BCOV'S FEYNMAN RULE OF QUINTIC 3-FOLDS

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Abstract. We prove the Bershadsky-Cecotti-Ooguri-Vafa's conjecture for all genus Gromov-Witten potentials of the quintic 3-folds, by identifying the Feynman graph sum with the NMSP stable graph sum via an R-matrix action. The Yamaguchi-Yau functional equations are direct consequences of the BCOV Feynman sum rule.

Contents

0. Introduction

The landmark work of Witten [Wit92] and Candelas-Ossa-Green-Parkes [CdOGP91] have initiated a new era of enumerating curves in projective (symplectic) manifolds. The mathematical foundation of this theory, called the Gromov-Witten (GW) theory, was laid by the work of Ruan-Tian [RT95] for semi-positive symplectic manifolds, and by Li-Tian and Behrend-Fantechi [LT98, BF97] for projective manifolds.

Since then, a central problem is to find the explicit formulae for all genus GW generating functions F_g of the distinguished CY threefold, the quintic threefold Q, among other CY threefolds. For genus zero case, F_0 is determined by the celebrated mirror symmetry conjecture [CdOGP91], which was mathematically proved by Givental [Giv96] and by Lian-Liu-Yau [LLY97]. For higher genus cases, Bershadsky-Cecotti-Ooguri-Vafa (BCOV) conjectured a Feynman rule for any CY threefold based on Super-Strings theories [BCOV94]. This rule gives an algorithm which effectively calculated the GW potential F_g for all $g > 0$, via the lower genus GW-potentials and finitely many (yet to be determined) initial conditions. BCOV's Feynman rule is a cornerstone in the GW theory of CY threefolds.

The main result of this paper is (see §0.2 for a more explicit statement):

Main Theorem. *The BCOV Conjecture for quintics holds for all genus.*

0.1. **Earlier developments.** Using Mirror Symmetry Conjecture, String theorists have computed the genus zero GW-invariants F_0 for many CY threefolds, by effectively evaluating certain variation of Hodge structures of the mirror CY at large complex structure limits, following the lead by Candelas et. al.. As we will be focusing on high genus GW-invariants, we will bypass listing any references along this line of development.

The theory developed in [BCOV94] is fundamental in the study of higher genus GW-invariants of CY threefolds. For a family of CY threefolds M, the authors used path integral

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to form a B-model topological partition function $\mathcal{F}_g^W(q,\bar{q})$ for its mirror CY threefold family W, which is a non-holomorphic extension of the GW potential $F_g^M(q)$. They further showed that this B-model topological partition function satisfies the *holomorphic anomaly equation*. Solving the equations and using mirror symmetry, they deduced their (BCOV) Feynman rule.

As will be demonstrated in the later part of the introduction, the BCOV's Feynman rule provides an effective algorithm to determine recursively all genus GW potentials of a CY threefold M, after the finite many ambiguity can be found at each g.

Huang-Klemm-Quackenbush in [HKQ09] has pushed the work of [BCOV94] further, demonstrating how to effectively find all initial conditions necessary for determining genus $g \le 51$ GW generating function F_g for the quintic threefold Q.

The task of mathematically proving these formulas (algorithms) for F_g has progressed as well. In [Kon93], Kontsevich showed how to use a hyperplane property of genus zero GW-invariants of Q to relate that of Q with that of $\mathbb{P}^4$, and to evaluate them using localization via the $\mathbb{C}^*$-action on $\mathbb{P}^4$. Based on this, the genus zero formula of Candelas for F_0 was proved independently by Givental [Giv96] and Lian-Liu-Yau [LLY97].

For F_1, Li-Zinger developed a theory of reduced genus one GW-invariants of the quintics, which made using $\mathbb{C}^*$ localization to evaluation F_1 possible [LZ09]. Shortly after, by overcoming daunting obstacles, Zinger in [Zi09], using the results proved by Zagier-Zinger [ZZ08], proved the explicit formula of F_1 obtained by BCOV. It is also worth mentioning that Kim and Lho [KL18] gave an independent proof of BCOV's formula for F_1.

Another line of attacks on F_g (for the quintic Q) is via using the algebraic relative GW-invariants and the degeneration formula of GW-invariants [Li01, Li02]. (For the symplectic version, see [LR01, IP04].) In [MP06] Maulik-Pandharipande developed an algorithm, which in principle can evaluate all genus GW-invariants of the quintic Q. They also proposed an alternative approach, which was simplified in [Wu18] for genus 2 and 3, after combined with that proposal in [Gat03]. In [FL17] via applying localization to a degeneration of $\mathbb{P}^4$ to Q, Fan-Lee obtained a recursive algorithm for F_g, depending on some initial conditions. In [GJR17], Guo-Janda-Ruan have proved that a localization formula via compactifying the moduli of stable maps with p-fields does give the F_2 of the quintic conjectured in [BCOV94].

0.2. **BCOV's Feynman rule.** Let $N_{g,d}$ be the genus g degree d GW-invariants of quintics Q. The genus g GW generating function (potential) F_g of the Q takes the form:

$$F_g(q) := \frac{5}{6}\delta_{g,0}\log q^3 - \frac{25}{12}\delta_{g,1}\log q + \sum_{d\ge 0} N_{g,d}\cdot q^d \tag{0.1}$$

Here the log term comes from the degree zero "unstable" contributions.

The genus zero F_0 can be computed by the celebrated genus zero mirror symmetry: Let

$$I(q,z) := z\sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d}(5H+mz)}{\prod_{m=1}^{d}(H+mz)^5} = \sum_{i=0}^{3} I_i(q)H^i z^{1-i};$$

be the I-function of the quintic threefold and let $J_i(q) := I_i(q)/I_0(q)$ for $i=0,\cdots,3$. Then

Theorem 0.1 (Mirror Theorem [Giv96, LLY97])**.** *The following formula for F_0 holds*

$$F_0(\mathrm{Q}) = \tfrac{5}{6}\big(\log \mathrm{Q}^3 - J_1(q)^3\big) + \tfrac{5}{2}\big(J_1(q)J_2(q) - J_3(q)\big), \quad \textit{with } \mathrm{Q} = q\exp J_1(q).$$

We now state BCOV's conjecture, which extends the mirror symmetry to higher genus. Let the three "propatators" introduced in [BCOV94] be $T^{\varphi\varphi}, T^{\varphi}$ and $T \in \mathbb{Q}[\![q]\!]$ (which are essentially the genus zero invariants and explicitly defined in (1.5), see also Remark 0.3). Let

$$Y = (1-5^5 q)^{-1} \quad \text{and} \quad I_{11} = 1 + q\tfrac{d}{dq}J_1. \tag{0.2}$$

For $2g-2+m>0$, we introduce the "normalized" GW potential following [YY04]

$$P_{g,m} := \frac{(5Y)^{g-1}(I_{11})^m}{(I_0)^{2g-2}}\big(\mathrm{Q}\frac{d}{d\mathrm{Q}}\big)^m F_g\Big|_{\mathrm{Q}=q\exp J_1} \in \mathbb{Q}[\![q]\!]. \tag{0.3}$$

We denote the state space by $H_{\mathbf{B}} := \mathrm{span}\{\psi, \varphi\}$, which is a linear span of the formal variables φ and ψ. Let G_g be the set of genus g stable (dual) graphs. For each $\Gamma \in G_g$, we define a contribution $\mathrm{Cont}_\Gamma^{\mathrm{BCOV}}$ via the following construction:

(i) at each edge, we place a bi-vector in $\mathbb{Q}[\![q]\!] \otimes H_{\mathbf{B}}^{\otimes 2}$*:*

$$T^{\varphi\varphi}\, \varphi \otimes \varphi + T^{\varphi}\, (\varphi \otimes \psi + \psi \otimes \varphi) + T\, \psi \otimes \psi;$$

(ii) at each vertex, we place a multi-linear map $H_{\mathbf{B}}^{\otimes(m+n)} \longrightarrow \mathbb{Q}[\![q]\!]$*:*

$$\varphi^{\otimes m} \otimes \psi^{\otimes n} \longmapsto P_{g,m,n} := \begin{cases} (2g{+}m{+}n{-}3)_n \cdot P_{g,m} & \text{if } 2g-2+m>0 \\ (n-1)!\big(\frac{\chi}{24}-1\big) & \text{if } (g,m)=(1,0) \end{cases} \tag{0.4}$$

where $\chi = -200$ *and* $(a)_k := a(a-1)\cdots(a-k+1)$*;*

(iii.) we apply the map (ii) at each vertex to the placements (i) at the edges incident to that vertex; we define $\mathrm{Cont}_\Gamma^{\mathbf{B}}$ *to be the product over all vertices and edges.*

Later, we will simply call (iii) the composition rule. The BCOV Conjecture is

Theorem 0.2 (BCOV's Feynman rule)**.** *For* $g > 1$*, the Feynman graph sum*

$$f_g^{\mathrm{BCOV}} := \sum_{\Gamma \in G_g} \frac{1}{|\operatorname{Aut}(\Gamma)|} \mathrm{Cont}_\Gamma^{\mathrm{BCOV}}, \tag{0.5}$$

which a priori is a power series in the Novikov variable q*, is a polynomial in* $X := \frac{-5^5 q}{1-5^5 q}$ *of degree at most* $3g-3$*.*

This polynomial is called the ambiguity in the physics literature. Once it is known, the F_g is determined entirely by the lower genus $F_{h<g}$. In Section 5, we will represent it via the quantization of a symplectic transformation on the "small" phase space $H_{\mathbf{B}}$.

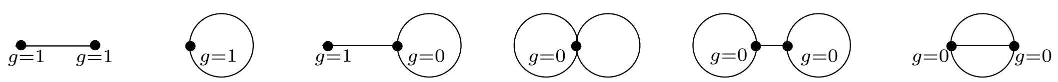
Remark 0.3. In [BCOV94], there are also freedoms in choosing the propagators, which were called "gauge". They conjectured that, the Feynman rule will hold with a suitable choice of gauge. In §1.1, we give the most general freedoms for the gauges (1.4) and their explicit roles in propagators (1.3). For this reason we regard Theorem 1 (given in §1.1) as the most general form of BCOV's conjectures, with insertions, and with gauges (1.4).

0.3. **The algorithm.** The BCOV's Feynman rule provides a recursive algorithm for determining F_g, up to finite ambiguity. The set G_g contains a distinguished "leading" graph Γ_g which has only a single genus g vertex with contribution P_g. Others $\Gamma \in G_g \backslash \{\Gamma_g\}$ contribute to products of $F_{g'<g}$ and propagators $\{T^{\varphi\varphi}, T^{\varphi}, T\}$, which are explicitly computable assuming all $F_{g'<g}$ are known. Then (0.5) implies that

$$P_g = -\sum_{\Gamma \in G_g \backslash \{\Gamma_g\}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \mathrm{Cont}_\Gamma^{\mathrm{BCOV}} + f_g^{\mathrm{BCOV}}, \quad \text{and} \quad \deg_X f_g^{\mathrm{BCOV}} \le 3g-3.$$

This way, F_g is determined explicitly once we have found f_g^{BCOV}, which has $3g-3$ unknown coefficients, as the constant term is given by the (known) degree zero GW-invariants.

To illustrate this, we apply the algorithm to find the genus two potential F_2 (A more detailed computation can be found in Appendix B.2). There are 6 stable graphs in $G_2 \backslash \{\Gamma_2\}$:



The BCOV's Feynman rule for $g=2$ gives us

$$\begin{aligned} -P_2 \;=\; & \frac{1}{2}\Big(T^{\varphi\varphi} P_{1,1}^2 + 2\,T^{\varphi} P_{1,0,1} P_{1,1} + T P_{1,0,1}^2\Big) + \frac{1}{2}\Big(T^{\varphi\varphi} P_{1,2} + T^{\varphi} P_{1,1} + T P_{1,0,2}\Big) \\ & + \frac{1}{2}\Big((T^{\varphi\varphi})^2 P_{1,1} + T^{\varphi\varphi} T^{\varphi} P_{1,0,1}\Big) + \frac{1}{8}\Big((T^{\varphi\varphi})^2 P_{0,4} + 4\,T^{\varphi}\Big) + \frac{1}{8}\,(T^{\varphi\varphi})^3 + \frac{1}{12}\,(T^{\varphi\varphi})^3 + f_2^{\mathrm{BCOV}}. \end{aligned}$$

As $N_{2,1}$, $N_{2,2}$ and $N_{2,3}$ can be calculated classically (see Appendix A), by using the definition of $P_{1,0,n}$ in (0.4) and the genus one mirror formula [Zi09, KL18, CGLZ18] [1]

$$P_{1,1} = -\frac{28}{3} \cdot q\frac{d}{dq}(\log I_0) - \frac{1}{2}T^{\varphi\varphi} - \frac{1}{12}X - \frac{107}{60}$$

we prove the genus two mirror formula conjectured in [BCOV94]:

Theorem 0.4. *Let $B := q\frac{d}{dq}(\log I_0)$. The genus two GW potential F_2 of quintics is*

$$F_2 = \frac{-I_0^2}{5(1-X)}\Big[\frac{350\,T}{9} + \Big(\frac{25\,X+535}{36} + \frac{700\,B}{9} + \frac{25\,T^{\varphi\varphi}}{6}\Big)T^{\varphi} + \frac{5}{24}(T^{\varphi\varphi})^3 + \frac{25\,B+X+4}{6}(T^{\varphi\varphi})^2$$
$$+\Big(\frac{65\,X^2+46\,X+2129}{1440} + \frac{25\,X+535}{36}\,B + \frac{350}{9}\,B^2\Big)T^{\varphi\varphi} + \Big(\frac{X^3}{240} - \frac{113\,X^2}{7200} - \frac{487\,X}{300} + \frac{11771}{7200}\Big)\Big].$$

0.4. **The strategy of the proof.** Our proof of BCOV's Feynman rules is via applying the NMSP theory, which was introduced in [NMSP1]. In its sequel [NMSP2], the property of the NMSP theory was further studied, and the conjecture on the Yamaguchi-Yau ring was proved. In this paper, we will continue to use the results proved in [NMSP1, NMSP2].

We begin our paper with stating the generalized BCOV's Feynman rule (Theorem 1). We then introduce a parallel Feynman rule, derived from the NMSP theory, which we call the NMSP Feynman rule (Theorem 2). We then state our Theorem 3, which says that the generalized BCOV's Feynman rule is equivalent to the NMSP Feynman rule.

In the first half of the paper, we will build the mentioned NMSP Feynman rule and prove Theorem 2. To build the NMSP Feynman rule, we use the NMSP theory and its $\mathbb{C}^*$ localization. As is shown in [NMSP1], the organization of the $\mathbb{C}^*$ localization of the NMSP theory is governed by a class of graphs, whose vertices are categorized into level 0, 1 and ∞; and among these three vertices, level 0 vertices are *GW-invariants of the quintic Q*. The key is that *the edges connecting level 0 vertices contribute (in the* NMSP *theory) exactly the BCOV propagators.* This leads us to introduce the "NMSP-[0] theory", given by summing the contributions from graphs in the NMSP theory whose vertices are of level 0.

In [NMSP2], we have identified the NMSP-$[0,1]$ theory (constructed in [NMSP2]) with the R matrix action on the CohFT of the union of the quintic Q with N points. We have proved the polynomiality of the NMSP-$[0,1]$ theory there. Based on these results, we prove the polynomiality of "NMSP-[0] theory" in Proposition 3.22 via Lemma 3.21 (proved in §4.3). We also identify (via the factorization (4.1)) the NMSP Feynman rule with the polynomiality of "NMSP-[0] theory", with the same controlled degree bound $3g-3$. So the NMSP Feynman rule is proved simultaneously.

In the second half of the paper, we write the generalized BCOV's Feynman rule in the form of the operator quantization of the symplectic transformation $R^{\mathbf{B}}$ on the B-model state space $H_{\mathbf{B}}$. Here the $R^{\mathbf{B}}$-matrix is exactly the restriction of the $R^{\mathbf{A}}$-matrix that appears in the NMSP Feynman rule(§5.1). We then introduce the "modified" Feynman rule via the factorization of the quantization action(§5.2). Compared with the NMSP's modified rule (§6.2), we prove that the generalized BCOV's Feynman rule is equivalent to the NMSP Feynman rule, hence proving Theorem 3. Theorem 2 and 3 imply Theorem 1 directly, and provide a mathematical proof of the BCOV's Feynman rule.

As a further remark (in §7), we will show that Yamaguchi-Yau's functional equations (7.4) and (7.5) for quintic Calabi-Yau threefold [2], can be derived from the operator formalism of the BCOV Feynman sum rule (Theorem 7.3). Indeed, J. Zhou and the authors of this paper will give a geometric proof that for a general Calabi-Yau threefold its BCOV Feynman rule implies Yamaguchi-Yau functional equations (7.4) and (7.5) (cf. [CGLZ]). For the quintic Calabi-Yau threefold, we include here a direct proof.

[1] See also Example B.3 for a short proof of the genus 1 mirror formula via BCOV's rule.

[2] They are also called Holomorphic Anomaly Equations in the literature, c.f. [LhP18, GJR18].

The paper is organized as follows. In §1, we make precise the statements of Theorem 1, 2 and 3. In §2, we recall the notion of CohFTs and R matrix actions. In §3 and §4, we prove the NMSP Feynman rule, the Theorem 2. In §5 and §6, we prove the equivalence of two Feynman rules, which is Theorem 3. In §7, we verify the Yamaguchi-Yau equations, and illustrate how to apply our main theorems to find lower genus $F_{g\le 3}$.

We believe that this approach should provide Feynman rules for complete intersection CY threefolds in products of weighted projective spaces. This is our work in progress.

1. The Main Theorems

In this section we give the statement of three theorems, respectively (i) generalized BCOV's Feynman rule, (ii) the NMSP Feynman rule, and (iii) their equivalence. We will prove (i) by showing (ii) and (iii), in next sections.

Following [YY04], let $D := q\frac{d}{dq}$ and we introduce the following generators [3]

$$A_p := \frac{D^p I_{11}(q)}{I_{11}(q)}, \quad B_p := \frac{D^p I_0(q)}{I_0(q)}, \quad X := \frac{-5^5 q}{1-5^5 q}. \tag{1.1}$$

It is proved in [YY04] that the *ring of five generators*

$$\mathcal{R} = \mathbb{Q}[A_1, B_1, B_2, B_3, X].$$

is closed under the differential operator D, and contains all the $A_{k\ge 4}$ and $B_{k\ge 2}$. Indeed, [4]

$$A_2 = 2B^2 - 2AB - 4B_2 - X\cdot\Big(A + 2B + \frac{2}{5}\Big), \quad B_4 = -X\cdot\Big(2\,B_3 + \frac{7}{5}\,B_2 + \frac{2}{5}\,B + \frac{24}{625}\Big), \tag{1.2}$$

where we always denote $A = A_1$ and know that $B = B_1$.

In [NMSP2], the *finite generation property* raised in [YY04] is proved. We state it now.

Theorem (Polynomial structure). *For $2g-2+m>0$, $P_{g,m}$ lies in the ring $\mathcal{R}$.*

1.1. **BCOV's Feynman rule with insertions in general gauge.** We now introduce a Feynman rule generalizing that in [BCOV94][5]. First we introduce the propagators

$$E^{\mathcal{G}}_{\psi} := B_1 + c_{1a}, \quad E^{\mathcal{G}}_{\varphi\varphi} := A + 2\,B_1 + c_{1b}, \quad E^{\mathcal{G}}_{\varphi\psi} := -B_2 - c_{1b}\,B_1 - c_2,$$
$$E^{\mathcal{G}}_{\psi\psi} := -B_3 + (B - X)\cdot B_2 - \frac{2}{5}\,B_1 X + c_{1b}\,B_1^2 - 2c_2\,B_1 + c_3, \tag{1.3}$$

which depend on the "gauge" $\mathcal{G} := (c_{1a}, c_{1b}, c_2, c_3)$, where

$$c_{1a},\ c_{1b} \in \mathbb{Q}[X]_1, \quad c_2 \in \mathbb{Q}[X]_2, \quad \text{and} \quad c_3 \in \mathbb{Q}[X]_3. \tag{1.4}$$

Here we denote by $\mathbb{Q}[X]_d$ the set of polynomials of degree $\le d$. In the papers [BCOV94, YY04], the propagators were chosen with the following special "gauge"

$$(T^{\varphi\varphi}, T^{\varphi}, T) := (E^{\mathcal{G}}_{\varphi\varphi}, E^{\mathcal{G}}_{\psi\varphi}, E^{\mathcal{G}}_{\psi\psi})|_{(c_{1b},c_2,c_3)=(\frac{3}{5},-\frac{2}{25},-\frac{4}{125})}. \tag{1.5}$$

Let $G_{g,n}$ be the set of stable graphs of genus g and n legs. Let $H_{\mathbf{B}} := \mathrm{span}\{\varphi,\psi\}$ be the B-model state space. We define the $\mathbf{B}$-master potential via the graph sum formula

$$f^{\mathbf{B},\mathcal{G}}_{g,m,n} = \langle \varphi^{\otimes m}, \psi^{\otimes n}\rangle^{\mathbf{B},\mathcal{G}}_{g,m+n} := \sum_{\Gamma\in G_{g,m+n}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\operatorname{Cont}^{\mathbf{B},\mathcal{G}}_{\Gamma}(\varphi^{\otimes m}, \psi^{\otimes n}). \tag{1.6}$$

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[3] Recall I_{11} was defined in (0.2). Here our choice of generators are slightly different from that in [YY04] and [HKQ09], which comes out naturally from our approach through A-model theory.

[4] Their proof relies on a "non-holomorphic completion" of the generators. For an algebraic proof of the first equation see [ZZ08, Lemma 3]. The second equation follows directly from the Picard-Fuch equation.

[5] See Appendix B.1 for a statement of this Feynman rule in the original language, and the relations with our version. See also §5 for the Feynman grasph sum as a geometric quantization.

Here for each $\Gamma \in G_{g,n}$, the contribution $\mathrm{Cont}_\Gamma^{\mathbf{B},\mathcal{G}}$ is defined via taking the product through all vertices by the composition rule by the following placements:

- at each of the first m or last n legs, we place a vector
$$\varphi - E_\psi^{\mathcal{G}} \cdot \psi \quad \text{or} \quad \psi \qquad \text{respectively;}$$
- at each edge, we place a bi-vector
$$E_{\varphi\varphi}^{\mathcal{G}}\, \varphi \otimes \varphi + E_{\varphi\psi}^{\mathcal{G}}\, (\varphi \otimes \psi + \psi \otimes \varphi) + E_{\psi\psi}^{\mathcal{G}}\, \psi \otimes \psi\,;$$
- at each vertex, we place a multi-linear map $:H_{\mathbf{B}}^{\otimes(m+n)} \longrightarrow \mathbb{Q}[\![q]\!]$:
$$\varphi^{\otimes m} \otimes \psi^{\otimes n} \mapsto \left\langle \varphi^{\otimes m}, \psi^{\otimes n} \right\rangle_{g,m+n}^{Q,\mathbf{B}} := P_{g,m,n},$$
where we recall $P_{g,m,n}$ is defined in (0.4).

Theorem 1 (Extended BCOV's Feynman rule with insertions in general gauge)**.** *For any gauge satisfying* (1.4)*, we have the following polynomial structure statement*
$$f_{g,m,n}^{\mathbf{B},\mathcal{G}} \in \mathbb{Q}[X]_{3g-3+m}.$$

By taking $g > 1$, $m = n = 0$ and picking the special gauge (1.5) in Theorem 1, one recovers Theorem 0.2 in the introduction.

Convention 1.1. In this paper ψ is the psi class of $\overline{\mathcal{M}}_{g,n}$, namely, the ancestor class.

Remark 1.2. After identification $\varphi = I_0 I_{11} H$, the correlation function $P_{g,m,n}$ matches the normalized GW correlator of quintic CY threefolds. Namely let $Y := 1 - X$, then
$$P_{g,m,n} = \frac{(5Y)^{g-1}}{I_0^{2g-2+m}} \left\langle \varphi^{\otimes m}, \psi^{\otimes n} \right\rangle_{g,m+n}^{Q}$$
except for the "exceptional" cases when $(g,m) = (1,0)$. Here
$$\left\langle \tau_1\psi_1^{k_1}, \cdots, \tau_n\psi_1^{k_n} \right\rangle_{g,n}^{Q} := \sum_d \mathrm{Q}^d \int_{[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\tau_1)\psi_1^{k_1} \cup \cdots \cup \mathrm{ev}_n^*(\tau_n)\psi_n^{k_n}.$$
For the exceptional cases, the BCOV's correlators
$$P_{1,0,n} = (n-1)!\left(\tfrac{\chi}{24} - 1\right)$$
differ from the corresponding GW correlators $\left\langle \psi^{\otimes n} \right\rangle_{1,n}^{Q} = (n-1)!\frac{\chi}{24}$ by a "correction term" $-(n-1)!$. This term is mysterious from the A-model side. In the proof of Theorem 3, we will see how this term comes into play.

1.2. **The** NMSP **Feynman rule.** Let $H_{\mathbf{A}}$ be the A-model state space:
$$H_{\mathbf{A}} := \mathrm{span}\{\varphi_0, \cdots, \varphi_3\}[\psi], \quad \varphi_i := I_0 I_{11} \cdots I_{ii} H^i \quad \text{for } i = 0, \cdots, 3.$$
We introduce the propagator matrix with gauge $\mathcal{G}$ by
$$R^{\mathbf{A},\mathcal{G}}(\psi)^{-1} = \mathrm{I} - \begin{pmatrix} 0 & \psi \cdot E_{1\varphi_2}^{\mathcal{G}} & \psi^2 \cdot E_{\varphi\psi}^{\mathcal{G}} & \psi^3 \cdot E_{1\psi^2}^{\mathcal{G}} \\ & 0 & \psi \cdot E_{\varphi\varphi}^{\mathcal{G}} & \psi^2 \cdot E_{1,\varphi\psi}^{\mathcal{G}} \\ & & 0 & \psi \cdot E_{1\varphi_2}^{\mathcal{G}} \\ & & & 0 \end{pmatrix} \in \mathrm{End}\, H_{\mathbf{A}}. \tag{1.7}$$
Here besides the BCOV's propagators (1.3), we introduce extra propagators
$$E_{1\varphi_2}^{\mathcal{G}} := E_\psi^{\mathcal{G}}, \quad E_{1,\varphi\psi}^{\mathcal{G}} := -E_\psi^{\mathcal{G}} \cdot E_{\varphi\varphi}^{\mathcal{G}} - E_{\varphi\psi}^{\mathcal{G}}, \quad E_{1\psi^2}^{\mathcal{G}} := -E_\psi^{\mathcal{G}} \cdot E_{\varphi\psi}^{\mathcal{G}} - E_{\psi\psi}^{\mathcal{G}}. \tag{1.8}$$

We define the A-model master potential via the following graph sum formula
$$f_{g;\mathbf{a},\mathbf{b}}^{\mathbf{A},\mathcal{G}} = \langle \varphi_{a_1}\psi^{b_1}, \cdots, \varphi_{a_n}\psi^{b_n} \rangle_{g,n}^{\mathbf{A},\mathcal{G}} := \sum_{\Gamma \in G_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \mathrm{Cont}_\Gamma^{\mathbf{A},\mathcal{G}}(\varphi_{a_l}\psi^{b_l}) \tag{1.9}$$

where for each stable graph Γ, the contribution $\mathrm{Cont}_\Gamma^{\mathbf{A},\mathcal{G}}$ is defined via taking the composition rule along the following placements[6]:

- at each leg l with insertion $\varphi_{a_l}\psi^{b_l}$, we place the vector
$$R^{\mathbf{A},\mathcal{G}}(\psi)^{-1}\varphi_{a_l}\psi^{b_l} \;\in\; H_{\mathbf{A}}\ ;$$
- at each edge, we place the bi-vector[7] in $H_{\mathbf{A}}\otimes H_{\mathbf{A}}$
$$V^{\mathbf{A},\mathcal{G}}(\psi,\psi') := \frac{1}{\psi+\psi'}\sum_i \Big(\varphi_i\otimes\varphi_{3-i} - R^{\mathbf{A},\mathcal{G}}(\psi)^{-1}\varphi_i\otimes R^{\mathbf{A},\mathcal{G}}(\psi)^{-1}\varphi_{3-i}\Big)\ ;$$
- at each vertex, we place the map
$$\tau_1(\psi)\otimes\cdots\otimes\tau_n(\psi)\mapsto \frac{(5Y)^{g-1}}{I_0^{2g-2+n}}\,\langle\tau_1(\psi_1),\cdots,\tau_n(\psi_n)\rangle_{g,n}^{Q}\,. \tag{1.10}$$

In particular, when $\mathbf{a}=1^m0^n$ and $\mathbf{b}=0^m1^n$, we define
$$f_{g,m,n}^{\mathbf{A},\mathcal{G}} = \langle\varphi_1^{\otimes m},(\varphi_0\psi)^{\otimes n}\rangle_{g,m+n}^{\mathbf{A},\mathcal{G}}. \tag{1.11}$$

Theorem 2 (A-model NMSP Feynman rule)**.** *For any*
$$c_{1a},\ c_{1b}\in\mathbb{Q}[X]_1,\quad c_2\in\mathbb{Q}[X]_2,\quad \text{and}\quad c_3\in\mathbb{Q}[X]_3, \tag{1.12}$$
we have the following polynomial structure statement
$$f_{g;\mathbf{a},\mathbf{b}}^{\mathbf{A},\mathcal{G}}\in\mathbb{Q}[X]_{3g-3+n-\sum_i b_i}.$$

Remark 1.3. Comparing with BCOV's Feynman rule, we see that in the A-model case

- the state space is is of higher dimension; and we have 6 (instead of 3) types of edge contributions (which we call extra propagators);
- there is no "correction term" in the $g=1$ vertex (see Remark 1.2 for more details);
- the master potential $f^{\mathbf{A},\mathcal{G}}$ is indeed the generating function of a CohFT $R^{\mathbf{A},\mathcal{G}}.\bar\Omega^Q$ (c.f. §2), where $\bar\Omega^Q$ is the normalized CohFT of quintics (c.f. §2.5.3).

Remark 1.4. The NMSP Feynman rule essentially says that, in the orbit of the R-matrix group action on the quintic CohFT, there exists a "special" subset $\{\Omega^{\mathbf{A},\mathcal{G}}:\mathcal{G}\}$ which is invariant under BCOV's "gauge" group, such that their genus g potential functions are simply degree $3g-3$ polynomials in X.

1.3. **BCOV's Feynman rule versus** NMSP **Feynman rule.** We now state our final result. Theorem 1 is a direct consequence of Theorem 2 and this result.

Theorem 3. *For $\star=A$ or B we introduce the master potential function*
$$f^{\star,\mathcal{G}}(\hbar,x,y) := \sum_{g,m,n}\hbar^{g-1}f_{g,m,n}^{\star,\mathcal{G}}x^my^n.$$
Then we have the identity
$$f^{\mathbf{A},\mathcal{G}}(\hbar,x,y) = f^{\mathbf{B},\mathcal{G}}(\hbar,x,y)-\ln(1-y).$$
Namely, the two types of graph sums are related by
$$f_{g,m,n}^{\mathbf{A},\mathcal{G}} = f_{g,m,n}^{\mathbf{B},\mathcal{G}}+\delta_{g,1}\delta_{m,0}(n-1)!.$$

[6] Indeed, the graph sum defined here is the $R^{\mathbf{A},\mathcal{G}}$-matrix action, see §2.3 for more details.

[7] A direct computation shows
$$V^{\mathbf{A},\mathcal{G}}(\psi,\psi') = E^{\mathcal{G}}_{\varphi\varphi}\,(\varphi_1\otimes\varphi_1)+E^{\mathcal{G}}_{\varphi\psi}\,(\varphi_1\otimes\varphi_0\psi'+\varphi_0\psi\otimes\varphi_1)+E^{\mathcal{G}}_{\psi\psi}\,(\varphi_0\psi\otimes\varphi_0\psi')$$
$$+\,E^{\mathcal{G}}_{1,\varphi\psi}\,(\varphi_0\otimes\varphi_1\psi'+\varphi_1\psi\otimes\varphi_0)+E^{\mathcal{G}}_{1\psi^2}\,(\varphi_0\otimes\varphi_0(\psi')^2+\varphi_0\psi^2\otimes\varphi_0)+E^{\mathcal{G}}_{1\varphi_2}\,(\varphi_0\otimes\varphi_2+\varphi_2\otimes\varphi_0).$$

Remark 1.5. Indeed, we will see that the graph sum definition of $f^{\star,\mathcal{G}}_{g,m,n}$ (for $\star = \mathbf{A}$ or $\mathbf{B}$) is equivalent to the following quantization of $R^\star$-matrix action:

$$f^{\star,\mathcal{G}}(h,x,y) := \log\left(\widehat{R}^{\star,\mathcal{G}} P^\star(\hbar;x,y)\right)$$

where the generating function $P^\star(\hbar;x,y)$ are defined via

$$P^{\mathbf{B}}(\hbar,x,y) = P^{\mathbf{A}}(\hbar,x,y) + \ln(1-y) := \sum_{g,m,n} \hbar^{g-1}\frac{x^m y^n}{m!n!}\cdot P_{g,m,n}. \tag{1.13}$$

See §5 for more details about the quantization of symplectic transformations.

2. Cohomological field theory and R-matrix action

In this section, we investigate the CohFTs and the R-matrix actions. We will follow closely the treatment developed by Pandharipande-Pixton-Zvionkone in [PPZ15].

We first fix notations. Let $Q\subset \mathbb{P}^4$ be a smooth quintic CY threefold; let $(\pi, \mathrm{ev}_{n+1}) : \mathcal{C}\to \overline{\mathcal{M}}_{g,n}(Q,d)\times Q$ be the universal family of the moduli of stable maps to Q, and let

$$\mathrm{pr}^Q_{g,n} : \overline{\mathcal{M}}_{g,n}(Q,d)\to \overline{\mathcal{M}}_{g,n} \quad \text{and} \quad \mathrm{pr}_k : \overline{\mathcal{M}}_{g,n+k}\to \overline{\mathcal{M}}_{g,n}$$

be the obvious the forgetful morphisms.

2.1. Definition of cohomological field theory.

We recall the definition of a CohFT introduced by Kontsevich-Manin [KM94].

Definition 2.1. A CohFT consists of a triple $(V,\eta,\mathbf{1})$, where V is an $\mathbb{F}$-linear space[8] for an integral domain $\mathbb{F}$, η is a non-degenerate (super) symmetric bilinear form $\eta : V\times V\to \mathbb{F}$, $\mathbf{1}\in V$ is called a unit, and S_n-equivariant maps

$$\Omega_{g,n} : V^{\otimes n}\to H^*(\overline{\mathcal{M}}_{g,n})\otimes \mathbb{A}, \quad g\ge 0,\ 2g-2+n>0,$$

where $\mathbb{A}$ is an $\mathbb{F}$-algebra, called the coefficient ring, such that, for any basis $\{e_k\}$ of V and its dual basis $\{e^k\}$[9], the maps $\Omega_{g,n}$ satisfy the following properties (axioms):

(a1) Fundamental Class Axiom:

$$\Omega_{0,3}(\mathbf{1},\tau_1,\tau_2) = \eta(\tau_1,\tau_2),$$
$$\Omega_{g,n+1}(\tau_{\mathbf{n}},\mathbf{1}) = (\mathrm{pr}_1)^*\Omega_{g,n}(\tau_{\mathbf{n}}), \qquad \tau_{\mathbf{n}} := (\tau_1,\cdots,\tau_n);$$

(a2) Splitting Axiom and Genus reduction axiom

$$s^*\Omega_{g_1+g_2,n_1+n_2}(\tau_{\mathbf{n}_1},\tau_{\mathbf{n}_2}) = \textstyle\sum_k \Omega_{g_1,n_1+1}(\tau_{\mathbf{n}_1},e^k)\cdot \Omega_{g_2,1+n_2}(e_k,\tau_{\mathbf{n}_2}),$$
$$r^*\Omega_{g+1,n}(\tau_{\mathbf{n}}) = \textstyle\sum_k \Omega_{g,n+2}(\tau_{\mathbf{n}},e^k,e_k).$$

Here s and r are the obvious gluing maps.

Example 2.2 (CohFT of the GW theory of X)**.** For a projective variety X, and a coefficient field $\mathbb{F}$, we introduce the triple and the maps by (with pr^X defined as like pr^Q)

$$V = H^*(X,\mathbb{F}); \quad (x,y) = \int_X x\cup y; \quad 1\in H^0(X,\mathbb{F});$$

$$\Omega^X_{g,n}(\tau_{\mathbf{n}}) := \sum_{d=0}^\infty q^d \big(\mathrm{pr}^X_{g,n}\big)_*\Big(\prod_{i=1}^n \mathrm{ev}_i^*\tau_i \cap [\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}\Big) \in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{F})[\![q]\!].$$

[8] By "a space over a domain $\mathbb{F}$" we mean a locally free $\mathbb{F}$ module.

[9] $\{e_k\}$ and $\{e^k\}$ satisfying $\eta(e_k,e^\ell) = (-1)^{\deg e_k \deg e^\ell}\eta(e^\ell,e_k) = \delta_{k\ell}$.

2.2. **Shift and direct sum of CohFTs.**

Definition 2.3 (The shifted CohFT Ω^τ of a given CohFT Ω.)**.** For $\tau \in V \otimes_{\mathbb{F}} \mathbb{A}$,

$$\Omega^\tau_{g,n}(\tau_{\mathbf{n}}) := \sum_{k\geq 0} \frac{1}{k!} \mathrm{pr}_{k*}\Omega_{g,n+k}(\tau_{\mathbf{n}}, \tau^k) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{A}),$$

with the same triple $(V, \eta, 1)$ of Ω. Here we assume that the infinite sum is well defined.

Definition 2.4 (The direct sum of CohFTs)**.** Let Ω^a and Ω^b be two CohFTs with identical coefficient ring $\mathbb{A}$. We define their direct sum to be the CohFT with triple $(V^a \oplus V^b, \eta^a \oplus \eta^b, \mathbf{1}^a \oplus \mathbf{1}^b)$, and with maps

$$(\Omega^a \oplus \Omega^b)_{g,n}(\tau_{\mathbf{n}}) = \Omega^a_{g,n}(\tau^a_{\mathbf{n}}) + \Omega^b_{g,n}(\tau^b_{\mathbf{n}}) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{A}),$$

where $\tau_i = (\tau^a_i, \tau^b_i) \in V^a \oplus V^b$. By iterating, we get a direct sum of finite copies of CohFTs. It is easy to check that the direct sum of CohFTs so defined satisfies all the CohFT axioms, and hence is a CohFT.

Example 2.5. Let Ω^X be as in Example 2.2. For two smooth projective varieties X_1 and X_2, we have $\Omega^{X_1 \sqcup X_2} = \Omega^{X_1} \oplus \Omega^{X_2}$.

2.3. ***R*-matrix action on CohFT.** The R-matrix was first introduced in [Giv01a, Giv01b] to compute higher genus equivariant GW invariants. Its lifting to CohFTs was studied in [Sh09, Te12]. In this section, we will mostly follow [PPZ15][10], with a slight generalization.

Let Ω be a CohFT with the triple $(V, \eta, \mathbf{1})$. We consider another triple $(V', \eta', \mathbf{1}')$, and a formal power series

$$R(z) = R_0 + R_1 z + R_2 z^2 + \cdots \in \mathrm{End}(V, V') \otimes \mathbb{A}[\![z]\!],$$

which satisfies the "symplectic condition": [11]

$$R^*(-z)R(z) = \mathrm{I} \in \mathrm{End}(V). \tag{2.1}$$

Notice that (2.1) implies that $R(z)$ is injective and $\dim_{\mathbb{F}} V \leq \dim_{\mathbb{F}} V'$.

We define the R-matrix action following [PPZ15]. Let $\Gamma \in G_{g,n}$ be a genus g stable graph with n legs. For each vertex v of Γ, we denote its genus by g_v and its valence by n_v. For each Γ we associate it the space $\overline{\mathcal{M}}_\Gamma := \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g_v,n_v}$, and define the contribution

$$\mathrm{Cont}_\Gamma : V'^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_\Gamma, \mathbb{A})$$

by the following construction

(1) at each leg l of Γ, we place a map
$$R^*(-\psi_l) : V' \longrightarrow V[\psi_l];$$
(2) at each edge $e = (v_1, v_2)$ of Γ, we place
$$\frac{\sum_\beta e_\beta \otimes e^\beta - \sum_\alpha R^*(-\psi_{(e,v_1)})e'_\alpha \otimes R^*(-\psi_{(e,v_2)})e'^\alpha}{\psi_{(e,v_1)} + \psi_{(e,v_2)}} \in V[\psi_{(e,v_1)}] \otimes V[\psi_{(e,v_2)}];$$
where $\{e_\beta\}$ and $\{e'_\alpha\}$ are bases of V and V', with dual bases $\{e^\beta\}$ and $\{e'^\alpha\}$ respectively;
(3) at each vertex v of Γ, we place
$$\Omega_{g_v,n_v} : V^{\otimes n_v} \longrightarrow H^*(\overline{\mathcal{M}}_{g_v,n_v}, \mathbb{A}).$$

Let $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ be the tautological morphism by gluing. We define

$$(R\Omega)_{g,n} = \sum_{\Gamma\in G_{g,n}} \frac{1}{|\operatorname{Aut}\Gamma|} \xi_{\Gamma*}(\mathrm{Cont}_\Gamma).$$

[10] In their paper, the authors give a careful proof that R-matrix actions preserve CohFTs.

[11] The symplectic condition is equivalent to : $\eta(v_1, v_2) = \eta'\big(R(z)v_1, R(-z)v_2\big)$, for $v_1, v_2 \in V$. It could not deduce $\eta(R^*(z)v_1, R^*(-z)v_2) = \eta'\big(v_1, v_2\big)$ when $\dim V \neq \dim V'$.

Let ψ_i be the ancestor psi classes of $\overline{\mathcal{M}}_{g,n+k}$. For the given R-matrix, we associate

$$T_R(z) = z\mathbf{1} - zR(-z)^*\mathbf{1}' \in z\mathbb{A}[\![z]\!] \otimes V;$$

we define its associated translation action by

$$T_R\Omega_{g,n}(-) = \sum_{k\geq 0} \frac{1}{k!}(\mathrm{pr}_k)_*\Omega_{g,n+k}(-, T_R(\psi_{n+1}), \cdots, T_R(\psi_{n+k})), \tag{2.2}$$

assuming that the infinite sum makes sense in $H^*(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{A}$. For example, if $\mathbb{A} = \mathbb{F}[\![q]\!]$ is endowed with q-adic topology and

$$T_R(z) \in z^2\mathbb{A}[\![z]\!] \otimes V + q\, z\, \mathbb{A}[\![z]\!] \otimes V, \tag{2.3}$$

then (2.2) automatically converges. We call (2.3) the q-adic conditions for T_R.

Definition 2.6. The R-matrix action on a CohFT Ω is defined by

$$R.\Omega := RT_R\Omega.$$

2.4. **Properties of CohFTs under R-matrix actions.** Following [PPZ15], we have

Theorem 2.7. *Let Ω be a CohFT with unit for the triple $(V, \eta, \mathbf{1})$. Let $\mathbb{A} = F[\![q]\!]$ be endowed with q-adic topology. We have the followings.*

(1) Let $(V', \eta', \mathbf{1}')$ be another triple. Suppose $R(z) \in \mathrm{End}(V, V')[\![z]\!]$ is symplectic and T_R satisfies the q-adic condition. Then $T_R\Omega$ is well-defined and is a CohFT, and $R.\Omega$ is also a CohFT. Furthermore, if $R^*(z) \in \mathrm{End}(V', V)[\![z]\!]$ is symplectic (which is equivalent to $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} V'$), $R.\Omega$ is a CohFT with unit $\mathbf{1}' \in V'$.

(2) Suppose $(V'', \eta'', \mathbf{1}'')$ is another vector space with pairing and unit. Suppose

$$R_a(z) \in \mathrm{End}(V, V')[\![z]\!] \quad \text{and} \quad R_b(z) \in \mathrm{End}(V', V'')[\![z]\!]$$

are symplectic, with T_{R_a}, T_{R_b} satisfying the q-adic conditions (2.3)*. Then as CohFTs on $(V'', \eta'', \mathbf{1}'')$*

$$(R_aR_b).\Omega = R_a.(R_b.\Omega).$$

Proof. All statements can be proved by exactly the same arguments as in [PPZ15, Prop 2.12 and 2.14], except that for the axioms on unit in (1), the identity $(RT\Omega)_{0,3}(\mathbf{1}, \tau_1, \tau_2) = \eta'(\tau_1, \tau_2)$ is shown in Lemma C.1. We leave other identities to readers. ☐

Remark 2.8. We remark that in [PPZ15] the authors used $V = V'$ and $R_0 = \mathrm{I}$. In the next section we will use R actions in the case $\dim_{\mathbb{F}} V < \dim_{\mathbb{F}} V'$. For more relations with [PPZ15], see Example C.2.

2.5. **Examples of CohFTs.** In this subsection, we list some CohFTs used in this paper.

We consider the following CohFTs that arise in the localization of the NMSP theory. As in [NMSP2], we pick a sufficiently large integer N; let $G = (\mathbb{C}^*)^{\mathrm{N}}$, and take $H^*(BG) = \mathbb{Q}[t_1, \cdots, t_{\mathrm{N}}]$ where t_α is the α-th equivariant generator. In this paper, after equivariant integration we will always specialize t_α to $-\zeta_{\mathrm{N}}^\alpha t$, where $\zeta_{\mathrm{N}} = e^{\frac{2\pi i}{\mathrm{N}}}$ is the primitive N-th root of unity. In this paper we always take $\mathbb{F} = \mathbb{Q}(\zeta_{\mathrm{N}})(t)$ and $\mathbb{A} = \mathbb{F}[\![q]\!]$.

2.5.1. *Twisted GW theory of a point $\Omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$ and its topological part $\omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$.* The triple is

$$\mathcal{H}_{\mathrm{pt}_\alpha} = H^0(\mathrm{pt}_\alpha), \quad (\cdot,\cdot)^{\mathrm{pt}_\alpha,\mathrm{tw}}, \quad \mathbf{1}_\alpha := 1 \in H^0(\mathrm{pt}_\alpha)$$

where the inner product is given by

$$(x, y)^{\mathrm{pt}_\alpha,\mathrm{tw}} := 5t_\alpha^{-4}\prod\nolimits_{\beta:\beta\neq\alpha}(t_\beta - t_\alpha)^{-1}xy = \tfrac{-5}{\mathrm{N}t_\alpha^3t^{\mathrm{N}}}xy.$$

Let $\mathbb{E}_{g,n}$ be the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$; the maps are

$$\Omega_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}}(\tau_{\mathbf{n}}) = (-1)^{1-g}\frac{e_T(\mathbb{E}_{g,n}^\vee \otimes (-t_\alpha))^5}{(-t_\alpha)^5}\frac{5t_\alpha}{e_T(\mathbb{E}_{g,n} \otimes 5t_\alpha)}\prod_{\beta:\beta\neq\alpha}\frac{e_T\big(\mathbb{E}_{g,n}^\vee \otimes (t_\beta - t_\alpha)\big)}{(t_\beta - t_\alpha)}\prod_i \tau_i.$$

This gives us the CohFT $\Omega_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}}$.

We introduce the CohFT $\omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$, which is the topological part of $\Omega_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}}$: the triple is $\mathcal{H}_{\mathrm{pt}_\alpha}$ with the same inner product and unit; the maps are defined by $\omega_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}}(1^{\otimes n}) = (\frac{-5}{\mathrm{N}t_\alpha^3 t^{\mathrm{N}}})^{1-g}$.

By [Mu83, FP00, Giv01a], we have

Proposition 2.9. *The following identity between CohFTs holds*

$$\Omega^{\mathrm{pt}_\alpha,\mathrm{tw}} = \Delta^{\mathrm{pt}_\alpha}.\omega^{\mathrm{pt}_\alpha,\mathrm{tw}}, \tag{2.4}$$

where the R-matrix $\Delta^{\mathrm{pt}_\alpha}$ is given by

$$\Delta^{\mathrm{pt}_\alpha}(z) = \exp\Big(\sum_{k>0} \frac{B_{2k}}{2k(2k-1)}\Big(\frac{5}{(-t_\alpha)^{2k-1}} + \frac{1}{(5t_\alpha)^{2k-1}} + \sum_{\beta\neq\alpha} \frac{1}{(t_\alpha - t_\beta)^{2k-1}}\Big) z^{2k-1}\Big).$$

Remark 2.10. We see that the topological CohFT $\omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$ has the same vector space as that of the CohFT $\Omega^{\mathrm{pt}_\alpha}$, but with different inner product. In fact if we define

$$\tilde{\Delta}^{\mathrm{pt}_\alpha}(z) := \sqrt{5/\mathrm{N}} \cdot t_\alpha^{-(3+\mathrm{N})/2}\, \Delta^{\mathrm{pt}_\alpha}(z),$$

then we have the CohFT identity[12]

$$\Omega^{\mathrm{pt}_\alpha,\mathrm{tw}} = \tilde{\Delta}^{\mathrm{pt}_\alpha}.\Omega^{\mathrm{pt}_\alpha}.$$

Convention 2.11. For simplicity, in the following we write Npt as the disjoint union of pt_α, $1 \le \alpha \le \mathrm{N}$, as in [NMSP2]. Accordingly, $\Omega^{\mathrm{Npt},\mathrm{tw}} = \oplus_{\alpha=1}^{\mathrm{N}} \Omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$, $\omega^{\mathrm{Npt},\mathrm{tw}} = \oplus_{\alpha=1}^{\mathrm{N}} \omega^{\mathrm{pt}_\alpha,\mathrm{tw}}$, etc.

2.5.2. *CohFT $\Omega^{Q,\mathrm{tw}}$ of the twisted GW theory of quintic threefold and the shifted CohFT $\Omega^{Q,\mathrm{tw},\tau}$.* Let Q be a smooth quintic CY threefold. The CohFT $\Omega^{Q,\mathrm{tw}}$ consists of the triple

$$\mathcal{H}_Q = H^*(Q), \quad (x,y)^{Q,\mathrm{tw}} = \int_Q \frac{xy}{\prod_{\alpha=1}^{\mathrm{N}}(H+t_\alpha)} = \int_Q \frac{xy}{-t^{\mathrm{N}}}, \quad \mathbf{1}_Q := 1 \in H^*(Q),$$

and the map

$$\Omega_{g,n}^{Q,\mathrm{tw}}(\tau_{\mathbf{n}}) = \sum_{d=0}^{\infty} q^d \cdot \mathrm{pr}^Q_{g,n*}\Big(\frac{\mathrm{ev}_1^*\tau_1 \cdots \mathrm{ev}_n^*\tau_n}{\prod_{\alpha=1}^{\mathrm{N}} e\big(R\pi_*\mathrm{ev}_{n+1}^*\mathcal{O}(1)\cdot t_\alpha\big)} \cap [\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{vir}}\Big).$$

Remark 2.12. By dimension reason one calculates

$$\Omega_{g,n}^{Q,\mathrm{tw}}(\tau_{\mathbf{n}}) = (-t^{\mathrm{N}})^{(g-1)}\Omega_{g,n}^{Q}(\tau_{\mathbf{n}})|_{q\mapsto q':=-q/t^{\mathrm{N}}}.$$

By the fundamental class axiom, if τ is a scalar multiple of the unit, $\Omega^\tau = \Omega$, for any CohFT Ω. In particular Ω^{pt} and $\Omega^{\mathrm{pt},\mathrm{tw}}$ are not affected by any shift. Also, for $\tau \in \mathcal{H}_Q \otimes_{\mathbb{F}} \mathbb{A}$, we denote by $\Omega^{Q,\mathrm{tw},\tau}$ the τ-shifted CohFT of $\Omega^{Q,\mathrm{tw}}$.

Convention 2.13. *By abuse of notations, we denote $\Omega^{Q,\mathrm{tw}} = \Omega^{Q,\mathrm{tw},\tau_Q(q')}$ from now on. Here $\tau_Q(q) := J_1(q)H = I_1(q)/I_0(q)H$ is the mirror map.*

2.5.3. *CohFT $\bar{\Omega}^{Q,\tau}$ of "normalized" shifted GW theory of the quintic threefold.* We consider the following "normalized" CohFT

$$\bar{\Omega}_{g,n}^{Q,\tau_Q}(-) := (5Y)^{g-1}\,\Omega_{g,n}^{Q,\tau_Q}(-), \qquad (v_1,v_2)^{\bar{Q}} = (5Y)^{-1}(v_1,v_2)^Q.$$

Then the graph sum defined in §1.2 is indeed an $R^{\mathbf{A},\mathcal{G}}$-action on the normalized quintic CohFT $\bar{\Omega}^{Q,\tau}$. In (1.10), the factor $(5Y)^{g-1}$ is from the above normalization factor, while $I_0^{-(2g-2+n)}$ is obtained by applying dilaton equations to the tail contributions (c.f. equation (C.2)).

Further, with the change of variable $q \mapsto q' := -q/t^{\mathrm{N}}$ and by adding the normalized factor $(-5Y/t^{\mathrm{N}})^{(1-g)}$, we can identify these two CohFTs:

$$\Omega_{g,n}^{Q,\mathrm{tw}}(-) = \Big[(-5Y/t^{\mathrm{N}})^{(1-g)} \cdot \bar{\Omega}_{g,n}^{Q,\tau_Q}(-)\Big]\Big|_{q\mapsto q':=-q/t^{\mathrm{N}}}. \tag{2.5}$$

---

[12] The T-action here is well-defined with suitable topology. We skip the argument as we don't need it.

2.5.4. *CohFT $\Omega^{\aleph}$ and Ω^{loc}.* The following CohFT $\Omega^{\aleph}$ is a fundamental object in this paper:

Definition 2.14. For $\aleph := Q \cup \mathrm{Npt}$, we define the CohFTs of the local theory to be

$$\Omega^{\aleph} := \Omega^{Q,\mathrm{tw}} \oplus \omega^{\mathrm{Npt},\mathrm{tw}}, \qquad \Omega^{\mathrm{loc}} := \Omega^{Q,\mathrm{tw}} \oplus \Omega^{\mathrm{Npt},\mathrm{tw}}$$

where the triples are both $\mathcal{H} := H^*(\aleph)$, with the pairing and the unit

$$(\cdot,\cdot)^M = (\cdot|_Q, \cdot|_Q)^{Q,\mathrm{tw}} + (\cdot|_{\mathrm{pt}}, \cdot|_{\mathrm{pt}})^{\mathrm{pt},\mathrm{tw}}, \qquad \mathbf{1} = \mathbf{1}_Q + \sum_\alpha \mathbf{1}_\alpha.$$

Here $\cdot|_Q : \mathcal{H} \to \mathcal{H}_0$ and $\cdot|_\alpha : \mathcal{H} \to \mathcal{H}_{\mathrm{pt}_\alpha} := H^*(\mathrm{pt}_\alpha)$ are the projections.

Convention. *Let $G = (\mathbb{C}^*)^{\mathrm{N}}$ act on $\mathbb{P}^{4+\mathrm{N}}$ via scaling the last N homogeneous coordinates of $\mathbb{P}^{4+\mathrm{N}}$. Let p the equivariant-hyperplane class in $H^2_G(\mathbb{P}^{4+\mathrm{N}})$. In this paper, we will view p^i as their images in $\mathcal{H}^{\mathrm{ev}} := H^{\mathrm{ev}}(\aleph, \mathbb{A}) \subset \mathcal{H}$.*

Now we recall some basic facts in the setup from [NMSP2, Sect. 1.1]. Considering the natural decomposition $\mathcal{H} = \mathcal{H}^{\mathrm{ev}} \oplus H^3(Q)$, we pick a basis $\{\phi_i := p^i\}_{i=0}^{\mathrm{N}+3}$ of $\mathcal{H}^{\mathrm{ev}}$ with dual basis

$$\{\phi^0, \cdots, \phi^{\mathrm{N}+3}\} = \Big\{\frac{p^3}{5}(p^{\mathrm{N}} - t^{\mathrm{N}}), \frac{p^2}{5}(p^{\mathrm{N}} - t^{\mathrm{N}}), \frac{p}{5}(p^{\mathrm{N}} - t^{\mathrm{N}}), \frac{1}{5}(p^{\mathrm{N}} - t^{\mathrm{N}}), \frac{p^{\mathrm{N}-1}}{5}, \frac{p^{\mathrm{N}-2}}{5}, \cdots, \frac{p^0}{5}\Big\}.$$

By using the above basis, let $[\mathrm{N}] := \{1, \cdots, \mathrm{N}\}$ we have

$$\mathbf{1}_\alpha = \frac{p^4}{t_\alpha^4} \prod_{\beta \neq \alpha} \frac{t_\beta + p}{t_\beta - t_\alpha} \quad \text{for } \alpha \in [\mathrm{N}], \quad \text{and} \quad H^j = \frac{p^j}{t^{\mathrm{N}}}(t^{\mathrm{N}} - p^{\mathrm{N}}) \quad \text{for } j = 0,1,2,3.$$

The Poincare dual of $\{1, H, H^2, H^3\} \cup \{\mathbf{1}_\alpha\}_{\alpha=1,\cdots,\mathrm{N}}$ is

$$\{\frac{-t^{\mathrm{N}}}{5}H^3, \frac{-t^{\mathrm{N}}}{5}H^2, \frac{-t^{\mathrm{N}}}{5}H, \frac{-t^{\mathrm{N}}}{5}H^0\} \cup \{\mathbf{1}^\alpha := \frac{\mathrm{N}t_\alpha^3 t^{\mathrm{N}}}{(-5)}\mathbf{1}_\alpha\}_{\alpha=1,\cdots,\mathrm{N}}.$$

Remark 2.15. In [NMSP2] we use the notation

$$[-]^{\bullet}_{g,n}, \qquad \text{where } \bullet = \text{“loc”}, \text{“}Q,\mathrm{tw}\text{” or “}\mathrm{pt}_\alpha,\mathrm{tw}\text{”} ,$$

to define certain classes. They are closely related to the CohFT notation $\Omega^\bullet$ used here, with a minor change: in $\Omega^\bullet_{g,n}(-)$ descendents are not allowed, while in $[-]^\bullet_{g,n}$ they are.

3. Expressing NMSP-$[0,1]$ theory via CohFTs

The moduli of NMSP fields and their localizations are constructed in [NMSP1]. In this paper we concentrate on the “NMSP-$[0,1]$ theory”: For $2g-2+n>0$ and $\tau_i \in \mathcal{H}[\![z]\!]$, we introduce

$$\big[\tau_1, \cdots, \tau_n\big]^{[0,1]}_{g,n} = \sum_{d \geq 0} (-1)^{d+1-g} q^d \sum_{\Theta \in G^{[0,1]}_{g,n,d}} \big(\mathrm{pr}^W_{g,n}\big)_* \Big(\prod_{i=1}^n \mathrm{ev}_i^* \tau_i \cdot \mathrm{Cont}_\Theta\Big) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{A}), \tag{3.1}$$

where $\mathrm{pr}^W_{g,n} : \mathcal{W}_{g,n,\mathbf{d}} \to \overline{\mathcal{M}}_{g,n}$ is the projection.and Cont_Θ are contributions from those NMSP localization graphs supported on $[0,1]$ (see [NMSP2, Sect. 0.3] for details).

Definition 3.1. We define $\Omega^{[0,1]}_{g,n}(\tau_1, \cdots, \tau_n) = \big[\tau_1, \cdots, \tau_n\big]^{[0,1]}_{g,n}$, for $\tau_i \in \mathcal{H}$.

We first quote the results proved in [NMSP2] in terms of the CohFT and R matrix actions. Let S^M, S^Q and S^{pt_α} be the S-matrices of the NMSP-$[0,1]$ theory, quintic and pt_α respectively,[13] and let $q' = -\frac{q}{t^{\mathrm{N}}}$. We define $R(z) \in \mathrm{End}\,\mathcal{H} \otimes \mathbb{A}[\![z]\!]$ via the Birkhoff factorization

$$S^M(q,z) \begin{pmatrix} \mathrm{diag}\{\Delta^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}} & \\ & 1 \end{pmatrix} = R(z) \begin{pmatrix} \mathrm{diag}\{S^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}} & \\ & S^Q(z) \end{pmatrix} \Big|_{q \mapsto q'} \tag{3.2}$$

Then the main results in [NMSP2, Theorem 3 and Theorem 4] is the following

[13] See [NMSP2, Sect. 1.3] for definitions of these S-matrices.

Theorem 3.2. *The* NMSP-$[0,1]$ *theory* $\Omega^{[0,1]}$*, with* $\mathbf{1}\in\mathcal{H}$ *and* $(\cdot,\cdot)^M$*, forms a CohFT satisfying*

$$\Omega^{[0,1]} \;=\; R.\Omega^{\aleph}.$$

Furthermore, for $2g-2+n>0$*, the* NMSP-$[0,1]$ *(ancestor) correlator*

$$\Big\langle \phi_{m_1}\psi^{k_1},\cdots,\phi_{m_n}\psi^{k_n}\Big\rangle_{g,n}^{[0,1]} = \int_{[\overline{\mathcal{M}}_{g,n}]}\psi_1^{k_1}\cdots\psi_n^{k_n}\cdot\Omega_{g,n}^{[0,1]}(\phi_{m_1},\cdots,\phi_{m_n})$$

is a q-polynomial of degree $\leq g-1+\frac{3g-3+\sum_{i=1}^n m_i}{\mathrm{N}}$.

A few remarks on Theorem 3.2 are in order. The whole argument [NMSP2, Sect. 3.5] is a composition of R-matrix actions on CohFTs

$$\Omega^{\aleph}=\Omega^{Q,\mathrm{tw}}\oplus\omega^{\mathrm{Npt,tw}}\xrightarrow{\ \mathrm{I}\oplus(\oplus_\alpha\Delta^{\mathrm{pt}_\alpha})\ }\Omega^{\mathrm{loc}}=\Omega^{Q,\mathrm{tw}}\oplus\Omega^{\mathrm{Npt,tw}}\xrightarrow{\ R^{\mathrm{loc}}\ }\Omega^{[0,1]}.$$

Here R^{loc} is the R matrix for torus localizations (defined in [NMSP2, Sect. 1.4]), and $\Delta^{\mathrm{pt}_\alpha}$ is from Grothendieck-Riemann-Roch(GRR) formula at pt_α(c.f.(2.4), see [Mu83, FP00, Giv01a]).

The q-adic condition for the GRR's R matrix holds since $\Delta^{\mathrm{pt}_\alpha}=1+O(z)$. The q-adic condition (2.3) for R^{loc} holds because its tail $T_{R^{\mathrm{loc}}}$ lies in $q\mathbb{A}\otimes V$ by [NMSP2, Remark 3.4]. Thus Theorem 2.7 implies $R.\Omega^{\aleph}=\Omega^{[0,1]}$, where R is the composition of these two actions

$$R(z):=R^{\mathrm{loc}}(z)\cdot(\mathrm{I}\oplus\Delta(z)^{\mathrm{Npt}})\in\mathrm{End}(\mathcal{H})\otimes\mathbb{A}[\![z]\!].$$

It satisfies the defining identity (3.2) by [NMSP2, Remark 3.6].

3.1. **$\Omega^{[0,1]}$-theory in terms of stable bipartite graphs of $\Omega^{[0]}$ and $\Omega^{[1]}$-theory.** In this section we decompose $\Omega^{[0,1]}$ into two subtheories, by restricting $R(z)$ action on small blocks.

Definition 3.3. For $\tau_i\in\mathcal{H}$ $(i=1,\cdots,n)$ and $\star=0$ or 1 , we define

$$\Omega^{[\star]}:=\begin{cases}R^{[0]}.\Omega^{Q,\mathrm{tw}} & \star=0\\ R^{[1]}.\omega^{\mathrm{Npt,tw}} & \star=1\end{cases},$$

where the $R^{[\star]}$-matrices are [14]

$$R^{[0]}(z)=R(z)|_Q:=R(z)|_{\mathcal{H}_Q}\in\mathrm{Hom}(\mathcal{H}_Q,\mathcal{H})[\![z]\!],$$
$$R^{[1]}(z)=R(z)|_{\mathrm{Npt}}:=R(z)|_{\mathcal{H}_{\mathrm{Npt}}}\in\mathrm{Hom}(\mathcal{H}_{\mathrm{Npt}},\mathcal{H})[\![z]\!].$$

Notice that here the state spaces (where the $R^{[\star]}$-matrix acting on):

$$\mathcal{H}_Q=\mathrm{span}\{\phi^i\}_{i=0}^3\oplus\mathcal{H}_Q^{\mathrm{odd}}\quad\text{and}\quad\mathcal{H}_{\mathrm{Npt}}=\mathrm{span}\{\phi_j\}_{j=4}^{\mathrm{N}+3}$$

have dimensions strictly less than that of $\mathcal{H}$.

Remark 3.4. By the definition, for $\star=0$ or 1, we see that

$$\Omega_{g,n}^{[\star]}(\tau_1,\cdots,\tau_n)\in H^*(\overline{\mathcal{M}}_{g,n})$$

is equal to the summation of those stable graph contributions in

$$(R.\Omega^{\aleph})_{g,n}(\tau_1,\cdots,\tau_n)$$

whose vertices are all labeled by $\star$.

Remark 3.5. In this paper, the R-matrices that we have used are all identity on odd classes and send even classes to even classes. Hence we will only describe their action on even classes.

In this paper a stable graph is a graph whose vertices are decorated by genus, such that $2g_v-2+n_v>0$, where $n_v:=|E_v|+|L_v|$ is the valence of the vertex v. A stable graph is called bipartite if each vertex is further decorated by (level) 1 or 0, and each edge connects vertices of different levels. Let $\Xi_{g,n}^{[0,1]}$ be the set of stable bipartite graphs, with total genus g and n many legs. For a stable bipartite graph Λ, we use $V(\Lambda)$, $E(\Lambda)$ to denote the set of its vertices, edges respectively; use $V_0(\Lambda)$ to denote its level 0 vertices, etc..

[14] It is easy to generalize such definition of subtheories to any R-matrix action on direct sum of CohFTs.

Theorem 3.6. *We have the following stable bipartite graph formula*

$$\Omega^{[0,1]}_{g,n}(\tau_1,\cdots,\tau_n)=\sum_{\Lambda\in\Xi^{[0,1]}_{g,n}}(\xi_\Lambda)_*\Big(\bigotimes_{v\in V_0(\Lambda)}\Omega^{[0]}_{g_v,n_v}\Big)\bigotimes\Big(\bigotimes_{v'\in V_1(\Lambda)}\Omega^{[1]}_{g_{v'},n_{v'}}\Big)$$
$$\Big(\bigotimes_{\substack{v\in V_0(\Lambda),\\ l\in L_v}}\tau_l,\quad\bigotimes_{\substack{v'\in V_1(\Lambda),\\ l'\in L_{v'}}}\tau_{l'},\quad\bigotimes_{\substack{e=(v,v')\\ \in E(\Lambda)}}V^{01}(\psi,\psi')\Big) \tag{3.3}$$

where the edge contribution $V^{01}(z,w)$ is given by [15]

$$V^{01}(z,w):=\textstyle\sum_{\alpha=1}^{\mathrm{N}}\frac{R^{[1]}(z)-R^{[1]}(-w)}{z+w}\mathbf{1}_\alpha\otimes R^{[1]}(w)\mathbf{1}^\alpha. \tag{3.4}$$

Proof. Just notice that for the graph sum formula of $[0,1]$-CohFT (via the R-matrix action on $\Omega^{\aleph}$), the contribution of an edge that connects a V_0 vertex and a V_1 vertex is given by

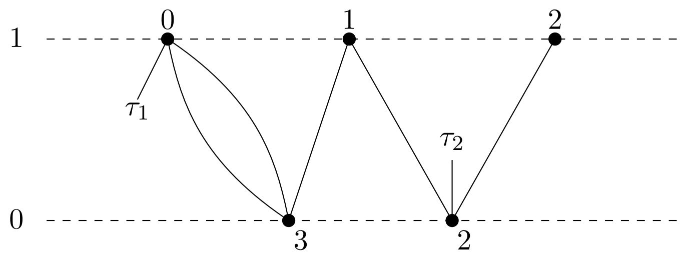
$$\begin{aligned}\mathrm{Cont}_{E_{01}}&=\textstyle\sum_{i=0}^{\mathrm{N}+3}\frac{-R^{[0]}(-z)^*\phi_i\otimes R^{[1]}(-w)^*\phi^i}{z+w}\\ &=\textstyle\sum_{\alpha=1}^{\mathrm{N}}\frac{R^{[0]}(-z)^*\big(R(z)-R(-w)\big)\mathbf{1}_\alpha\otimes\mathbf{1}^\alpha}{z+w}=\big(R^{[0]}(-z)^*\otimes R^{[1]}(-w)^*\big)\cdot V^{01}(z,w),\end{aligned}$$

where we have used the symplectic condition,

$$R^{[0]}(-z)^*R^{[1]}(z)=0,\qquad R^{[1]}(-z)^*R^{[1]}(z)=\mathrm{I}\in\mathrm{End}(\mathcal{H}_{\mathrm{Npt}}).$$

The graph sum formula then follows from the definition of the R-matrix action. ☐

Example 3.7. The following is an example of a stable bipartite graph of total genus 9 and two insertions (τ_1,τ_2):



Convention 3.8. In the remainder of this and the next section, we use $K\in\{L,Y,X,I_k,A_i,B_i\}$ to mean the function $K|_{q\mapsto q'}$ of $q':=-q/t^{\mathrm{N}}$. For example, $L=(1+5^5\frac{q}{t^{\mathrm{N}}})^{\frac{1}{\mathrm{N}}}$.

Convention 3.9. From now on, we assume N is a prime.

3.2. **Polynomiality of** [1]**-theory.** Let $R(z)=\sum_k R_kz^k$ and

$$V(z,w)=\textstyle\sum_{k,l}V_{kl}z^kw^l:=\sum_j\frac{1}{z+w}(\phi_j\otimes\phi^j-R(z)^{-1}\phi_j\otimes R(w)^{-1}\phi^j).$$

We start by a key Lemma proved in [NMSP2, Appendix C]:

Lemma 3.10 ([NMSP2, Lemma C.1])**.** *Let $k,l\geq 0$, $a=0,1,\cdots,\mathrm{N}+3$ and $\alpha,\beta\in[\mathrm{N}]$. We consider the entries*

$$(R_k)^\alpha_a:=L^{\frac{\mathrm{N}+3}{2}}\cdot L_\alpha^{-a+k}(\mathbf{1}^\alpha,R_k^*\phi_a),\quad\text{with } L_\alpha:=\zeta_{\mathrm{N}}^\alpha\cdot t\cdot L.$$

- *For the R-matrix, we have that $(R_k)^\alpha_a$ does not depend on α, and*
$$(R_k)_a:=(R_k)^\alpha_a\in\mathbb{Q}[X]_{k+\lfloor\frac{a}{\mathrm{N}}\rfloor}. \tag{3.5}$$
- *For the V-matrix, we have that $V_{kl}|_{\mathrm{Npt}\times\mathrm{Npt}}$ is of the following form*
$$V_{kl}|_{\mathrm{Npt}\times\mathrm{Npt}}=L^{-3}t^{\mathrm{N}}\sum_{\alpha,\beta}\sum_s L_\alpha^{s-k}L_\beta^{2-s-l}\cdot(V_{kl})^{\alpha\beta;s}\,\mathbf{1}_\alpha\otimes\mathbf{1}_\beta,$$
such that the entries $(V_{kl})^{\alpha\beta;s}$ are independent of α,β and
$$(V_{kl})^{\alpha\beta;s}\in\mathbb{Q}[X]_{k+l+1}. \tag{3.6}$$

[15] The basis $\{\mathbf{1}_\alpha\}$ and $\{\mathbf{1}^\alpha\}$ in (3.4) can be replaced by any basis of $\mathcal{H}_{\mathrm{Npt}}$ and its dual. Also since the 0 and 1 are symmetric here, we can define V^{01} in terms of $R^{[1]}$-matrix as well.

Definition 3.11. Let $\star = [0], [1]$ or $[0,1]$, we introduce the $\star$-potential for $\mathbf{a} = (a_1, \cdots, a_n)$, $\mathbf{b} = (b_1, \cdots, b_n)$

$$f^{[\star]}_{g,(\mathbf{a},\mathbf{b})} := \int_{\overline{\mathcal{M}}_{g,n}} \Big(\prod_{i=1}^{n} \psi_i^{b_i}\Big) \cup \Omega^{[\star]}_{g,n}(\phi_{a_1}, \cdots, \phi_{a_n}).$$

Here $r := \frac{1}{\mathrm{N}}(|\mathbf{a}| + |\mathbf{b}| - n)$, and $|\mathbf{a}| := \sum a_i$.

Our goal is to study the $[0]$ theory, using the $[1]$ and the $[0,1]$ theories. We first study the $[1]$ theory by considering additional "special" insertions:

$$f^{[1]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} := L^{\sum_{i=1}^m a_i'} \cdot \int_{\overline{\mathcal{M}}_{g,n+m}} \prod_{i=1}^{n} \psi_i^{b_i} \prod_{j=1}^{m} \psi_{n+j}^{b_j'} \cdot$$
$$\Omega^{[1]}_{g,n+m}\Big(\phi_{a_1}, \cdots, \phi_{a_n}, R(\psi_{n+1})\bar{\phi}_{a_1'}, \cdots, R(\psi_{n+m})\bar{\phi}_{a_l'}\Big),$$

where $\mathbf{a} \in \{0, \cdots, \mathrm{N}{+}3\}^{\times n}$, $\mathbf{a}' \in [\mathrm{N}]^{\times m}$, and $\{\bar{\phi}_a := L^{-(\mathrm{N}+3)/2} \sum_\alpha (-t_\alpha)^a \mathbf{1}_\alpha\}_{a=1}^{\mathrm{N}}$ is the "normalized" basis[16]. Let $\lfloor \frac{\mathbf{a}}{\mathrm{N}} \rfloor := \sum_l \left\lfloor \frac{a_l}{\mathrm{N}} \right\rfloor$

Proposition 3.12. *Let $\mathrm{N} > 3g - 3 + n + m$ be sufficiently large.*

(1) If $r := \frac{1}{\mathrm{N}}(|\mathbf{a}| + |\mathbf{a}'| + |\mathbf{b}| + |\mathbf{b}'| - n - m) \in \mathbb{Z}$, then

$$(Y/t^{\mathrm{N}})^{g-1+r} \cdot f^{[1]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} \in \mathbb{Q}[X]$$

is a polynomial in X of degree no more than

$$3g - 3 + n + m - |\mathbf{b}| - |\mathbf{b}'| + \lfloor \frac{\mathbf{a}}{\mathrm{N}} \rfloor.$$

(2) Otherwise, $f^{[1]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} = 0$.

Proof. By definition of $\Omega^{[1]} := R^{[1]}.\omega^{\mathrm{Npt,tw}}$, the $[1]$-potential is given by the sum of the stable graph contributions. For each graph Γ, the contribution is given via applying the composition rule to the following placements:

(1) at each leg with insertion $\phi_a \psi^b$ (one of the first n legs), we put

$$R^*(-\psi_l)\phi_a \psi_l^b|_{\mathrm{Npt}} = \sum_{\alpha,k} L^{-\frac{\mathrm{N}+3}{2}} L_\alpha^{a-k} (R_k)^\alpha_a (-1)^k \psi_l^{k+b} \mathbf{1}_\alpha;$$

(2) at each special leg with insertion $R(\psi_{l'})\bar{\phi}_{a'_{l'}}\psi^{b'_{l'}}$ (one of the last m legs), we put

$$R(-\psi_{l'})^* R(\psi_{l'})\bar{\phi}_{a'_{l'}} \psi_{l'}^{b'_{l'}} = \sum_\alpha L^{-\frac{\mathrm{N}+3}{2}} L^{a'_{l'}} (-t_\alpha)^{a'_{l'}} \mathbf{1}_\alpha \psi_{l'}^{b'_{l'}};$$

(3) at each edge, we put a bi-vector

$$V(z,w)|_{\mathrm{Npt}\times\mathrm{Npt}} = \sum_{\alpha,\beta} \sum_{k,l,s} L^{-3} t^{\mathrm{N}} L_\alpha^{1+s-k} L_\beta^{1-s-l} \, (V_{kl})^{\alpha\beta;s+1} \, \mathbf{1}_\alpha \otimes \mathbf{1}_\beta; \tag{3.7}$$

(4) at each vertex of genus g with n-legs, we put a map

$$(-) \mapsto \sum_\alpha L^{\frac{\mathrm{N}+3}{2}(2g_v - 2 + n)} \frac{1}{s!} (\mathrm{pr}_s)_* \, \omega^{\mathrm{pt}_\alpha,\mathrm{tw}}_{g,n+s}\Big(-, T_\alpha(\psi)^{\otimes s}\Big),$$

where $T_\alpha(z) = L^{\frac{\mathrm{N}+3}{2}} \sum_{k\geq 1} (\mathbf{1}^\alpha, R_k^* \mathbf{1})(-z)^{k+1} \mathbf{1}_\alpha = \sum_{k\geq 1} L_\alpha^{-k} \, (R_k)_0^\alpha \, (-z)^{k+1} \mathbf{1}_\alpha$.

Denote L_v (resp. L_v') the set of ordinary legs (resp. special legs respectively) over v. We estimate the degree of the legs, edges, tails contributions at each vertex of level pt_α. By using Lemma 3.10, we obtain that:

[16] Recall the flat basis of $\mathcal{H}_{\mathrm{Npt}}$ is given by $\{\phi_a := p^a = \sum_\alpha (-t_\alpha)^a \mathbf{1}_\alpha\}_{a=4}^{\mathrm{N}+3}$, and notice that we can choose $\{\sum_\alpha (-t_\alpha)^a \mathbf{1}_\alpha\}_{a=k+1}^{k+\mathrm{N}}$ as a basis of $\mathcal{H}_{\mathrm{Npt}}$ for any k.

- the factor involving L_α, L and Y is

$$(-t_\alpha)^{(\mathrm{N}+3)(g-1)} L^{\frac{\mathrm{N}+3}{2}(2g_v-2)} \prod_{l\in L_v} L_\alpha^{a_l-k_l}\cdot \prod_{l'\in L'_v} L_\alpha^{a_{l'}}\cdot \prod_{\substack{f=(e,v),\\ e\in E_v}} (t\cdot L)^{\frac{\mathrm{N}}{2}} L_\alpha^{1+s_f-k_f}\cdot \prod_t L_\alpha^{-k_t}$$
$$= L_\alpha^{\sum_{l\in L_v}(a_l+b_l-1)+\sum_{l'\in L'_v}(a_{l'}+b_{l'}-1)+\sum_{e\in E_v} s_{(e,v)}} L^{\mathrm{N}(g_v-1+\frac{|E_v|}{2})} t^{\mathrm{N}(g_v-1+\frac{|E_v|}{2})},$$

where we have used $n_v = |L_v|+|L'_v|+|E_v|$, and

$$\textstyle\sum_t k_t+\sum_f k_f+\sum_l (k_l+b_l)+\sum_{l'} b'_{l'} = 3g_v-3+n_v;$$

- the total X-degree of the tail, edge and leg contributions at the vertex is at most

$$\begin{aligned}&\textstyle\sum_t k_t + \sum_{f=(e,v),e\in E_v}(k_f+\frac{1}{2})+\sum_{l\in L_v}(k_l+\lfloor\frac{a_l}{\mathrm{N}}\rfloor)\\ &= 3g_v-3+n_v+\frac{|E_v|}{2}+\textstyle\sum_{l\in L_v}\lfloor\frac{a_l}{\mathrm{N}}\rfloor-\sum_{l\in L_v} b_l-\sum_{l'\in L'_v} b'_{l'}.\end{aligned}\tag{3.8}$$

For each graph we may forget the hour decoration of each vertex to obtain an "hour-free graph". For each vertex v in an "hour-free" graph, we may sum up its all possible hours $\alpha=1,\cdots,\mathrm{N}$ and extract a multiplicative factor $L_\alpha^{r_v}$ with

$$r_v := \tfrac{1}{\mathrm{N}}\big(\textstyle\sum_{l\in L_v}(a_l+b_l-1)+\sum_{l'\in L'_v}(a_{l'}+b_{l'}-1)+\sum_{e\in E_v} s_{(e,v)}\big).$$

By fixing a choice in each summand of (1)-(4) above, such extraction can be done for all vertices at once. Since $\sum_\alpha L_\alpha^k$ vanishes unless $\mathrm{N}|k$, we see that if some $r_v\notin\mathbb{Z}$, the decomposition summand of $(1)-(4)$ contributed by an "hour-free graph" vanishes.

At each edge $e=(v_1,v_2)$, by the form of (3.7), we see $s_{(e,v_1)}+s_{(e,v_2)}=0$. This gives

$$r := \textstyle\sum_v r_v = \frac{1}{\mathrm{N}}(|\mathbf{a}|+|\mathbf{a}'|+|\mathbf{b}|+|\mathbf{b}'|-n-m).$$

The argument above proves the second statement.

Now we evaluate the contributions of all the vertices together. After multiplying them over all vertices we have

(1) the factor involving L and Y (using $L_\alpha^{\mathrm{N}}=(t\,L)^{\mathrm{N}}=t^{\mathrm{N}}\cdot Y^{-1}$) becomes

$$(t\,L)^{\mathrm{N}\sum_v r_v}(t\,L)^{\sum_v \mathrm{N}(g_v-1+|E_v|/2)} = (Y/t^{\mathrm{N}})^{-(g-1+r)};$$

(2) the total degree of X of contributions of Γ is the sum of (3.8) over all vertices v, which equals

$$3g-3+n+m+\lfloor\frac{\mathbf{a}}{\mathrm{N}}\rfloor-|\mathbf{b}|-|\mathbf{b}'|.$$

Multiply (1) with (2), and sum over all graphs. The first statement is proved. $\square$

3.3. **Vanishing properties of [0]-theory.** Recall that we have computed

$$R^{[0]}(z)^*\mathbf{1} = \varphi_0+O(z^{\mathrm{N}-3}),\quad\text{and}\quad R^{[0]}(z)^*p = zB\cdot\varphi_0+\varphi_1+O(z^{\mathrm{N}-2}),\tag{3.9}$$

in [NMSP2, Example 5.2]. Furthermore, $R^{[0]}(z)^*$ satisfies the following "QDE"[17]:

$$z\,DR^{[0]}(z)^* = R^{[0]}(z)^*\cdot A^M - A^Q\cdot R^{[0]}(z)^*.\tag{3.10}$$

We have the following general property for $R^{[0]}(z)$:

[17] We recall the explicit formulas for $A^Q\in\operatorname{End}\mathcal{H}_Q$ and $A^M\in\operatorname{End}\mathcal{H}$ that were proved in [NMSP2]

$$A^Q=\begin{pmatrix}0&&&\\ I_{11}&0&&\\ &I_{22}&0&\\ &&I_{11}&0\end{pmatrix}\quad\text{and}\quad (A^M)^i_j=\begin{cases}1 & \text{if } i=j+1\\ c_{j+1}q-\delta_{i,4}t^{\mathrm{N}} & \text{if } i=j-\mathrm{N}+1\\ 0 & \text{otherwise}\end{cases}$$

where $(c_j)_{j=\mathrm{N},\cdots,\mathrm{N}+4}=(120,770,1345,770,120)$. See [NMSP2, Sect. 1.5 and Appendix A] for more details.

Lemma 3.13. *We introduce the mod-*N *degrees by letting*

$$\deg \psi = 1, \quad \deg \varphi_j = j = \deg \phi_j.$$

*Then, $R^{[0]}$-matrix preserves the mod-*N *degree. Furthermore, let $\bar{j} := j - \mathrm{N}\lfloor \frac{j}{\mathrm{N}} \rfloor$ and $\varphi_j := 0$ for $j > 3$, we have the following key property:*

$$R^{[0]}(z)^* \phi_j = c'_j q^{\lfloor \frac{j}{\mathrm{N}} \rfloor} \cdot \varphi_{\bar{j}} + O(z^{\bar{j}-3}) \quad \text{for } j = 0, \cdots, \mathrm{N}+3, \tag{3.11}$$

where $(c'_j)_{j=0,\cdots,\mathrm{N}+3} = (1, \cdots, 1, -120, -890, -2235, -3005)$.

Proof. Recall $R^{[0]}(z) = S^M(z) S^Q(q', z)$. Since the local and global S-matrices preserve mod-N degrees, the $R^{[0]}$-matrix preserves the mod-N degree as well. Furthermore, because $\deg \varphi_i \leq 3$, we obtain the $O(z^{\bar{j}-3})$ in (3.11). The leading term is from (3.10). □

The shape of $R^{[0]}$ gives us control on $f^{[0]}_{g,(\mathbf{a},\mathbf{b})}$. The followings are the most direct ones.

Lemma 3.14. *If $r \notin \mathbb{Z}$ then $f^{[0]}_{g,(\mathbf{a},\mathbf{b})} = 0$.*

Proof. First by Lemma 3.13, each edge(in the $R^{[0]}$ action on $\Omega^{Q,\mathrm{tw}}$) contributes the mod-N degree 2. Secondly, observe that, for quintic CohFT, $\int_{\overline{\mathcal{M}}_{g,n}} \Omega^Q_{g,n}(\otimes_{i=1}^n \phi_{a_i} \psi^{b_i}) = 0$ unless $\sum_i (a_i + b_i) = n$. The same statement holds when $\Omega^Q_{g,n}$ is substituted by $\Omega^{Q,\mathrm{tw}}_{g,n}$, by Remark 2.12. The lemma follows by summing up the mod-N degrees over vertices and edges in arbitrary graph defining $\Omega^{[\star]} = R^{[0]}.\Omega^{Q,\mathrm{tw}}$. □

We will assmue r is an integer in the remainder of this paper.

Lemma 3.15. *Suppose $\mathrm{N} > 3g - 3 + 3n$. Let $\mathbf{a} := (\bar{a}_1, \cdots, \bar{a}_n)$ with*

$$\bar{a}_j := a_j - \mathrm{N}\lfloor \tfrac{a_j}{\mathrm{N}} \rfloor, \quad \text{and} \quad r^{\sim} := r - \lfloor \tfrac{\mathbf{a}}{\mathrm{N}} \rfloor = \tfrac{|\bar{\mathbf{a}}| + |\mathbf{b}| - n}{\mathrm{N}}.$$

We have $r^{\sim} \in \mathbb{Z}_{\geq 0}$; and if $r^{\sim} \neq 0$ then $f^{[0]}_{g,(\mathbf{a},\mathbf{b})} = 0$. Namely,

$$|\bar{\mathbf{a}}| + |\mathbf{b}| \neq n \quad \Longrightarrow \quad f^{[0]}_{g,(\mathbf{a},\mathbf{b})} = 0.$$

Proof. By $\mathrm{N} > 3g-3+3n$ and the stability condition $3g-3+n \geq 0$, we have $-\mathrm{N} < -2n \leq -n$. Since $r^{\sim} = r - \lfloor \frac{\mathbf{a}}{\mathrm{N}} \rfloor$ is an integer we must have $r' = \frac{|\bar{\mathbf{a}}|+|\mathbf{b}|-n}{\mathrm{N}} \geq \lfloor \frac{-n}{\mathrm{N}} \rfloor = 0$. This proves the first statement.

Next we prove the vanishing result. By definition, if $r^{\sim} > 0$

$$|\bar{\mathbf{a}}| = r^{\sim} \cdot \mathrm{N} - (|\mathbf{b}| - n) \geq \mathrm{N} - (|\mathbf{b}| - n).$$

By definition of R-matrix action, we write $f^{[0]}_{g,(\mathbf{a},\mathbf{b})}$ as a sum of stable graph contributions. At each vertex v the contribution is of the form

$$\int_{\overline{\mathcal{M}}_{g_v,n_v}} \Omega^Q_{g_v,n_v}\Big(\bigotimes_{l \in L_v} R^{[0]}(-\psi_l)^* \phi_{a_l} \psi^{b_l} \bigotimes_{f=(e,v), e \in E_v} C_f(\psi_f) \Big),$$

where C_f is from edge contributions. By using (3.11), we see that, if $r^{\sim} > 0$, the total degree of psi-classes of all vertices is at least

$$|\bar{\mathbf{a}}| - 3n + |\mathbf{b}| \geq \mathrm{N} - 2n. \tag{3.12}$$

On the other hand, the graph contribution vanishes if for any vertex v,

$$\textstyle\sum_{l \in L_v} (\bar{a}_l - 3 + b_l) > 3g_v - 3 + n_v.$$

Hence it vanishes if

$$|\bar{\mathbf{a}}| - 3n + |\mathbf{b}| > \textstyle\sum_v (3g_v - 3 + n_v) = 3g - 3 + n - |E|.$$

By the condition $\mathrm{N} > 3g - 3 + 3n$ and (3.12) we finish the proof. □

Corollary 3.16. *If $f^{[0]}_{g,(\mathbf{a},\mathbf{b})}$ is nonzero, we have*

$$r := \frac{1}{\mathrm{N}}(|\mathbf{a}|+|\mathbf{b}|-n) = \lfloor \frac{\mathbf{a}}{\mathrm{N}} \rfloor := \sum_l \lfloor \frac{a_l}{\mathrm{N}} \rfloor = \#\{i : a_i \geq \mathrm{N}\}.$$

Corollary 3.17. *If $f^{[0]}_{g,(\mathbf{a},\mathbf{b})}$ is nonzero, we have*

$$g-1+r \leq 3g-3+r+n-|\mathbf{b}|. \tag{3.13}$$

Proof. If $g \geq 1$, (3.13) follows from the non-vanishing condition $|\mathbf{b}| \leq n$. If $g=0$, we have the non-vanishing condition $|\mathbf{b}| \leq 3g-3+n = n-3$. Hence $g-1+r = -1+r < r-3+n-|\mathbf{b}|$. We finish the proof. □

3.4. **Polynomiality of** [0]**-theory.** In the last subsection, we want to give the similar degree estimate for [0]-theory as what we have done for [1]-theory in Proposition 3.12.

We introduce the [0]-potential with special insertions:

$$f^{[0]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} := \int_{\overline{\mathcal{M}}_{g,n+m}} \Omega^{[0]}_{g,n+m}\Big(\bigotimes_{l=1}^{n} \phi_{a_l}\psi_i^{b_l}, \bigotimes_{l'=1}^{m} E_{a'_{l'},b'_{l'}}(\psi_{n+l'})\Big),$$

where the indices $\mathbf{a} \in \{0,\cdots,\mathrm{N}{+}3\}^{\times n}$, $\mathbf{a}' \in [\mathrm{N}]^{\times m}$, $\mathbf{b} \in \mathbb{Z}_{\geq 0}^{\times n}$, $\mathbf{b}' \in \mathbb{Z}_{\geq 0}^{\times m}$ and

$$E_{a',b'}(\psi) := L^{-a'} \cdot \mathrm{Coef}_{z^{b'}} \tfrac{1}{\psi+z}\big(R(\psi)-R(-z)\big)\bar{\phi}^{a'} \tag{3.14}$$

with the dual basis $\{\bar{\phi}^{a'} {:=} L^{\frac{(\mathrm{N}+3)}{2}} \sum_\alpha (-t_\alpha)^{-a'} \mathbf{1}^\alpha\}_{a'=1}^{\mathrm{N}}$ of the "normalized" basis $\{\bar{\phi}_i\}_{a=1}^{\mathrm{N}}$.

Using $V^{01}(z,w) = \sum_{a=1}^{\mathrm{N}} \frac{R(z)-R(-w)}{z+w}\bar{\phi}^a \otimes R(w)\bar{\phi}_a$, one has

$$V^{01}(z,w) = \textstyle\sum_{a=1}^{\mathrm{N}} \sum_{b\geq 0} E_{a,b}(z) w^b \otimes L^a\, R(w)\bar{\phi}_a. \tag{3.15}$$

Lemma 3.18. *We have $L^{-a+k}(\phi_a, R_k\bar{\phi}^b) \in \mathbb{Q}(t^{\mathrm{N}})[X]_{k+\lfloor \frac{a}{\mathrm{N}} \rfloor}$ and*

$$(\phi_a, R_k\bar{\phi}^b) = 0 \quad \textit{if} \quad a-k \neq b \mod \mathrm{N}. \tag{3.16}$$

Proof. It follows from Lemma 3.10 and

$$(\phi_a, R_k\bar{\phi}^b) = \textstyle\sum_\alpha (-t_\alpha)^{-b} L_\alpha^{a-k}(R_k)_a^\alpha = \sum_\alpha (\zeta_{\mathrm{N}}^\alpha t)^{-b}(\zeta_{\mathrm{N}}^\alpha t\, L)^{a-k}(R_k)_a^\alpha.$$

Here we have used $\sum_\alpha (\zeta_{\mathrm{N}}^\alpha)^m = 0$ unless $\mathrm{N}|m$ because N is a prime. □

Lemma 3.19. *We have $f^{[0]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} = 0$, unless*

$$r := \tfrac{1}{\mathrm{N}}(|\mathbf{a}|+|\mathbf{b}|-|\mathbf{a}'|-|\mathbf{b}'|-n+s) \in \mathbb{Z}.$$

Proof. Recall the mod-N degree introduced in Lemma 3.13. Apply (3.16) to $R_k\bar{\phi}^b = \sum_{s=1}^{\mathrm{N}+3}(R_k\bar{\phi}^b,\phi_s)\phi^s$ one sees the mod-N degree of $R_k\bar{\phi}^b$ is $3-(k+a)$. One then calculates the mod-N degree of $E_{a,b}$ is $2-a-b$. The same reasoning as proof of Lemma 3.14 applies. □

When $s=0$, we have $f^{[0]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} = f^{[0]}_{g,(\mathbf{a},\mathbf{b})}$ and $r = \frac{1}{\mathrm{N}}(|\mathbf{a}|+|\mathbf{b}|-n)$.

Definition 3.20. For any (g,n), we introduce a statement

$$\mathfrak{S}_{g,n} \;=\; \text{“}\,\forall \mathbf{a} \in \{0,1,2,3\}^{\times n}, \forall \mathbf{b} \in \mathbb{Z}^n_{\geq 0} \quad (Y/t^{\mathrm{N}})^{g-1} \cdot f^{[0]}_{g,(\mathbf{a},\mathbf{b})} \in \mathbb{Q}[X]_{3g-3+n-|\mathbf{b}|}\,\text{”}.$$

We also introduce stronger statements

$$\begin{aligned}\mathfrak{S}'_{g,n} \;=\; &\text{“}\,\forall k,s \geq 0 \text{ with } \ell+s=n, \forall \mathbf{a} \in \{0,1,2,3,\mathrm{N},\cdots,\mathrm{N}{+}3\}^{\times \ell}, \mathbf{a}' \in [\mathrm{N}]^{\times s}, (\mathbf{b},\mathbf{b})' \in \mathbb{Z}^n_{\geq 0}\\ &(Y/t^{\mathrm{N}})^{g-1+r+s} \cdot f^{[0]}_{g,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')} \in \mathbb{Q}[X]_{3g-3+\ell+2s+\lfloor \frac{\mathbf{a}}{\mathrm{N}} \rfloor-|\mathbf{b}|+|\mathbf{b}'|}\,\text{”}.\end{aligned} \tag{3.17}$$

One of the main result in the next section [18], is the following lemma.

[18] See §4.3 for the proof

Lemma 3.21. *Suppose (g,n) satisfies $2g-2+n>0$. Let $\mathrm{N}>3g+n$. If the statement $\mathfrak{S}_{h,m}$ holds for any $(h,m)<(g,n)$ and $3h+m\le 3g+n$. Then for any (h,m) such that $(h,m)<(g,n)$ and $3h+m\le 3g+n$, the statement $\mathfrak{S}'_{h,m}$ holds.*

By using the above two lemmas, we prove

Proposition 3.22. *Let $\mathrm{N}>3g+n$. Then*

$$\forall \mathbf{a},\mathbf{b}\in[\mathrm{N}+3]^{\times n},\qquad (Y/t^{\mathrm{N}})^{g-1+r}\cdot f^{[0]}_{g,(\mathbf{a},\mathbf{b})}\in\mathbb{Q}[X]_{3g-3+n+\lfloor\frac{\mathbf{a}}{\mathrm{N}}\rfloor-|\mathbf{b}|}. \tag{3.18}$$

Proof. By definition of $R^{[0]}$-matrix action, $f^{[0]}_{0,(\vec{a},\vec{b})}$ is equal to a graph sum. In case $(g,n)=(0,3)$, there is only one graph with a single vertex and with no psi-classes insertions. In this case $r=\sum_l\lfloor\frac{a_l}{\mathrm{N}}\rfloor$. By the property of $\mathrm{Coef}_{z^0}R^{[0]}(z)$ (c.f. (3.9) and (3.11)), one calculates (for any a_1,a_2,a_3)

$$(Y/t^{\mathrm{N}})^{0-1+r}\cdot f^{[0]}_{0,(\mathbf{a},0)}=(Y/t^{\mathrm{N}})^{r}\cdot(Y/t^{\mathrm{N}})^{-1}\cdot I_0^2I_{11}^2I_{22}\cdot C\,q^r=C\cdot X^r$$

for a $C\in\mathbb{Q}$ (which is a product of c_j's defined as in 4.15). This is a polynomial in X of degree $0-3+3+r$. Here we have used $\langle H,H,H\rangle^Q=I_{22}/I_{11}$ and $\langle 1,H,H^2\rangle^Q=1$ (c.f. [NMSP2, Appendix A]).

We now prove the proposition by induction on (g,n) under the lexicographical order. We will use Proposition 3.12. Fix g,n such that $2g-2+n>0$, and from induction hypothesis assume (3.18) holds for any $(h,m)<(g,n)$. Then $\mathfrak{S}_{h,m}$ holds for any $(h,m)<(g,n)$ and $3g+m\le 3g+n$.By Lemma 3.21, $\mathfrak{S}'_{h,m}$ holds for any $(h,m)<(g,n)$ and also $3h+m\le 3g+n$.

Now for any $\mathbf{a},\mathbf{b}\in[\mathrm{N}+3]^{\times n}$ we consider the $[0,1]$-potential $f^{[0,1]}_{g,(\mathbf{a},\mathbf{b})}$. Suppose $f^{[0,1]}_{g,(\mathbf{a},\mathbf{b})}$ vanishes, (3.18) holds. Suppose $f^{[0,1]}_{g,(\mathbf{a},\mathbf{b})}\neq 0$. By Corollary 3.16, $g-1+r\in\mathbb{Z}$. Since $\mathrm{N}>3g-3+n\ge\sum_i b_i\ge 0$, we have $\frac{3g-3+n-\sum b_i}{\mathrm{N}}<1$, which implies

$$\lfloor g-1+\tfrac{3g-3+\sum_{i=1}^n a_i}{\mathrm{N}}\rfloor=g-1+r.$$

By Theorem 3.2, $f^{[0,1]}_{g,(\mathbf{a},\mathbf{b})}\neq 0$ is a polynomial in X of degree $\deg_X f^{[0,1]}_{g,(\mathbf{a},\mathbf{b})}\le g-1+r\le 3g-3+n-|\mathbf{b}|+r$ (by (3.13)).

On the other hand we apply (3.3) to this $[0,1]$-potential. There is a bipartite graph with only a single genus g level 0 vertex, which we call the leading graph. It suffices to prove that for any non-leading graph Γ, the contribution

$$\mathrm{Cont}_\Gamma\in\mathbb{Q}[X]_{3g-3+n-|\mathbf{b}|+r}.$$

Indeed, every [0] vertex of any non-leading graph is applicable for the statement $\mathfrak{S}'_{h,m}$. Apply (3.15) first, and Proposition 3.12 at V_1, and (3.17) at V_2, we obtain

- the degree of the total contribution of Γ in X is given by (with $n_v:=|E_v|+|L_v|$)

$$\sum_{v\in V_0}\Big(3g_v-3+n_v+|E_v|+\sum_{l\in L_v}\big(\lfloor\frac{a_l}{\mathrm{N}}\rfloor-b_l\big)+\sum_{e\in E_v}b'_{(e,v)}\Big)+$$
$$+\sum_{v\in V_1}\Big(3g_v-3+n_v+\sum_{l\in L_v}\big(\lfloor\frac{a_l}{\mathrm{N}}\rfloor-b_l\big)-\sum_{e\in E_v}b'_{(e,v)}\Big)\le 3g-3+n+\lfloor\frac{\mathbf{a}}{\mathrm{N}}\rfloor-|\mathbf{b}|;$$

- the total factor involving (Y/t^{N}) is given by

$$(Y/t^{\mathrm{N}})^{g-1+r}\cdot\prod_{v\in V_0}(Y/t^{\mathrm{N}})^{-(g_v-1+r_v+s)}\prod_{v\in V_1}(Y/t^{\mathrm{N}})^{-(g_v-1+r_v)}=1.$$

This finishes the induction. $\square$

4. From NMSP [0]-theory to the CohFT $\Omega^{\mathbf{A}}$ via R^X-action

Definition 4.1. We define the CohFT $\Omega^{\mathbf{A},\vec{0}}$ via the following R-matrix action

$$\Omega^{\mathbf{A},\vec{0}} := R^{\mathbf{A},\vec{0}}.\Omega^{Q,\mathrm{tw}}, \qquad \text{with} \qquad R^{\mathbf{A},\vec{0}}(z) := R^{\mathbf{A},\mathcal{G}}|_{c_{1a}=c_{1b}=c_2=c_3=0},$$

where $R^{\mathbf{A},\mathcal{G}}$ defined in (1.7). For $a_i = 0,1,2,3$ $(i = 1,\cdots,n)$, We introduce

$$f^{\mathbf{A},\vec{0}}_{g;\mathbf{a},\mathbf{b}} := (-5Y/t^{\mathrm{N}})^{h-1} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{b_1}\cdots\psi_n^{b_n} \cup \Omega^{\mathbf{A},\vec{0}}_{g,n}(\varphi_{a_1},\cdots,\varphi_{a_n}).$$

Remark 4.2. By the relation (2.5) , we see that this definition matches with the graph sum definition in §1.2 (equation (1.9)), with gauge $c_{1a} = c_{1b} = c_2 = c_3 = 0$ and with $q \mapsto q'$.

In this section we will prove the polynomiality of the $\Omega^{\mathbf{A},\vec{0}}$-theory, via the polynomiality of the [0]-theory. In the end, we will prove Theorem 2.

To extract information from the NMSP-[0] theory, we consider the following matrix factorization which defines $R^X(z)$: [19]

$$R^{[0]}(z) = R^X(z)\cdot R^{\mathbf{A},\vec{0}}(z), \tag{4.1}$$

where the matrix is under the following basis

$$\{\varphi_i\}_{i=0}^3 \xrightarrow{R^{\mathbf{A},\vec{0}}} \{\varphi_i\}_{i=0}^3 \xrightarrow{R^X} \{\phi_a\}_{a=0}^{\mathrm{N}+3}.$$

(Recall $\varphi_i := I_0 I_{11}\cdots I_{ii} H^i$ for $i = 0,1,2,3$.) By definition, we see

$$\Omega^{[0]} = R^X.\Omega^{\mathbf{A},\vec{0}}.$$

4.1. Properties of R^X.

The advantage of the factorization (4.1) is:

Lemma 4.3. *Let $R^X(z)$ be the matrix defined by (4.1), then*

$$R^X(z)^*\phi_0 = \varphi_0 + O(z^{\mathrm{N}-3}) \quad \text{and} \quad T_{R^X}(z) := z\varphi_0 - zR^X(-z)^*\phi_0 = O(z^{\mathrm{N}-2}). \tag{4.2}$$

Furthermore, the following properties hold for $k < \mathrm{N}-3$.

1. *if $j \neq k+a \mod \mathrm{N}$, then $(\phi_j, R^X_k\varphi^a) = 0$;*
2. *if $j < \mathrm{N}$, we have $(\phi_j, R^X_k\varphi^a) \in X\,\mathbb{Q}[X]_{k-1}$;*
3. *if $j \geq \mathrm{N}$, we have $(\phi_j, R^X_k\varphi^a) \in q\,\mathbb{Q}[X]_k$;*
4. *the $C^X(z) \in \mathrm{End}(\mathcal{H}_Q,\mathcal{H})[z]$ defined below satisfies $R^X(-z)^*C^X(z) = \mathrm{I}_Q \in \mathrm{End}\,\mathcal{H}_Q$,*

$$C^X(z) := \begin{pmatrix} 1 & -z\cdot\frac{24X}{625} & z^2\cdot\frac{24X}{625} & z^3\cdot\left(-\frac{576X^2}{390625}-\frac{24X}{625}\right) \\ 0 & 1 & -z\cdot\frac{202X}{625} & z^2\cdot\left(\frac{4848X^2}{390625}+\frac{226X}{625}\right) \\ 0 & 0 & 1 & -z\cdot\frac{649X}{625} \\ 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where the dots represent zeros.

Proof. The formula (4.2) follows from the definition of $R^X(z)$ (4.1), the formula (3.9) and the special form of $R^{\mathbf{A},\vec{0}}(z)^*$ by (1.7).

To prove the properties for $R^X(z)^*\phi_i$ $(i > 0)$, recall the QDE (3.10) for $R^{[0]}(z)$ is

$$zDR^{[0]}(z)^* = R^{[0]}(z)^*\cdot A^M - A^Q\cdot R^{[0]}(z)^*.$$

Together with the definition $R^{[0]}(z)^* = R^{\mathbf{A},\vec{0}}(z)^*R^X(z)^*$, we obtain

$$zD(R^{\mathbf{A},\vec{0}}(z)^*R^X(z)^*) = (R^{\mathbf{A},\vec{0}}(z)^*R^X(z)^*)\cdot A^M - A^Q\cdot(R^{\mathbf{A},\vec{0}}(z)^*R^X(z)^*).$$

---

[19] Since $R^{\mathbf{A},\vec{0}}$ is invertable, such matrix $R^X(z)$ exists and can be calculated.

Then we see that $R^X(z)^*$ satisfies

$$R^X(z)^* \cdot A^M = zD\big(R^X(z)^*\big) + A^X(z) \cdot R^X(z)^*, \tag{4.3}$$

where a direct computation via Yamaguchi-Yau's relations (1.2) shows that

$$A^X(z) := R^{\mathbf{A},\vec{0}}(-z)\Big[(zD + A^Q)R^{\mathbf{A},\vec{0}}(z)^*\Big] = \begin{pmatrix} 0 & 0 & 0 & -\frac{24\,X}{625}z^4 \\ 1 & 0 & -\frac{2\,X}{5}z^2 & 0 \\ 0 & 1 & -Xz & 0 \\ 0 & 0 & 1 & -Xz \end{pmatrix}. \tag{4.4}$$

Hence we have an algorithm which recursively compute $R^X(z)^*\phi_i$ $(i > 0)$ from $R^X(z)^*\phi_0 = \varphi_0 + O(z^{\mathrm{N}-3})$. Furthermore, since the matrix in the algorithm always increases the mod N degree (see definition in Lemma 3.13) by 1 simultaneously, we see the first three statements hold. The last one is obtained by direct computation of the leading term of R^X (see Appendix D) and using the vanishing properties of R^X in the first three statements. □

4.2. **Polynomiality of $\Omega^{\mathbf{A},\vec{0}}$-theory.**

Lemma 4.4. *We have $f^{\mathbf{A},\vec{0}}_{h;\mathbf{a},\mathbf{b}} = 0$ when $\sum_i(a_i + b_i) \neq n$.*

Proof. Just notice that the $R^{\mathbf{A},\vec{0}}$ action preserve degrees mod N. □

Proposition 4.5. *Assume $\mathfrak{S}_{h,m}$ holds for any $(h,m) < (g,n)$ and $3h + m \leq 3g + n$. Then*

$$\forall (h,m) < (g,n), 3h + m \leq 3g + n, \forall \mathbf{a} \in \{0,1,2,3\}^{\times m}, f^{\mathbf{A},\vec{0}}_{h;\mathbf{a},\mathbf{b}} \in \mathbb{Q}[X]_{3h-3+m-\sum_i b_i}. \tag{4.5}$$

Proof. For $a_i \in \{0,1,2,3\}$ we define

$$\tilde{f}^{[0]}_{h;\mathbf{a},\mathbf{b}} := \int_{\overline{\mathcal{M}}_{h,m}} (R^X.\Omega^{\mathbf{A},\vec{0}})_{h,m}(C^X(\psi_1)\varphi_{a_1}\psi_1^{b_1}, \cdots, C^X(\psi_m)\varphi_{a_m}\psi_m^{b_m}),$$

where recall that $C^X(z) := \sum_{k=0}^3 C^X_k z^k$ is defined in §4.1 such that

$$R^X(-z)^* C^X(z) = \mathrm{I}_Q.$$

Furthermore, C^X_k satisfies the following property

$$(\phi^j, C^X_k\varphi_a) \text{ vanishes, if } j > 3 \text{ or } j \neq k + a \qquad \text{and} \qquad (\phi^j, C^X_k\varphi_a) \in \mathbb{Q}[X]_k.$$

By the above property of C^X_k and the condition $\mathfrak{S}_{h,m}$, we see

$$\tilde{f}^{[0]}_{h;\mathbf{a},\mathbf{b}} \in (Y/t^{\mathrm{N}})^{-(h-1)}\mathbb{Q}[X]_{3h-3+m-\sum_i b_i}. \tag{4.6}$$

We note that in the stable graph summation formula of $\tilde{f}^{[0]}_{h;\mathbf{a},\mathbf{b}}$ via the R^X-matrix action on $\Omega^{\mathbf{A},\vec{0}}$, there is this "leading" graph that is a single genus h vertex with m-insertions $\varphi_{a_1}\psi_1^{b_1}, \cdots, \varphi_{a_m}\psi_1^{b_m}$. This graph contributes to $f^{\mathbf{A},\vec{0}}_{h;\mathbf{a},\mathbf{b}}$. The contribution of any other non-leading graph Γ will be of the form

$$\Big(\bigotimes_{v\in V(\Gamma)} f^{\mathbf{A},\vec{0}}_{g_v,n_v}\Big)\Big(\bigotimes_{i=1}^{m}\varphi_{a_i}\psi_i^{b_i} \ \otimes \bigotimes_{e\in E(\Gamma)} V_X(e)\Big), \tag{4.7}$$

where $f^{\mathbf{A},\vec{0}}_{g_v,n_v} : H_{\mathbf{A}}^{\otimes n_v} \to Q[\![q]\!]$ are the linear maps $f^{\mathbf{A},\vec{0}}_{g_v,n_v}(-) := \langle - \rangle^{\mathbf{A},\vec{0}}_{g_v,n_v}$ and

$$V_X(e) := \frac{\sum_{i=0}^3 \varphi_i \otimes \varphi^i - \sum_{j=0}^{\mathrm{N}+3} R^X(-\psi_{v_1})^*\phi_j \otimes R^X(-\psi_{v_2})^*\phi^j}{\psi_{v_1} + \psi_{v_2}}$$

is a bivector[20] in $H_{\mathbf{A}} \otimes H_{\mathbf{A}}$, for v_1 and v_2 incident to the edge e. Furthermore, by Lemma 4.3 and by using $\varphi^i = (5Y/t^{\mathrm{N}})^{-1}\varphi_{3-i}$, we have the following degree estimate:

$$\forall k_1, k_2 \qquad (Y/t^{\mathrm{N}}) \cdot \mathrm{Coef}_{\psi_{v_1}^{k_1}\psi_{v_2}^{k_2}} V_X(e) \quad \in \quad H^{\otimes 2}_{\mathbf{A}}[X]_{k_1+k_2+1}. \tag{4.8}$$

[20] We have used $(\varphi_i, \varphi_{3-i}) = 5I_0^2 I_{11}^2 I_{22} = 5Y$.

We now prove the polynomiality by induction. First we see for $(h,m)=(0,3)$, the "leading" graph is the only graph. The theorem for this case follows directly.

Next we assume the polynomiality (4.5) for all genus h' with m' insertions with $(h',m')<(h,m)$ and $3h'+m'\le 3g+n$. Recall (4.6) is equal to the graph sum of (4.7). For a "non-leading" graph Γ,

(1) the factor of (4.7) associated to Γ involving Y is in total
$$\prod_v (Y/t^{\mathrm{N}})^{-(g_v-1)}\prod (Y/t^{\mathrm{N}})^{-E}=(Y/t^{\mathrm{N}})^{-(h-1)};$$

(2) the X-degree of (4.7) associated to Γ is in total
$$\begin{aligned}&\sum_v(3g_v-3+n_v-\sum_{e\in E_v}b_{(e,v)}-\sum_{i\in L_v}b_i)+\sum_{e=(v_1,v_2)}(b_{(e,v_1)}+b_{(e,v_2)}+1)\\ =&(\sum_v 3g_v)-3|V|+3|E|+m-\sum_i b_i=3h-3+m-\sum_i b_i;\end{aligned}$$

as desired. This finishes the induction and proves $f^{\mathbf{A},\vec{0}}_{h;\mathbf{a},\mathbf{b}}\in\mathbb{Q}[X]_{3h-3+m-\sum_i b_i}$. □

4.3. **Proof of Lemma 3.21.** We state the lemma we will use to prove Lemma 3.21.

Lemma 4.6. *We have the following degree estimate: whenever $b''<\mathrm{N}-3$*
$$(Y/t^{\mathrm{N}})^{-r_E}\cdot \mathrm{Coef}_{z^{b''}}\Big(\varphi^{a''},R^X(-z)^*E_{a',b'}(z)\Big)\in\mathbb{Q}[X]_{b+b'+1},\tag{4.9}$$
where $r_E:=\frac{1}{\mathrm{N}}(a'+b'+a''+b''-\mathrm{N}-2)$ and the LHS of (4.9) vanishes unless $r_E\in\mathbb{Z}$.

Proof of Lemma 3.21. Assume $\mathfrak{S}_{h,m}$ holds for all $(h,m)<(g,n)$ and $3h+m\le 3g+n$. By Proposition 4.5
$$f^{\mathbf{A},\vec{0}}_{h;\mathbf{a},\mathbf{b}}\in\mathbb{Q}[X]_{3h-3+m-\sum_i b_i}\quad\forall (h,m)<(g,n),3h+m\le 3g+n,\ a_i=0,1,2,3.$$
We now look at the statement of $\mathfrak{S}'_{h,m}$, under the assumption $(h,m)<(g,n)$ and $3h+m\le 3g+n$. Let $\mathbf{a}\in\{0,1,2,3,\mathrm{N},\cdots,\mathrm{N}+3\}^{\times\ell}$ and $m=\ell+s$. By $\Omega^{[0]}=R^X.\Omega^{\mathbf{A},\vec{0}}$, we obtain
$$f^{[0]}_{h,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')}=\int_{\overline{\mathcal{M}}_{h,\ell+s}}(R^X.\Omega^{\mathbf{A},\vec{0}})_{h,\ell+s}\Big(\phi_{\mathbf{a}}\psi^{\mathbf{b}},E_{\mathbf{a}',\mathbf{b}'}(\psi)\Big).\tag{4.10}$$
By applying the R^X-action, for each stable graph $\Gamma\in G_{h,m}$, the contribution to (4.10) consists of (using Lemma 4.3)[21]

- at each leg $l\in L$, we have an insertion [22]
$$\Big(\varphi^{\bar a_l-k_l},\psi_l^{b_l+k_l}(-1)^{k_l}(R^X_{k_l})^*\phi_{a_l}\Big)\in\psi_l^{b_l+k_l}(Y/t^{\mathrm{N}})^{-\lfloor\frac{a_l}{\mathrm{N}}\rfloor}\mathbb{Q}[\![X]\!]_{k_l+\lfloor\frac{a_l}{\mathrm{N}}\rfloor};$$
- at each leg $l'\in L'$, we have an insertion [23]
$${\psi_{l'}}^{b''_{l'}}\mathrm{Coef}_{z^{b''_{l'}}}\Big(\varphi^{a''_{l'}},R^X(-z)^*E_{a'_{l'},b'_{l'}}(z)\Big)\in{\psi_{l'}}^{b''_{l'}}(Y/t^{\mathrm{N}})^{r_{E_{l'}}}\cdot\mathbb{Q}[\![X]\!]_{b''_{l'}+b'_{l'}+1};$$
 where $r_{E_{l'}}=\frac{1}{\mathrm{N}}(a'_{l'}+b'_{l'}+a''_{l'}+b''_{l'}-\mathrm{N}-2)$;
- at each edge $e=(v_1,v_2)$, we have a bivector $V_X(e)$.

Here we have used the degree estimate in Lemma 4.6. Further we see

[21] We will denote the set of first m (last s) legs by L (L' respectively).

[22] Here we have used $k_l\le k_l+b_l\le 3h-3+m$, otherwise the contribution vanishes by dimension reason. Hence we have $k_l\le 3g-3+n<\mathrm{N}-3$, and by this condition we see that the only integer $a\in[0,3]$ making $a_l-k_l\equiv a(\mathrm{mod}\,\mathrm{N})$ is $a=\bar a_l-k_l$ (i.e. we must have $k_l\le\bar a_l\le 3$).

[23] Here we have used $b''_{l'}\le 3h-3+m<\mathrm{N}-3$ for the same reason as above.

(1) the total factor of (Y/t^{N}) in the contribution of graph Γ to (4.10) is given by

$$\prod_l \Big(\frac{Y}{t^{\mathrm{N}}}\Big)^{-\lfloor\frac{a_l}{\mathrm{N}}\rfloor}\prod_{l'\in L'}\Big(\frac{Y}{t^{\mathrm{N}}}\Big)^{r_{E_{l'}}}\prod_v\Big(\frac{Y}{t^{\mathrm{N}}}\Big)^{1-h_v}\prod_e\Big(\frac{Y}{t^{\mathrm{N}}}\Big)^{-1}=\Big(\frac{Y}{t^{\mathrm{N}}}\Big)^{-(r+s+h-1)},$$

where we have used $\sum_{l'}(a''_{l'}+b''_{l'})+\sum_l(\bar a_l+b_l)=\ell+s$[24] and

$$\sum_{l'\in L'}r_{E_{l'}}+r+s-\sum_l\lfloor\tfrac{a_l}{\mathrm{N}}\rfloor=\tfrac{1}{\mathrm{N}}\sum_{l'}(a'_{l'}+b'_{l'}+a''_{l'}+b''_{l'}-\mathrm{N}-2)$$
$$+\tfrac{1}{\mathrm{N}}(|\bar{\mathbf{a}}|+|\mathbf{b}|-|\mathbf{a}'|-|\mathbf{b}'|-\ell+s)+s=0;$$

(2) the total degree of the contribution of graph Γ to (4.10) in X is no more than

$$\sum_v\Big(3h_v-3+n_v-\sum_{l\in L_v}(b_l+k_l)-\sum_{l'\in L'_v}b''_{l'}-\sum_{e\in E_v}k_{(e,v)}$$
$$+\sum_{l\in L_v}(k_l+\lfloor\frac{a_l}{\mathrm{N}}\rfloor)+\sum_{l'\in L'_v}(b''_{l'}+b'_{l'}+1)\Big)+\sum_e(k_{(e,v_1)}+k_{(e,v_2)}+1)$$
$$=3h-3+\ell+2s+\lfloor\frac{\mathbf{a}}{\mathrm{N}}\rfloor-|\mathbf{b}|+|\mathbf{b}'|.$$

Here we have used the degree estimate (4.8).

To summarize we obtain

$$Y^{h-1+r+s}\cdot f^{[0]}_{h,(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')}\in\mathbb{Q}[X]_{3h-3+\ell+2s+\lfloor\frac{\mathbf{a}}{\mathrm{N}}\rfloor-|\mathbf{b}|+|\mathbf{b}'|}.$$

This finishes the proof of Lemma 3.21, and therefore (3.18) is correct by Proposition 3.22. □

Lemma 4.7. *We have* $\big(\phi^a,\mathrm{Coef}_{z^b}E_{a',b'}(z)\big)=0$ *unless* $a'+a+b'+b=2\mod\mathrm{N}$.

Proof. By the definition (3.14) of $E_{a',b'}$,

$$\big(\phi_a,\mathrm{Coef}_{z^b}E_{a',b'}(z)\big)=L^{(\mathrm{N}+3)/2}\sum_\alpha L_\alpha^{-a'}(-1)^{b'}\big(\phi_a,R_{b'+b+1}\mathbf{1}^\alpha\big).\tag{4.11}$$

By Lemma 3.10 it vanishes unless $a'=a-(b'+b+1)\mod\mathrm{N}$. □

Proof of Lemma 4.6. The vanishing result follows from Lemma 4.3 and 4.7.

For the degree estimate, we consider three cases:

(1) For $a=4,\cdots,\mathrm{N}-1$: by Lemma 4.3 for any $k'<\mathrm{N}-3$, (note $a''=0,1,2,3$ [22])

$$(-1)^{k'}(R^X_{k'})_a=(\varphi^{a''},\mathrm{Coef}_{z^{k'}}R^X(-z)^*\phi_a)\neq0\quad\text{only if } a=a''+k'.\tag{4.12}$$

When it is nonzero, it is a degree k' polynomial in X. This implies that [25]

$$(Y/t^{\mathrm{N}})^{-r_E}\cdot\mathrm{Coef}_{z^{b''}}(\varphi^{a''},R^X(-z)^*\phi_a)\big(\phi^a,E_{a',b'}(z)\big)\in\mathbb{Q}[X]_{b'+b''+1}.\tag{4.13}$$

(2) For $a=0,\cdots,3$: by Lemma 4.3 for any $k'<\mathrm{N}-3$ we still have (4.12). Further, when it is nonzero, it is a degree k' polynomial in X. This implies that [26]

$$(Y/t^{\mathrm{N}})^{-r_E}\cdot\mathrm{Coef}_{z^{b''}}(\varphi^{a''},R^X(-z)^*\phi_a)\big(\phi^a,E_{a',b'}(z)\big)\in\mathbb{Q}[X]_{b'+b''+2}.\tag{4.14}$$

(3) For $a=\mathrm{N},\cdots,\mathrm{N}+3$: by Lemma 4.3, for any $k'<\mathrm{N}-3$,

$$(\varphi^{a''},\mathrm{Coef}_{z^{k'}}R^X(-z)^*\phi_a)\neq0\quad\text{only if } a-\mathrm{N}=a''+k'.$$

When it is nonzero, it is a degree k' polynomial in X multiplied by q. This implies (by argument similar to (2))

$$(Y/t^{\mathrm{N}})^{-r_E}\cdot\mathrm{Coef}_{z^{b''}}(\varphi^{a''},R^X(-z)^*\phi_a)\big(\phi^a,E_{a',b'}(z)\big)\in q\,\mathbb{Q}[X]_{b'+b''+1}.$$

---

[24] This identity follows from Lemma 4.4 and the fact that V_X has cohomology degree two (by Lemma 4.3, see also for (D.3) the explicit formula).

[25] By using (4.11), $\phi^a=\phi_{\mathrm{N}+3-a}/5$ and the property (3.5) of $(R_k)_a$, we have

$$\mathrm{Coef}_{z^b}\big(\phi^a,E_{a',b'}(z)\big)=\tfrac{\mathrm{N}}{5}(Y/t^{\mathrm{N}})^{r_E}(-1)^{b'}(R_{b+b'+1})_{\mathrm{N}+3-a}\in(Y/t^{\mathrm{N}})^{r_E}\mathbb{Q}[X]_{b+b'+1}.$$

Then each contribution to LHS of (4.13) has degree $\le(b+b'+1)+k=b'+b''+1$ (here $b''=b+k'$).

[26] By applying $\phi^a=(\phi_{\mathrm{N}+3-a}-t^{\mathrm{N}}\phi_{3-a})/5$ in (4.14), the term $\phi_{\mathrm{N}+3-a}$ contributes the same formula as (4.13), except that by (3.5) the X degree bound is increased by $1=\lfloor\frac{\mathrm{N}+3-a}{\mathrm{N}}\rfloor$. The second term $t^{\mathrm{N}}\phi_{3-a}$ contributes $\mathrm{Coef}_{z^b}\big(t^{\mathrm{N}}\phi_{3-a},E_{a',b'}(z)\big)=\sum_\alpha(\frac{Y}{t^{\mathrm{N}}})^{r_E}(-1)^{b'}(R_{b+b'+1})_{3-a}\cdot Y$. With (4.12) we obtain (4.14).

Sum up the process deducing (1),(2),(3). One obtains that, the LHS of (4.9) equals

$$\mathrm{Contri}(1)+\sum_{\substack{j+k=b'+b''+1,\\ 0\le k\le b'',0\le a\le 3}}(-1)^{b'+k}\frac{\mathrm{N}}{5}\Big((R_k^X)_a(R_j)_{\mathrm{N}+3-a}-Y(R_k^X)_a(R_j)_{3-a}+(Y/t^{\mathrm{N}})(R_k^X)_{\mathrm{N}+a}(R_j)_{3-a}\Big).$$

where we denote by $(R_k^X)_i:=\big(\varphi^{a''},(R_k^X)^*\phi_i\big)$, and Contri(1) is a sum of terms of form (4.13) in case (1) (thus lies in $Q[X]_{b'+b''+1}$). By (2) and (3) the rest terms lie in $\mathbb{Q}[X]_{b'+b''+2}$. Now we want to prove the top degree term indeed vanishes. The argument is similar to the one in the proof of [NMSP2, Appendix C]. Recall [NMSP2, (C.4)], for $a=\mathrm{N},\cdots,\mathrm{N}+3$ we have

$$\mathrm{Coef}_{X^{k+1}}(R_k)_a=\frac{c'_a}{5^5}\mathrm{Coef}_{X^k}(R_k)_{a-\mathrm{N}},\quad (c'_a)_{a=\mathrm{N}}^{\mathrm{N}+3}=(-120,-890,-2235,-3005). \tag{4.15}$$

Similar property holds for R^X by using the explicit formula (D.2) : [27]

$$\text{for } a=\mathrm{N},\cdots,\mathrm{N}+3\qquad \mathrm{Coef}_{X^k}\Big(q^{-1}\cdot(R_k^X)_a\Big)=c'_a\cdot\mathrm{Coef}_{X^k}(R_k^X)_{a-\mathrm{N}}.$$

Then we obtain for $a=0,1,2,3$ and for $j+k=b'+b''+1$

$$\begin{aligned}&\mathrm{Coef}_{X^{j+k+1}}\Big(-Y\,(R_k^X)_a(R_j)_{3-a}+(R_k^X)_a(R_j)_{\mathrm{N}+3-a}+Y/t^{\mathrm{N}}(R_k^X)_{\mathrm{N}+a}(R_j)_{3-a}\Big)\\ &=\Big(1+\frac{c'_{\mathrm{N}+a}}{5^5}+\frac{c'_{\mathrm{N}+3-a}}{5^5}\Big)\cdot\mathrm{Coef}_{X^{b'+b''-1}}\Big((R_k^X)_a(R_j)_{3-a}\Big)=0.\end{aligned}$$

where we have used $Y=1-X$ and $5^5Yq=t^{\mathrm{N}}X$. Hence the true degree in X is decreased by 1 and then we finish the proof. □

4.4. **Choice of gauge and finish the proof of Theorem 2.** We consider the following symplectic transformation:

$$\mathcal{G}(z)^{-1}=I-\begin{pmatrix}0 & z\cdot c_{1a} & z^2\cdot c_2 & z^3\cdot c'_3\\ & 0 & z\cdot c_{1b} & z^2\cdot c'_2\\ & & 0 & z\cdot c_{1a}\\ & & & 0\end{pmatrix}, \tag{4.16}$$

where $c'_2=-c_{1a}c_{1b}-c_2$ and $c'_3=-c_{1a}c_2-c_3$. Then we are able to recover the family of R-matrices $R^{\mathbf{A},\mathcal{G}}(z)$ defined in (1.7) via

$$R^{\mathbf{A},\mathcal{G}}(z)^{-1}=R^{\mathbf{A},\vec{0}}(z)^{-1}\cdot\mathcal{G}(z)^{-1} \tag{4.17}$$

where the family of propagators $E^{\mathcal{G}}_{**}$ in $R^{\mathbf{A},\mathcal{G}}(z)$ is related with the propagators $E^{\vec{0}}_{**}:=E^{\mathcal{G}=\mathbf{0}}_{**}$ in $R^{\mathbf{A},\vec{0}}(z)$ by the following

$$\begin{aligned}&E^{\mathcal{G}}_{\psi}=E^{\vec{0}}_{\psi}+c_{1a},\qquad E^{\mathcal{G}}_{\varphi\varphi}=E^{\vec{0}}_{\varphi\varphi}+c_{1b},\qquad E^{\mathcal{G}}_{\varphi\psi}=E^{\vec{0}}_{\varphi\psi}-c_{1b}\,E^{\vec{0}}_{\psi}-c_2,\\ &E^{\mathcal{G}}_{\psi\psi}=E^{\vec{0}}_{\psi\psi}+c_{1b}\,(E^{\vec{0}}_{\psi})^2-2\,c_2\,E_{\psi}+c_3.\end{aligned}$$

Proof of Theorem 2. Recall we have proved (Proposition 4.5)[28]

$$f^{\mathbf{A},\vec{0}}_{g;\mathbf{a},\mathbf{b}}\in\mathbb{Q}[X]_{3g-3+n-\sum_i b_i}.$$

Via (4.17), we define the CohFT

$$\Omega^{\mathbf{A},\mathcal{G}}:=\mathcal{G}.\Omega^{\mathbf{A},\vec{0}}=R^{\mathbf{A},\mathcal{G}}.\Omega^{Q,\mathrm{tw}}.$$

Then we see the A-model master potential (1.11) is indeed its generating function[29]

$$f^{\mathbf{A},\mathcal{G}}_{g;\mathbf{a},\mathbf{b}}=(-5Y/t^{\mathrm{N}})^{g-1}\int_{\overline{\mathcal{M}}_{h,n}}\psi_1^{b_1}\cdots\psi_n^{b_n}\cup\Omega^{\mathbf{A},\mathcal{G}}_{g,n}(\varphi_{a_1},\cdots,\varphi_{a_n}). \tag{4.18}$$

[27] Note that $k<b''<\mathrm{N}-3$ implies $k\le 3$, for the same reason as stated in footnote 22.

[28] The assumption in the statement of Proposition 4.5 is no longer needed after finishing the induction.

[29] See more explanations at the end of this proof.

We claim that with the condition (1.12), the $\mathcal{G}$-action will not change the polynomiality. We can write down the graph sum formula for $\Omega^{\mathbf{A},\mathcal{G}} := \mathcal{G}.\Omega^{\mathbf{A},\vec{0}}$ via the $\mathcal{G}$-action. For each graph Γ, the contribution Cont_Γ to (4.18) is given by the following construction

- at each leg l with insertion $\varphi_{a_l}\psi_l^{b_l}$, we put $\sum_k \mathcal{G}_k^*(-\psi_l)^k\varphi_{a_l}\psi_l^{b_l}$;
- at each edge $e=(v_1,v_2)$, we put
$$V^{\mathcal{G}}(\psi_{v_1},\psi_{v_2}) := \tfrac{1}{\psi_{v_1}+\psi_{v_2}}(\mathrm{I}-\mathcal{G}(\psi_{v_1})^{-1}\mathcal{G}(-\psi_{v_2})) = Y^{-1}\cdot \textstyle\sum_{k,l} V_{kl}^{\mathcal{G}}\psi_{v_1}^k\psi_{v_2}^l,$$
where $\deg_X G_k^* = k$, $\deg_X V_{kl} = k+l+1$ and we have used $\varphi^i = (5Y/t^{\mathrm{N}})^{-1}\varphi_{3-i}$ in the last equility.

The total factor involving $(-5Y/t^{\mathrm{N}})$ in Cont_Γ is
$$(-5Y/t^{\mathrm{N}})^{-E}\textstyle\prod_v(-5Y/t^{\mathrm{N}})^{-(g_v-1)} = (-5Y/t^{\mathrm{N}})^{-(g-1)},$$
and the X-degree of total contribution of Cont_Γ is
$$\begin{aligned}\textstyle\sum_v \big(3g_v-3+n_v-\sum_{l\in L_v}(k_l+b_l)-\sum_{e\in E_v}k_{(e,v)}\big)+\sum_{l\in L}k_l\\ +\textstyle\sum_{e=(v_1,v_2)\in E}(k_{(e,v_1)}+k_{(e,v_2)}+1) = 3g-3+n-\sum_{l\in L}b_l.\end{aligned}$$
This proves
$$f_{g;\mathbf{a},\mathbf{b}}^{\mathbf{A},\mathcal{G}} \in \mathbb{Q}[X]_{3g-3+n-\sum_i b_i}. \tag{4.19}$$

Pick t such that $t^{\mathrm{N}}=-1$ and substitute it into (4.19), then $q'=q$ and $\Omega_{g,n}^{Q,\mathrm{tw},\tau_Q(q')} = \Omega_{g,n}^{Q,\tau_Q(q)}$ by Remark 2.12. By using the identification (2.5), the definition (4.18) matches (1.11), and (4.19) becomes the statement of Theorem 2. □

5. BCOV's Feynman graph sum via geometric quantization

In this section, we view BCOV's Feynman graph sum as the quantization of a symplectic transformation $R^{\mathbf{B}}$, which is a restriction of our A-model propagator matrix $R^{\mathbf{A}}$ in the smaller phase space.

Convention 5.1. In this and the next section, we will omit the supscript $\mathcal{G}$ in $\Omega^{\star,\mathcal{G}}$, $R^{\star,\mathcal{G}}$,$f^{\star,\mathcal{G}}$, $E_{**}^{\mathcal{G}}$, etc..

5.1. **Quantization of the symplectic transformation in the small phase space.** Let $\{v_i\}_{i=0,1,2,3}=\{\varphi_3 z^{-2},-\varphi_2 z^{-1},\varphi_1,z\}$, with inner product given by
$$v_i\cdot v_j := \frac{I_0^2}{5Y}\cdot \mathrm{Res}_{z=0}(v_i|_{z\mapsto -z},v_j) = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&&\end{pmatrix}.$$
We consider the 4-dimensional symplectic subspace
$$H_S := \mathrm{span}\{v_i\}\subset \mathcal{H}_Q[z,z^{-1}]\otimes\mathbb{A}.$$
By the explicit formula of the propagator matrix $R^{\mathbf{A}}$, we see
$$R^{\mathbf{A}}(z)H_S\subset H_S.$$
Hence we can restrict the symplectic transformation $R^{\mathbf{A}}(z)$ to subspace H_S, which is denoted by $R^{\mathbf{B}}$. Under the symplectic basis $\{v_i\}_{i=0,1,2,3}$, we have
$$R^{\mathbf{B}} = \begin{pmatrix} A & B\\ C & D\end{pmatrix} := \begin{pmatrix} 1&&&\\ -E_\psi & 1&&\\ -E_{\varphi\psi} & -E_{\varphi\varphi} & 1 &\\ E_{1\psi^2} & E_{1,\varphi\psi} & E_\psi & 1\end{pmatrix}. \tag{5.1}$$

For a vector in H_S under the symplectic basis $\{v_i\}$, we write it in the form
$$\vec{v} = (\mathbf{p},\mathbf{x}) = p_y v_0 + p_x v_1 + x\, v_2 + y\, v_3 \in H_S$$
We define the quantization of the symplectic transformation $R^{\mathbf{B}}$ as follows:

Definition 5.2. We introduce the following quadratic form over H_S:

$$\mathbf{Q}(\mathbf{x},\mathbf{p})=(D^{-1}\mathbf{x})\cdot\mathbf{p}'-\frac{1}{2}(D^{-1}C\mathbf{p}')\cdot\mathbf{p}'=\ (\mathbf{p}')^t\begin{pmatrix}1&0\\E_\psi&1\end{pmatrix}\mathbf{x}+\frac{1}{2}(\mathbf{p}')^t\begin{pmatrix}E_{\varphi\psi}&E_{\varphi\varphi}\\E_{\psi\psi}&E_{\varphi\psi}\end{pmatrix}\mathbf{p}'.$$

The quantization $\widehat{R}^{\mathbf{B}}$ is defined via the following Feynman integral[30]

$$(\widehat{R}^{\mathbf{B}}F)(\hbar,\mathbf{x}):=\ln\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{\frac{1}{\hbar}(\mathbf{Q}(\mathbf{x},\mathbf{p}')-\mathbf{x}'\cdot\mathbf{p}')+F(\hbar,\mathbf{x}')}d\mathbf{x}'d\mathbf{p}'.\tag{5.2}$$

The standard argument of Fourier transform deduces the following (we refer the reader to [CPS13, Sect. 1.4] for detailed discussion of the geometric quantization).

Lemma 5.3. *We have the following operator form for* $\widehat{R}^{\mathbf{B}}$

$$(\widehat{R}^{\mathbf{B}}F)(\hbar,\mathbf{x})=\ln\Big(e^{\hbar V^B(\partial_\mathbf{x},\partial_\mathbf{x})}e^{F(\hbar,D^{-1}\mathbf{x})}\Big),\tag{5.3}$$

where the differential operator is defined by

$$V^{\mathbf{B}}(\partial_\mathbf{x},\partial_\mathbf{x}):=-\frac{1}{2}\left(\partial_x,\partial_y\right)\left(D^{-1}C\right)\begin{pmatrix}\partial_x\\\partial_y\end{pmatrix}=\frac{1}{2}\left(\partial_x,\partial_y\right)\begin{pmatrix}E_{\varphi\psi}&E_{\psi\psi}\\E_{\varphi\varphi}&E_{\varphi\psi}\end{pmatrix}\begin{pmatrix}\partial_x\\\partial_y\end{pmatrix}.\tag{5.4}$$

Now our Theorem 1 has an equivalent statement

Theorem 5.4 (BCOV's Feynman rule)**.** *Recall* $P^{\mathbf{B}}(\hbar,x,y)$ *is defined in* (1.13)*. The quantization of* $R^{\mathbf{B}}$ *acting on* $P^{\mathbf{B}}$ *defines the B-model master potential function, with the form*

$$f^{\mathbf{B}}(\hbar,x,y):=\widehat{R}^{\mathbf{B}}P^{\mathbf{B}}(\hbar,x,y)=\textstyle\sum_{g,m,n}\hbar^{g-1}x^my^n\cdot f^{\mathbf{B}}_{g,m,n}.\tag{5.5}$$

Then for each (g,m,n)*,* $f^{\mathbf{B}}_{g,m,n}$ *is a degree* $3g-3+m$ *polynomials in* X*.*

5.2. **Modified Feynman rule.** We introduce the following modified B-model correlators

$$\tilde{P}_{g,m,n}=\big\langle(\varphi-E_\psi\psi)^{\otimes m},\psi^{\otimes n}\big\rangle^{Q,\mathbf{B}}_{g,m+n},\tag{5.6}$$

and their generating function

$$\tilde{P}^{\mathbf{B}}(\hbar,x,y):=\sum_{g,m,n}\hbar^{g-1}\frac{x^my^n}{m!n!}\cdot\tilde{P}_{g,m,n}=\ P^{\mathbf{B}}(\hbar,x,y-E_\psi\,x).\tag{5.7}$$

It is not hard to see, if we replace $P_{g,m,n}$ in the BCOV's Feynman rule by $\tilde{P}_{g,m,n}$, then the Feynman rule Theorem 1 will still hold if we replace $\{E_{\varphi\varphi},E_{\varphi\psi},E_{\psi\psi},E_\psi\}$ by [31]

$$\begin{aligned}\tilde{E}_{\varphi\varphi}=&\ E_{\varphi\varphi},\quad \tilde{E}_{\varphi\psi}=\ E_\psi E_{\varphi\varphi}+E_{\varphi\psi},\\ \tilde{E}_{\psi\psi}=&\ E_\psi^2E_{\varphi\varphi}+2E_\psi E_{\varphi\psi}+E_{\psi\psi},\quad \tilde{E}_\psi=0.\end{aligned}\tag{5.8}$$

More precisely, the Feynman graph sum is given by the following quantization

$$f^{\mathbf{B}}(\hbar,x,y)=\widehat{R}^{\mathbf{B}}|_{E_{**}\mapsto\tilde{E}_{**}}\tilde{P}^{\mathbf{B}}(\hbar,x,y).\tag{5.9}$$

Indeed, the change of variables (5.7) can be written as a quantization

$$\tilde{P}^{\mathbf{B}}(\hbar,x,y)=\widehat{\mathcal{E}}^{\mathbf{B}}P^{\mathbf{B}}(\hbar,x,y)$$

of the symplectic transformation $\mathcal{E}^{\mathbf{B}}$ defined by[32]

$$\mathcal{E}^{\mathbf{B}}=\begin{pmatrix}1&&&\\-E_\psi&1&&\\&0&1&\\&&E_\psi&1\end{pmatrix}.\tag{5.10}$$

Then the modified B-model propagator matrix $R^{\mathbf{B}}|_{E_{**}\mapsto\tilde{E}_{**}}$ is given by

$$R^{\mathbf{B}}|_{E_{**}\mapsto\tilde{E}_{**}}=\tilde{R}^{\mathbf{B}}:=R^{\mathbf{B}}\cdot(\mathcal{E}^{\mathbf{B}})^{-1},$$

which matches (5.8).

---

[30] This is a finite dimensional Gaussian integral, hence it is well-defined.

[31] To generalize Yamaguchi-Yau equations, similar modified propagators were defined in [AL07] .

[32] We can see for this case $C=0$ and by (5.4) there is no edge contribution.

6. From NMSP Feynman rule to BCOV's Feynman rule

We have proved Theorem 2 in §4 and established the NMSP Feynman rule. In this section, we will prove the equivalence of NMSP Feynman rule and BCOV's Feynman rule (Theorem 3). This will finish the proof of the BCOV's Feynman rule.

Notice that the A-model state space $H_{\mathbf{A}}$ has a higher dimension, with the B-model one $H_{\mathbf{B}}$ as its subspace. In particular, we have 3 more extra propagators as edge contributions. We first deal with the edge that contributes a bivector $1 \otimes \varphi_2$ (with propagator $E_{1\varphi_2} = E_\psi$). The idea is to consider the similar factorization of the symplectic transformation as in §5.2.

6.1. **Decomposition of $R^{\mathbf{A}}$-matrix and modified quintic theory.** We consider the following matrix factorization of $R^{\mathbf{A}}$-matrix:

$$R^{\mathbf{A}}(z) = \tilde{R}^{\mathbf{A}}(z) \cdot \mathcal{E}^{\mathbf{A}}(z), \tag{6.1}$$

where (recall $E_{1\varphi_2} := E_\psi$)

$$\mathcal{E}^{\mathbf{A}}(z) := \mathrm{I} + z \begin{pmatrix} 0 & E_{1\varphi_2} & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & E_{1\varphi_2} \\ & & & & 0 \end{pmatrix} \in \operatorname{End} H_{\mathbf{A}}. \tag{6.2}$$

The modified quintic CohFT is defined via

$$\tilde{\Omega}^Q := \mathcal{E}^{\mathbf{A}}.\Omega^Q.$$

Notice that here $\tilde{\Omega}^Q$ theory depends on the choice of the gauge $\mathcal{G}$. (Recall by Convention 5.1, we always omit the supscript $\mathcal{G}$ in this section.)

Convention 6.1. In this section, we will not distinguish the Ω^Q and the twisted theory $\Omega^{Q,\mathrm{tw}}$. We identify them by setting $t = 1$ in this section.

Definition 6.2. For the following coordinate

$$\mathbf{t} = x\,\varphi_1 + y\,\varphi_0\psi + a\,\varphi_1\psi + b\,\varphi_0\psi^2 + c\,\varphi_0 \in H_{\mathbf{A}},$$

we introduce modified normalized A-model potential for the quintic 3-fold

$$\tilde{P}^{\mathbf{A}}(\hbar;\mathbf{t}) = \tilde{P}^{\mathbf{A}}(\hbar,x,y,a,b,c) := \sum_{g,n} \frac{\hbar^{g-1}}{n!} \frac{(5Y)^{g-1}}{I_0^{2g-2+n}} \int_{\overline{\mathcal{M}}_{g,n}} \tilde{\Omega}^Q_{g,n}(\mathbf{t}^n). \tag{6.3}$$

In particular, we define

$$\tilde{P}^{\mathbf{A}}(\hbar;x,y) := \tilde{P}^{\mathbf{A}}(\hbar;x\,\varphi_1 + y\,\varphi_0\psi).$$

Lemma 6.3. *String and dilaton equations hold for the theory $\tilde{\Omega}^Q$.*

Proof. By the result of [Lee03], the R-matrix action preserve tautological equations. Hence the $\tilde{\Omega}^Q$ theory satisfies string and dilation equation as well. ☐

Proposition 6.4. *We have the following relation*

$$\tilde{P}^{\mathbf{A}}(\hbar,x,y) = \tilde{P}^{\mathbf{B}}(\hbar,x,y) - \ln(1-y). \tag{6.4}$$

Proof. By Lemma 6.3, we can use dilaton equations to remove the $\varphi_0\psi$ insertions. Namely, both sides of (6.4) satisfy[33]

$$\tfrac{\partial}{\partial y} f = \Big(2\hbar\tfrac{\partial}{\partial \hbar} + x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}\Big) f + \tfrac{\chi_Q}{24}, \qquad \chi_Q = -200.$$

It suffices to prove

$$\tilde{P}^{\mathbf{A}}(\hbar,x,0) = \tilde{P}^{\mathbf{B}}(\hbar,x,0). \tag{6.5}$$

Now we apply the graph sum formula to $\tilde{\Omega}^Q := \mathcal{E}^{\mathbf{A}}\Omega^Q$. Notice that when there is an insertion $\varphi_2 = I_0 I_{11} I_{22} H^2$, the quintic correlators are zero unless $g = 0$ (which is from degree 0 contribution). It is not hard to see that in our case (the leg insertions are all φ_1's), the stable graph will contribute zero unless it is a loop with l-vertices: at each vertex there is exactly one φ_1 leg insertion and several $-E_{1\varphi_2}\varphi_0\psi$ insertions, at each edge the bivector is $E_{1\varphi_2}\varphi_0 \otimes \varphi_2$. This

[33] one can check that the B-model correlators satisfy dilaton equations directly.

only contributes to $g = 1$ potential. Denoted by P_1^E the generating function of such "loop type" contribution, we have

$$\tilde{P}^{\mathbf{A}}(\hbar, x, 0) = P^{\mathbf{A}}(\hbar, x, -E_{1\varphi_2}x) + P_1^E(x).$$

By using the dilaton equation for each vertex[34] of the "loop type" graph, we obtain

$$\begin{aligned} P_1^E(x) &= \sum_{\substack{\Gamma \text{ is a loop with } l \\ \text{vertices and } n+l \text{ legs}}} \frac{x^{l+n}}{l!n!} \frac{\mathrm{Cont}_\Gamma}{|\operatorname{Aut}\Gamma|} = \sum_{l>0} \frac{(l-1)!}{l} (E_{1\varphi_2}x)^l \prod_{i=1}^{l} \sum_{n_i \geq 0} (-E_{1\varphi_2}x)^{n_i} \\ &= -\ln\Big(1 - \frac{E_{1\varphi_2}x}{1+E_{1\varphi_2}x}\Big) = \ln(1 + E_{1\varphi_2}x). \end{aligned} \tag{6.6}$$

In the second equality above we used that there are $(l-1)!$ choices when we put l different vertices in a loop. Together with the following relations

$$\tilde{P}^{\mathbf{B}}(\hbar, x, y) = P^{\mathbf{B}}(\hbar, x, y - E_{1\varphi_2}\, x), \quad \text{and} \quad P^{\mathbf{A}}(\hbar, x, y) = P^{\mathbf{B}}(\hbar, x, y) - \ln(1-y),$$

We obtain (6.5), and hence finish the proof of this proposition. $\square$

Remark 6.5. We can see the symplectic transformation (5.10) in §5.2 is exactly the restriction of the $\mathcal{E}^{\mathbf{A}}$-action to the B-model state space.

Next, we will use string equations proved in Lemma 6.3, to write down any $\tilde{\Omega}^Q$-theory invariants in terms of $\tilde{\Omega}^Q$-theory invariants with only insertions φ and ψ. In this way, we deal with the remaining two "extra" propagators.

6.2. **Modified propagators and operator formalism for the quantization action.** By the definition of $\tilde{R}$-matrix and the $\tilde{\Omega}^Q$ (c.f. (6.1) and (6.2)), we see that the CohFT $\Omega^{\mathbf{A}}$ is equal to the $\tilde{R}^Q(z)$-action on the CohFT $\tilde{\Omega}^Q$:

$$\Omega^A = \tilde{R}^Q . \tilde{\Omega}^Q. \tag{6.7}$$

Extending §5.2, for the edge contribution of $\tilde{R}^Q$-action, we have the modified propagators

$$\begin{aligned} &\tilde{E}_{\varphi\varphi} = E_{\varphi\varphi}, \qquad \tilde{E}_{\varphi\psi} = E_\psi E_{\varphi\varphi} + E_{\varphi\psi}, \\ &\tilde{E}_{1,\varphi\psi} = E_{1,\varphi\psi}, \qquad \tilde{E}_{1\psi^2} = E_{1\psi^2} + E_\psi E_{1,\varphi\psi}, \\ &\tilde{E}_{\psi\psi} = E_\psi^2 E_{\varphi\varphi} + 2E_\psi E_{\varphi\psi} + E_{\psi\psi}, \qquad \tilde{E}_\psi = 0. \end{aligned} \tag{6.8}$$

(Note $\tilde{E}_{**}$'s are $\tilde{E}^{\mathcal{G}}_{**}$'s defined via the same formulas.) Using (6.8), we write down the differential operator form of NMSP A-model potential and BCOV's B-model potential.

Proposition 6.6. *For $\star = \mathbf{A}$ or $\mathbf{B}$ and $u = x\varphi_1 + y\varphi_0\psi$, we have*

$$\exp\big(f^\star(\hbar, x, y)\big) = \exp\big(\hbar \cdot \tilde{V}^\star(\partial_{\mathbf{t}}, \partial_{\mathbf{t}})\big) \exp\big(\tilde{P}^\star(\hbar; \mathcal{E}\mathbf{t})\big)|_{\mathbf{t}=R^\star(\psi)^{-1}u(\psi)}$$

where the $\tilde{V}$-operator is defined by

$$\begin{aligned} &\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}}, \partial_{\mathbf{t}}) := \frac{1}{2}\tilde{E}_{\varphi\varphi}\frac{\partial^2}{\partial x^2} + \tilde{E}_{\varphi\psi}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\tilde{E}_{\psi\psi}\frac{\partial^2}{\partial y^2}, \\ &\tilde{V}^{\mathbf{A}}(\partial_{\mathbf{t}}, \partial_{\mathbf{t}}) := \tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}}, \partial_{\mathbf{t}}) + \tilde{V}^E(\partial_{\mathbf{t}}, \partial_{\mathbf{t}}), \quad \tilde{V}^E(\partial_{\mathbf{t}}, \partial_{\mathbf{t}}) := \tilde{E}_{1,\varphi\psi}\frac{\partial^2}{\partial a \partial c} + \tilde{E}_{1\psi^2}\frac{\partial^2}{\partial b \partial c}. \end{aligned}$$

Here the operator $\tilde{V}^E$ corresponds to edge contributions with extra propagators.

Proof. For the case $\star = \mathbf{A}$, the formula follows from (4.18), (6.7) and Givental's quantization formula [Giv01a]. For the case $\star = B$, the formula follows from the operator form of the B-model quantization formuma (5.3) and (5.5). $\square$

Lemma 6.7. *We have*

$$e^{\tilde{P}^{\mathbf{A}}(\hbar, x, y, a, b, c)} = e^{\frac{c}{1-y}(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})} e^{\tilde{P}^{\mathbf{A}}(\hbar, x, y)}.$$

---

[34] Suppose there are n_i ψ-insertions at the i-th vertices ($i = 1, \cdots, l$), by forgotting all the ψ-insertions we get a factor $n_i!$.

Proof. By string equations, we have[35]

$$\frac{\partial}{\partial c}e^{\tilde{P}^{\mathbf{A}}(\hbar,x,y,a,b,c)} = \Big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + y\frac{\partial}{\partial c}\Big)e^{\tilde{P}^{\mathbf{A}}(\hbar,x,y,a,b,c)}.$$

Then the Lemma follows from the initial condition

$$\tilde{P}^{\mathbf{A}}(\hbar,x,y,a,b,c)|_{c=0} = \tilde{P}^{\mathbf{A}}(\hbar,x,y).$$

This proves the Lemma. □

6.3. **Finish the proof of Theorem 3.** We first prove two identities.

Lemma 6.8. *For any $f(x,y)$, the following identities hold:*

$$e^{\hbar\tilde{V}^E(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\frac{c}{1-y}(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y})}f(x,y)\Big|_{a,b,c=0} = \sum_{k\geq 0}E(\partial_{\mathbf{t}})^k f(x,y), \tag{6.9}$$

$$e^{-\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}(1-y)^{-1}e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}(1-y) = \sum_{k\geq 0}E(\partial_{\mathbf{t}})^k f(x,y), \tag{6.10}$$

where $E(\partial_{\mathbf{t}}) := \frac{\hbar}{1-y}(\tilde{E}_{1,\varphi\psi}\frac{\partial}{\partial x} + \tilde{E}_{1\psi^2}\frac{\partial}{\partial y})$.

Proof. For the first identity, we have that the LHS of (6.9)

$$\begin{aligned}
&= \sum_n \frac{\hbar^n}{(n!)^2}\Big(\tilde{E}_{1,\varphi\psi}\frac{\partial^2}{\partial a\partial c} + \tilde{E}_{1\psi^2}\frac{\partial^2}{\partial b\partial c}\Big)^n\Big(\frac{c}{1-y}(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})\Big)^n f(x,y)\Big|_{a,b,c=0}\\
&= \sum_n \frac{\hbar^n}{n!}\Big(\tilde{E}_{1,\varphi\psi}\frac{\partial}{\partial a} + \tilde{E}_{1\psi^2}\frac{\partial}{\partial b}\Big)^n\Big(\frac{1}{1-y}(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})\Big)^n f(x,y)\Big|_{a,b,c=0}\\
&= \sum_n \Big(\frac{\hbar^n}{1-y}(\tilde{E}_{1,\varphi\psi}\frac{\partial}{\partial x} + \tilde{E}_{1\psi^2}\frac{\partial}{\partial y})\Big)^n f(x,y).
\end{aligned}$$

Here in the second equality we have used the following: when expanding the differential operators as power series, the contribution is non-zero only if $\tilde{V}^E(\partial_{\mathbf{t}},\partial_{\mathbf{t}})$ and $\frac{c}{1-y}(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})$ appear in the form of the same powers.

For the second identity, by using $\tilde{E}_{\varphi\psi} + \tilde{E}_{1,\varphi\psi} = 0$, $\tilde{E}_{\psi\psi} + \tilde{E}_{1\psi^2} = 0$, and

$$\begin{aligned}
e^{-\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}(1-y)e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})} &= e^{\mathrm{ad}_{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}}(1-y)\\
&= (1-y) - [\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}}),(1-y)] = (1-y) + \hbar(\tilde{E}_{\varphi\psi}\tfrac{\partial}{\partial x} + \tilde{E}_{\psi\psi}\tfrac{\partial}{\partial y}),
\end{aligned}$$

we obtain $(1-y)^{-1}e^{-\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}(1-y)e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})} = (1-E(\partial_{\mathbf{t}}))f(x,y)$, which is equivalent to (6.10). □

By the above two identities, we obtain the following key Lemma.

Lemma 6.9. *For any $f(x,y)$ we have*

$$(1-y)e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})+\hbar\tilde{V}^E(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\frac{c}{1-y}(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y})}\frac{f(x,y)}{1-y}\Big|_{a,b,c=0} = e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}f(x,y). \tag{6.11}$$

Proof. Since $\tilde{V}^{\mathbf{B}}$ commutes with $\tilde{V}^E$, (6.9) and (6.10) imply

$$\mathrm{LHS} = (1-y)\,e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}\textstyle\sum_{k\geq 0}E(\partial_{\mathbf{t}})^k(1-y)^{-1}f(x,y) = \mathrm{RHS}.$$

This proves the lemma. □

Now we finish the last step of the proof of Theorem 3. By setting $f(x,y) = e^{\tilde{P}^{\mathbf{B}}(\hbar,x,y)}$ in (6.11) and by using Proposition 6.4, we have

$$e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})+\hbar\tilde{V}^E(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\frac{c}{1-y}(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y})}e^{\tilde{P}^{\mathbf{A}}(\hbar,x,y)}\Big|_{a,b,c=0} = (1-y)^{-1}\,e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\tilde{P}^{\mathbf{B}}(\hbar,x,y)}.$$

Then by Lemma 6.7, the identity becomes

$$e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})+\hbar\tilde{V}^E(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\tilde{P}^{\mathbf{A}}(\hbar,x,y,a,b,c)}\big|_{a,b,c=0} = (1-y)^{-1}\,e^{\hbar\tilde{V}^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}})}e^{\tilde{P}^{\mathbf{B}}(\hbar,x,y)}.$$

Together with Proposition 6.6 we complete the proof.

[35] Here since there is no φ_2-insertions, the unstable contribution does not appear in the equation.

7. Reduction of generators, Yamaguchi-Yau's equations and examples

The modified propagators (5.8) and (6.8) were introduced to remove the $(1,\varphi_2)$ edges in the NMSP rule in order to prove Theorem 3. As a by-product, we find that four specific modified propagators give exactly Yamaguchi-Yau's generators, which generate a subring containing the normalized quintic potentials $P_{g>1}$.

Theorem 7.1. *We consider the following modified propagators as generators* [36]

$$\begin{aligned} \mathcal{E}_1 &:= \tilde{E}^{\vec{0}}_{\varphi\varphi} = A + 2B, \quad \mathcal{E}_2 = \tilde{E}^{\vec{0}}_{\varphi\psi} = -B_2 + B(A+2B), \\ \mathcal{E}_3 &= \tilde{E}^{\vec{0}}_{\psi\psi} = -B_3 - (B+X)\,B_2 + (A+2B)\,B^2 - \frac{2}{5}X\,B, \end{aligned} \tag{7.1}$$

and we introduce the subring which is closed under the differential operator D:

$$\tilde{\mathcal{R}} := \mathbb{Q}[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, X] \subset \mathcal{R}.$$

Then for $2g-2+m+n>0$, the $\tilde{P}_{g,m,n}$ defined in (5.6) *lie in $\mathcal{R}$. In particular, we have the reduction of generators which was originally conjectured in* [YY04]*:*

$$P_g \in \tilde{\mathcal{R}} \quad \textit{for} \;\; g>1 \tag{7.2}$$

Remark 7.2. Notice that

$$\tilde{E}^{\mathcal{G}}_{\varphi\varphi} = \mathcal{E}_1 + c_1, \quad \tilde{E}^{\mathcal{G}}_{\varphi\psi} = \mathcal{E}_2 + c_2, \quad \tilde{E}^{\mathcal{G}}_{\psi\psi} = \mathcal{E}_3 + c_3.$$

Hence the subring $\tilde{\mathcal{R}}$ is also independent of the choice of gauge.

Proof of Theorem 7.1. First, we prove $\tilde{\mathcal{R}}$ is closed under D [37]

$$D\mathcal{E}_1 = -X\left(\mathcal{E}_1 - \frac{2}{5}\right) - \mathcal{E}_1^2 + 2\mathcal{E}_2, \quad D\mathcal{E}_2 = -X\,\mathcal{E}_2 - \mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_3, \quad D\mathcal{E}_3 = \frac{24}{625}X - X\,\mathcal{E}_3 - \mathcal{E}_2^2.$$

Next by using the dilaton equation, $\tilde{P}_{g,m,n} = (2g-3+m+n)\tilde{P}_{g,m,n-1}$, we see

$$\tilde{P}_{g,m} \in \tilde{\mathcal{R}} \quad \Rightarrow \quad \tilde{P}_{g,m,n} \in \tilde{\mathcal{R}}. \tag{7.3}$$

Now we prove $\tilde{P}_{g,m} \in \tilde{\mathcal{R}}$ by induction. Initially we have

$$\tilde{P}_{1,0,1} = \chi/24 - 1 \quad \text{and} \quad \tilde{P}_{0,3} = 1 \quad \in \quad \tilde{\mathcal{R}}.$$

Assume $\tilde{P}_{h,l} \in \tilde{\mathcal{R}}$ for $(h,l) < (g,m)$. By using the modified Feynman rule (see §5.2), for $2g-2+m>0$, we have $f^{B,\vec{0}}_{g,m} \in \mathbb{Q}[X]_{3g-3+m}$ is equal to the sum over contributions of stable graphs $\Gamma \in G_{g,m}$.

Except for the "leading graph" (which has a single genus g vertex with m-legs), the vertices in the other graphs all satisfy $(g_v, n_v) < (g,m)$. By induction assumptions and (7.3), these vertices contributions $\tilde{P}_{g_v,m_v,n_v}$ all lie in the ring $\tilde{\mathcal{R}}$. Together with that the edge contributions $\mathcal{E}_k \in \tilde{\mathcal{R}}$ for $k=1,2,3$, we deduce $\tilde{P}_{g,m} \in \mathcal{R}$ and finish the induction. □

Theorem 7.3. *The Yamaguchi-Yau equations hold:*

$$-\,\partial_A P_g = \frac{1}{2}P_{g-1,2} + \frac{1}{2}\sum_{g_1+g_2=g} P_{g_1,1}P_{g_2,1}, \tag{7.4}$$

$$\Big(-2\partial_A + \partial_B + (A+2B)\partial_{B_2} + \big((B-X)(A+2B) - B_2 - \tfrac{2}{5}X\big)\partial_{B_3}\Big)P_g = 0. \tag{7.5}$$

Indeed, the second equation (7.5) *is equivalent to the reduction of generators* (7.2).

---

[36] Our generator $\mathcal{E}_k$ is related with the v_i defined in [YY04] as follows: $v_1 = -\mathcal{E}_1$, $v_2 = -\mathcal{E}_2$, $v_3 = \mathcal{E}_3 - X\,\mathcal{E}_2$. In a sense, we give a geometric explanation for Yamaguchi-Yau's generators v_i: they are edge contributions (propagators) of the modified Feynmann rule introduced in §5.2.

[37] This follows from a direct computation by using the relations (1.2), which is proved in [YY04]. See also (4.4) which gives equivalent relations.

Proof. In the end, we prove (7.4). By using Theorem 5.4, the definition of $V^{\mathbf{B}}$ (5.4) and the definition of quantization action (5.3) we have

$$\exp P^{\mathbf{B}}(\hbar,x,y)=e^{-\hbar V^{\mathbf{B}}(\partial_{\mathbf{x}},\partial_{\mathbf{x}})}\exp f^{\mathbf{B}}(\hbar,x,y). \tag{7.6}$$

Note both sides lie in the ring $\mathcal{R}[[\hbar,\hbar^{-1},x,y]]$. By applying the partial derivative $\partial\in\mathrm{span}\{\partial_A,\partial_B,\partial_{B_2},\partial_{B_3}\}$ on both sides of (7.6), we see

$$-\partial P^{\mathbf{B}}(\hbar,x,y)\exp P^{\mathbf{B}}(\hbar,x,y)=\hbar\,\partial V^{B}(\partial_{\mathbf{x}},\partial_{\mathbf{x}})e^{-\hbar V^{B}(\partial_{\mathbf{x}},\partial_{\mathbf{x}})}\exp f^{\mathbf{B}}(\hbar,x,y), \tag{7.7}$$

where we have used $[\partial V^B,V^B]=0$, $\partial f^{\mathbf{B}}=0$, and we recall

$$V^{\mathbf{B}}(\partial_{\mathbf{t}},\partial_{\mathbf{t}}):=\tfrac{1}{2}E_{\varphi\varphi}\tfrac{\partial^2}{\partial x^2}+E_{\varphi\psi}\tfrac{\partial^2}{\partial x\partial y}+\tfrac{1}{2}E_{\psi\psi}\tfrac{\partial^2}{\partial y^2}$$

with E_{**} defined in (1.3). We claim (7.7) will give us PDEs for $P_{g,m,m}$: Let $\partial=\partial_A$ we have $\partial_A V^B=\frac{1}{2}\partial_x^2$. Then (7.7) becomes the following PDE

$$-\partial_A P^{\mathbf{B}}(\hbar,x,y)=\frac{1}{2}\partial_x^2 P^{\mathbf{B}}(\hbar,x,y)+\frac{1}{2}\Big(\partial_x P^{\mathbf{B}}(\hbar,x,y)\Big)^2. \tag{7.8}$$

In particular by setting $x=y=0$, for $g>2$ the coefficient of $\hbar^{g-1}$ gives exactly (7.4). Let ∂ be the differential operator on the LHS of (7.5), we see it kills $V^{\mathbf{B}}$. By using similar argument, we deduce (7.5). $\square$

The proof of Theorem 7.1 indeed gives another algorithm which computes the genus g potential P_g recursively from the lower genus potentials, by using the modified Feynman rule (5.9). The advantage of this algorithm is that only four generators/propagators (instead of five) are involved, expressing P_g in simpler terms.

For any $g>1$, suppose the master potential is given by

$$f_g^{\mathbf{A},\vec{0}}=f_g^{\mathbf{B},\vec{0}}=f_g(X):=\textstyle\sum_{k=0}^{3g-3}f_{g,k}X^k,$$

then one can solve the genus g "normalized" GW potential P_g from the low genus by using (either NMSP or BCOV's, modified or original) graph sum formulae.

Example 7.4. In terms of the generators (7.1), a maple program gives

$$\begin{aligned}
P_2=\ &\frac{350\,\mathcal{E}_3}{9}+\frac{25\,\mathcal{E}_1\,\mathcal{E}_2}{6}+\frac{5\,\mathcal{E}_1^3}{24}+\frac{625\,\mathcal{E}_2}{36}+\frac{25\,\mathcal{E}_1^2}{24}+\frac{25\,X\,\mathcal{E}_2}{36}+\frac{X\,\mathcal{E}_1^2}{6}+\frac{13\,X^2\mathcal{E}_1}{288}+\frac{167\,X\,\mathcal{E}_1}{720}+\frac{625\,\mathcal{E}_1}{288}\\
&+f_2(X),\quad \text{with } f_2(X)=-\frac{1}{240}X^3-\frac{41}{3600}X^2+\frac{5759}{3600}X-\frac{25}{144}; \qquad (7.9)\\
P_3=\ &\frac{8225\mathcal{E}_3^2}{27}+\frac{275\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3}{3}+\frac{29375\mathcal{E}_2\mathcal{E}_3}{108}+\frac{185\mathcal{E}_1^3\mathcal{E}_3}{24}+\frac{575\mathcal{E}_1^2\mathcal{E}_3}{24}+\frac{29375\mathcal{E}_1\mathcal{E}_3}{864}-\frac{10450\mathcal{E}_2^3}{81}-\frac{3595\mathcal{E}_1^2\mathcal{E}_2^2}{72}\\
&-\frac{3575\mathcal{E}_1\mathcal{E}_2^2}{54}+\frac{14375\mathcal{E}_2^2}{288}-\frac{35\mathcal{E}_1^4\mathcal{E}_2}{3}-\frac{4075\mathcal{E}_1^3\mathcal{E}_2}{144}-\frac{8125\mathcal{E}_1^2\mathcal{E}_2}{432}+\frac{15625\mathcal{E}_1\mathcal{E}_2}{1728}-\frac{5\mathcal{E}_1^6}{4}-\frac{25\mathcal{E}_1^5}{6}-\frac{3125\mathcal{E}_1^4}{576}\\
&-\frac{15625\mathcal{E}_1^3}{5184}+X\cdot\Big(\frac{1175\mathcal{E}_2\mathcal{E}_3}{108}+\frac{39\mathcal{E}_1^2\mathcal{E}_3}{8}+\frac{7849\mathcal{E}_1\mathcal{E}_3}{2160}-\frac{1397\mathcal{E}_1\mathcal{E}_2^2}{54}+\frac{2773\mathcal{E}_2^2}{2160}-\frac{1687\mathcal{E}_1^3\mathcal{E}_2}{144}-\frac{16163\mathcal{E}_1^2\mathcal{E}_2}{1080}\\
&-\frac{21433\mathcal{E}_1\mathcal{E}_2}{8640}-\frac{23\mathcal{E}_1^5}{12}-\frac{3107\mathcal{E}_1^4}{720}-\frac{5893\mathcal{E}_1^3}{1728}-\frac{82091\mathcal{E}_1^2}{86400}\Big)+X^2\cdot\Big(\frac{611\mathcal{E}_1\mathcal{E}_3}{864}-\frac{1603\mathcal{E}_2^2}{864}-\frac{1897\mathcal{E}_1^2\mathcal{E}_2}{432}\\
&-\frac{4363\mathcal{E}_1\mathcal{E}_2}{2880}-\frac{731\mathcal{E}_1^4}{576}-\frac{14609\mathcal{E}_1^3}{8640}-\frac{51473\mathcal{E}_1^2}{86400}\Big)-X^3\cdot\Big(\frac{325\mathcal{E}_1\mathcal{E}_2}{576}+\frac{2305{\mathcal{E}_1}^3}{5184}+\frac{4337{\mathcal{E}_1}^2}{17280}\Big)-D^2P_2\cdot\frac{\mathcal{E}_1}{2}\\
&+DP_2\cdot\Big(\frac{19\mathcal{E}_2}{3}+\frac{\mathcal{E}_1^2}{2}+\frac{25\mathcal{E}_1}{12}-\frac{11X\,\mathcal{E}_1}{12}\Big)+\frac{47\mathcal{E}_3}{3}+\mathcal{E}_1\mathcal{E}_2+\frac{25\mathcal{E}_2}{6}+\frac{X^2\mathcal{E}_1}{12}+X\left(\frac{13\mathcal{E}_2}{2}+\frac{\mathcal{E}_1^2}{2}+\frac{19\mathcal{E}_1}{12}\right)\\
&+f_3(X),\quad \text{with } f_3(X)=\textstyle\sum_{i=1}^6 f_{3,i}X^i+\frac{125}{36288}. \qquad (7.10)
\end{aligned}$$

Here $f_{g,0}=(5)^{g-1}N_{g,0}$ is computed by using (A.1) and the ambiguity polynomial $f_2(X)$ is deduced from the lower degree GW invariants computed in Appendix A.

These formulae (7.9), (7.10) match the physicists' predictions [BCOV94, YY04] for the potential functions of the quintic 3-folds up to the "ambiguity" $\{f_{g=3,k}\}$.

Appendix A. Low degree GW-invariants

Recall $N_{g,d}$ are the genus g and degree d GW-invariants of quintic threefolds. The degree zero invariants are computed in [FP00] :

$$N_{g,0} = \frac{(-1)^g \cdot \chi \cdot |B_{2g}| \cdot |B_{2g-2}|}{2 \cdot 2g \cdot (2g-2) \cdot (2g-2)!} \tag{A.1}$$

In this appendix, we will show

Proposition A.1. *The low degree genus two GW-invariants are given by*

$$N_{1,1} = \frac{2875}{12}, \quad N_{2,1} = \frac{575}{48}, \quad N_{2,2} = \frac{5125}{2}, \quad \text{and} \quad N_{2,3} = \frac{7930375}{6}.$$

We let $Q \subset \mathbb{P}^4$ be a general quintic threefold. For a smooth curve $E \subset Q$, we denote by $N_{E/Q}$ the normal bundle of E in Q, and call E rigid if $h^0(N_{E/Q}) = 0$.

We let $f : C \to Q$ be a stable map from a genus 2 curve C to Q of degree $d \le 3$. We let $E = f(C)$ be the image curve.

Lemma A.2. *Let the notation be as stated. Then E either is a smooth rigid rational curve or a smooth rigid elliptic curve.*

Proof. Because $\deg f \le 3$, the image curve E has degree at most three. In case E is a union of rational curves, by [Kat86, JK08], E is irreducible, smooth and rigid.

Now suppose E contains an elliptic curve. As elliptic curves in $\mathbb{P}^4$ have degree at least 3, E is irreducible and has degree three. Thus E must be an irreducible component of $Q \cap L$, the intersection of Q with a plane $L \subset \mathbb{P}^4$. This way, $Q \cap L = E \cup E'$, where E' is rational and of degree 2. By the rigidity proved in [Kat86, JK08], there is no infinitesimal deformation of E' in Q. As E' determines L, there is no infinitesimal deformation of E in Q, thus $E \subset Q$ is rigid. □

We recall the following results from [FP00, Pan99]. We let $C_0(h, d)$ be the contribution to $N_{h,de}$ from a rigid degree e smooth rational curve $E \subset Q$. Then for any $d \ge 1$,

$$\sum_{h=0}^{\infty} C_0(h,1)\, t^{2h} = \Big(\frac{\sin(t/2)}{t/2}\Big)^{-2} \quad \text{and} \quad C_0(h,d) = d^{2h-3} C_0(h,1).$$

In particular, for any $d \ge 1$, $C_0(1,1) = \frac{1}{12d}$ and $C_0(2,d) = \frac{d}{240}$.

We let $C_1(h,1)$ be the contribution to $N_{1+h,e}$ from a rigid degree e smooth elliptic curve $E \subset Q$. Then

$$C_1(h,1) = 0.$$

Proof of Proposition A.1. By multiple cover formula of $N_{0,d}$, and the known $N_{0,d\le 3}$, we see that the general quintic Q has exactly $n_1 = 2,875$, $n_2 = 609,250$ and $n_3 = 317,206,375$ many degree one, two and three rational curves, all rigid, smooth, and mutually disjoint. Applying the proceeding arguments, we get

$$N_{2,1} = n_1 C_0(2,1), \quad N_{2,2} = n_1 C_0(2,2) + n_2 C_0(2,1), \quad N_{2,3} = n_1 C_0(2,3) + n_3 C_0(2,1).$$

Plugging the numbers, we get $N_{2,1} = \frac{575}{48}$, $N_{2,2} = \frac{5125}{2}$, and $N_{2,3} = \frac{7930375}{6}$. We obtain $N_{1,1} = \frac{2875}{12}$ for the same reason. □

Appendix B. Original forms of Feynman rules in the paper of BCOV

The original form of Feynman graphs in [BCOV94] took a slightly different shape of edges, with certain freedom of gauges. We present BCOV's original form, and the generalization with insertions in the original style in this section, for the readers who are more familiar with the B-model theory. We also give $g = 1$ and 2 examples in the original forms.

B.1. **Original statement of BCOV's Feynman rule.** In [BCOV94] the authors considered all g B-model topological partition function $\mathcal{F}_g^W(q,\bar{q})$ for an arbitrary compact Calabi-Yau threefold W. Its definition uses path integral, and it is a non-holomorphic extension of the GW potential $F_g^M(q)$ of the mirror Calabi-Yau threefold M of W:

$$\lim_{\bar{q}\to 0} \mathcal{F}_g^W(q,\bar{q}) = F_g^M(q). \tag{B.1}$$

One of the primary result in [BCOV94] is that $\mathcal{F}_g^W$ satisfies "holomorphic anomaly equation"(HAE). When W is a one-dimensional mirror family, the equation is

$$\partial_{\bar{q}}\mathcal{F}_g^W(q,\bar{q}) = \frac{1}{2}C_{\bar{q}}^{qq}\Big(D_q^2\mathcal{F}_{g-1}^W(q,\bar{q}) + \sum_{g_1+g_2=g} D_q\mathcal{F}_{g_1}^W(q,\bar{q})D_q\mathcal{F}_{g_2}^W(q,\bar{q})\Big),$$

where D_q is certain covariant derivative and $C_{\bar{q}}^{qq}$ is certain three point function (Yukawa coupling) that can be calcuated by B side special geometry. Using integrations by parts, [BCOV94, Sect. 6] solves HAE and express its solutions $\mathcal{F}_g^W$ via Feynman rules. We state here the BCOV's Feynman rules for the limit F_g^M (B.1), with M being the quintic 3-fold.

BCOV's Feynman graph: *For any $g>1$, we consider the set G_g^{BCOV} of genus g stable graphs with three types of edges: solid lines; half dotted half solid lines, and dotted lines. For each graph Γ, we do the following:*

Edge*: at each edge drawn as solid lines, half dotted lines and dotted lines, we place one of the opagators $(T^{\varphi\varphi}, T^{\varphi}, T)$ defined in* (1.5) *respectively;*

Vertex*: at each vertex of genus g, with m solid half edges and n dotted half edges, we place $P_{g,m,n}$ (defined in* (0.4)*).*

We define Cont_Γ *to be the product of the edge and the vertex placements; and define*

$$f_g^{\mathrm{BCOV}} := \sum_{\Gamma\in G_g^{\mathrm{BCOV}}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\operatorname{Cont}_\Gamma .$$

Conjecture B.1. *For $g>1$, f_g^{BCOV} is a degree $3g-3$ polynomial in X.*

This original BCOV's rule can be generalized to allow legs:

BCOV's Feynman graph with legs: *We consider the set $G_{g,n}^{\mathrm{BCOV}}$ of genus g, n-leg stable graphs with three types of edges (as above) and two types of legs: solid half lines and dotted half lines. Besides what we do for edges and vertices as above, furthermore*

Leg*: at each leg, we place one of the following* 2*-types of propogators*

$$E_\varphi := 1, \quad \textit{and} \quad -E_\psi^{c_{1a}} := -B - c_{1a} \tag{B.2}$$

according to the types of the edge: φ goes with solid half line and ψ goes with half dotted line. Here c_{1a} can be any polynomial of X with degree no more than 1*.*

We define Cont_Γ *to be the product of the legs, edges and vertices placements, and define*

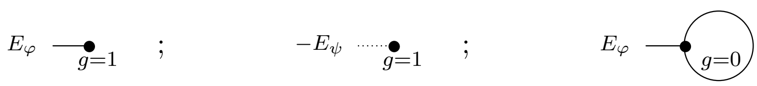
$$f_{g,n}^{\mathrm{BCOV}} := \sum_{\Gamma\in G_{g,n}^{\mathrm{BCOV}}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\operatorname{Cont}_\Gamma .$$

Conjecture B.2. *For $2g-2+n>0$, $f_{g,n}^{\mathrm{BCOV}}$ is a degree $3g-3+n$ polynomial in X.*

By setting $m=0$ and picking the gauge $(c_{1b},c_2,c_3)=(\frac{3}{5},-\frac{2}{25},-\frac{4}{125})$ in Theorem 1, we recover the statement in Conjecture B.2; furthermore by setting $n=0$ we recover the statement in Conjecture B.1.

B.2. **Example of BCOV's original Feynman rule.** We illustrate how BCOV's Feynman rules compute genus g GW potential from lowers genus GW potentials.

Example B.3 $(g=1,n=1)$**.** In this case, the BCOV's Feynman graphs are



.

BCOV's rule gives us (note by definition (0.4), $P_{1,0,1} = \frac{\chi}{24} - 1$.)

$$E_\varphi \cdot P_{1,1} + (-E_\psi) \cdot P_{1,0,1} + \frac{1}{2} E_\varphi \cdot T^{\varphi\varphi} \cdot P_{0,3} = f_{1,1}^{\mathrm{BCOV}}(X) \ \in \ \mathbb{Q}[X]_1. \tag{B.3}$$

By using the initial data $N_{1,0}, N_{1,1}$ (see Appendix A), and setting $c_{1a} = 0$ we obtain

$$f_{1,1}^{\mathrm{BCOV}}(X) = -\tfrac{1}{12}X - \tfrac{107}{60} = P_{1,1} - \tfrac{28}{3} \cdot (-B) + \tfrac{1}{2}\Big(A + 2B + \tfrac{3}{5}\Big).$$

Hence we solve $P_{1,1}$ that matches Zinger's formula [Zi09] (also c.f. [KL18], [CGLZ18])

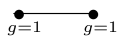
$$P_{1,1} = -\tfrac{1}{2}A - \tfrac{31}{3}B - \tfrac{1}{12}X - \tfrac{25}{12}. \tag{B.4}$$

Example B.4 ($g = 2, n = 0$)**.** In this case, the BCOV's Feynman rule becomes

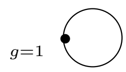
$$\begin{aligned} &P_2 + \frac{1}{2} T^{\varphi\varphi} P_{1,1}^2 + \frac{1}{2} T^{\varphi\varphi} P_{1,2} + \frac{1}{2} (T^{\varphi\varphi})^2 P_{1,1} + \frac{1}{8} (T^{\varphi\varphi})^2 P_{0,4} + \frac{1}{8} (T^{\varphi\varphi})^3 + \frac{1}{12} (T^{\varphi\varphi})^3 \\ &\quad + \frac{\chi}{24} T^{\varphi} P_{1,1} + \frac{1}{2} \frac{\chi}{24} T^{\varphi} T^{\varphi\varphi} + \frac{1}{2} \frac{\chi}{24} (\frac{\chi}{24} - 1) T \ = \ f_2^{\mathrm{BCOV}}(X) \ \in \ \mathbb{Q}[X]_3, \end{aligned} \tag{B.5}$$

accroding to the BCOV's Feynman graphs listed below:

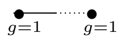
$P_2\,,$



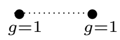
$\frac{1}{2} P_{1,1}^2 \cdot T^{\varphi\varphi}\,,$



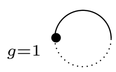
$\frac{1}{2} P_{1,2} \cdot T^{\varphi\varphi}\,,$



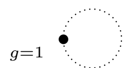
$P_{1,1} \cdot T^{\varphi} \cdot \Big(\frac{\chi}{24} - 1\Big)\,,$



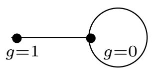
$\frac{1}{2} \Big(\frac{\chi}{24} - 1\Big)^2 \cdot T\,,$



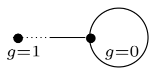
$P_{1,1} \cdot T^{\varphi}\,,$



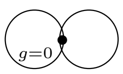
$\frac{1}{2} \Big(\frac{\chi}{24} - 1\Big) \cdot T\,,$



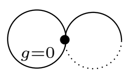
$\frac{1}{2} P_{1,1} \cdot (T^{\varphi\varphi})^2 \cdot P_{0,3}\,,$



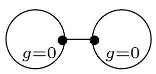
$\frac{1}{2} \Big(\frac{\chi}{24} - 1\Big) \cdot T^{\varphi} \cdot P_{0,3} \cdot T^{\varphi\varphi}\,,$



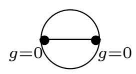
$\frac{1}{8} P_{0,4} \cdot (T^{\varphi\varphi})^2\,,$



$\frac{1}{2} P_{0,3} \cdot T^{\varphi\varphi} \cdot T^{\varphi}\,,$



$\frac{1}{8} P_{0,3}^2 \cdot (T^{\varphi\varphi})^3$



$\frac{1}{12} P_{0,3}^2 \cdot (T^{\varphi\varphi})^3$

The list of stable $g = 2$ decorated graphs, thirteen of them.

By using the genus 1 formula (B.4), the divisor equation $P_{1,2} = (D - A)P_{1,1}$, together with the initial data $N_{2,0}, N_{2,1}, N_{2,2}, N_{2,3}$[38], one obtains

$$f_2^{\mathrm{BCOV}}(X) = -\tfrac{1}{240}\, X^3 + \tfrac{113}{7200}\, X^2 + \tfrac{487}{300}\, X - \tfrac{11771}{7200}.$$

Hence one solves from (B.5)

$$\begin{aligned} -P_2 = &-\frac{350\,B_3}{9} - \Big(\frac{25\,A}{6} + \frac{425\,B}{9} + \frac{625}{36}\Big) B_2 + \frac{5\,A^3}{24} + \frac{65\,A^2 B}{12} + \frac{1045\,AB^2}{18} + \frac{865\,B^3}{9} \\ &+ \frac{25}{144} + \Big(\frac{A^2}{6} + \frac{49\,AB}{36} + \frac{167\,A}{720} + \frac{37\,B^2}{18} - \frac{1811\,B}{120} - \frac{475\,B_2}{12} - \frac{5759}{3600}\Big) X \\ &+ \frac{25\,A^2}{24} + \frac{775\,AB}{36} + \frac{350\,B^2}{9} + \frac{625}{288}\,(A + 2B) + \Big(\frac{13\,A}{288} + \frac{13\,B}{144} + \frac{41}{3600}\Big) X^2 + \frac{X^3}{240}. \end{aligned}$$

This is exactly the formula in Theorem 0.4. Here we just rewrite the propagators in terms of Yamaguchi-Yau's generators via (1.5).

[38] These are originally conjectured by physicists by using some "boundary" behavior of F_g. A mathematical computation of them is put in Appendix A.

Appendix C. Remarks on the R-matrix actions on CohFTs

C.1. **Unit axiom.** We prove that the R-matrix action preserves the unit axiom if it is invertible, as stated in Theorem 2.7.

Lemma C.1. *Let Ω be a CohFT with the triple $(V,\eta,\mathbf{1})$. We consider another triple $(V',\eta',\mathbf{1}')$ with $\dim_F V'=\dim_F V$, and a symplectic transformation $R(z)\in\mathrm{End}(V,V')\otimes\mathbb{A}[\![z]\!]$ acting on Ω. We have*

$$R.\Omega_{0,3}(\mathbf{1}',\alpha,\beta)=\eta'(\alpha,\beta).$$

Proof. By definition of the R-matrix action, we have

$$R.\Omega_{0,3}(\mathbf{1}',\alpha,\beta)=\sum_{k\geq 0}\frac{1}{k!}(\mathrm{pr}_k)_*\Omega_{0,3+k}\Big(R_0^{-1}\mathbf{1}',R_0^{-1}\alpha,R_0^{-1}\beta,(T_0\psi)^k\Big),$$

where $T(z):=z\,\mathbf{1}-R^{-1}(z)\mathbf{1}'=T_0z+O(z^2)$, and hence the second equality holds simply for dimensional reason.

Let ω be the topological part of Ω (i.e. the part of degree zero classes, c.f. [PPZ15]). By the axiom of CohFT, it is uniquely determined by the quantum product. In particular,

$$\omega_{0,n+2}(\tau_{\mathbf{n}},\beta_1,\beta_2)=\textstyle\sum_\alpha\omega_{0,n+1}(\tau_{\mathbf{n}},e_\alpha)\cdot\omega_{0,3}(e^\alpha,\beta_1,\beta_2)=\omega_{0,n+1}(\tau_{\mathbf{n}},\beta_1*\beta_2),$$

where we have used the spliting axiom in the first equality and the definition of the quantum product in the second equality. Hence

$$\begin{aligned}R.\Omega_{0,3}(\mathbf{1}',\alpha,\beta)&=\textstyle\sum_{k\geq 0}\frac{1}{k!}\omega_{0,3+k}\Big(\,R_0^{-1}\mathbf{1}',\,R_0^{-1}\alpha,\,R_0^{-1}\beta,(\,T_0)^k\Big)(\mathrm{pr}_k)_*(\psi_4\cdots\psi_{k+3})\\&=\omega_{0,3}\Big(\textstyle\sum_k R_0^{-1}\mathbf{1}'*(\,T_0)^{*k},R_0^{-1}\alpha,R_0^{-1}\beta\Big)\\&=\omega_{0,3}\Big(\mathbf{1},R_0^{-1}\alpha,R_0^{-1}\beta\Big)=\Big(R_0^{-1}\alpha,R_0^{-1}\beta\Big),\end{aligned}$$

where we have used $\int_{\overline{\mathcal{M}}_{0,3+k}}\psi_1\cdots\psi_k=k!$ and $\sum_{k\geq 0}R_0^{-1}\mathbf{1}'*(T_0)^{*k}=(\mathbf{1}-T_0)*\sum_{k\geq 0}(T_0)^{*k}=\mathbf{1}$ in the third equality; and the fundamental class axiom in the last equality. Furthermore, since R_0 is invertible, R_0^{-1} is symplectic as well, hence we finish the proof. ☐

C.2. **Dilaton flow.** Let Ω be an arbitrary CohFT with triple $(V,\eta,\mathbf{1})$ and the coefficient adic ring $\mathbb{A}=F[\![q]\!]$. Let $R(z)\in\mathrm{End}(V,V')\otimes\mathbb{A}[\![z]\!]$ be symplectic.

We consider an arbitrary nonzero “scaling constant” $c\in 1+q\mathbb{A}$, and we let

$$\tilde{R}^{-1}(z)=c^{-1}R^{-1}(z)\quad\text{and}\quad\tilde{T}(z)=z(\mathbf{1}-\tilde{R}^{-1}(z)\mathbf{1}'). \tag{C.1}$$

For any $2g-2+n>0$, using $\mathrm{pr}_{1*}\psi_{n+1}=2g-2+n$, we have

$$\begin{aligned}T_R\Omega_{g,n}(-)&=\textstyle\sum_{k\geq 0}\frac{1}{k!}\mathrm{pr}_{k*}\Omega_{g,n+k}(-,T(\psi)^k)\\&=\textstyle\sum_{\ell,m\geq 0}\frac{1}{\ell!m!}\mathrm{pr}_{\ell+m*}\Omega_{g,n+\ell+m}(-,[(1-c)\psi\mathbf{1}]^\ell,[c\tilde{T}(\psi)]^m)\\&=\textstyle\sum_{m\geq 0}\frac{1}{m!}c^{-(2g-2+n)}\mathrm{pr}_{m*}\Omega_{g,n+m}\Big((-),\tilde{T}(\psi)^m\Big).\end{aligned} \tag{C.2}$$

We see that, usually if (2.2) converges , then (C.2) converges as well. For example, if (2.3) holds , then $\tilde{T}(z)$ also lies in $z^2\mathbb{A}[\![z]\!]\otimes V+q\,z\,\mathbb{A}[\![z]\!]\otimes V$. Then $c-1\in q\mathbb{A}$ makes the infinite sum converges in the q-adic topology.

In the end, we give an example that how the Dilaton flow relates the R-matrix actions with general R_0 to the one defined in [PPZ15] for the semi-simple cases.

Example C.2. For a semi-simple CohFT $\Omega_{g,n}$, we can state Givental-Teleman's reconstruction theorem in a slightly different form: there exists an R-matrix such that

$$\Omega=R.(\Omega_{\mathrm{pt}}^{\oplus n}),\qquad R=R_0+R_1z+\cdots\in\mathrm{End}\,F^n\otimes\mathbb{A}[\![z]\!]$$

where the state space of Ω is still F^n as a linear space; the unit of Ω is also the same one:

$$\mathbf{1}:=\textstyle\sum_{\alpha=1}^n e_\alpha,\qquad e_\alpha\text{ is the unit of each copy of }I_{\mathrm{pt}};$$

and the pairing is different in general, which we will denote by $(\cdot,\cdot)^{\mathrm{tw}}$.

Indeed, since we require R to be symplectic, the pairing of Ω is indeed determined by R_0 (note the pairing in $\Omega_{\mathrm{pt}}^{\oplus n}$ is the standard pairing of F^n) . Let $c_\alpha := (e_\alpha, R_0^{-1}\mathbf{1})$, and

$$\Psi := \mathrm{diag}(\{c_\alpha^{-1}\}_{\alpha=1}^n), \qquad \bar{e}_\alpha := \Psi e_\alpha = c_\alpha^{-1} e_\alpha.$$

Then the inner product of Ω is given by

$$(e_\alpha, e_\beta)^{\mathrm{tw}} := \delta_{\alpha\beta} c_\alpha^2 \qquad \text{or} \qquad (\bar{e}_\alpha, \bar{e}_\beta)^{\mathrm{tw}} := \delta_{\alpha\beta}.$$

We define the normalized R-matrix via

$$\tilde{R}(z) = R(z)\Psi^{-1} = \mathrm{I} + O(z),$$

which is indeed the R-matrix defined in [PPZ15]. By using Dilaton flow, one checks

$$\Omega \;=\; R.\big(\Psi.(I_{\mathrm{pt}}^{\oplus n})\big) \;=\; R.\omega,$$

where the Ψ-matrix transforms the trivial CohFT $I_{\mathrm{pt}}^{\oplus n}$ with standard pairing, to the topological part ω of Ω with the twisted pairing $(,)^{\mathrm{tw}}$

$$\omega_{g,n}(e_{\alpha_1}, \cdots, e_{\alpha_n}) = \delta_{\alpha_1,\cdots,\alpha_n} c_\alpha^{-(2g-2)}.$$

Here $\delta_{\alpha_1,\cdots,\alpha_n} = 1$ if $\alpha_1 = \cdots = \alpha_n$, otherwise it is zero.

Appendix D. Explicit formulae for R-matrices

First we give the explicit formulae for the leading terms of $R^{[0]}$. We hope they can make the arguments in the §3 and §4 more clear, though we do not really use them in our proof.

Lemma D.1. *We have the following explicit formula (with understanding that $R^{[0]}(z)$ is identity operator on odd classes* [39]*)*

$$
\begin{aligned}
R^{[0]}(z)^* = &\begin{pmatrix} I_0 & & & \\ & I_0 I_{11} & & \\ & & I_0 I_{11} I_{22} & \\ & & & I_0 I_{11}^2 I_{22} \end{pmatrix} \cdot \Bigg[\left(\begin{array}{ccccccccccccc}
1 & & & & & & & & & -120q & & & \\
 & 1 & & & & & & & & & -890q & & \\
 & & 1 & & & & & & & & & -2235q & \\
 & & & 1 & & & & \cdots & & & & & -3005q
\end{array}\right) \\
&+ z \left(\begin{array}{ccccccccccccc}
0 & B & & & & & & & & \begin{smallmatrix}-q(890B\\+120)\end{smallmatrix} & & & \\
 & 0 & A+2B & & & & & & & & \begin{smallmatrix}-2235q(A+2B)\\-1010q\end{smallmatrix} & & \\
 & & 0 & B-X & & & & & & & & \begin{smallmatrix}-3005q(B-X)\\-120q\end{smallmatrix} & \\
 & & & 0 & -2X & & & \cdots & \cdots & & & &
\end{array}\right) \\
&+ z^2 \left(\begin{array}{ccccccccccccc}
0 & 0 & B_2 & & & & & & & \begin{smallmatrix}-5q\,(447B_2+\\202B+24)\end{smallmatrix} & & & \\
 & 0 & 0 & \begin{smallmatrix}(A+2B)(B-X)\\-B_2-\frac{2}{5}X\end{smallmatrix} & & & & & & & \begin{smallmatrix}3005q\,\big(B_2+\frac{2}{5}X-\frac{226}{601}\\-(A+2B)(B+\frac{649}{601}-X)\big)\end{smallmatrix} & & \\
 & & 0 & 0 & \begin{smallmatrix}-(2B-2X\\+\frac{7}{5})X\end{smallmatrix} & & & & & & & & \\
 & & & 0 & 0 & -6X+\frac{17}{5} & & \cdots & \cdots & & & &
\end{array}\right) \\
&+ z^3 \left(\begin{array}{ccccccccccccc}
0 & 0 & 0 & B_3 & & & & & & \begin{smallmatrix}-5q\,(601B_3+649B_2\\+226B+24)\end{smallmatrix} & & & \\
 & 0 & 0 & 0 & \begin{smallmatrix}2(A+2B)(B-X+\frac{7}{10})\\-2B_2-\frac{4}{5}X+\frac{2}{5}\end{smallmatrix} & & & & & & & & \\
 & & 0 & 0 & 0 & \begin{smallmatrix}-6(B-X+\frac{7}{10})X\\+\frac{17}{5}(B-X)+\frac{9}{5}\end{smallmatrix} & & & & & & & \\
 & & & 0 & 0 & 0 & \begin{smallmatrix}\frac{2}{5}(60X^2\\-66X+13)\end{smallmatrix} & \cdots & \cdots & & & &
\end{array}\right) \Bigg] \\
&+ O(z^4),
\end{aligned}
\tag{D.1}
$$

under the basis $\{\phi_j\}_{j=0}^{\mathrm{N}+3}$ and $\{H^i\}_{i=0}^3$, where $\cdots$ are all zeros.

Proof. By using the QDE (3.10) of $R^{[0]}(z)$ and the initial data $R^{[0]}(z)^*\mathbf{1}$ in (3.9), $R^{[0]}(z)$ can be computed recursively. A direct computation shows this lemma. □

Next, we give the explicit formulae for the leading terms of $R^X(z)$, as defined in (4.1).

[39] in this paper all operators from $\mathcal{H}_Q$ to $\mathcal{H}$ or conversely $\mathcal{H}$ to $\mathcal{H}_Q$ are assumed to be identity on odd classes. Thus we only describe their action on even classes.

Lemma D.2. *We have the following explicit form of $R^X(z)^* : \mathcal{H} \to \mathcal{H}_Q[\![z]\!] \otimes A$ in terms of basis $\{\phi_j\}$ for $\mathcal{H}$ and $\{\varphi_i\}_{i=0}^3$ respectively:*

$$R^X(z)^* = \left(\begin{array}{cccccccccc} 1 & & & & & & -120q & & & \\ & 1 & & & & & & -890q & & \\ & & 1 & & & & & & -2235q & \\ & & & 1 & \cdots & \cdots & & & & -3005q \end{array}\right) \tag{D.2}$$
$$+ z \cdot \left(\begin{array}{lllllllll} 0 & 0 & & & & \cdots & -120q & & \\ & 0 & 0 & & & \cdots & & -1010q & \\ & & 0 & -X & & \cdots & & & 5q(601X-649) \\ & & & 0 & -2X & \cdots & & & \end{array}\right)$$
$$+ z^2 \cdot \left(\begin{array}{lllllllll} 0 & 0 & 0 & & & & \cdots & -120q & \\ & 0 & 0 & -\frac{2}{5}X & & & \cdots & & -2q(601X-565) \\ & & 0 & 0 & (10X-7)\cdot X/5 & & \cdots & & \\ & & & 0 & 0 & (30X-17)\cdot X/5 & \cdots & & \end{array}\right)$$
$$+ z^3 \cdot \left(\begin{array}{lllllllll} 0 & 0 & 0 & 0 & & & & \cdots & -120q \\ & 0 & 0 & 0 & (2X-1)\cdot 2X/5 & & & \cdots & \\ & & 0 & 0 & 0 & -(30X^2-38Y+9)\cdot X/5 & & \cdots & \\ & & & 0 & 0 & 0 & -2X(60X^2-66X+13)/5 & \cdots & \end{array}\right) + O(z^4),$$

where $\cdots$ are all zeros.

Proof. The R^X-matrix can be computed by using the algorithm introduced in the proof of Lemma 4.3, which starts from $R^X(z)^*\phi_0 = 1 + O(z^{\mathrm{N}-3})$ and computes $R^X(z)^*\phi_j (j > 0)$ recursively by using the equation (4.3). □

Corollary D.3. *Recall $V_X(z,w) := \frac{\sum_{i=0}^3 \varphi_i \otimes \varphi^i - \sum_{j=0}^{\mathrm{N}+3} R^X(-z)^*\phi_j \otimes R^X(-w)^*\phi^j}{z+w}$, we have*

$$Y \cdot V_X(z,w) = \tag{D.3}$$
$$\begin{pmatrix} -\frac{\left(24\,w^2-24\,zw+24\,z^2\right)X}{625} & -\frac{(24\,w-24\,z)X}{625} & -\frac{24\,X}{625} & 0 \\ \frac{(24\,w-24\,z)X}{625} & -\frac{202\,X}{625} & 0 & 0 \\ -\frac{24\,X}{625} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sum_i O(z^i)O(w^{\mathrm{N}-i}).$$

Proof. By Lemma 4.3, the coefficients of $z^i w^j$ for $i + j < \mathrm{N}$ is non-zero only when $i + j \le 3$. Then the matrix can be computed directly by (D.2). □

List of Symbols

N	a prime that will be taken large
t_α	$t_\alpha = -\zeta_{\mathrm{N}}^\alpha t$ for $\alpha = 1, \cdots, \mathrm{N}$, where ζ_{N} is the primitive N-th root of unity
p	the equivariant hyperplane class $c_1(\mathcal{O}_{\mathbb{P}^{4+\mathrm{N}}}(1))$
H	the hyperplane class of the quintic 3-fold Q
$\mathbb{F}$, $\mathbb{A}$	the base field $\mathbb{F} = \mathbb{Q}(t)$ and coefficient ring $\mathbb{A} = \mathbb{Q}(t)[\![q]\!]$ for all CohFTs
$\tau_{\mathbf{n}}$	we abbreviate $\tau_{\mathbf{n}} := (\tau_1, \cdots, \tau_n)$
Y, X	rational functions of q: $Y = (1 - 5^5 q)^{-1}$, $X := 1 - Y$
I_{ii}	Let $D := q\frac{d}{dq}$, then $I_{11} = I_{33} = 1 + DJ_1$, $I_{22} = Y/I_0^2 I_{11}^2$
A_k,B_k	Yamaguchi-Yau's generators (1.1), we abbreviate $A := A_1$, $B := B_1$
$\mathcal{G}$	the gauge $\mathcal{G} := (c_{1a}, c_{1b}, c_2, c_3)$ satisfying (1.4), as a group action it is (4.16)
$E_{**}^{\mathcal{G}}$	BCOV's propogators (1.3) and extra propagators (1.8)
$\tilde{E}_{**}$	modified propogators defined in (5.8) and (6.8) (see Convention 5.1)
ϕ_i	the basis $\{\phi_i := p^i\}_{i=0}^{\mathrm{N}+3}$ of $\mathcal{H}$ with dual basis $\{\phi^i\}_{i=0}^{\mathrm{N}+3}$.
φ_i	the normalized basis $\{\varphi_i := I_0 \cdots I_{ii} H^i\}$ of $\mathcal{H}_Q$ with dual basis $\{\varphi^i\}_{i=0}^3$
ψ_k	in this paper ψ_k always denote ancestors, i.e. pullback of psi-classes in $\overline{\mathcal{M}}_{g,n}$

$\mathbb{Q}[X]_d$	the set of polynomials in X of degree $\le d$, by definition $\mathbb{Q}[X]_d = \mathbb{Q}[Y]_d$
$H_{\mathbf{A}}, H_{\mathbf{B}}$	the state spaces: $H_{\mathbf{A}} := \mathrm{span}\{\varphi_0, \cdots, \varphi_3\}[\psi]$ and $H_{\mathbf{B}} := \mathrm{span}\{\psi, \varphi\} \subset H_{\mathbf{A}}$
$R^{\mathbf{A}}, R^{\mathbf{B}}$	the symplectic transformation defined in (1.7) and (5.1)
$P^{\mathbf{A}}, P^{\mathbf{B}}$	the type $\mathbf{A}, \mathbf{B}$ "normalized" generating function for the quintic theory (1.13)
$f^{\mathbf{A}}, f^{\mathbf{B}}$	the generating function for the type $\mathbf{A}, \mathbf{B}$ "master" theory (1.9),(1.6)
$\aleph$	the union of quintic and N points: $\aleph = Q \cup \mathrm{Npt}$, where $\mathrm{Npt} := \cup_{\alpha=1}^{\mathrm{N}}\{\mathrm{pt}_\alpha\}$
$\mathcal{H}$	the state space $H^*(\aleph, \mathbb{Q})$ with unit $\mathbf{1}$ and pairing $(\ ,\ )^M$ (c.f. Def. 2.14)
Ω^X	the CohFT defined by the Gromov-Witten class of a projective variety X
ω^X	the topological part of the CohFT Ω^X (restriction to $H^0(\overline{\mathcal{M}}_{g,n})$)
$\Omega^\aleph$	the CohFT $\Omega^\aleph := \Omega^{Q,\mathrm{tw}} \oplus \omega^{\mathrm{Npt},\mathrm{tw}}$, where $\Omega^{\bullet,\mathrm{tw}}$ are CohFTs of certain twisted theories which naturally appear in the localization (c.f. Sect. 2.5)
$\Omega^{\mathbf{A},\mathcal{G}}$	the CohFT $\Omega^{\mathbf{A},\mathcal{G}} := R^{\mathbf{A},\mathcal{G}}.\Omega^{Q,\mathrm{tw}}$ of type $\mathbf{A}$ "master" theory
$R(z)$	the R-matrix action (3.2), which transform $\Omega^\aleph$ to the $[0,1]$-CohFT $\Omega^{[0,1]}$
$R^{[0]}, R^{[1]}$	the restriction of the R-matrix to $\mathcal{H}_Q$, $\mathcal{H}_{\mathrm{Npt}}$ respectively
$\Omega^{[0]}, \Omega^{[1]}$	the CohFT defined via $R^{[0]}$, $R^{[1]}$-action : $\Omega^{[0]} := R^{[0]}.\Omega^{Q,\mathrm{tw}}$, $\Omega^{[1]} := R^{[1]}.\omega^{\mathrm{Npt},\mathrm{tw}}$
$f^{[0]}, f^{[1]}$	the generating function for the $[0],[1]$-theory (Def. 3.11)
pr_k	the map $\overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ defined by forgetting last k markings
$\mathrm{pr}_{g,n}^W$	the natural projection $\mathcal{W}_{g,n,\mathbf{d}} \to \overline{\mathcal{M}}_{g,n}$
$\mathrm{pr}_{g,n}^Q$	the natural projection $\overline{\mathcal{M}}_{g,n}(Q,d) \to \overline{\mathcal{M}}_{g,n}$

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