\Gamma,e|\mathfrak{B}}(\pi_1^*\tilde\sigma_{\Gamma,\nu})) \xrightarrow{0^!_{\pi_2,\mathrm{loc}}} A_*(D(\pi_1^*\tilde\sigma_{\Gamma,\nu})) \xrightarrow{0^!_{\pi_1}} A_*(D(\tilde\sigma_{\Gamma,\nu})). \tag{6.15}$$

By (6.12), the ϵ in (6.13) (cf. (6.14)) factors through $A_*(\mathfrak{A}_{\Gamma,e|\mathfrak{B}}(\tilde\sigma_{\Gamma,\nu}))$, giving the $\tilde\epsilon$ in (6.15). Since $D(\pi_1^*\tilde\sigma_{\Gamma,\nu})$ is proper, the last arrow $0^!_{\pi_1}$ is the ordinary Gysin map of the bundle stack π_1.

Proposition 6.3. *Let $\jmath : D(\sigma_{\Gamma,\nu}) \to D(\tilde\sigma_{\Gamma,\nu})$ be the inclusion, then*

$$f^!_{\mathrm{loc}}[\mathfrak{C}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}] = \jmath_*[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(D(\tilde\sigma_{\Gamma,\nu})).$$

Proof. We quote the relative version of cosection-localized pullback in [CKL, Prop. 2.11], stated in [CKL, Remark 2.12]. The proof of [CKL, Prop. 2.11] applies word for word to our case, such as $\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}$ satisfies the "virtually smooth" condition in [CKL, (2.1)] because of (6.1). The cosection setup is also consistent. Proposition 6.3 follows. □

We are now ready to prove Proposition 5.5. Let

$$\mathfrak{A}^\mu_{\bar e} = \mathfrak{A}_{\bar e} \times_{\mathcal{W}_{\bar e}} \mathcal{W}^\mu_{\bar e} \quad \text{and} \quad \mathfrak{B}^\mu = \mathfrak{B} \times_{\mathcal{Y}_{\Gamma,\nu,e}} \mathcal{Y}^\mu_{\Gamma,\nu,e}.$$

By Lemma 5.3, $\mathfrak{A}^\mu_{\bar e}$ is a bundle stack over $\mathcal{W}_{\bar e}$, where the latter is irreducible. Thus for a rational number c, $[\mathfrak{A}_{\bar e}] = c \cdot [\mathfrak{A}^\mu_{\bar e}]$. Because the second square in (5.7) is Cartesian, using (6.9), we conclude that

$$[\mathfrak{B}] = [\mathfrak{C}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}] = [\mathfrak{A}_{\bar e} \times_{\mathcal{W}_{\bar e}} \mathcal{Y}_{\Gamma,\nu,e}] = c \cdot [\mathfrak{A}^\mu_{\bar e} \times_{\mathcal{W}_{\bar e}} \mathcal{Y}_{\Gamma,\nu,e}] = c \cdot [\mathfrak{B}^\mu].$$

Therefore by (6.9),

$$f^!_{\mathrm{loc}}([\mathfrak{C}_{\mathcal{Y}_{\Gamma,\nu,e}/\mathcal{D}_{\Gamma,\nu}}]) = f^!_{\mathrm{loc}}([\mathfrak{B}]) = c \cdot f^!_{\mathrm{loc}}([\mathfrak{B}^\mu]). \tag{6.16}$$

Let $\kappa : \mathcal{W}_\Gamma^\mu \to \mathcal{W}_{\Gamma'}$ be induced by the β (in (5.7)); let $\tilde\kappa : \mathcal{W}_\Gamma^\sim \to \mathcal{W}_{\Gamma'}^-$ be that induced by κ, as defined in (5.8). Let

$$\theta : \mathfrak{A}_{\Gamma,e}|_{\mathcal{W}_\Gamma^\mu} = h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{Y}_{\Gamma,\nu,e}})|_{\mathcal{W}_\Gamma^\mu} \longrightarrow \kappa^* h^1/h^0(\mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}})$$

be induced by (6.3) and the identity before (6.3); it is a smooth morphism. We claim that (as cycles)

$$[\mathfrak{C}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}}] = \theta^*[\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}] \in Z_*\big(h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}})\big). \tag{6.17}$$

To prove (6.17), we introduce a new stack $\mathcal{D}_{\Gamma',\nu,\diamond}$ consisting of objects $(\xi,\rho_\diamond)$, where $\xi = (\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\cdots) \in \mathcal{D}_{\Gamma',\nu}(S)$, and a nowhere vanishing $\rho_\diamond \in H^0(\omega_{\mathcal{C}}^{\log} \otimes \mathcal{L}^{\vee\otimes 5})|_{\mathcal{R}})$, where $\mathcal{R} \subset \mathcal{C}$ is the section of the marking associated with the distinguished 1_φ-leg of Γ'. (The distinguished leg is the one added after trimming the edge e; see definition 5.1.)

For any family $(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\nu,\phi^e,\rho^e)$ in $\mathcal{Y}^\mu_{\Gamma,\nu,e}(S)$, let $\mathcal{R}\subset\mathcal{C}$ be the section of nodes that separate $\mathcal{C}$ into $\mathcal{C}^\diamond$ and $\mathcal{C}^e$ (cf. (5.1)). Then by adding $\rho|_{\mathcal{R}}$ to the family (5.1) we obtain a family in $\mathcal{D}_{\Gamma',\nu,\diamond}$. This defines the morphism ζ_1 below. Let α shown below be the morphism defined similarly. They form the (left) commutative diagram

$$\begin{array}{ccccc} \mathcal{W}_\Gamma^\mu & \xrightarrow{\kappa} & \mathcal{W}_{\Gamma'} & \xrightarrow{=} & \mathcal{W}_{\Gamma'} \\ \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow \\ \mathcal{Y}^\mu_{\Gamma,\nu,e} & \xrightarrow{\zeta_1} & \mathcal{D}_{\Gamma',\nu,\diamond} & \xrightarrow{\zeta_2} & \mathcal{D}_{\Gamma',\nu} \end{array} \tag{6.18}$$

Let ζ_2 be the forgetful morphism. This morphism fits into the right commutative diagram above. Because for family $(\mathcal{C},\cdots,\phi^e,\rho^e)$ in $\mathcal{Y}^\mu_{\Gamma,\nu,e}(S)$, $\phi^e|_{\mathcal{R}} = 0$, one checks directly that the left square above is a fiber product.

By its construction, ζ_2 is smooth. Thus

$$\begin{array}{ccc} \mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu,\diamond}} & \longrightarrow & \mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} \\ \downarrow & & \downarrow \\ h^1/h^0(\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu,\diamond}}) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}) \end{array} \tag{6.19}$$

is a fiber product. This implies

$$\mathbb{T}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}} \cong \kappa^*\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu,\diamond}} \quad \text{and} \quad \mathfrak{C}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}} \cong \kappa^*\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu,\diamond}}. \tag{6.20}$$

By (6.19) and (6.20), the following square is a fiber product:

$$\begin{array}{ccc} \mathfrak{C}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}} & \longrightarrow & \kappa^*\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} \\ \downarrow & & \downarrow \\ h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma^\mu/\mathcal{Y}^\mu_{\Gamma,\nu,e}}) & \longrightarrow & \kappa^*h^1/h^0(\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}). \end{array} \tag{6.21}$$

We next look at their deformation complexes. To begin with, the family version of (6.4) gives an exact sequence

$$\kappa^*\alpha^*\mathbb{T}_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}} \longrightarrow \mathcal{O}b_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}} \longrightarrow \kappa^*\mathcal{O}b_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} \longrightarrow 0. \tag{6.22}$$

Note that $\mathbb{T}_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}}$ is an invertible sheaf whose fibers are $(\omega^{\log}_{\mathcal{C}} \otimes \mathcal{L}^{\vee\otimes 5})|_{\mathcal{R}}$. This sequence is the cohomology of the top row in

$$\begin{array}{ccccccc} \mathbb{T}_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}} & \longrightarrow & \kappa^*\mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} & \xrightarrow{+1} & \\ \| & & \uparrow \phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}} & & \uparrow \phi^\vee_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} & & \\ \mathbb{T}_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}}[-1] & \longrightarrow & \mathbb{T}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}} & \longrightarrow & \kappa^*\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}} & \xrightarrow{+1} & . \end{array} \tag{6.23}$$

Here the upper row is induced by the derived push-forward of the family version of (6.3) and the lower row by (6.18) and (6.20). Hence both rows are distinguished triangles. The arrow $\phi^\vee_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}$ is induced by the ordinary construction and $\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}$ is induced by the same process which derives the first vertical arrow in (7.11) using (7.10)'s blow up construction. Both vertical arrows use direct image cone constructions. The commutativity of the second square in (6.23) follows from the natural arrow between the two universal families and the two evaluations maps directly.

Taking h^1/h^0 of the diagram we obtain

$$\begin{array}{ccccc} T_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}} & \longrightarrow & h^1/h^0(\mathbb{E}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}) & \xrightarrow{\theta} & \kappa^*h^1/h^0(\mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}) \\ \| & & \uparrow h^1/h^0(\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}) & & \uparrow h^1/h^0(\phi^\vee_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}) \\ T_{\mathcal{D}_{\Gamma',\nu,\diamond}/\mathcal{D}_{\Gamma',\nu}} & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}) & \longrightarrow & \kappa^*h^1/h^0(\mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}). \end{array} \tag{6.24}$$

By [BF, Prop. 2.7], both rows are exact sequences of cone stacks. Therefore the second square of (6.24) is a fiber product. By Proposition 7.5, we know that $\tilde\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}$(in (6.1)) is ν-equivalent to $\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}$ (cf. [CL1, Def. 2.9]), thus the cycle $[\mathfrak{C}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}]$ induced by $\tilde\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}$ is identical to that induced by $\phi^\vee_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}}$ (cf. [CL1, Prop. 2.10] and [CL1, Lemm. 2.3]). Combining the above with (6.21), the claim (6.17) is proved.

We consider π_1^μ (compare with π_1 in (6.10))

$$\pi_1^\mu := \pi|_{\mathcal{W}^\mu_{\Gamma|\mathfrak{B}}} : \mathcal{W}^\mu_{\Gamma|\mathfrak{B}} := \mathcal{W}^\mu_\Gamma \times_{\mathcal{Y}_{\Gamma,\nu,e}} \mathfrak{B} = \mathcal{W}^\mu_\Gamma \times_{\mathcal{W}_{\bar e}} \mathfrak{A}^\mu_{\bar e} \longrightarrow \mathcal{W}^\mu_\Gamma,$$

where $\mathfrak{A}^\mu_{\bar e} = \mathfrak{A}_{\bar e} \times_{\mathcal{W}_{\bar e}} \mathcal{W}^\mu_{\bar e}$. Let

$$\psi : \mathcal{W}^\mu_{\Gamma|\mathfrak{B}} = \mathcal{W}^\mu_\Gamma \times_{\mathcal{W}_{\bar e}} \mathfrak{A}^\mu_{\bar e} \longrightarrow \mathcal{W}^\mu_\Gamma$$

be the first projection.

Then by the definition of $\tilde\epsilon$ (cf. (6.15) and (6.14)),

$$\tilde\epsilon[\mathfrak{B}^\mu] = [\mathfrak{C}_{\mathcal{W}^\mu_\Gamma/\mathcal{Y}^\mu_{\Gamma,\nu,e}} \times_{\mathcal{W}^\mu_\Gamma} \mathfrak{A}^\mu_{\bar e}] = \psi^*\theta^*[\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}].$$

Applying $0^!_{\pi_1^*\tilde\sigma_{\Gamma,\nu},\mathrm{loc}}$, we obtain

$$0^!_{\pi_1^*\tilde\sigma_{\Gamma,\nu},\mathrm{loc}}\big(\psi^*\theta^*[\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}]\big)=\psi^*\tilde\kappa^*\big(0^!_{\tilde\sigma_{\Gamma',\nu},\mathrm{loc}}[\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma',\nu}}]\big)=\psi^*\tilde\kappa^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

(Recall that $\tilde\kappa:\mathcal{W}_\Gamma^\sim\to\mathcal{W}_{\Gamma'}^-$ is defined in (5.8).) By adding

$$0^!_{\pi_1}(\psi^*\tilde\kappa^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}})=\tilde\kappa^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\mathcal{W}_\Gamma^\sim=A_*D(\tilde\sigma_{\Gamma,\nu}),$$

we have proven that

$$f^!_{\mathrm{loc}}[\mathfrak{B}^\mu]=\tilde\kappa^*[\mathcal{W}_{\Gamma'}]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*D(\tilde\sigma_{\Gamma,\nu}).$$

This proves Proposition 5.5.

7. Appendix

Let $\mathcal{X}$ be an Artin stack; let $\pi:\mathcal{C}\to\mathcal{X}$ be a flat family of twisted nodal curves, and let $\mathcal{V}\to\mathcal{C}$ be a smooth morphism of quasi-projective type. We denote by $C(\pi_*\mathcal{V})$ the groupoid defined as follows: for any scheme S, $C(\pi_*\mathcal{V})(S)$ consists of all (σ,s), where $\sigma:S\to\mathcal{X}$ is a morphism, $\mathcal{C}_\sigma=\mathcal{C}\times_\mathcal{X}S$ and $\mathcal{V}_\sigma=\mathcal{V}\times_\mathcal{C}\mathcal{C}_\sigma$, and $s:S\to\mathcal{V}_\sigma$ is an S-morphism (a section of $\mathcal{V}_\sigma\to S$). An arrow between two objects (σ,s) and (σ',s') is an arrow between σ and σ' so that $s=s'$ under the induced isomorphism $\mathcal{V}_\sigma\cong\mathcal{V}_{\sigma'}$.

We abbreviate $\mathcal{W}=C(\pi_*\mathcal{V})$. Let $\pi_\mathcal{W}:\mathcal{C}_\mathcal{W}\to\mathcal{W}$ be the pullback of $\mathcal{C}\to\mathcal{X}$, and let $\mathrm{ev}:\mathcal{C}_\mathcal{W}\to\mathcal{V}$ be the tautological evaluation map (induced by the section s), which fits into the commutative diagrams

$$\begin{array}{ccccc}\mathcal{W} & \xleftarrow{\pi_\mathcal{W}} & \mathcal{C}_\mathcal{W} & \xrightarrow{\mathrm{ev}} & \mathcal{V}\\ \downarrow & & \downarrow & & \downarrow\\ \mathcal{X} & \longleftarrow & \mathcal{C} & = & \mathcal{C}.\end{array}\tag{7.1}$$

Applying the projection formula to $\pi_\mathcal{W}^*\mathbb{T}_{\mathcal{W}/\mathcal{X}}\cong\mathbb{T}_{\mathcal{C}_\mathcal{W}/\mathcal{C}}\to\mathrm{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}}$ and using $\mathbb{T}_{\mathcal{W}/\mathcal{X}}\longrightarrow R\pi_{\mathcal{W}*}\pi_\mathcal{W}^*\mathbb{T}_{\mathcal{W}/\mathcal{X}}$, we obtain

$$\phi^\vee_{\mathcal{W}/\mathcal{X}}:\mathbb{T}_{\mathcal{W}/\mathcal{X}}\longrightarrow\mathbb{E}_{\mathcal{W}/\mathcal{X}}:=R\pi_{\mathcal{W}*}\mathrm{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}}.\tag{7.2}$$

By [CL2, Prop. 1.1], (7.2) is a perfect obstruction theory.[5]

In the following, we assume $\mathcal{V}\to\mathcal{C}$ is a (fixed) vector bundle. We consider two cases. The first case is when $\mathcal{V}=\mathcal{V}_1\oplus\mathcal{V}_2$ is a direct sum of two vector bundles. We continue to denote $\mathcal{W}=C(\pi_*\mathcal{V})$. We introduce $\mathcal{W}_i=C(\pi_*\mathcal{V}_i)$. Then the direct sum $\mathcal{V}=\mathcal{V}_1\oplus\mathcal{V}_2$ induces a morphism $\mathcal{W}\longrightarrow\mathcal{W}_1\times_\mathcal{X}\mathcal{W}_2$, which by a direct check is an isomorphism.

There is another way to interpret this isomorphism. Let $\mathcal{C}_{\mathcal{W}_2}:=\mathcal{C}\times_\mathcal{X}\mathcal{W}_2$; use (the same) $\pi:\mathcal{C}_{\mathcal{W}_2}\to\mathcal{W}_2$ to denote its projection; and denote $\mathcal{V}_{1,\mathcal{W}_2}=\mathcal{V}_1\times_\mathcal{C}\mathcal{C}_{\mathcal{W}_2}$.

Lemma 7.1. *We have canonical isomorphisms* $\mathcal{W}\cong C(\pi_*(\mathcal{V}_{1,\mathcal{W}_2}))\cong\mathcal{W}_1\times_\mathcal{X}\mathcal{W}_2$.

[5]This construction of $\phi^\vee_{\mathcal{W}/\mathcal{X}}$ applies to arbitrary representable $\mathcal{V}\to\mathcal{C}$. We restrict ourselves to the bundle case for notational simplicity.

Let $q_2 : \mathcal{W} \to \mathcal{W}_2$ be the projection, as in the above lemma. Let $\phi^\vee_{\mathcal{W}/\mathcal{W}_2}, \phi^\vee_{\mathcal{W}/\mathcal{X}}, \phi^\vee_{\mathcal{W}_2/\mathcal{X}}$ be similarly defined perfect obstruction theories shown as follows,

$$\begin{array}{ccccccc} \mathbb{E}_{\mathcal{W}/\mathcal{W}_2} & \longrightarrow & \mathbb{E}_{\mathcal{W}/\mathcal{X}} & \longrightarrow & q_2^*\mathbb{E}_{\mathcal{W}_2/\mathcal{X}} & \xrightarrow{+1} & \\ \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}/\mathcal{W}_2}} & & \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}/\mathcal{X}}} & & \uparrow{\scriptstyle \phi^\vee_{\mathcal{W}_2/\mathcal{X}}} & & \\ \mathbb{T}_{\mathcal{W}/\mathcal{W}_2} & \longrightarrow & \mathbb{T}_{\mathcal{W}/\mathcal{X}} & \longrightarrow & q_2^*\mathbb{T}_{\mathcal{W}_2/\mathcal{X}} & \xrightarrow{+1} & , \end{array} \tag{7.3}$$

where the top line is the d.t. induced by $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, and the lower line is induced by $\mathcal{W} \to \mathcal{W}_2 \to \mathcal{X}$.

Proposition 7.2. *The diagram* (7.3) *is a morphism between d.t.s.*

Proof. We form the diagram

$$\begin{array}{ccccc} \mathcal{C}_{\mathcal{W}} & \xrightarrow{\text{ev}} & \mathcal{V} & \xrightarrow{\gamma_1} & \mathcal{V}_1 \\ \downarrow{\scriptstyle \tilde{q}_2} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow \\ \mathcal{C}_{\mathcal{W}_2} & \xrightarrow{\text{ev}_2} & \mathcal{V}_2 & \longrightarrow & \mathcal{C}, \end{array} \tag{7.4}$$

where γ_i are projections induced by the direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$; and $\tilde{q}_2$ is the lift of $q_2 : \mathcal{W} \to \mathcal{W}_2$. This diagram induces a homomorphism between d.t.s

$$\begin{array}{ccccccc} \text{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{V}_2} & \longrightarrow & \text{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}} & \longrightarrow & \tilde{q}_2^*\text{ev}_2^*\mathbb{T}_{\mathcal{V}_2/\mathcal{C}} & \xrightarrow{+1} & \\ \uparrow{\scriptstyle \psi} & & \uparrow & & \uparrow & & \\ \mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}_{\mathcal{W}_2}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}} & \longrightarrow & \tilde{q}_2^*\mathbb{T}_{\mathcal{C}_{\mathcal{W}_2}/\mathcal{C}} & \xrightarrow{+1} & . \end{array} \tag{7.5}$$

As $\mathcal{C} \to \mathcal{X}$ is flat, the second row is equal to the pull back via $\pi_{\mathcal{W}} : \mathcal{C}_{\mathcal{W}} \to \mathcal{W}$ of the tangent complexes d.t. of the triple $\mathcal{W} \to \mathcal{W}_2 \to \mathcal{X}$. Applying the projection formula to (7.5), we obtain the following morphism of d.t.s

$$\begin{array}{ccccccc} R\pi_{\mathcal{W}*}\text{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{V}_2} & \longrightarrow & R\pi_{\mathcal{W}*}\text{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}} & \longrightarrow & R\pi_{\mathcal{W}*}\tilde{q}_2^*\text{ev}_2^*\mathbb{T}_{\mathcal{V}_2/\mathcal{C}} & \xrightarrow{+1} & \\ \uparrow{\scriptstyle \psi_1} & & \uparrow{\scriptstyle \psi_2} & & \uparrow{\scriptstyle \psi_3} & & \\ \mathbb{T}_{\mathcal{W}/\mathcal{W}_2} & \longrightarrow & \mathbb{T}_{\mathcal{W}/\mathcal{X}} & \longrightarrow & q_2^*\mathbb{T}_{\mathcal{W}_2/\mathcal{X}} & \xrightarrow{+1} & \end{array}$$

Note that by definition, $\mathbb{E}_{\mathcal{W}_2/\mathcal{X}} = R\pi_{\mathcal{W}*}\text{ev}_2^*\mathbb{T}_{\mathcal{V}_2/\mathcal{C}}$ and $\mathbb{E}_{\mathcal{W}/X} = R\pi_{\mathcal{W}*}\text{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}}$. Because of the identity

$$R\pi_{\mathcal{W}*}\tilde{q}_2^*\text{ev}_2^*\mathbb{T}_{\mathcal{V}_2/\mathcal{C}} = q_2^*R\pi_{\mathcal{W}*}\text{ev}_2^*\mathbb{T}_{\mathcal{V}_2/\mathcal{C}},$$

we see that $\psi_2 = \phi^\vee_{\mathcal{W}/\mathcal{X}}$ and $\psi_3 = q_2^*\phi^\vee_{\mathcal{W}_2/\mathcal{X}}$.

It remains to be shown that $\psi_1 = \phi^\vee_{\mathcal{W}/\mathcal{W}_2}$. Observe that ψ_1 is induced by the left square in (7.4), and that square is identical to the left square in

$$
\begin{array}{ccccc}
\mathcal{C}_{\mathcal{W}} & \xrightarrow{\mathrm{ev}'} & \mathcal{V}_{1,\mathcal{W}_2} & \xrightarrow{\mathrm{pr}} & \mathcal{V}_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_{\mathcal{W}_2} & = & \mathcal{C}_{\mathcal{W}_2} & \longrightarrow & \mathcal{C}.
\end{array} \tag{7.6}
$$

Here ev' is the universal evaluation associated with the canonical $\mathcal{W} \cong C(\pi_*(\mathcal{V}_{1,\mathcal{W}_2}))$. Thus we have $\mathrm{pr} \circ \mathrm{ev}' = \gamma_1 \circ \mathrm{ev}$, where $\gamma_1 : \mathcal{V} \to \mathcal{V}_1$ is defined in (7.4).

Since $\mathcal{V}_1 \to \mathcal{C}$ is a bundle and hence flat, we have $\mathbb{T}_{\mathcal{V}/\mathcal{V}_2} \cong \gamma_1^* \mathbb{T}_{\mathcal{V}_1/\mathcal{C}}$; thus the arrow ψ_1 equals

$$
\mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}_{\mathcal{W}_2}} \longrightarrow \mathrm{ev}^* \gamma_1^* \mathbb{T}_{\mathcal{V}_1/\mathcal{C}} = \mathrm{pr}^* (\mathrm{ev}')^* \mathbb{T}_{\mathcal{V}_1/\mathcal{C}} = (\mathrm{ev}')^* \mathbb{T}_{\mathcal{V}_{1,\mathcal{W}_2}/\mathcal{C}_{\mathcal{W}_2}}. \tag{7.7}
$$

Here the last isomorphism comes from $\mathrm{pr}^* \mathbb{T}_{\mathcal{V}_1/\mathcal{C}} \cong \mathbb{T}_{\mathcal{V}_{1,\mathcal{W}_2}/\mathcal{C}_{\mathcal{W}_2}}$, as $\mathcal{V}_1 \to \mathcal{C}$ is smooth. On the other hand, it is evident that (7.7) is induced by ev'. Therefore,

$$
\mathbb{E}_{\mathcal{W}/\mathcal{W}_2} := R\pi_{\mathcal{W}*} (\mathrm{ev}')^* \mathbb{T}_{\mathcal{V}_{1,\mathcal{W}_2}/\mathcal{C}_{\mathcal{W}_2}} = R\pi_{\mathcal{W}*} \mathrm{ev}^* \mathbb{T}_{\mathcal{V}/\mathcal{V}_2},
$$

and $\psi_1 = \phi^\vee_{\mathcal{W}/\mathcal{W}_2}$. This proves the proposition. □

Remark 7.3. *The natural diagram* (7.3) *is commutative when $\mathcal{V}_1$ and $\mathcal{V}_2$ are arbitrary Artin stacks representable and quasi-projective over $\mathcal{C}$, and $\mathcal{V}_1 \to \mathcal{C}$ is flat. The proof is identical to the above.*

The second case is when there is a (scheme) section of nodes $\mathcal{R} \subset \mathcal{C}$ that decomposes $\mathcal{C}$ into a union of two $\mathcal{X}$-families $\mathcal{C}_1$ and $\mathcal{C}_2$. We denote the (same) projection by $\pi : \mathcal{C}_i \to \mathcal{X}$. Let $\mathcal{V}_i = \mathcal{V}|_{\mathcal{C}_i} (= \mathcal{V} \times_{\mathcal{C}} \mathcal{C}_i)$, and define $\mathcal{W}_1 = C(\pi_* \mathcal{V}_1)$. Let

$$
\phi^\vee_{\mathcal{W}_1/\mathcal{X}} : \mathbb{T}_{\mathcal{W}_1/\mathcal{X}} \longrightarrow \mathbb{E}_{\mathcal{W}_1/\mathcal{X}}
$$

be the similarly defined perfect obstruction theory. Note that for any S-family (σ, s) in $\mathcal{W}(S)$, letting $\mathcal{C}_{1,\sigma} = \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_\sigma$, then the family $(\sigma, s|_{\mathcal{C}_{1,\sigma}})$ is a family in $\mathcal{W}_1(S)$. This defines a morphism

$$
\tau : \mathcal{W} \longrightarrow \mathcal{W}_1. \tag{7.8}
$$

To proceed, we rewrite τ along the line of a similar construction. For $i = 1$ and 2, let

$$
\mathcal{C}_{i,\mathcal{W}_1} = \mathcal{C}_i \times_{\mathcal{X}} \mathcal{W}_1, \quad \text{and} \quad \mathcal{V}_{i,\mathcal{W}_1} = \mathcal{V}_i \times_{\mathcal{C}_i} \mathcal{C}_{i,\mathcal{W}_1},
$$

with $\pi : \mathcal{C}_{i,\mathcal{W}_1} \to \mathcal{W}_1$ to be the projection.

Let $\mathcal{S}_1 \in \Gamma(\mathcal{V}_{1,\mathcal{W}_1})$ be the universal section of $\mathcal{W}_1$. Let $\tilde{\mathcal{R}} = \mathcal{R} \times_{\mathcal{C}} \mathcal{C}_{1,\mathcal{W}_1}$ be the section (of $\mathcal{C}_{1,\mathcal{W}_1} \to \mathcal{W}_1$) associated with $\mathcal{R} \subset \mathcal{C}$. Then $\mathcal{S}_1|_{\tilde{\mathcal{R}}}$ is a section of $\mathcal{V}_{1,\mathcal{W}_1}|_{\tilde{\mathcal{R}}}$. Using $\mathcal{R} = \mathcal{C}_1 \cap \mathcal{C}_2$, we have $\mathcal{R} \times_{\mathcal{C}} \mathcal{C}_{1,\mathcal{W}_2} = \mathcal{R} \times_{\mathcal{C}} \mathcal{C}_{2,\mathcal{W}_1}$. As $\mathcal{V}_1$ and $\mathcal{V}_2$ are restrictions of $\mathcal{V}$, $\mathcal{S}_1|_{\tilde{\mathcal{R}}}$ is also a section of $\mathcal{V}_{2,\mathcal{W}_1}|_{\tilde{\mathcal{R}}} = \mathcal{V}_{1,\mathcal{W}_1}|_{\tilde{\mathcal{R}}}$. Let $\Sigma \subset \mathcal{V}_{2,\mathcal{W}_1}$ be the substack $\Sigma = \mathcal{S}_1|_{\tilde{\mathcal{R}}} \subset \mathcal{V}_{2,\mathcal{W}_1}|_{\tilde{\mathcal{R}}} \subset \mathcal{V}_{2,\mathcal{W}_1}$. Let $\mathrm{Bl}_\Sigma(\mathcal{V}_{2,\mathcal{W}_1})$ be the blowing-up of $\mathcal{V}_{2,\mathcal{W}_1}$ along Σ. Let

$$\mathcal{V}_{2/1} = \mathrm{Bl}_\Sigma(\mathcal{V}_{2,\mathcal{W}_1}) - \{\text{the proper transform of } \mathcal{V}_{2,\mathcal{W}_1}|_{\tilde{\mathcal{R}}} \subset \mathcal{V}_{2,\mathcal{W}_1}\}. \tag{7.9}$$

Let $\pi : \mathcal{V}_{2/1} \to \mathcal{W}_1$ be the induced projection; we define

$$\mathcal{W}_{2/1} = C(\pi_* \mathcal{V}_{2/1}). \tag{7.10}$$

Note that $\mathcal{V}_{2/1}$ is smooth over $\mathcal{C}_{2,\mathcal{W}_1}$.

We now construct a canonical (restriction) $\mathcal{W}_1$-morphism $\imath : \mathcal{W} \to \mathcal{W}_{2/1}$. Given any $\phi : S \to \mathcal{W}$ associated with $(\sigma, s) \in \mathcal{W}(S)$, restricting s to $\mathcal{C}_1 \times_{\mathcal{X}} S$ gives a family $(\sigma, s|_{\mathcal{C}_1 \times_{\mathcal{X}} S}) \in \mathcal{W}_1(S)$ associated with the morphism $\tau(\phi) : S \to \mathcal{W}_1$. The other part $s|_{\mathcal{C}_2 \times_{\mathcal{X}} S}$ is a section of the bundle

$$\mathcal{V}_2 \times_{\mathcal{X}} S = (\tau(\phi))^*(\mathcal{V}_{2,\mathcal{W}_1}) = \mathcal{V}_{2,\mathcal{W}_1} \times_{\tau(\phi),\mathcal{W}_1} S.$$

Because $s|_{\mathcal{C}_1 \times_{\mathcal{X}} S}$ and $s|_{\mathcal{C}_2 \times_{\mathcal{X}} S}$ are identical along $\mathcal{R} \times_{\mathcal{X}} S$, the section $s|_{\mathcal{C}_2 \times_{\mathcal{X}} S}$ lifts to a section of $(\tau(\phi))^*(\mathcal{V}_{2/1})$. This defines a morphism $\imath(\phi) : S \longrightarrow \mathcal{W}_{2/1}$, commuting with $\phi : S \to \mathcal{W}$, $\tau : \mathcal{W} \to \mathcal{W}_1$, and the projection $\mathcal{W}_{2/1} \to \mathcal{W}_1$. As $\imath(\phi)$ is canonical, it defines a $\mathcal{W}_1$-morphism $\imath : \mathcal{W} \longrightarrow \mathcal{W}_{2/1}$.

Lemma 7.4. *The morphism $\imath$ is an isomorphism. Let* $\mathrm{pr} : \mathcal{W}_{2/1} \to \mathcal{W}_1$ *be the tautological projection, then* $\mathrm{pr} \circ \imath = \tau$.

Proof. The proof follows directly from the construction. □

In the following, we will not distinguish between $\mathcal{W}$ and $\mathcal{W}_{2/1}$ because of $\imath$. Because $\mathcal{W} = \mathcal{W}_{2/1} \to \mathcal{W}_1$, $\mathcal{W} \to \mathcal{X}$ and $\mathcal{W}_1 \to \mathcal{X}$ all use the construction stated at the beginning of the Appendix, we have perfect obstruction theories $\phi^\vee_{\bullet/\bullet}$ shown as follows:

$$\begin{array}{ccccccc}
\mathbb{E}_{\mathcal{W}/\mathcal{W}_1} & \xrightarrow{\lambda_1} & \mathbb{E}_{\mathcal{W}/\mathcal{X}} & \xrightarrow{\lambda_2} & \tau^*\mathbb{E}_{\mathcal{W}_1/\mathcal{X}} & \xrightarrow{+1} & \\
\uparrow \phi^\vee_{\mathcal{W}/\mathcal{W}_1} & & \uparrow \phi^\vee_{\mathcal{W}/\mathcal{X}} & & \uparrow \tau^*\phi^\vee_{\mathcal{W}_1/\mathcal{X}} & & \\
\mathbb{T}_{\mathcal{W}/\mathcal{W}_1} & \longrightarrow & \mathbb{T}_{\mathcal{W}/\mathcal{X}} & \longrightarrow & \tau^*\mathbb{T}_{\mathcal{W}_1/\mathcal{X}} & \xrightarrow{+1} & .
\end{array} \tag{7.11}$$

Here the lower sequence is the one induced by $\mathcal{W} = \mathcal{W}_{2/1} \to \mathcal{W}_1 \to \mathcal{X}$. The arrow λ_1 is induced by the canonical composite $\mathcal{V}_{2/1} \to \mathcal{V}$, and λ_2 by the restriction of sheaves (bundles) $\mathcal{V} \to \mathcal{V}_2$.

Proposition 7.5. *The two rows in* (7.11) *are d.t.s; the two squares in* (7.11) *are commutative. Further, taking base change of* (7.11) *via any $\xi \in \mathcal{W}(\mathbb{C})$ and taking long exact sequences of cohomology groups of the two rows, the vertical arrows induce a morphism between the two complexes of vector spaces.*

Proof. We denote by $\mathrm{ev}_1 : \mathcal{C}_{1,\mathcal{W}} \to \mathcal{V}_1$ and $\mathrm{ev} : \mathcal{C}_{\mathcal{W}} \to \mathcal{V}$ the obvious evaluation maps. We have the following obvious fiber diagram

$$\begin{array}{ccccc} \mathcal{C}_{1,\mathcal{W}} & \xrightarrow{\mathrm{ev}_1} & \mathcal{V}_1 & \longrightarrow & \mathcal{C}_1 \\ \downarrow j & & \downarrow & & \downarrow \\ \mathcal{C}_{\mathcal{W}} & \xrightarrow{\mathrm{ev}} & \mathcal{V} & \longrightarrow & \mathcal{C}, \end{array}$$

where the vertical arrows are closed embeddings. This implies that the square

$$\begin{array}{ccc} \mathbb{T}_{\mathcal{C}_{1,\mathcal{W}}/\mathcal{C}_1} & \xrightarrow{d(\mathrm{ev}_1)} & \mathrm{ev}_1^*\mathcal{V}_1 \\ \downarrow u_1 & & \downarrow u_2 \\ j^*\mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}} & \xrightarrow{j^*d(\mathrm{ev})} & j^*\mathrm{ev}^*\mathcal{V} \end{array} \tag{7.12}$$

is commutative. Since $\mathcal{C}_{1,\mathcal{W}} \subset \mathcal{C}_{\mathcal{W}}$ is the fiber product of $\mathcal{C}_1 \subset \mathcal{C}$ with $\mathcal{W} \to \mathcal{X}$, that $\mathcal{C}$ and $\mathcal{C}_1$ are flat over $\mathcal{X}$ implies that $\mathbb{T}_{\mathcal{C}_{1,\mathcal{W}}/\mathcal{C}_1}$ and $\mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}}$ are pullbacks of $\mathbb{T}_{\mathcal{W}/\mathcal{X}}$; thus u_1 is an isomorphism. Similarly, u_2 is an isomorphism. This implies that the following square is commutative

$$\begin{array}{ccc} \mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}} & \xrightarrow{d(\mathrm{ev})} & \mathrm{ev}^*\mathbb{T}_{\mathcal{V}/\mathcal{C}} = \mathrm{ev}^*\mathcal{V} \\ \downarrow u_1^{-1}\circ j^* & & \downarrow u_2^{-1}\circ j^* \\ j_*\mathbb{T}_{\mathcal{C}_{1,\mathcal{W}}/\mathcal{C}_1} & \xrightarrow{d(\mathrm{ev}_1)} & j_*\mathrm{ev}_1^*\mathbb{T}_{\mathcal{V}_1/\mathcal{C}_1} = j_*\mathrm{ev}_1^*\mathcal{V}_1. \end{array}$$

Let $\zeta : \mathcal{V}_{2/1} \to \mathcal{V}_{2,\mathcal{W}_1} \to \mathcal{V}_2 \to \mathcal{V}$ be the composite of the obvious morphisms. Then we have the commutative square

$$\begin{array}{ccc} \mathcal{C}_{2,\mathcal{W}} := \mathcal{C}_2 \times_{\mathcal{X}} \mathcal{W} & \xrightarrow{ev_{2/1}} & \mathcal{V}_{2/1} \\ \downarrow & & \downarrow \zeta \\ \mathcal{C}_{\mathcal{W}} & \xrightarrow{ev} & \mathcal{V}. \end{array}$$

Here $\mathrm{ev}_{2/1}$ is defined using the universal section of $\mathcal{W}_{2/1}(= \mathcal{W})$.

The above two squares induce the following two commutative squares of objects in $D^b(\mathcal{O}_{\mathcal{C}_{\mathcal{W}}})$: (letting $\pi_2 : \mathcal{C}_{2,\mathcal{W}} \to \mathcal{W}$ be the projection, and letting $\jmath_1 : \mathcal{C}_{1,\mathcal{W}_1} \times_{\mathcal{W}_1} \mathcal{W} \to \mathcal{C}_{\mathcal{W}}$ and $\jmath_2 : \mathcal{C}_{2,\mathcal{W}} \to \mathcal{C}_{\mathcal{W}}$ be the obvious inclusions)

$$\begin{array}{ccccc} \jmath_{2*}ev_{2/1}^*\mathbb{T}_{\mathcal{V}_{2/1}/\mathcal{C}_{2,\mathcal{W}_1}} & \longrightarrow & ev^*\mathbb{T}_{\mathcal{V}/\mathcal{C}} & \longrightarrow & \jmath_{1*}ev_1^*\mathbb{T}_{\mathcal{V}_1/\mathcal{C}_1} \\ \uparrow & & \uparrow & & \uparrow \\ \jmath_{2*}\pi_2^*\mathbb{T}_{\mathcal{W}/\mathcal{W}_1} = \mathbb{T}_{\mathcal{C}_{2,\mathcal{W}}/\mathcal{C}_{2,\mathcal{W}_1}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_{\mathcal{W}}/\mathcal{C}} & \longrightarrow & \jmath_{1*}\tilde{\tau}^*\mathbb{T}_{\mathcal{C}_{1,\mathcal{W}_1}/\mathcal{C}_1}, \end{array} \tag{7.13}$$

where $\tilde{\tau} : \mathcal{C}_{1,\mathcal{W}} \to \mathcal{C}_{1,\mathcal{W}_1}$ is the projection lifting $\tau : \mathcal{W} \to \mathcal{W}_1$. (cf. (7.8).)

Taking $\pi : \mathcal{C}_{\mathcal{W}} \to \mathcal{W}$ and $\pi_1 : \mathcal{C}_{1,\mathcal{W}_1} \to \mathcal{W}_1$ as the respective projections, letting $\bar{ev}_1 : \mathcal{C}_{1,\mathcal{W}_1} \to \mathcal{V}_1$ be the evaluation using the universal section of $\mathcal{W}_1$, and applying

$R\pi_*$ to (7.13), we obtain commutative diagrams

$$\begin{array}{ccccc} R\pi_{2*}ev_{2/1}^*\mathbb{T}_{\mathcal{V}_{2/1}/\mathcal{C}_{2,\mathcal{W}_1}} & \longrightarrow & R\pi_*ev^*\mathbb{T}_{\mathcal{V}/\mathcal{C}} & \longrightarrow & \tau^*R\pi_{1*}\bar{ev}_1^*\mathbb{T}_{\mathcal{V}_1/\mathcal{C}_1} \\ \uparrow & & \uparrow & & \uparrow \\ R\pi_{2*}\pi_2^*\mathbb{T}_{\mathcal{W}/\mathcal{W}_1} & \longrightarrow & R\pi_*\mathbb{T}_{\mathcal{C}_\mathcal{W}/\mathcal{C}} & \longrightarrow & \tau^*R\pi_{1*}\mathbb{T}_{\mathcal{C}_{1,\mathcal{W}_1}/\mathcal{C}_1} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{W}/\mathcal{W}_1} & \longrightarrow & \mathbb{T}_{\mathcal{W}/\mathcal{X}} & \longrightarrow & \tau^*\mathbb{T}_{\mathcal{W}_1/\mathcal{X}}. \end{array} \tag{7.14}$$

Note that the first row of (7.14) is identical to the first row of (7.11), and the three composed vertical arrows in (7.14) are $\phi^\vee_{\mathcal{W}/\mathcal{W}_1}$, $\phi^\vee_{\mathcal{W}/\mathcal{X}}$ and $\tau^*\phi^\vee_{\mathcal{W}_1/\mathcal{X}}$ in (7.11).

On the other hand, we have canonical $ev_{2/1}^*\mathbb{T}_{\mathcal{V}_{2/1}/\mathcal{C}_{2,\mathcal{W}_1}} \cong ev^*(\mathcal{V}|_{\mathcal{C}_2}(-\mathcal{R}))$ (due to the blowing up construction). The first row of (7.13) equals

$$0 \longrightarrow ev^*(\mathcal{V}|_{\mathcal{C}_2}(-\mathcal{R})) \longrightarrow ev^*\mathcal{V} \longrightarrow ev^*(\mathcal{V}|_{\mathcal{C}_1}) \longrightarrow 0, \tag{7.15}$$

and thus is a distinguished triangle. Therefore, the first row of (7.14), which is the first row of (7.11), is a distinguished triangle. Finally, the rest of the proposition is implied by the commutativity of the following diagram

$$\begin{array}{ccc} h^0(\mathbb{E}_{\mathcal{W}_1/\mathcal{X}}|_{\tau(\xi)}) & \longrightarrow & h^1(\mathbb{E}_{\mathcal{W}/\mathcal{W}_1}|_\xi) \\ \uparrow{\scriptstyle h^1(\phi^\vee_{\mathcal{W}_1/\mathcal{X}}|_\xi)} & & \uparrow{\scriptstyle h^1(\phi^\vee_{\mathcal{W}/\mathcal{W}_1}|_\xi)} \\ h^0(\mathbb{T}_{\mathcal{W}_1/\mathcal{X}}|_{\tau(\xi)}) & \longrightarrow & h^1(\mathbb{T}_{\mathcal{W}/\mathcal{W}_1}|_\xi) \end{array}$$

which can be checked by Čech cohomology description of the obstruction class assignment. We leave this to the reader. □

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