Differential Geometry and its Applications 66 (2019) 109–154

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Generalized Ricci flow II: Existence for complete noncompact manifolds

Yi Li [1]

School of Mathematics and Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing 211189, China

ARTICLE INFO

Article history:
Received 11 January 2018
Received in revised form 25 January 2019
Accepted 30 May 2019
Available online xxxx
Communicated by F. Fang

MSC:
primary 54C40, 14E20
secondary 46E25, 20C20

Keywords:
Steady gradient Ricci solitons
Geometric flow
Ricci-flat metrics

ABSTRACT

In this paper, we continue to study the generalized Ricci flow. We give a criterion on steady gradient Ricci soliton on complete and noncompact Riemannian manifolds that is Ricci-flat, and then introduce a natural flow whose stable points are Ricci-flat metrics. Modifying the argument used by Shi and List, we prove the short time existence and higher order derivatives estimates.

1. Introduction

Ricci-flat metrics play an important role in geometry and physics. For compact Kähler manifold with trivial first Chern class, the existence of a (Kähler) Ricci-flat metric was proved by Yau in his famous paper [28] on the Calabi conjecture. In the Riemannian setting, Ricci-flat metrics are stationary solutions of the Ricci flow introduced by Hamilton [13] as a powerful tool, together with Perelman's breakthrough [22–24], to study the Poincaré conjecture.

In the study of the singularities of the Ricci flow, Ricci solitons naturally arises as the self-similar solutions. From the definition, a Ricci-flat metric is indeed a Ricci soliton.

E-mail address: yilicms@gmail.com.

[1] Yi Li is partially supported by "the Fundamental Research Funds for the Central Universities" (China) No. 3207019208.

https://doi.org/10.1016/j.difgeo.2019.05.010

1.1. Compact steady gradient Ricci solitons

In particular, we consider a steady gradient Ricci soliton which is a triple (M, g, f), where M is a smooth manifold, g is a Riemannian metric on M and f is a smooth function, such that

$$\mathrm{Ric}_g + \nabla_g^2 f = 0 \quad \text{or} \quad R_{ij} + \nabla_i \nabla_j f = 0. \tag{1.1}$$

Hamilton [15] showed that on a compact manifold any steady gradient Ricci soliton must be Ricci-flat; this, together with Perelman's result [22] that any compact Ricci soliton is necessarily a gradient Ricci soliton, implies that any compact steady Ricci soliton must be Ricci-flat (cf. [5,22])

1.2. Complete noncompact steady Ricci solitons

Now we suppose (M, g, f) is a complete noncompact steady gradient Ricci soliton. The simplest example is Hamilton's cigar soliton or Witten's black hole ([9,11]), which is the complete Riemann surface $(\mathbf{R}^2, g_{\mathrm{cs}})$ where $g_{\mathrm{cs}} := (dx \otimes dx + dy \otimes dy)/(1+x^2+y^2)$. If we define $f(x, y) := -\ln(1+x^2+y^2)$, then $\mathrm{Ric}_{g_{\mathrm{cs}}} + \nabla^2_{g_{\mathrm{cs}}} f = 0$. The cigar soliton is rotationally symmetric, has positive Gaussian curvature, and is asymptotic to a cylinder near infinity; moreover, up to homothety, the cigar soliton is the uniques rotationally symmetric gradient Ricci soliton of positive curvature on $\mathbf{R}^2$ (cf. [9,11]). The classification of two-dimensional complete compact steady gradient Ricci solitons was achieved by Hamilton [15], which states that Any complete noncompact steady gradient Ricci soliton with positive Gaussian curvature is indeed the cigar soliton.

The cigar soliton can be generalized to a rotationally symmetric steady gradient Ricci soliton in higher dimensions on $\mathbf{R}^n$. The resulting solitons are referred to be Bryant's solitons (see [11] for the construction), which is rotationally symmetric and has positive Riemann curvature operator. Other examples of steady gradient Ricci solitons were constructed by Cao [4] and Ivey [16].

For three-dimensional case, Perelman [22] conjectured a classification of complete noncompact steady gradient Ricci soliton with positive sectional curvature which satisfies a non-collapsing assumption at infinity. Namely, a three-dimensional complete and noncompact steady gradient Ricci soliton which is nonflat and κ-noncollapsed, is isometric to the Bryant soliton up to scaling. Under some extra assumptions, it was proved in [1,6,7]. A complete proof was recently achieved by Brendle [2] and its generalization can be found in [3].

Another important result is Chen's result [8] saying that any complete noncompact steady gradient Ricci soliton has nonnegative scalar curvature. For certain cases, the lower bounded for the scalar curvature can be improved [10,12]. When the scalar curvature of a complete steady gradient Ricci soliton achieves its minimum, Petersen and William [25] proved that such a soliton must be Ricci-flat. On the other hand, if a complete noncompact steady gradient Ricci soliton has positive Ricci curvature and its scalar curvature achieves its maximum, then it must be diffeomorphic to the Euclidean space with the standard metric ([15,5]); in particular, in this case, such a soliton is Ricci-flat.

To remove the curvature condition, we can prove the following

Proposition 1.1. *Suppose M is a compact or complete noncompact manifold of dimension n. Then the following conditions are equivalent:*

(i) *there exists a Ricci-flat Riemannian metric on M;*
(ii) *there exist real numbers α, β, a smooth function ϕ on M, and a Riemannian metric g on M such that*

$$0 = -R_{ij} + \alpha \nabla_i \nabla_j \phi, \quad 0 = \Delta_g \phi + \beta |\nabla_g \phi|_g^2. \tag{1.2}$$

The proof is given in subsection 2.1.

Remark 1.2. In the compact case, the second condition in (1.2) can be removed. However, in the complete noncompact case, the second condition in (1.2) is necessarily. For example, the cigar soliton is a steady gradient Ricci soliton with nonzero scalar curvature $4/(1+x^2+y^2)$.

The equation (1.2) suggests us to study the parabolic flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha\nabla^2_{g(t)}\phi(t), \quad \partial_t\phi_t = \Delta_{g(t)}\phi(t) + \beta\left|\nabla_{g(t)}\phi(t)\right|^2_{g(t)}. \tag{1.3}$$

The system (1.3) is similar to the gradient flow of Perelman's entropy functional $\mathcal{W}$ [22]. Let $\mathfrak{Met}(M)$ denote the space of smooth Riemannian metrics on a compact smooth manifold M of dimension m. We define Perelman's entropy functional $\mathcal{W}: \mathfrak{Met}(M)\times C^\infty(M)\times \mathbf{R}^+ \longrightarrow \mathbf{R}$ by

$$\mathcal{W}(g,f,\tau) := \int_M \left[\tau\left(R_g + |\nabla_g f|^2_g\right) + f - m\right]\frac{e^{-f}}{(4\pi\tau)^{m/2}}\,dV_g, \tag{1.4}$$

where dV_g stands for the volume form of g. Perelman's showed that the gradient flow of (1.4) is

$$\begin{aligned} &\partial_t g(t) = -2\mathrm{Ric}_{g(t)} - 2\nabla^2_{g(t)} f(t), \\ &\partial_t f(t) = -\Delta_{g(t)} f(t) - R_{g(t)} + \frac{m}{2\tau(t)}, \\ &\frac{d}{dt}\tau(t) = -1; \end{aligned} \tag{1.5}$$

moreover, the entropy $\mathcal{W}$ is nondecreasing along (1.5). Since $\mathcal{W}$ is diffeomorphic invariant, i.e., $\mathcal{W}(\Phi^* g, \Phi^* f, \tau) = \mathcal{W}(g,f,\tau)$ for any diffeomorphisms Φ on M, it follows that the system (1.5) is equivalent to

$$\begin{aligned} &\partial_t g(t) = -2\mathrm{Ric}_{g(t)}, \\ &\partial_t f(t) = -\Delta_{g(t)} f(t) + \left|\nabla_{g(t)} f(t)\right|^2_{g(t)} - R_{g(t)} + \frac{m}{2\tau(t)}, \\ &\frac{d}{dt}\tau(t) = -1; \end{aligned} \tag{1.6}$$

Thus, (1.3) is a mixture of (1.5) and (1.6). There also are lots of interesting generalized Ricci flows, for example, see [14,17–21].

1.3. A parabolic flow

In this paper, we consider a class of Ricc flow type parabolic differential equation:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha_1\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t) + 2\alpha_2\nabla^2_{g(t)}\phi(t), \tag{1.7}$$

$$\partial_t\phi(t) = \Delta_{g(t)}\phi(t) + \beta_1\left|\nabla_{g(t)}\phi(t)\right|^2_{g(t)} + \beta_2\phi(t), \tag{1.8}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants. When $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \phi(t) = 0$, the system (1.7)–(1.8) is exactly the Ricci flow introduced by Hamilton [13]. When $\alpha_2 = \beta_1 = \beta_2 = 0$, it reduces to List's flow [19]. Recently, Hu and Shi [27] introduced a static flow on complete noncompact manifold that is similar to our flow. The main result is

Theorem 1.3. *Let (M, g) be an m-dimensional complete and noncompact Riemannian manifold with $|\mathrm{Rm}_g|^2 \leq k_0$ on M and ϕ a smooth function on M satisfying $|\phi|^2 + |\nabla_g \phi|^2_g \leq k_1$ and $|\nabla^2_g \phi|^2_g \leq k_2$. Then there exists a positive constant T, depending only on $m, k_0, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2$, such that the $\star$-regular $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-flow (1.7)–(1.8) with the initial data (g, ϕ) has a smooth solution $(g(t), \phi(t))$ on $M \times [0, T]$ and satisfies the following curvature estimate. For any nonnegative integer n, there exist uniform positive constants C_k, depending only on $m, n, k_0, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2$, such that*

$$\left|\nabla^n_{g(t)} \mathrm{Rm}_{g(t)}\right|^2_{g(t)} \leq \frac{C_n}{t^n}, \quad \left|\nabla^{n+2}_{g(t)} \phi(t)\right|^2_{g(t)} \leq \frac{C_n}{t^n}$$

on $M \times [0, T]$.

For the definition of regular flow and $\star$-regular flow, see Definition 2.11 and Section 3.

1.4. Notions and convenience

Manifolds are denote by $M, N, \cdots$. If g is a Riemannian metric on M, we write $\mathrm{Rm}_g, \mathrm{Ric}_g, R_g, \nabla_g$, and dV_g the Riemann curvature, Ricci curvature, scalar curvature, Levi-Civita connection, and volume form of g, respectively. We always omit the time variable t in concrete computations. For a family of Riemannian metrics, we denote by Δ_t and dV_t the corresponding Beltrami-Laplace operator and volume form respectively.

If $\mathcal{P}$ and $\mathcal{Q}$ are two quantities (may depend on time) satisfying $\mathcal{P} \leq C\mathcal{Q}$ for some *positive uniform* constant C, then we set $\mathcal{P} \lesssim \mathcal{Q}$. Similarly, we can define $\mathcal{P} \approx \mathcal{Q}$ if $\mathcal{P} \lesssim \mathcal{Q}$ and $\mathcal{Q} \lesssim \mathcal{P}$.

We also use the Einstein summation for tensor fields; for example,

$$\langle a, b\rangle_g = a_{ij} b^{ij} := \sum_{1 \leq i,j \leq m} a_{ij} b^{ij} = \sum_{1 \leq i,j,k,\ell \leq m} g^{ik} g^{j\ell} a_{ij} b_{k\ell}$$

for any two 2-tensor fields $a = (a_{ij})$ and $b = (b_{ij})$ on a Riemannian manifold (M, g) of dimension m.

If A and B are two tensor fields on a Riemannian manifold (M, g) we denote by $A * B$ any quantity obtained from $A \otimes B$ by one or more of these operations (a slightly different from that in [9]):

(1) summation over pairs of matching upper and lower indices,
(2) multiplication by constants depending only on the dimension of M and the ranks of A and B.

We also denote by A^k any k-fold product $A \cdots A$. The above product $\langle a, b\rangle_g$ can be written as $\langle a, b\rangle_g = a * b$; in order to stress the metric g, we also write it as $\langle a, b\rangle_g = g^{-1} * g^{-1} * a * b$.

2. A parabolic geometric flow

In this section we introduce a parabolic geometric flow motivated by (1.2). At first we will prove Proposition 1.1

2.1. A characterization of Ricci-flat metrics

Recall that a steady gradient Ricci soliton is a triple (M, g, f) satisfying (1.1).

Proposition 2.1. (See also Proposition 1.1) *Suppose M is a compact or complete noncompact manifold of dimension m. Then the following conditions are equivalent:*

(i) *there exists a Ricci-flat Riemannian metric on M;*
(ii) *there exist real numbers α, β, a smooth function ϕ on M, and a Riemannian metric g on M such that*

$$0 = -R_{ij} + \alpha\nabla_i\nabla_j\phi, \quad 0 = \Delta_g\phi + \beta|\nabla_g\phi|_g^2. \tag{2.1}$$

Proof. One direction (i)⇒(ii) is trivial, since we can take $\alpha = \beta = \phi = 0$. In the following we assume that the equation (2.1) holds for some α, β, and ϕ, g. When M is compact, a result of Hamilton [15] tells us that g must be Ricci-flat. Now we assume that M is complete noncompact.

Taking the trace of the first equation in (2.1), we get

$$R_g = \alpha\Delta_g\phi. \tag{2.2}$$

In particular,

$$R_g = -\alpha\beta|\nabla_g\phi|_g^2. \tag{2.3}$$

Hence, if $\alpha\beta \geq 0$, then $R_g \leq 0$; on the other hand, by a result of Chen [8], we know that any complete noncompact steady gradient Ricci soliton has nonnegative scalar curvature. Together with those two inequalities, we must have $R_g = 0$ and $\nabla_g\phi = 0$ by (2.3). Consequently, from (2.1), we see that $R_{ij} = 0$.

To deal with the case $\alpha\beta < 0$, we take the derivative ∇^i on the first equation of (2.1): $0 = -\frac{1}{2}\nabla_j R_g + \alpha\Delta_g\nabla_j\phi$ since $\nabla^i R_{ij} = \frac{1}{2}\nabla_j R_g$. According to the identity $\Delta_g\nabla_j\phi = \nabla_j\Delta_g\phi + R_{jk}\nabla^k\phi$, we arrive at $0 = -\frac{1}{2}\nabla_j R_g + \alpha(\nabla_j\Delta_g\phi + R_{jk}\nabla^k\phi)$. Using (2.1) and (2.3), we obtain

$$0 = \nabla_j\left[\left(\frac{\alpha^2}{2} - \alpha\beta\right)|\nabla_g\phi|_g^2 - \frac{1}{2}R_g\right] = \frac{\alpha(\alpha-\beta)}{2}\nabla_j|\nabla_g\phi|_g^2.$$

In the case $\alpha\beta < 0$, we must have $\alpha \neq 0, \beta \neq 0$, and $\alpha \neq \beta$, so the above identity yields $|\nabla_g\phi|_g^2 = c$ for some constant c, and hence $R_g = -\alpha\beta c$ using again (2.1). From the proved identity $0 = -\frac{1}{2}\nabla_j R_g + \alpha\Delta_g\nabla_j\phi$, we obtain $\Delta_g\nabla_j\phi = 0$. Consequently $0 = 2\nabla^j\phi\,\Delta_g\nabla_j\phi = \Delta_g|\nabla_g\phi|_g^2 - 2|\nabla_g^2\phi|_g^2 = 0 - 2|\nabla_g^2\phi|_g^2$ and then $|\nabla_g^2\phi|_g^2 = 0$. In particular, $\nabla_i\nabla_j\phi = 0$ and hence $R_{ij} = 0$. In each case, we get a Ricci-flat metric. □

2.2. Evolution equations

Motivated by Proposition 2.1, we consider a class of Ricc flow type parabolic differential equation:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha_1\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t) + 2\alpha_2\nabla_{g(t)}^2\phi(t), \tag{2.4}$$

$$\partial_t\phi(t) = \Delta_{g(t)}\phi(t) + \beta_1\left|\nabla_{g(t)}\phi(t)\right|_{g(t)}^2 + \beta_2\phi(t), \tag{2.5}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants. When $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \phi(t) = 0$, the system (1.7)–(1.8) is exactly the Ricci flow introduced by Hamilton [13]. When $\alpha_2 = \beta_1 = \beta_2 = 0$, it reduces to List's flow [19].

To compute evolution equations for (2.4)–(2.5), we recall variation formulas stated in [9]. Consider a flow $\partial_t g_{ij} = h_{ij}$ where h is a family of symmetric 2-tensor fields. Then

$$\partial_t g^{ij} = -g^{ik}g^{j\ell}h_{k\ell}, \quad \partial_t\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}\right),$$

$$\partial_t R_{ijk}^\ell = \frac{1}{2}g^{\ell p}\left(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\right),$$

$$\partial_t R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \right),$$

$$\partial_t R = -\Delta_t \mathrm{tr} h + \mathrm{div}\left(\mathrm{div} h\right) - \langle h, \mathrm{Ric} \rangle, \quad \partial_t dV_t \;=\; \frac{1}{2} \mathrm{tr} h \, dV_t.$$

We now take $h_{ij} := -2R_{ij} + 2\alpha_1 \nabla_i \phi \nabla_j \phi + 2\alpha_2 \nabla_i \nabla_j \phi$.

Lemma 2.2. *Under (2.4)–(2.5), we have*

$$\partial_t \Gamma^k_{ij} = -\nabla_i R_j{}^k - \nabla_j R_i{}^k + \nabla^k R_{ij} + 2\alpha_1 \nabla_i \nabla_j \phi \cdot \nabla^k \phi + \alpha_2 \nabla^k \nabla_i \nabla_j \phi - \alpha_2 \left(R_i{}^k{}_j{}^p \nabla_p \phi + R_j{}^k{}_i{}^p \nabla_p \phi \right).$$

Proof. Compute

$$\partial_t \Gamma^k_{ij} = -\nabla_i R_j{}^k - \nabla_j R_i{}^k + \nabla^k R_{ij} + 2\alpha_1 \nabla_i \nabla_j \phi \cdot \nabla^k \phi + \alpha_2 g^{k\ell} \left(\nabla_i \nabla_j \nabla_\ell \phi + \nabla_j \nabla_i \nabla_\ell \phi - \nabla_\ell \nabla_i \nabla_j \phi \right).$$

According to the Ricci identity, we obtain the desired result. □

Lemma 2.3. *Under (2.4)–(2.5), we have*

$$\partial_t R_{ij} \;=\; \Delta_t R_{ij} - 2R_{ik} R^k{}_j + 2R_{pijq} R^{pq} - 2\alpha_1 R_{pijq} \nabla^p \phi \nabla^q \phi + 2\alpha_1 \Delta_{g(t)} \phi \cdot \nabla_i \nabla_j \phi - 2\alpha_1 \nabla_i \nabla_k \phi \nabla^k \nabla_j \phi + \alpha_2 \left(R_i{}^p \nabla_p \nabla_j \phi + R_j{}^p \nabla_p \nabla_i \phi + \nabla_p R_{ij} \nabla^p \phi \right).$$

Proof. Note that

$$\partial_t R_{ij} = -\frac{1}{2} \Delta_t h_{ij} - \frac{1}{2} \nabla_i \nabla_j \left(g^{pq} h_{pq} \right) + \frac{1}{2} g^{pq} \left(\nabla_p \nabla_j h_{iq} + \nabla_p \nabla_i h_{jq} \right).$$

Denote by I_i, $i = 1, 2, 3, 4$, the ith term on the right-hand side of the above equation. For I_1 we have

$$I_1 = \Delta_t R_{ij} - \alpha_1 \Delta_t \nabla_i \phi \nabla_j \phi - \alpha_1 \nabla_i \phi \Delta_t \nabla_j \phi - 2\alpha_1 \nabla_k \nabla_i \phi \nabla^k \nabla_j \phi - \alpha_2 \Delta_t \left(\nabla_i \nabla_j \phi \right).$$

Since $|\nabla_{g(t)} \phi|^2_{g(t)}$ is a function, it follows $\nabla_i \nabla_j |\nabla_{g(t)} \phi(t)|^2_{g(t)} = \nabla_j \nabla_i |\nabla_{g(t)} \phi(t)|^2_{g(t)}$. Hence

$$I_2 = \nabla_i \nabla_j R - \alpha_1 \left(\nabla_i \nabla_j \nabla_k \phi + \nabla_j \nabla_i \nabla_k \phi \right) \nabla^k \phi - 2\alpha_1 \nabla_i \nabla_k \phi \nabla_j \nabla^k \phi - \alpha_2 \nabla_i \nabla_j \left(\Delta_t \phi \right).$$

The symmetry of I_3 and I_4 allows us to consider only one term, saying for example I_3. Since $\nabla_p \nabla_j \nabla_i \phi = \nabla_p \nabla_i \nabla_j \phi = \nabla_i \nabla_p \nabla_j \phi - R^k_{pij} \nabla_k \phi$, we have

$$I_3 = -\frac{1}{2} \nabla_i \nabla_j R + R_{pijq} R^{pq} - R_{ik} R^k{}_j + \alpha_1 \left[\nabla_i \nabla_j \nabla_p \phi \nabla^p \phi - R_{pijq} \nabla^p \phi \nabla^q \phi \right] + \alpha_1 \left[\Delta_t \phi \nabla_i \nabla_j \phi + \nabla_i \nabla_p \phi(t) \nabla_j \nabla^p \phi + \nabla_i \phi \Delta_t \nabla_j \phi \right] + \alpha_2 \nabla^q \nabla_j \left(\nabla_i \nabla_q \phi \right).$$

Consequently, we arrive at

$$\partial_t R_{ij} = \Delta_t R_{ij} - 2R_{ik} R^k{}_j + 2R_{pijq} R^{pq} - 2\alpha_1 R_{pijq} \nabla^p \phi \nabla^q \phi + 2\alpha_1 \Delta_t \phi \cdot \nabla_i \nabla_j \phi - 2\alpha_1 \nabla_i \nabla_k \phi \nabla^k \nabla_j \phi + \Lambda,$$

where

$$\begin{aligned}\Lambda &:= -\alpha_2\Delta_t\left(\nabla_i\nabla_j\phi\right) - \alpha_2\nabla_i\nabla_j\left(\Delta_t\phi\right) + \alpha_2\nabla^q\nabla_j\left(\nabla_i\nabla_q\phi\right) + \alpha_2\nabla^q\nabla_i\left(\nabla_j\nabla_q\phi\right)\\ &= \alpha_2\left[\Delta_t\nabla_i\nabla_j\phi - \nabla_i\nabla_j\Delta_t\phi - 2R_{ipjq}\nabla^p\nabla^q\phi - \nabla_i R_{jp}\nabla^p\phi - \nabla_j R_{ip}\nabla^p\phi + 2\nabla_p R_{ij}\nabla^p\phi\right]\end{aligned}$$

where we used the contract Bianchi identity $\nabla^p R_{jk\ell p} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}$ in the last line. The final step is to simplify the difference $[\Delta_{g(t)}, \nabla_i\nabla_j]\phi(t)$. According to the Ricci identity, we get $\Lambda = \alpha_2(R_{i\ell}\nabla^\ell\nabla_j\phi + R_{j\ell}\nabla^\ell\nabla_i\phi + \nabla_\ell R_{ij}\nabla^\ell\phi)$. □

Lemma 2.4. *Under (2.4)–(2.5), we have*

$$\begin{aligned}\partial_t R_{g(t)} \quad &= \quad \Delta_{g(t)}R_{g(t)} + 2\left|\mathrm{Ric}_{g(t)}\right|^2_{g(t)} + 2\alpha_1|\Delta_{g(t)}\phi(t)|^2_{g(t)} - 2\alpha_1\left|\nabla^2_{g(t)}\phi(t)\right|^2_{g(t)}\\ &- 4\alpha_1\left\langle \mathrm{Ric}_{g(t)}, \nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t)\right\rangle_{g(t)} + \alpha_2\left\langle\nabla_{g(t)}R_{g(t)}, \nabla_{g(t)}\phi(t)\right\rangle_{g(t)}.\end{aligned}$$

Proof. By the above formula for $\partial_t R_{g(t)}$, we obtain

$$\begin{aligned}\partial_t R_{g(t)} &= \Delta_{g(t)}R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2_{g(t)} + 2\alpha_1|\Delta_{g(t)}\phi(t)|^2_{g(t)} - 2\alpha_1\left|\nabla^2_{g(t)}\phi(t)\right|^2_{g(t)}\\ &\quad - 2\alpha_1 R^{ij}\nabla_i\phi(t)\nabla_j\phi(t) - 2\alpha_1\left(\Delta_{g(t)}\nabla_i\phi(t) - \nabla_i\Delta_{g(t)}\phi(t)\right)\nabla^i\phi(t) + I\end{aligned}$$

where $I := -2\alpha_2\Delta_t(\Delta_t\phi) + 2\alpha_2\nabla^i\nabla^j(\nabla_i\nabla_j\phi) - 2\alpha_2 R^{ij}\nabla_i\nabla_j\phi$. By the Ricci identity we get $I = \alpha_2\langle\nabla_t R, \nabla\phi\rangle$ and the desired formula. □

Following Hamilton, we introduce the tensor field

$$B_{ijk\ell} := -g^{pr}g^{qs}R_{ipjq}R_{kr\ell s}. \tag{2.6}$$

Note that $B_{ji\ell k} = B_{ijk\ell}$ and $B_{ijk\ell} = B_{k\ell ij}$.

Lemma 2.5. *Under (2.4)–(2.5), we have*

$$\begin{aligned}\partial_t R_{ijk\ell} \quad &= \quad \Delta_t R_{ijk\ell} + 2(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}) - ({R_i}^p R_{pjk\ell} + {R_j}^p R_{ipk\ell}\\ &\quad + \ {R_k}^p R_{ijp\ell} + {R_\ell}^p R_{ijkp}) + 2\alpha_1\left[\nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi - \nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi\right]\\ &+ \ \alpha_2\left[\nabla^p R_{ijk\ell}\nabla_p\phi - {R_{ijk}}^p\nabla_p\nabla_\ell\phi + {R^p}_{jk\ell}\nabla_i\nabla_p\phi + {R_i}^p{}_{k\ell}\nabla_j\nabla_p\phi + {R_{ij}}^p{}_\ell\nabla_k\nabla_p\phi\right].\end{aligned}$$

Proof. Recall the evolution equation

$$\partial_t R^\ell_{ijk} = \frac{1}{2}g^{\ell p}\left(\nabla_i\nabla_k h_{jp} + \nabla_j\nabla_p h_{ik} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_k h_{ip} - R^q_{ijk}h_{qp} - R^q_{ijp}h_{kq}\right)$$

where $\partial_t g_{ij} = h_{ij}$. Applying the above formula to $h_{ij} = -2R_{ij} + 2\alpha_1\nabla_i\phi\nabla_j\phi + 2\alpha_2\nabla_i\nabla_j\phi$ implies $\partial_t R^\ell_{ijk} = I_1 + I_2 + I_3$, where

$$\begin{aligned}I_1 &:= g^{\ell p}\left(\nabla_i\nabla_p R_{jk} + \nabla_j\nabla_k R_{ip} - \nabla_i\nabla_k R_{jp} - \nabla_j\nabla_p R_{ik} + R^q_{ijk}R_{qp} + R^q_{ijp}R_{kq}\right),\\ I_3 &:= g^{\ell p}\left[R^q_{ijk}\left(-\alpha_1\nabla_q\phi\nabla_p\phi - \alpha_2\nabla_q\nabla_p\phi\right) + R^q_{ijp}\left(-\alpha_1\nabla_k\phi\nabla_q\phi - \alpha_2\nabla_k\nabla_q\phi\right)\right],\end{aligned}$$

and I_2 := the rest terms. According to [9,13] we have

$$I_1 = \Delta_t R^{\ell}_{ijk} + g^{pq}\left(R^r_{ijp}R^{\ell}_{rqk} - 2R^r_{pik}R^{\ell}_{jqr} + 2R^{\ell}_{pir}R^r_{jqk}\right) - R_i{}^r R^{\ell}_{rjk} - R_j{}^r R^{\ell}_{irk} - R_k{}^r R^{\ell}_{ijr} + R_r{}^{\ell} R^r_{ijk}.$$

It can be showed that

$$I_2 = -\alpha_1 R^q_{ijk}\nabla_q\phi\nabla^{\ell}\phi + \alpha_1 R_{ij}{}^{\ell q}\nabla_q\phi\nabla_k\phi + 2\alpha_1\left(\nabla_i\nabla^{\ell}\phi\nabla_k\nabla_j\phi - \nabla_i\nabla_k\phi\nabla_j\nabla^{\ell}\phi\right) + \alpha_2\left(\nabla_i\nabla_k\nabla_j\nabla^{\ell}\phi + \nabla_j\nabla^{\ell}\nabla_i\nabla_k\phi - \nabla_j\nabla_k\nabla_i\nabla^{\ell}\phi - \nabla_i\nabla^{\ell}\nabla_j\nabla_k\phi\right);$$

together with I_3, we arrive at

$$I_2 + I_3 = -2\alpha_1 R^q_{ijk}\nabla_q\phi\nabla^{\ell}\phi + 2\alpha_1\left(\nabla_i\nabla^{\ell}\phi\nabla_k\nabla_j\phi - \nabla_i\nabla_k\phi\nabla_j\nabla^{\ell}\phi\right) + \alpha_2\left(\nabla^q R_{ijk}{}^{\ell} - R_k{}^{\ell}{}_j{}^q\nabla_i\nabla_q\phi - R^{\ell}{}_{ki}{}^q\nabla_j\nabla_q\phi - R_{ijk}{}^q\nabla_q\nabla^{\ell}\phi - R_{ij}{}^{\ell q}\nabla_k\nabla_q\phi\right)$$

where we used the Ricci identity and the formula $\nabla_i R_k{}^{\ell}{}_j{}^q + \nabla_j R^{\ell}{}_{ki}{}^q = -\nabla^q R_{ijk}{}^{\ell}$. Replacing ℓ by s in I_1, I_2, I_3, we have $\partial_t R^s_{ijk} = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$, where we denote by $\tilde{I}_i$ the corresponding terms; hence

$$\partial_t R_{ijk\ell} = (-2R_{\ell s}R^s_{ijk} + g_{\ell s}\tilde{I}_1) + 2\alpha_1 R^s_{ijk}\nabla_{\ell}\phi\nabla_s\phi + 2\alpha_2 R^s_{ijk}\nabla_{\ell}\nabla_s\phi + g_{\ell s}(\tilde{I}_2 + \tilde{I}_3).$$

The first bracket on the right-hand side follows from Hamilton's computation [9,13]; the rest terms can be computed from the expressions for $\tilde{I}_2$ and $\tilde{I}_3$. □

Next we compute evolution equations for $\phi(t)$.

Lemma 2.6. *Under (2.4)–(2.5), we have*

$$\begin{aligned}\partial_t|\nabla_{g(t)}\phi(t)|^2_{g(t)} &= \Delta_{g(t)}|\nabla_{g(t)}\phi(t)|^2_{g(t)} + 2\beta_2|\nabla_{g(t)}\phi(t)|^2_{g(t)} - 2\left|\nabla^2_{g(t)}\phi(t)\right|^2_{g(t)} \\ &- 2\alpha_1|\nabla_{g(t)}\phi(t)|^4_{g(t)} + (4\beta_1 - 2\alpha_2)\left\langle\nabla_{g(t)}\phi(t)\otimes\nabla_{g(t)}\phi(t), \nabla^2_{g(t)}\phi(t)\right\rangle_{g(t)}.\end{aligned}$$

Proof. Using (2.5) we have $\partial_t\nabla_i\phi = \Delta_t\nabla_i\phi - R_{ij}\nabla^j\phi + 2\beta_1\nabla^j\phi\nabla_i\nabla_j\phi + \beta_2\nabla_i\phi(t)$, where we use the identity $\nabla_i\Delta_t\phi = \Delta_t\nabla_i\phi - R_{ij}\nabla^j\phi$. Using (2.4) we then get

$$\begin{aligned}\partial_t|\nabla_{g(t)}\phi(t)|^2_{g(t)} &= -2\alpha_1|\nabla_{g(t)}\phi(t)|^4_{g(t)} - 2\alpha_2\nabla^i\nabla^j\phi(t)\nabla_i\phi(t)\nabla_j\phi(t) \\ &+ 2\nabla^i\phi(t)\Delta_{g(t)}\nabla_i\phi(t) + 4\beta_1\nabla_i\nabla_k\phi(t)\nabla^i\phi(t)\nabla^k\phi(t) + 2\beta_2|\nabla_{g(t)}\phi(t)|^2_{g(t)}\end{aligned}$$

which implies the desired equation. □

Lemma 2.7. *Under (2.4)–(2.5), we have*

$$\begin{aligned}\partial_t(\nabla_i\nabla_j\phi) &= \Delta_t(\nabla_i\nabla_j\phi) + 2R_{pijq}\nabla^p\nabla^q\phi + \beta_2\nabla_i\nabla_j\phi - R_{ip}\nabla^p\nabla_j\phi - R_{jp}\nabla^p\nabla_i\phi \\ &- 2\alpha_1|\nabla_{g(t)}\phi(t)|^2_{g(t)}\nabla_i\nabla_j\phi + (2\beta_1 - \alpha_2)\nabla_k\phi\nabla_k\nabla_i\nabla_j\phi \\ &+ 2\beta_1\nabla_i\nabla^k\phi\nabla_j\nabla_k\phi + 2(\beta_1 - \alpha_2)R_{pijq}\nabla^p\phi\nabla^q\phi.\end{aligned}$$

Proof. Compute $\partial_t(\nabla_i\nabla_j\phi) = \nabla_i\nabla_j(\partial_t\phi) - \partial_t\Gamma^k_{ij}\cdot\partial_k\phi$. Using (2.5) we have

$$\begin{aligned}\nabla_i\nabla_j(\partial_t\phi) &= \nabla_k\nabla_i\nabla_j\nabla^k\phi - R^\ell_{ikj}\nabla_\ell\nabla^k\phi + R^k_{ik\ell}\nabla_j\nabla^\ell\phi - \nabla_i R_{j\ell}\nabla^\ell\phi \\ &- R_{j\ell}\nabla_i\nabla^\ell\phi + \beta_2\nabla_i\nabla_j\phi + 2\beta_1\left(\nabla_i\nabla^k\phi\nabla_j\nabla_k\phi + \nabla^k\phi\nabla_i\nabla_j\nabla_k\phi\right).\end{aligned}$$

Since $\nabla_k\nabla_i\nabla_j\nabla^k\phi = \Delta_t(\nabla_i\nabla_j\phi) - \nabla^k R^\ell_{ikj}\nabla_\ell\phi - R^\ell_{ikj}\nabla^k\nabla_\ell\phi$, we have

$$\begin{aligned}\nabla_i\nabla_j(\partial_t\phi) &= \Delta_t(\nabla_i\nabla_j\phi(t)) - R_{i\ell}\nabla^\ell\nabla_j\phi - R_{j\ell}\nabla^\ell\nabla_i\phi + \beta_2\nabla_i\nabla_j\phi - (\nabla_i R_{j\ell} \\ &+ \nabla_j R_{i\ell} - \nabla_\ell R_{ij})\nabla^\ell\phi - 2R^\ell_{ikj}\nabla_\ell\nabla^k\phi + 2\beta_1\left(\nabla_i\nabla^k\phi\nabla_j\nabla_k\phi + \nabla^k\phi\nabla_i\nabla_j\nabla_k\phi\right).\end{aligned}$$

Using Lemma 2.2 implies

$$\begin{aligned}\partial_t(\nabla_i\nabla_j\phi) &= \Delta_t(\nabla_i\nabla_j\phi) - 2R_{ikj\ell}\nabla^k\nabla^\ell\phi - R_{i\ell}\nabla^\ell\nabla_j\phi - R_{j\ell}\nabla^\ell\nabla_i\phi + \beta_2\nabla_i\nabla_j\phi \\ &+ 2\beta_1\nabla_i\nabla^k\phi\nabla_j\nabla_k\phi + 2\beta_1\nabla^k\phi\nabla_i\nabla_j\nabla_k\phi \\ &- 2\alpha_1|\nabla_{g(t)}\phi(t)|^2_{g(t)}\nabla_i\nabla_j\phi - \alpha_2\nabla^k\phi\nabla_k\nabla_i\nabla_j\phi + 2\alpha_2 R_{ikj\ell}\nabla^\ell\phi\nabla^k\phi.\end{aligned}$$

Now Lemma 2.7 follows from $\nabla_i\nabla_j\nabla_k\phi = \nabla_k\nabla_i\nabla_j\phi - R^\ell_{ikj}\nabla_\ell\phi$. □

Lemma 2.8. *Under (2.4)–(2.5), we have*

$$\begin{aligned}\partial_t(\nabla_i\phi\nabla_j\phi) &= \Delta_t(\nabla_i\phi(t)\nabla_j\phi) - \nabla^k\phi\left(R_{ik}\nabla_j\phi + R_{jk}\nabla_i\phi\right) \\ &- 2\nabla_i\nabla^k\phi\nabla_j\nabla_k\phi + 2\beta_2\nabla_i\phi\nabla_j\phi + 2\beta_1\nabla^k\phi\left(\nabla_i\phi\nabla_j\nabla_k\phi(t) + \nabla_j\phi\nabla_i\nabla_k\phi\right).\end{aligned}$$

Proof. From the evolution equation for $\nabla_i\phi(t)$ obtained in the proof of Lemma 2.6, we get

$$\begin{aligned}\partial_t(\nabla_i\phi\nabla_j\phi) &= \nabla_j\phi\Big(\Delta_t\nabla_i\phi - R_{ik}\nabla^k\phi + 2\beta_1\nabla^k\phi\nabla_i\nabla_k\phi + \beta_2\nabla_i\phi\Big) \\ &+ \nabla_i\phi\Big(\Delta_t\nabla_j - R_{jk}\nabla^k\phi + 2\beta_1\nabla^k\phi\nabla_j\nabla_k\phi + \beta_2\nabla_j\phi\Big)\end{aligned}$$

which implies the equation. □

2.3. Regular flows on compact manifolds

Let $(g(t),\phi(t))_{t\in[0,T)}$ be the solution of (2.4)–(2.5) on a compact m-manifold M with the initial value $(\tilde{g},\tilde{\phi})$. Define

$$\tilde{c} := \max_M |\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}}, \quad D := \frac{1}{4}|2\beta_1 - \alpha_2|^2 - \alpha_1. \tag{2.7}$$

Proposition 2.9. *Suppose $(g(t),\phi(t))_{t\in[0,T)}$ is the solution of (2.4)–(2.5) on a compact m-manifold M with the initial value $(\tilde{g},\tilde{\phi})$. Then we have*

(1) **Case 1:** $4\beta_1 - 2\alpha_2 = 0$.
   (1.1) *If $\alpha_1 > 0$ and $\beta_2 > 0$, then*

$$|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \frac{\tilde{c}\beta_2 e^{2\beta_2 t}}{\tilde{c}\alpha_1 e^{2\beta_2 t} + (\beta_2 - \tilde{c}\alpha_1)}.$$

(1.2) *If* $\alpha_1 > 0$ *and* $\beta_2 \le 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}$.
(1.3) *If* $\alpha_1 = 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}e^{2\beta_2 t}$.
(1.4) *If* $\alpha_1 < 0$ *and* $\beta_2 \le 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}/(1 + 2\alpha_1\tilde{c}t)$.
(1.5) *If* $\alpha_1 < 0$ *and* $\beta_2 > 0$, *then*

$$\left|\nabla_{g(t)}\phi(t)\right|^2_{g(t)} \le \frac{\tilde{c}\beta_2 e^{2\beta_2 t}}{\beta_2 + \tilde{c}\alpha_1\left(e^{2\beta_2 t} - 1\right)}.$$

(2) **Case 2:** $4\beta_1 - 2\alpha_2 \ne 0$**.**
(2.1) *If* $D < 0$ *and* $\beta_2 > 0$, *then*

$$|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \frac{\tilde{c}\beta_2 e^{2\beta_2 t}}{-\tilde{c}De^{2\beta_2 t} + (\beta_2 + \tilde{c}D)}.$$

(2.2) *If* $D < 0$ *and* $\beta_2 \le 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}$.
(2.3) *If* $D = 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}e^{2\beta_2 t}$.
(2.4) *If* $D > 0$ *and* $\beta_2 \le 0$, *then* $|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le \tilde{c}/(1 - 2D\tilde{c}t)$.
(2.5) *If* $D >$ *and* $\beta_2 > 0$, *then*

$$\left|\nabla_{g(t)}\phi(t)\right|^2_{g(t)} \le \frac{\tilde{c}\beta_2 e^{2\beta_2 t}}{\beta_2 - \tilde{c}D\left(e^{2\beta_2 t} - 1\right)}.$$

Proof. For any time t, we have $\langle\nabla\phi\otimes\nabla\phi, \nabla^2\phi\rangle \le |\nabla\phi|^2|\nabla^2\phi|$. By Lemma 2.6 we have $\partial_t|\nabla\phi|^2 \le \Delta|\nabla\phi|^2 + 2\beta_2|\nabla\phi|^2 - 2|\nabla^2\phi|^2 - 2\alpha_1|\nabla\phi|^4 + |4\beta_1 - 2\alpha_2||\nabla\phi|^2|\nabla^2\phi|$. For convenience, set $u := |\nabla\phi|^2$ and $v := |\nabla^2\phi|$. Then

$$\partial_t u \le \Delta_t u + 2\beta_2 u - 2v^2 - 2\alpha_1 u^2 + |4\beta_1 - 2\alpha_2|uv.$$

(1) Case 1: $4\beta_1 - 2\alpha_2 = 0$. In this case, the above inequality becomes

$$\partial_t u \le \Delta_t u + 2\beta_2 u - 2\alpha_1 u^2.$$

If $\alpha_1 \ge 0$, then $\partial_t u \le \Delta_t u + 2\beta_2 u$ and $\partial_t\left(e^{-2\beta_2 t}u\right) = e^{-2\beta_2 t}\left(-2\beta_2 u + \partial_t u\right) \le \Delta\left(e^{-2\beta_2 t}u\right)$ from which we obtain $u \le \tilde{c}e^{2\beta_2 t}$ by the maximum principle.

If $\alpha_1 < 0$ and $\beta_2 \le 0$, then $\partial_t u \le \Delta_t u - 2\alpha_1 u^2$ and $u_t \le \frac{u(0)}{1+2\alpha_1 u(0)t} \le \frac{\tilde{c}}{1+2\alpha_1\tilde{c}t}$. On the other hand, if $\beta_2 > 0$, then $u \le \tilde{c}\beta_2 e^{2\beta_2 t}/[\beta_2 + \tilde{c}\alpha_1(e^{2\beta_2 t} - 1)]$ since the solution to the ordinary differential equation

$$U'(t) = 2\beta_2 U(t) - 2\alpha_1 U^2(t), \quad U(0) = \tilde{c}$$

is of the form

$$U(t) = \frac{\tilde{c}\beta_2 e^{2\beta_2 t}}{\beta_2 + \tilde{c}\alpha_1(e^{2\beta_2 t} - 1)}.$$

(2) Case 2: $4\beta_1 - 2\alpha_2 \ne 0$. Using the inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$ for any $a, b \ge 0$ and any positive number ϵ, we obtain

$$\partial_t u \le \Delta_t u + 2\beta_2 u - \left(\epsilon|4\beta_1 - 2\alpha_2| - 2\right)v^2 + \left(\frac{|4\beta_1 - 2\alpha_2|}{4\epsilon} - 2\alpha_1\right)u^2.$$

Choosing $\epsilon := 2/|4\beta_1 - 2\alpha_2|$ implies $\partial_t u \le \Delta_t u + 2\beta_2 u + 2Du^2$, where D is given in (2.7). This is just the case (1) if we replace α_1 by $-D$. The following discussion can be obtained. $\square$

Corollary 2.10. *Suppose $(g(t), \phi(t))_{t\in[0,T)}$ is the solution of (2.4)–(2.5) on a compact m-manifold M with the initial value $(\tilde{g}, \tilde{\phi})$. If $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy one of the conditions*

(i) $\beta_2 \le 0$ *and* $4\alpha_1 \ge (2\beta_1 - \alpha_2)^2$, *or*
(ii) $\beta_2 > 0$ *and* $\frac{4}{\tilde{c}}\beta_2 + |2\beta_1 - \alpha_2|^2 \ge 4\alpha_1 > (2\beta_1 - \alpha_2)^2$,

then

$$|\nabla_{g(t)}\phi(t)|^2_{g(t)} \le C \tag{2.8}$$

on $M \times [0, T)$, where C is a positive constant depending only on α_1, β_2, and $|\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}}$.

In particular, we recover List's result [19] for $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (4, 0, 0, 0)$. Anther example is $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\alpha_1, 2\beta_1, \beta_1, 0)$, where $\alpha_1 \ge 0$ and $\beta_1 \in \mathbf{R}$.

Definition 2.11. We say the flow (2.4)–(2.5) is *regular*, if the constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the conditions (i) or (ii) in Corollary 2.10. An $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-flow is $\star$-*regular* if the associated $(\alpha_1, 0, \beta_1 - \alpha_2, \beta_2)$-flow (see Proposition 2.12) is regular.

Clearly that there are no relations between regular flows and $\star$-regular in general. For example, $(1, 1, 0, 0)$-flow is regular but not $\star$-regular, while $(1, 1, 2, 0)$-flow is $\star$-regular but not regular.

2.4. Reduction to $(\alpha_1, 0, \beta_1, \beta_2)$-flow

Let $(\bar{g}(t), \bar{\phi}(t))$ be the solution of (2.4)–(2.5); that is,

$$\begin{aligned}
\partial_t \bar{g}(t) &= -2\mathrm{Ric}_{\bar{g}(t)} + 2\alpha_1 \nabla_{\bar{g}(t)}\bar{\phi}(t) \otimes \nabla_{\bar{g}(t)}\bar{\phi}(t) + 2\alpha_2 \nabla^2_{\bar{g}(t)}\bar{\phi}(t),\\
\partial_t \bar{\phi}(t) &= \Delta_{\bar{g}(t)}\bar{\phi}(t) + \beta_1 |\nabla_{\bar{g}(t)}\bar{\phi}(t)|^2_{\bar{g}(t)} + \beta_2 \bar{\phi}(t).
\end{aligned}$$

Consider a 1-parameter family of diffeomorphisms $\Phi(t) : M \to M$ by

$$\frac{d}{dt}\Phi(t) = -\alpha_2 \nabla_{\bar{g}(t)}\bar{\phi}(t), \quad \Phi(0) = \mathrm{Id}_M. \tag{2.9}$$

The above system of ODE is always solvable. Define

$$g(t) := [\Phi(t)]^* \bar{g}(t), \quad \phi(t) := [\Phi(t)]^* \bar{\phi}(t). \tag{2.10}$$

Then

$$\begin{aligned}
\partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t),\\
\partial_t \phi(t) &= \Delta_{g(t)}\phi(t) + (\beta_1 - \alpha_2)|\nabla_{g(t)}\phi(t)|^2_{g(t)} + \beta_2 \phi(t).
\end{aligned}$$

Proposition 2.12. *Under a 1-parameter family of diffeomorphisms given by (2.9), any solution of an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-flow is equivalent to a solution of $(\alpha_1, 0, \beta_1 - \alpha_2, \beta_2)$-flow.*

2.5. De Turck's trick

By Proposition 2.12, we suffice to study $(\alpha_1, 0, \beta_1, \beta_2)$-flow:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \tag{2.11}$$

$$\partial_t \phi(t) = \Delta_{g(t)}\phi(t) + \beta_1 \left|\nabla_{g(t)}\phi(t)\right|^2_{g(t)} + \beta_2\phi(t). \tag{2.12}$$

Let $(M, \tilde{g})$ be an m-dimensional compact or complete noncompact Riemannian manifold with $\tilde{g} = \tilde{g}_{ij}dx^i \otimes dx^j$ and $\tilde{\phi}$ a smooth function on M. Let $(\hat{g}(t), \hat{\phi}(t))_{t\in[0,T]}$ be a solution of (2.11)–(2.12) with the initial data $(\tilde{g}, \tilde{\phi})$, that is,

$$\partial_t \hat{g}(t) = -2\mathrm{Ric}_{\hat{g}(t)} + 2\alpha_1 \nabla_{\hat{g}(t)}\hat{\phi}(t) \otimes \nabla_{\hat{g}(t)}\hat{\phi}(t), \tag{2.13}$$

$$\partial_t \hat{\phi}(t) = \Delta_{\hat{g}(t)}\hat{\phi}(t) + \beta_1 \left|\nabla_{\hat{g}(t)}\hat{\phi}(t)\right|^2_{\hat{g}(t)} + \beta_2\hat{\phi}(t) \tag{2.14}$$

with $(\hat{g}(0), \hat{\phi}(0)) = (\tilde{g}, \tilde{\phi})$.

Notation 2.13. If $\hat{g}(t)$ is a time-dependent Riemannian metric, its components are written as $\hat{g}_{ij}$ or $\hat{g}_{ij}(x,t)$ when we want to indicate space and time. The corresponding components of $\mathrm{Rm}_{\hat{g}(t)}, \mathrm{Ric}_{\hat{g}(t)}$ and $\nabla_{\hat{g}(t)}$ are $\hat{R}_{ijk\ell}, \hat{R}_{ij}$ and $\hat{\nabla}_i$, respectively. In the form of local components, we always omit space and time variables for convenience.

Locally, the system (2.13)–(2.14) is of the form

$$\partial_t \hat{g}_{ij} = -2\hat{R}_{ij} + 2\alpha_1\hat{\nabla}_i\hat{\phi}\hat{\nabla}_j\hat{\phi}, \quad \partial_t\hat{\phi} = \hat{\Delta}\hat{\phi} + \beta_1|\hat{\nabla}\hat{\phi}|^2_{\hat{g}} + \beta_2\hat{\phi} \tag{2.15}$$

with $(\hat{g}_{ij}(0), \hat{\phi}(0)) = (\tilde{g}_{ij}, \tilde{\phi})$. The system (2.15) is not strictly parabolic even for the case $\alpha_1 = \beta_1 = \beta_2$. As in the Ricci flow (see [26]) we consider one-parameter family of diffeomorphisms $(\Psi_t)_{t\in[0,T]}$ on M as follows: Let

$$g(t) := \Psi_t^*\hat{g}(t) = g_{ij}(x,t)dx^i \otimes dx^j, \quad \phi(t) := \Psi_t^*\hat{\phi}(t), \quad t \in [0,T] \tag{2.16}$$

and $\Psi_t(x) := y(x,t)$ be the solution of the quasilinear first order system

$$\partial_t y^\alpha = \frac{\partial}{\partial x^k}y^\alpha \cdot g^{\beta\gamma}\left(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma}\right), \quad y^\alpha(x,0) = x^\alpha, \tag{2.17}$$

where Γ and $\tilde{\Gamma}$ are Christoffel symbols of g and $\hat{g}$ respectively. As in [19,26], we have

$$\partial_t g_{ij} = -2R_{ij} + 2\alpha_1\nabla_i\phi\nabla_j\phi + \nabla_iV_j + \nabla_jV_i, \quad V_i := g_{ik}g^{\beta\gamma}\left(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma}\right). \tag{2.18}$$

Similarly, we have $\partial_t\phi(x,t) = \Delta\phi(x,t) + \beta_1|\nabla\phi|^2_g + \beta_2\phi(x,t) + \langle V, \nabla\phi\rangle_g$. Here, Δ and ∇ are Laplacian and Levi-Civita connection of g accordingly. Hence, under the one-parameter family of diffeomorphisms $(\Psi_t)_{t\in[0,T]}$ on M, (2.15) is equivalent to

$$\partial_t g_{ij} = -2R_{ij} + 2\alpha_1\nabla_i\phi\nabla_j\phi + \nabla_iV_j + \nabla_jV_i, \tag{2.19}$$

$$\partial_t\phi = \Delta\phi + \beta_1|\nabla\phi|^2_g + \beta_2\phi + \langle V, \nabla\phi\rangle_g \tag{2.20}$$

with $(g_{ij}(0), \phi(0)) = (\tilde{g}_{ij}, \tilde{\phi})$.

Lemma 2.14. *the system (2.19)–(2.20) is strictly parabolic. Moreover*

$$\begin{aligned}\partial_t g_{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{ij} + g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{j\alpha q\beta} + g^{\alpha\beta}g_{jp}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} \\ &\quad + \frac{1}{2}g^{\alpha\beta}g^{pq}\Big(\tilde{\nabla}_i g_{p\alpha}\tilde{\nabla}_j g_{q\beta} + 2\tilde{\nabla}_\alpha g_{jp}\tilde{\nabla}_q g_{i\beta} - 2\tilde{\nabla}_\alpha g_{jp}\tilde{\nabla}_\beta g_{iq} \\ &\quad - 2\tilde{\nabla}_j g_{p\alpha}\tilde{\nabla}_\beta g_{iq} - 2\tilde{\nabla}_i g_{p\alpha}\tilde{\nabla}_\beta g_{jq}\Big) + 2\alpha_1\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi, \end{aligned} \tag{2.21}$$

$$\partial_t\phi = g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + \beta_1|\tilde{\nabla}\phi|^2_g + \beta_2\phi. \tag{2.22}$$

Proof. The first equation (2.21) directly follows from the computations made in [19,26] and the only difference is the sign of the Riemann curvature tensors used in this paper. To (2.22), we first observe that $\Delta\phi + \langle V, \nabla\phi\rangle_g = g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi$ as showed in [19]; then $\partial_t\phi = g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + \beta_1 g^{ij}\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi + \beta_2\phi$ since $\nabla\phi = d\phi = \tilde{\nabla}\phi$. □

3. Complete and noncompact case

In this section we study the flow (2.4)–(2.5) on complete and noncompact Riemannian manifolds. The main result of this paper is

Theorem 3.1. *Let (M, g) be an m-dimensional complete and noncompact Riemannian manifold with $|\mathrm{Rm}_g|^2_g \leq k_0$ on M, where k_0 is a positive constant, and let ϕ be a smooth function on M satisfying $|\phi|^2 + |\nabla_g\phi|^2_g \leq k_1$ and $|\nabla^2_g\phi|^2_g \leq k_2$. Then there exists a constant $T = T(m, k_0, k_1) > 0$, depending only on m and k_0, k_1, such that any $\star$-regular $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-flow (2.4)–(2.5) has a smooth solution $(g(t), \phi(t))$ on $M \times [0, T]$ and satisfies the following curvature estimate. For any nonnegative integer n, there exist constants $C_k > 0$, depending only on m, n, k_0, k_1, k_2, such that*

$$\left|\nabla^k_{g(t)}\mathrm{Rm}_{g(t)}\right|^2_{g(t)} \leq \frac{C_k}{t^k}, \quad \left|\nabla^k_{g(t)}\phi(t)\right|^2_{g(t)} \leq \frac{C_k}{t^?} \tag{3.1}$$

on $M \times [0, T]$.

By Proposition 2.12, we suffice to study a regular $(\alpha_1, 0, \beta_1, \beta_2)$-flow:

$$\begin{aligned}\partial_t\hat{g}(t) &= -2\mathrm{Ric}_{\hat{g}(t)} + 2\alpha_1\nabla_{\hat{g}(t)}\hat{\phi}(t)\otimes\nabla_{\hat{g}(t)}\hat{\phi}(t), \\ \partial_t\hat{\phi}(t) &= \Delta_{\hat{g}(t)}\hat{\phi}(t) + \beta_1|\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} + \beta_2\hat{\phi}(t),\end{aligned}$$

where $(\tilde{g}, \tilde{\phi}) = (\hat{g}(0), \hat{\phi}(0))$ is a fixed pair consisting of a Riemannian metric $\hat{g}$ and a smooth function $\hat{\phi}$. According to De Turck's trick, the above system of parabolic partial differential equations are reduced to (2.10)–(2.20).

Suppose that $D \subset M$ is a domain with boundary ∂D a compact smooth $(m-1)$-dimensional submanifold of M, and the closure $\overline{D} := D \cup \partial D$ is a compact subset of M. We shall shove the following Dirichlet boundary problem:

$$\begin{aligned}\partial_t g_{ij} &= -2R_{ij} + 2\alpha_1\nabla_i\phi\nabla_j\phi + \nabla_i V_j + \nabla_j V_i, \quad \text{in } D\times[0,T], \\ \partial_t\phi &= \Delta\phi + \beta_1|\nabla\phi|^2_g + \beta_2\phi + \langle V, \nabla\phi\rangle_g, \quad \text{in } D\times[0,T], \\ (g_{ij}, \phi) &= (\tilde{g}_{ij}, \tilde{\phi}), \quad \text{on } D_T,\end{aligned} \tag{3.2}$$

where

$$D_T := (D \times \{0\}) \cup (\partial D \cup [0,T]) \tag{3.3}$$

stands for the parabolic boundary of the domain $D \times [0,T]$.

Consider the assumption

$$|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \le k_0, \quad |\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{\phi}} \le k_1. \tag{3.4}$$

3.1. Zeroth order estimates

Suppose that (g_{ij}, ϕ) is a solution of (3.2). For each positive integer n, define

$$u = u(x,t) := g^{\alpha_1\beta_1}\tilde{g}_{\beta_1\alpha_2}g^{\alpha_2\beta_2}\tilde{g}_{\beta_2\alpha_3}\cdots g^{\alpha_n\beta_n}\tilde{g}_{\beta_n\alpha_1} \tag{3.5}$$

on $D \times [0,T]$.

Lemma 3.2. *If $|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \le k_0$, then the function $u = u(x,t)$ satisfies*

$$\partial_t u \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta u + 2nm\sqrt{k_0}u^{1+\frac{1}{n}} \quad \text{in } D \times [0,T], \quad u = m \quad \text{on } D_T. \tag{3.6}$$

Proof. The proof is identically similar to that of [19,26]; for completeness, we give a self-contained proof. Since $\partial_t g^{ij} = -g^{ik}g^{j\ell}\partial_t g_{k\ell}$ and $\tilde{\nabla}_\beta g^{ij} = -g^{ip}g^{jq}\tilde{\nabla}_\beta g_{pq}$, it follows from (2.21) that

$$\begin{aligned} \partial_t g^{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} + g^{\alpha\beta}g^{j\ell}g^{ik}\tilde{\nabla}_\alpha g^{ik}\tilde{\nabla}_\beta g_{k\ell} + g^{\alpha\beta}g^{ik}\tilde{\nabla}_\alpha g^{j\ell}\tilde{\nabla}_\beta g_{k\ell} \\ &- g^{\alpha\beta}g^{ik}g^{j\ell}g_{kp}\tilde{g}^{pq}\tilde{R}_{\ell\alpha q\beta} - g^{\alpha\beta}g^{ik}g^{j\ell}g_{p\ell}\tilde{g}^{pq}\tilde{R}_{k\alpha q\beta} - 2\alpha_1 g^{ik}g^{j\ell}\tilde{\nabla}_k\phi\tilde{\nabla}_\ell\phi + \frac{1}{2}g^{\alpha\beta}g^{pq}g^{j\ell} \\ &\left(2\tilde{\nabla}_\alpha g_{p\ell}\tilde{\nabla}_\beta g_{kq} + 2\tilde{\nabla}_\ell g_{p\alpha}\tilde{\nabla}_\beta g_{kq} + 2\tilde{\nabla}_k g_{p\alpha}\tilde{\nabla}_\beta g_{q\ell} - 2\tilde{\nabla}_\alpha g_{\ell p}\tilde{\nabla}_q g_{k\beta} - \tilde{\nabla}_k g_{p\alpha}\tilde{\nabla}_\ell g_{q\beta}\right). \end{aligned} \tag{3.7}$$

Choosing a normal coordinate system such that $\tilde{g}_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i\delta_{ij}$, we conclude from (3.7) that

$$\begin{aligned} \partial_t g^{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} - \frac{2\tilde{\nabla}_\ell g_{ik}\tilde{\nabla}_\ell g_{jk}}{\lambda_i\lambda_j\lambda_k\lambda_\ell} - \frac{\tilde{R}_{ikjk}}{\lambda_i\lambda_k} - \frac{\tilde{R}_{ikjk}}{\lambda_j\lambda_k} - \frac{2\alpha_1\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi}{\lambda_i\lambda_j} \\ &+ \frac{1}{\lambda_i\lambda_j\lambda_k\lambda_\ell}\left(\tilde{\nabla}_k g_{\ell j}\tilde{\nabla}_k g_{i\ell} + \tilde{\nabla}_j g_{\ell k}\tilde{\nabla}_k g_{i\ell} + \tilde{\nabla}_i g_{\ell k}\tilde{\nabla}_k g_{\ell j} - \tilde{\nabla}_k g_{j\ell}\tilde{\nabla}_\ell g_{ik} - \frac{1}{2}\tilde{\nabla}_i g_{\ell k}\tilde{\nabla}_j g_{\ell k}\right). \end{aligned} \tag{3.8}$$

From $u = \sum_{i=1}^n (1/\lambda_i)^n$, it is not hard to see that

$$\begin{aligned} \partial_t u &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta u - \frac{n}{\lambda_k}\sum_{a=0}^{n-2}\frac{1}{\lambda_i^{n-2-a}\lambda_j^a}\left(\tilde{\nabla}_k g_{ij}\right)^2 - \frac{2n}{\lambda_i^n\lambda_k}\tilde{R}_{ikik} \\ &- \frac{2\alpha_1 n}{\lambda_i^{n+1}}|\nabla_{\tilde{g}}\phi|^2_{\tilde{g}} - \frac{n}{2\lambda_i^{n+1}\lambda_k\lambda_\ell}\left(\tilde{\nabla}_k g_{i\ell} + \tilde{\nabla}_\ell g_{ik} - \tilde{\nabla}_i g_{\ell k}\right)^2. \end{aligned} \tag{3.9}$$

Thanks to $|\mathrm{Rm}_{\tilde{g}}|_{\tilde{g}} \le \sqrt{k_0}$ and (3.9) we get $\partial_t u \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta u + 2n\sqrt{k_0}(\sum_{j=1}^m 1/\lambda_j)u$. According to Hölder's inequality we have $\sum_{j=1}^m \lambda_j^{-1} \le (\sum_{j=1}^m \lambda_j^{-n})^{1/n}(\sum_{j=1}^m 1^{n'})^{1/n'} = mu^{1/n}$ with $\frac{1}{n} + \frac{1}{n'} = 1$; therefore $\partial_t u \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta u + 2nm\sqrt{k_0}u^{1+\frac{1}{n}}$. □

As showed in [19,26], the lower bound of g_{ij} now directly follows from Lemma 3.2.

Lemma 3.3. *If $|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \leq k_0$, then, for any $\delta \in (0,1)$, we have*

$$g(t) \geq (1-\delta)\tilde{g} \tag{3.10}$$

on $D \times [0, T_-(\delta, m, k_0)]$, where $T_-(\delta, m, k_0) := \frac{1}{2\sqrt{k_0}}(\frac{1}{m})^{1+1/n}[1-(\frac{1}{2})^{1/n}]$, n is a positive integer satisfying $\frac{\ln(2m)}{\ln(1/(1-\delta))} \leq n < \frac{\ln(2m)}{\ln(1/(1-\delta))} + 1$.

Since we consider the regular $(\alpha_1, 0, \beta_1, \beta_2)$-flow, we conclude from Corollary 2.10 that $|\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} \leq C$, where C is a positive constant depending only on α_1, β_2, and $|\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}}$. Following the arguments in [19,26], we have an upper bound of g_{ij}.

Lemma 3.4. *If $|\widetilde{\mathrm{Rm}}|^2_{\tilde{g}} \leq k_0$ and $|\tilde{\nabla}\tilde{\phi}|^2_{\tilde{g}} \leq k_1$, then, for any $\theta > 0$, we have*

$$g(t) \leq (1+\theta)\tilde{g} \tag{3.11}$$

on $D \times [0, T_+(\theta, m, k_0, k_1, \alpha_1, \beta_2)]$, where $T_+(\delta, m, k_0, k_1, \alpha_1, \beta_2)$ is a positive constant depending only on $\theta, m, k_0, k_1, \alpha_1$, and β_2.

From Lemma 3.3 and Lemma 3.4, we have

Theorem 3.5. *Suppose that $|\widetilde{\mathrm{Rm}}|^2_{\tilde{g}} \leq k_0$ and $|\tilde{\nabla}\tilde{\phi}|^2_{\tilde{g}} \leq k_1$ on M. If $(g(t), \phi(t))$ is a solution of (3.2), then, for any $\epsilon \in (0,1)$, we have*

$$(1-\epsilon)\tilde{g} \leq g(t) \leq (1+\epsilon)\tilde{g} \tag{3.12}$$

on $\overline{D} \times [0, T(\epsilon, m, k_0, k_1, \alpha_1, \beta_2)]$, where $T(\epsilon, m, k_0, k_1, \alpha_1, \beta_2)$ is a positive constant depending only on $\epsilon, m, k_0, k_1, \alpha_1, \beta_2$.

3.2. Existence of the De Turck flow

We establish the short time existence of the De Turck flow (3.2) on the whole manifold M. Fix a point $x_0 \in M$ and let $B_{\tilde{g}}(x_0, r)$ be the metric ball of radius r centered at x_0 with respect to the metric $\tilde{g}$.

Lemma 3.6. *Given positive constants r, δ, T. Suppose that $(g(t), \phi(t))$ is a solution of (3.2) on $B_{\tilde{g}}(x_0, r+\delta) \times [0, T]$, that is,*

$$\begin{aligned}
\partial_t g_{ij} &= -2R_{ij} + 2\alpha_1 \nabla_i\phi\nabla_j\phi + \nabla_i V_j + \nabla_j V_i, \quad \text{in } B_{\tilde{g}}(x_0, r+\delta) \times [0,T],\\
\partial_t \phi &= \Delta\phi + \beta_1|\nabla\phi|^2_g + \beta_2\phi + \langle V, \nabla\phi\rangle_g, \quad \text{in } B_{\tilde{g}}(x_0, r+\delta)\times[0,T],\\
(g_{ij}, \phi) &= (\tilde{g}_{ij}, \tilde{\phi}), \quad \text{on } D_T,
\end{aligned}$$

and $|\widetilde{\mathrm{Rm}}|^2_{\tilde{g}} \leq k_0, |\tilde{\nabla}\tilde{\phi}|^2_{\tilde{g}} \leq k_1$ on M. If

$$\left(1 - \frac{1}{80000(1+\alpha_1^2+\beta_1^2)m^{10}}\right)\tilde{g} \leq g(t) \leq \left(1 + \frac{1}{80000(1+\alpha_1^2+\beta_1^2)m^{10}}\right)\tilde{g} \tag{3.13}$$

on $B_{\tilde{g}}(x_0, r+\delta) \times [0,T]$, then there exists a positive constant $C = C(m, r, \delta, T, \tilde{g}, k_1)$ depending only on $m, r, \delta, T, \tilde{g}$, and k_1, such that

$$|\tilde{\nabla} g|^2_{\tilde{g}} \leq C, \quad |\tilde{\nabla}\phi|^2_{\tilde{g}} \leq C \tag{3.14}$$

on $B_{\tilde g}(x_0, r+\frac{\delta}{2})\times[0,T]$.

Proof. Using the $*$-notion, we can write (2.21) as

$$\partial_t g_{ij} = g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta g_{ij} + g^{-1}*g*\widetilde{\mathrm{Rm}} + g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g + 2\alpha_1\tilde\nabla_i\phi\tilde\nabla_j\phi.$$

Then

$$\begin{aligned}\partial_t\tilde\nabla g_{ij} = g^{\alpha\beta}\left(\nabla\tilde\nabla_\alpha\tilde\nabla_\beta g_{ij}\right) &+ g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla\tilde\nabla g + 2\alpha_1\tilde\nabla_i\phi\tilde\nabla\tilde\nabla_j\phi\\ &+ g^{-1}*g^{-1}*\tilde\nabla g*g*\widetilde{\mathrm{Rm}} + g^{-1}*\tilde\nabla g*\widetilde{\mathrm{Rm}}\\ &+ g^{-1}*g*\tilde\nabla\widetilde{\mathrm{Rm}} + g^{-1}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g;\end{aligned}\tag{3.15}$$

since $\tilde\nabla\tilde\nabla_\alpha\tilde\nabla_\beta g_{ij} = \tilde\nabla_\alpha\tilde\nabla_\beta\tilde\nabla g_{ij} + g*\tilde\nabla\widetilde{\mathrm{Rm}} + \tilde\nabla g*\widetilde{\mathrm{Rm}}$, we conclude from (3.15) that

$$\begin{aligned}\partial_t\tilde\nabla g_{ij} = g^{\alpha\beta}\left(\tilde\nabla_\alpha\tilde\nabla_\beta\tilde\nabla g_{ij}\right) &+ g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla\tilde\nabla g + 2\alpha_1\tilde\nabla_i\phi\tilde\nabla\tilde\nabla_j\phi\\ &+ g^{-1}*g^{-1}*\tilde\nabla g*g*\widetilde{\mathrm{Rm}} + g^{-1}*\tilde\nabla g*\widetilde{\mathrm{Rm}}\\ &+ g^{-1}*g*\tilde\nabla\widetilde{\mathrm{Rm}} + g^{-1}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g.\end{aligned}\tag{3.16}$$

It follows from (3.16) that

$$\begin{aligned}\partial_t|\tilde\nabla g|^2_{\tilde g} &= g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta|\tilde\nabla g|^2_{\tilde g} - 2g^{\alpha\beta}\left\langle\tilde\nabla_\alpha\tilde\nabla g, \tilde\nabla_\beta\tilde\nabla g\right\rangle_{\tilde g} + \tilde\nabla g*\tilde\nabla\phi*\tilde\nabla\tilde\nabla\phi\\ &+ \widetilde{\mathrm{Rm}}*g^{-1}*g^{-1}*g*\tilde\nabla g*\tilde\nabla g + \widetilde{\mathrm{Rm}}*g^{-1}*\tilde\nabla g*\tilde\nabla g + g^{-1}*g*\tilde\nabla\widetilde{\mathrm{Rm}}*\tilde\nabla g\\ &+ g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla\tilde\nabla g + g^{-1}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g.\end{aligned}\tag{3.17}$$

Since the closure $\overline{B_{\tilde g}(x_0, r+\delta)}$ is compact, we have

$$|\tilde\nabla\widetilde{\mathrm{Rm}}|_{\tilde g}\lesssim 1\tag{3.18}$$

on $\overline{B_{\tilde g}(x_0, r+\delta)}$, where $\lesssim$ depends on $r, \delta, \tilde g$. From (3.13) we get

$$\frac{1}{2}\tilde g\le g(t)\le 2\tilde g\tag{3.19}$$

on $B_{\tilde g}(x_0, r+\delta)\times[0,T]$. According to (3.18) and (3.19), we arrive at

$$\widetilde{\mathrm{Rm}}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g,\ \widetilde{\mathrm{Rm}}*g^{-1}*\tilde\nabla g*\tilde\nabla g,\ \tilde\nabla\widetilde{\mathrm{Rm}}*g^{-1}*g*\tilde\nabla g\lesssim|\tilde\nabla g|_{\tilde g},\tag{3.20}$$

where $\lesssim$ depends on $m, r, \delta, \tilde g$. From the explicit formulas we can see that $\tilde\nabla g*\tilde\nabla\phi*\tilde\nabla\tilde\nabla\phi\le 4\alpha_1 m^3|\tilde\nabla g|_{\tilde g}|\tilde\nabla\phi|_{\tilde g}|\tilde\nabla\tilde\nabla\phi|_{\tilde g}$, where we used a normal coordinate system of $\tilde g$. Similarly, $g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla\tilde\nabla g\le 72m^5|\tilde\nabla g|^2_{\tilde g}|\tilde\nabla\tilde\nabla g|_{\tilde g}$ and $g^{-1}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g\le 144m^6|\tilde\nabla g|^4_{\tilde g}$. Thus

$$\begin{aligned}\tilde\nabla g*\tilde\nabla\phi*\tilde\nabla\tilde\nabla\phi &\le 4\alpha_1 m^3|\tilde\nabla g|_{\tilde g}|\tilde\nabla\phi|_{\tilde g}|\tilde\nabla\tilde\nabla\phi|_{\tilde g},\\ g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla\tilde\nabla g &\le 72m^5|\tilde\nabla g|^2_{\tilde g}|\tilde\nabla\tilde\nabla g|_{\tilde g},\\ g^{-1}*g^{-1}*g^{-1}*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g*\tilde\nabla g &\le 144m^6|\tilde\nabla g|^4_{\tilde g}.\end{aligned}\tag{3.21}$$

Furthermore, using(3.19), we get

$$g^{\alpha\beta}\left\langle\tilde{\nabla}_\alpha\tilde{\nabla}g,\tilde{\nabla}_\beta\tilde{\nabla}g\right\rangle_{\tilde{g}}=g^{\alpha\beta}\tilde{g}^{ik}\tilde{g}^{j\ell}\tilde{g}^{\gamma\delta}\tilde{\nabla}_\alpha\tilde{\nabla}_\gamma g_{ij}\tilde{\nabla}_\beta\tilde{\nabla}_\delta g_{k\ell}\geq\frac{1}{2}|\tilde{\nabla}\tilde{\nabla}g|^2_{\tilde{g}}. \tag{3.22}$$

Substituting (3.20), (3.21), and (3.22) into (3.16) implies

$$\begin{aligned}\partial_t|\tilde{\nabla}g|^2_{\tilde{g}}\leq{}& g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla}g|^2_{\tilde{g}}-|\tilde{\nabla}^2g|^2_{\tilde{g}}+C_1|\tilde{\nabla}g|^2_{\tilde{g}}+C_1|\tilde{\nabla}g|_{\tilde{g}}\\&+72m^5|\tilde{\nabla}g|^2_{\tilde{g}}|\tilde{\nabla}^2g|_{\tilde{g}}+144m^6|\tilde{\nabla}g|^4_{\tilde{g}}+4\alpha_1m^3|\tilde{\nabla}g|_{\tilde{g}}|\tilde{\nabla}\phi|_{\tilde{g}}|\tilde{\nabla}^2\phi|_{\tilde{g}}\end{aligned} \tag{3.23}$$

for some positive constant C_1 depending only on $m, r, \delta, \tilde{g}$.

Using (2.22)and the Ricci identity, we have

$$\begin{aligned}\partial_t\tilde{\nabla}_k\phi={}&g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j(\tilde{\nabla}_k\phi)-g^{ij}\tilde{R}_{kijp}\tilde{\nabla}^p\phi+\tilde{\nabla}_kg^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi\\&+\beta_1\tilde{\nabla}_kg^{ij}\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi+2\beta_1g^{ij}\tilde{\nabla}_j\phi\tilde{\nabla}_i\tilde{\nabla}_k\phi+\beta_2\tilde{\nabla}_k\phi.\end{aligned} \tag{3.24}$$

The evolution equation (3.24) gives us the following equation

$$\begin{aligned}\partial_t|\tilde{\nabla}\phi|^2_{\tilde{g}}={}&g^{ij}\left(\tilde{\nabla}_i\tilde{\nabla}_j(\tilde{\nabla}_k\phi)\cdot2\tilde{g}^{k\ell}\tilde{\nabla}_\ell\phi\right)-2g^{ij}\tilde{g}^{k\ell}\tilde{R}_{kijp}\tilde{\nabla}_\ell\phi\tilde{\nabla}^p\phi\\&+2\tilde{g}^{k\ell}\tilde{\nabla}_kg^{ij}\tilde{\nabla}_\ell\phi\tilde{\nabla}_i\tilde{\nabla}_j\phi+2\beta_1\tilde{g}^{k\ell}\tilde{\nabla}_kg^{ij}\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi\tilde{\nabla}_\ell\phi\\&+4\beta_1\tilde{g}^{k\ell}g^{ij}\tilde{\nabla}_j\phi\tilde{\nabla}_\ell\phi\tilde{\nabla}_i\tilde{\nabla}_k\phi+2\beta_2\tilde{g}^{k\ell}\tilde{\nabla}_k\phi\tilde{\nabla}_\ell\phi.\end{aligned} \tag{3.25}$$

The identity $g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j|\tilde{\nabla}\phi|^2_{\tilde{g}}=g^{ij}(\tilde{\nabla}^i\tilde{\nabla}_j(\tilde{\nabla}_k\phi)\cdot2\tilde{g}^{k\ell}\tilde{\nabla}_\ell\phi)+2g^{ij}\tilde{g}^{k\ell}\tilde{\nabla}_i\tilde{\nabla}_k\phi\tilde{\nabla}_j\tilde{\nabla}_\ell\phi$ together with (3.18), implies that

$$g^{ij}\left(\tilde{\nabla}_i\tilde{\nabla}_j(\tilde{\nabla}_k\phi)\cdot2\tilde{g}^{k\ell}\tilde{\nabla}_\ell\phi\right)\leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j|\tilde{\nabla}\phi|^2_{\tilde{g}}-|\tilde{\nabla}^2\phi|^2_{\tilde{g}}. \tag{3.26}$$

Using (3.18) again and substituting (3.26) into (3.25), we arrive at

$$\begin{aligned}\partial_t|\tilde{\nabla}\phi|^2_{\tilde{g}}\leq{}&g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j|\tilde{\nabla}\phi|^2_{\tilde{g}}-|\tilde{\nabla}^2\phi|^2_{\tilde{g}}+C_2|\tilde{\nabla}\phi|^2_{\tilde{g}}+8m^3|\tilde{\nabla}g|_{\tilde{g}}|\tilde{\nabla}\phi|_{\tilde{g}}|\tilde{\nabla}^2\phi|_{\tilde{g}}\\&+8\beta_1m^3|\tilde{\nabla}g|_{\tilde{g}}|\tilde{\nabla}\phi|^3_{\tilde{g}}+16\beta_1m^2|\tilde{\nabla}\phi|^2_{\tilde{g}}|\tilde{\nabla}^2\phi|_{\tilde{g}}+2\beta_2|\tilde{\nabla}\phi|^2_{\tilde{g}}.\end{aligned} \tag{3.27}$$

As in [19], we consider the vector-valued tensor field

$$\mathbf{\Theta}(t):=(g(t),\phi(t)) \tag{3.28}$$

and define $\tilde{\nabla}^k\mathbf{\Theta}(t):=\left(\tilde{\nabla}^kg(t),\tilde{\nabla}^k\phi(t)\right)$ with $|\tilde{\nabla}^k\mathbf{\Theta}(t)|^2_{\tilde{g}}:=|\tilde{\nabla}^kg(t)|^2_{\tilde{g}}+|\tilde{\nabla}^k\phi(t)|^2_{\tilde{g}}$. From (3.23) and (3.27), we obtain

$$\begin{aligned}\partial_t|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\leq{}&g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}-|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}}+C_3|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}+C_3|\tilde{\nabla}\mathbf{\Theta}|_{\tilde{g}}\\&+(80+4\alpha_1+16\beta_1)m^5|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}|\tilde{\nabla}^2\mathbf{\Theta}|_{\tilde{g}}+(144+8\beta_1)m^6|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}},\end{aligned} \tag{3.29}$$

where C_3 is positive constant depending on $m, r, \delta, \tilde{g}$ and β_2. The inequality (3.29) is similar to the equation (11) in page 247 of [26] and the equation (3.28) in page 36 of [19], so that the proof is essentially without change anything. However, for our flow, we want to find the positive constant $\epsilon>0$ with $(1-\epsilon)\tilde{g}\leq g\leq(1+\epsilon)\tilde{g}$ on $B_{\tilde{g}}(x_0, r+\delta)\times[0,T]$ such that both functions $|\tilde{\nabla}g|^2_{\tilde{g}}$ and $|\tilde{\nabla}\phi|^2_{\tilde{g}}$ are bounded from above. The mentioned positive constant ϵ depends on m, α_1, β_1, and β_2; the aim of the following computations is to find an explicit formula for ϵ. We shall follow Shi's idea but do more slight work on calculus, in particular on the positive uniform constants we are going to obtain. By the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b \in \mathbf{R}, \quad \epsilon > 0, \tag{3.30}$$

we have $(80 + 4\alpha_1 + 16\beta_1)m^5|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}|\tilde{\nabla}^2\mathbf{\Theta}|_{\tilde{g}} \leq \frac{1}{2}|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} + 2(40 + 2\alpha_1 + 8\beta_1)^2 m^{10}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}}$ and $C_3|\tilde{\nabla}\mathbf{\Theta}|_{\tilde{g}} \leq C_3\frac{1+|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}}{2} = \frac{C_3}{2} + \frac{C_3}{2}|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}$. As a consequence of (3.29), we conclude that

$$\begin{aligned}\partial_t|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} - \frac{1}{2}|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} + C_4|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} + C_4 + \big(3344 + 8\alpha_1^2 \\ + 320\alpha_1 + 64\alpha_1\beta_1 + 128\beta_1^2 + 1280\beta_1 + 8\beta_1\big)\, m^{10}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}}\end{aligned} \tag{3.31}$$

for some positive constant C_4 depending only on $m, r, \delta, \tilde{g}$ and β_2.

Given $\epsilon := \frac{1}{Am^{10}} \leq \frac{1}{2}$, where A is a positive constant depending on α_1, β_1 and chosen later. If we choose a normal coordinate system so that $\tilde{g}_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i\delta_{ij}$, then we have

$$1 - \epsilon \leq \lambda_k \leq 1 + \epsilon, \quad \frac{1}{2} \leq \lambda_k \leq 2, \quad k = 1, \cdots, m. \tag{3.32}$$

Define

$$n := \frac{1}{\epsilon}, \quad a := \frac{n}{4}, \tag{3.33}$$

and

$$\varphi = \varphi(x,t) := a + \sum_{1\leq k\leq m}\lambda_k^n, \quad (x,t) \in B_{\tilde{g}}(x_0, r+\delta) \times [0,T]. \tag{3.34}$$

By the formula (16) in page 248 of [26], we can compute

$$\begin{aligned}\partial_t\varphi = n\sum_{1\leq k,\alpha,\beta\leq m}\lambda_k^{n-1}g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{kk} + 2n\sum_{1\leq k,\alpha\leq m}\lambda_k^{n-1}\frac{\lambda_k}{\lambda_\alpha}\tilde{R}_{k\alpha k\alpha} \\ + \frac{n}{2}\sum_{1\leq k,\alpha,p\leq m}\frac{\lambda_k^{n-1}}{\lambda_\alpha\lambda_p}\Big(\tilde{\nabla}_k g_{p\alpha}\tilde{\nabla}_k g_{p\alpha} + 2\tilde{\nabla}_\alpha g_{kp}\tilde{\nabla}_p g_{k\alpha} - 2\tilde{\nabla}_\alpha g_{kp}\tilde{\nabla}_\alpha g_{kp} \\ - 2\tilde{\nabla}_k g_{p\alpha}\tilde{\nabla}_\alpha g_{kp} - 2\tilde{\nabla}_k g_{p\alpha}\tilde{\nabla}_\alpha g_{kp}\Big).\end{aligned} \tag{3.35}$$

Since the second and the third on the right-hand side of (3.35) is bounded from above by $4n(1+\epsilon)^n m^2\sqrt{k_0}$ and $\frac{n}{2}m^3(1+\epsilon)^{n-1}4(4\times 2+1)|\tilde{\nabla}g|^2_{\tilde{g}} = 18nm^3(1+\epsilon)^{n-1}|\tilde{\nabla}g|^2_{\tilde{g}}$, respectively, it follows that

$$\partial_t\varphi \leq n\sum_{1\leq k,\alpha,\beta\leq m}\lambda_k^{n-1}g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{kk} + C_5 + 18nm^3(1+\epsilon)^{n-1}|\tilde{\nabla}g|^2_{\tilde{g}} \tag{3.36}$$

where C_5 is a positive constant depending only on m, A, and k_0. On the other hand, from (3.36) and the equation (19) in page 249 of [26], we have

$$\partial_t\varphi \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + C_5 + \left[18m^3n(1+\epsilon)^{n-1} - \frac{n(n-1)}{2}(1-\epsilon)^{n-2}\right]|\tilde{\nabla}g|^2_{\tilde{g}}. \tag{3.37}$$

Instead of the inequalities (20), (21), and (22) in page 249 of [26], we will prove uniform inequalities as follows (recall that $\epsilon := 1/n$):

(a) For any $n \geq 2$ and any m, we have

$$18m^3 n(1+\epsilon)^{n-1} \leq \frac{54m^3}{n+1}n^2. \tag{3.38}$$

(b) For any $n \geq 2$, we have

$$(1-\epsilon)^{n-2} \geq \frac{1}{4}. \tag{3.39}$$

(c) For any $n \geq 2$, we have

$$\frac{n(n-1)}{2}(1-\epsilon)^{n-2} \geq \frac{n^2}{8}. \tag{3.40}$$

To prove (3.38), we write $18m^3 n(1+\epsilon)^{n-1} = \frac{18m^3 n^2}{n+1}(1+\frac{1}{n})^n$; since the function $(1+1/x)^x$, $x > 0$, is in increasing in x, it follows that $18m^3 n(1+\epsilon)^{n-1} \leq \frac{18m^3 n^2}{n+1}e \leq \frac{54m^3}{n+1}n^2$. Since the function $(1-1/x)^x$, $x \geq 2$, is increasing in x, we obtain $(1-\epsilon)^{n-2} = (1-\frac{1}{n})^{n-2} \geq (1-\frac{1}{n})^n = 1/4$. From the proof of (3.39), together with (3.38), we arrive at $\frac{n(n-1)}{2}(1-\epsilon)^{n-2} \geq \frac{n(n-1)}{2}\frac{n^2}{(n-1)^2}\frac{1}{4} \geq \frac{n^2}{8}$. Substituting (3.38) and (3.40) into (3.37) implies $\partial_t \varphi \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + C_5 - (\frac{1}{8} - \frac{54m^3}{n+1})n^2|\tilde{\nabla}g|^2_{\tilde{g}}$; choosing

$$n \geq 864m^{10} > 864m^3, \tag{3.41}$$

we find that

$$\partial_t \varphi \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + C_5 - \frac{n^2}{16}|\tilde{\nabla}g|^2_{\tilde{g}}. \tag{3.42}$$

As the equation (24) in page 249 of [26], we have

$$\begin{aligned}\partial_t\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) \leq{}& g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) - 2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\tilde{\nabla}_\beta|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} - \frac{1}{2}\varphi|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} \\ &+ \left(3344 + 8\alpha_1^2 + 320\alpha_1 + 64\alpha_1\beta_1 + 128\beta_1^2 + 1280\beta_1 + 8\beta_1\right) \times \\ & m^{10}\varphi|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} + C_4\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} + C_4\varphi + C_5|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} - \frac{n^2}{16}|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}|\tilde{\nabla}g|^2_{\tilde{g}}.\end{aligned} \tag{3.43}$$

According to Corollary 2.10, we get $|\nabla\phi|^2_g \lesssim 1$ and hence $|\tilde{\nabla}\phi|^2_{\tilde{g}} = \tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi \leq 2|\nabla\phi|^2_g \lesssim 1$ where $\lesssim$ depends on α_1, β_2 and k_1. Consequently, (3.43) can be written as

$$\begin{aligned}\partial_t\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) \leq{}& g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) - 2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\tilde{\nabla}_\beta|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} - \frac{1}{2}\varphi|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} \\ &+ \left(3344 + 8\alpha_1^2 + 320\alpha_1 + 64\alpha_1\beta_1 + 128\beta_1^2 + 1280\beta_1 + 8\beta_1\right) \times \\ & m^{10}\varphi|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} + C_4\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} + C_4\varphi + C_6|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} - \frac{n^2}{16}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}},\end{aligned} \tag{3.44}$$

where C_6 is a positive constant depending only on $m, A, k_0, n, \alpha_1, \beta_2$, and k_1. From (3.32) and (3.34),

$$a + m(1-\epsilon)^n \leq \varphi \leq a + m(1+\epsilon)^n \tag{3.45}$$

on $B_{\tilde{g}}(x_0, r+\delta) \times [0, T]$, we arrive at (recall from (3.41) that $n = Am^{10}$ with $A \geq 864$) $Cm^{10}\varphi \leq n^2/(2A/C)$, where

$$C := 3344 + 8\alpha_1^2 + 320\alpha_1 + 64\alpha_1\beta_1 + 128\beta_1^2 + 1280\beta_1 + 8\beta_1. \tag{3.46}$$

If we choose

$$A \ge 16C, \tag{3.47}$$

then

$$Cm^{10}\varphi|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} - \frac{n^2}{16}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} \le -\frac{n^2}{32}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}}. \tag{3.48}$$

On the other hand, by the argument in the proof of (28) in page 250 of [26], we have

$$-2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\tilde{\nabla}_\beta|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} \le \frac{\varphi}{2}|\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} + \frac{288n^2m^{10}}{\varphi}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}}. \tag{3.49}$$

Plugging (3.48) and (3.49) into (3.44) implies

$$\begin{aligned}\partial_t\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) + \frac{288n^2m^{10}}{\varphi}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}}\\ - \frac{n^2}{32}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} + C_4\varphi + (C_4 + C_6)\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}},\end{aligned} \tag{3.50}$$

because $\varphi \ge a = \frac{n}{4} \ge 1$. According to $\varphi \ge a = \frac{n}{4} = \frac{A}{4}m^{10}$, we conclude that $28n^2m^{10}/\varphi \le 1152nm^{10} = n^2/(A/1152) \le n^2/64$, where we choose

$$A \ge 1152 \times 64 = 73728. \tag{3.51}$$

Consequently,

$$\partial_t\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\left(\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}\right) - \frac{n^2}{64}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} + C_4\varphi + (C_4 + C_6)\varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}. \tag{3.52}$$

Using the following inequality $\varphi \le a + m(1+\epsilon)^n \le \frac{n}{4} + 3m \le (\frac{1}{4} + \frac{3}{A})n \le \frac{18435}{73728}n \le 0.26n$, by (3.51), we get $\frac{n^2}{64}|\tilde{\nabla}\mathbf{\Theta}|^4_{\tilde{g}} \ge \frac{1}{5}\varphi^2|\tilde{\nabla}g|^4_{\tilde{g}}$. Defining

$$\psi := \varphi|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}}, \tag{3.53}$$

we obtain from the above the inequality and (3.52) that

$$\partial_t\psi \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta - \frac{1}{5}\psi^2 + (C_4 + C_6)\psi + C_4 n \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\psi - \frac{1}{10}\psi^2 + C_7 \tag{3.54}$$

on $B_{\tilde{g}}(x_0, r+\delta) \times [0, T]$, for some positive constant C_7 depending only m, A, k_0, n, α_1, β_2, and k_1. Using the cutoff function and going through the argument in [26], we can prove that $|\tilde{\nabla}\mathbf{\Theta}|^2_{\tilde{g}} \lesssim 1$ on $B_{\tilde{g}}(x_0, r+\frac{\delta}{2}) \times [0, T]$, where $\lesssim$ depends on $m, r, \delta, T, \tilde{g}, k_1$. Note that $(1 - \frac{1}{Am^{10}})\tilde{g} \le g \le (1 + \frac{1}{Am^{10}}\tilde{g}$, where $A \ge \max(73728, 16C)$. From the definition (3.46), we can estimate $C \le 4212 + 200\alpha_1^2 + 740\beta_1^2$. Then we may choose $A = 80000(1 + \alpha_1^2 + \beta_1^2)$. □

By the same method we can prove the higher order derivatives estimates for g.

Lemma 3.7. *Under the assumption in Lemma 3.6 where we furthermore assume $|\phi|^2 \le k_1$, for any nonnegative integer n, there exist positive constants $C_n = C(m, n, r, \delta, T, \tilde{g}, k_1)$ depending only on $m, n, r, \delta, T, \tilde{g}$, and k_1, such that*

$$|\tilde{\nabla}^n g|^2_{\tilde{g}} \le C_n, \quad |\tilde{\nabla}^n \phi|^2_{\tilde{g}} \le C_n \tag{3.55}$$

on $B_{\tilde{g}}(x_0, r + \frac{\delta}{n+1}) \times [0, T]$.

Proof. We prove this lemma by induction on n. If $n = 0$, using (3.19) we have $|g|^2_{\tilde{g}} = \tilde{g}^{ik}\tilde{g}^{j\ell}g_{ij}g_{k\ell} \le 4m$ on $B_{\tilde{g}}(x_0, r+\delta) \times [0, T]$. Since $|\tilde{\phi}|^2_{\tilde{g}} \le k_1$, it follows from (3.14) that $|\phi|^2_{\tilde{g}} \lesssim 1$ on $B_{\tilde{g}}(x_0, r + \frac{\delta}{2}) \times [0, T]$, where $\lesssim$ depends only on $m, r, \delta, T, \tilde{g}, k_1$ and is independent on x_0. Now we consider the annulus $B_{\tilde{g}}(x_0, r+\delta) \setminus B_{\tilde{g}}(x_0, r + \frac{\delta}{2})$. For any x in this annulus, we can find a small ball $B_{\tilde{g}}(x, \delta') \subset B_{\tilde{g}}(x_0, r+\delta) \setminus B_{\tilde{g}}(x_0, r+\frac{\delta}{2})$. Using (3.14) again, we have $|\phi|^2_{\tilde{g}} \lesssim 1$ on $B_{\tilde{g}}(x, \frac{\delta'}{2}) \times [0, T]$, where $\lesssim$ depends only on $m, r, \delta, T, \tilde{g}, k_1$ and is independent on x. In particular, $|\phi|^2_{\tilde{g}}(x) \lesssim 1$ on $[0, T]$. Hence $|\phi|^2_{\tilde{g}} \lesssim 1$ on $B_{\tilde{g}}(x_0, r+\delta) \times [0, T]$, where $\lesssim$ depends only on $m, r, \delta, T, \tilde{g}, k_1$ and is independent on x_0.

If $n = 1$, the estimate (3.55) was proved in Lemma 3.6. We now suppose that $n \ge 2$ and

$$|\tilde{\nabla}^k g|^2_{\tilde{g}} \le C_k, \quad |\tilde{\nabla}^k \phi|^2_{\tilde{g}} \le C_k, \quad k = 0, 1, \cdots, n-1, \tag{3.56}$$

on $B_{\tilde{g}}(x_0, r + \frac{\delta}{k+1}) \times [0, T]$. According to (3.39) in [19], we have

$$\begin{aligned}\partial_t \tilde{\nabla}^n g = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}^n g &+ \sum_{\ell_1+\ell_2=n+2,\ 1\le \ell_s \le n+1} \tilde{\nabla}^{\ell_1}\phi * \tilde{\nabla}^{\ell_2}\phi \\ &+ \sum_{\sum_{s=1}^{n+2} k_s \le n+2,\ 0\le k_s\le n+1} \tilde{\nabla}^{k_1} g * \cdots * \tilde{\nabla}^{k_{n+2}} g * P_{k_1\cdots k_{n+2}}\end{aligned} \tag{3.57}$$

where $P_{k_1\cdots k_{n+2}}$ is a polynomial in $g, g^{-1}, \widetilde{\mathrm{Rm}}, \cdots, \tilde{\nabla}^n\widetilde{\mathrm{Rm}}$. Similarly, from (2.22) we can show that

$$\begin{aligned}\partial_t \tilde{\nabla}^n \phi = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}^n \phi &+ \sum_{\sum_{s=1}^{n} k_s+\ell_1+\ell_2\le n+2,\ 0\le k_s\le n,\ 0\le \ell_1,\ell_2\le n+1} \\ &\tilde{\nabla}^{k_1} g * \cdots * \tilde{\nabla}^{k_n} g * \tilde{\nabla}^{\ell_1}\phi * \tilde{\nabla}^{\ell_2}\phi * P_{k_1\cdots k_n \ell_1 \ell_2}.\end{aligned} \tag{3.58}$$

Using the notion $\boldsymbol{\Theta}$ defined in (3.28), we conclude from (3.57) and (3.58) that

$$\begin{aligned}\partial_t \tilde{\nabla}^n \boldsymbol{\Theta} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}^n \boldsymbol{\Theta} \\ &+ \sum_{\sum_{s=1}^{n+2} k_s \le n+2,\ 0\le k_s\le n+1} \tilde{\nabla}^{k_1}\boldsymbol{\Theta} * \cdots * \tilde{\nabla}^{k_{n+2}}\boldsymbol{\Theta} * P_{k_1\cdots k_s}.\end{aligned} \tag{3.59}$$

The above equation is exact the equation (3.41) in [19] or the equation (69) in page 254 of [26], and, following the same argument, we obtain $|\tilde{\nabla}^n \boldsymbol{\Theta}|^2_{\tilde{g}} \lesssim 1$ on $B_{\tilde{g}}(x_0, r + \frac{\delta}{n+1}) \times [0, T]$, where $\lesssim$ depends only on $n, m, r, \delta, T, \tilde{g}, k_1$. □

Fix a point $x_0 \in M$ and choose a family of domains $(D_k)_{k\in\mathbf{N}}$ on M such that for each k, ∂D_k is a compact smooth $(m-1)$-dimensional submanifold of M and

$$\bar{D}_k = D_k \cup \partial D_k \text{ is a compact subset of } M, \quad B_{\tilde{g}}(x_0, k) \subset D_k.$$

By the same argument used in [26], together with Theorem 3.5 and Lemma 3.7, we have

Theorem 3.8. *Suppose that $(M,\tilde g)$ is a smooth complete Riemannian manifold of dimension m with $|\widetilde{\mathrm{Rm}}|^2_{\tilde g}\le k_0$ and $\tilde\phi$ be a smooth function satisfying $|\phi|^2+|\tilde\nabla\phi|^2_{\tilde g}\le k_1$ on M. There exists a positive constant $T=T(m,k_0,k_1,\alpha_1,\beta_1,\beta_2)$ such that the flow*

$$\begin{aligned}\partial_t g_{ij} &= -2R_{ij}+2\alpha_1\nabla_i\phi\nabla_j\phi+\nabla_iV_j+\nabla_jV_i,\\ \partial_t\phi &= \Delta\phi+\beta_1|\nabla\phi|^2_g+\beta_2\phi+\langle V,\nabla\phi\rangle_g,\\ (g(0),\phi(0)) &= (\tilde g,\tilde\phi),\end{aligned}$$

has a smooth solution $(g(t),\phi(t))$ on $M\times[0,T]$ that satisfies the estimate

$$\left(1-\frac{1}{80000(1+\alpha_1^2+\beta_1^2)m^{10}}\right)\tilde g\le g(t)\le\left(1+\frac{1}{80000(1+\alpha_1^2+\beta_1^2)m^{10}}\right)\tilde g$$

on $M\times[0,T]$. Moreover $|\phi(t)|^2\lesssim 1$ where $\lesssim$ depends on $m,k_0,\alpha_1,\beta_1,\beta_2$.

Proof. By the regularity of the flow and applying Corollary to D_k, we have $|\nabla\phi|^2_g\lesssim 1$ on D_k, where $\lesssim$ depends only on k_1,α_1,β_2; then $|\tilde\nabla\phi|^2_{\tilde g}\lesssim 1$ on D_k, where $\lesssim$ depends only on $m,k_1,\alpha_1,\beta_1,\beta_2$. Letting $k\to\infty$, we see that $|\tilde\nabla\phi|^2_{\tilde g}\lesssim 1$ on M, where $\lesssim$ depends only on $m,k_1,\alpha_1,\beta_1,\beta_2$. In particular, $|\phi(t)|^2\lesssim 1$ on M, where $\lesssim$ depends only on $m,k_0,\alpha_1,\beta_1,\beta_2$. □

3.3. First order derivative estimates

Let $\tilde\phi$ be a smooth function on a smooth complete Riemannian manifold $(M,\tilde g)$ of dimension m. Assume

$$|\widetilde{\mathrm{Rm}}|^2_{\tilde g}\le k_0,\quad |\tilde\phi|^2+|\tilde\nabla\tilde\phi|^2_{\tilde g}\le k_1,\quad |\tilde\nabla^2\tilde\phi|^2_{\tilde g}\le k_2 \tag{3.60}$$

on M. Let $(g(t),\phi(t)),T$ be obtained in Theorem 3.8 and

$$\delta:=\frac{1}{80000(1+\alpha_1^2+\beta_1^2)m^{10}}. \tag{3.61}$$

Then

$$(1-\delta)\tilde g\le g(t)\le(1+\delta)\tilde g \tag{3.62}$$

on $M\times[0,T]$. As in [26], define

$$h_{ij}:=g_{ij}-\tilde g,\quad H_{ij}:=\frac{1}{\delta}h_{ij}. \tag{3.63}$$

Then $\partial_t h_{ij}=g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta h_{ij}+A_{ij}$, where

$$\begin{aligned}A_{ij} \;=\; & g^{\alpha\beta}g_{ip}\tilde g^{pq}\tilde R_{j\alpha q\beta}+g^{\alpha\beta}g_{jp}\tilde g^{pq}\tilde R_{i\alpha q\beta}+2\alpha_1\tilde\nabla_i\phi\tilde\nabla_j\phi+\frac12 g^{\alpha\beta}g^{pq}\\ &\left(\tilde\nabla_i h_{p\alpha}+2\tilde\nabla_\alpha h_{jp}\tilde\nabla_q h_{i\beta}-2\tilde\nabla_\alpha h_{jp}\tilde\nabla_\beta h_{iq}-2\tilde\nabla_j h_{p\alpha}\tilde\nabla_\beta h_{iq}-2\tilde\nabla_i h_{p\alpha}\tilde\nabla_\beta h_{jq}\right).\end{aligned} \tag{3.64}$$

From $\delta<1/2$ and (3.60) we have from (3.64) that

$$-\left(8m\sqrt{k_0}+20|\tilde\nabla h|^2_{\tilde g}\right)\tilde g\le A_{ij}\le\left(8m\sqrt{k_0}+20|\tilde\nabla h|^2_{\tilde g}\right)\tilde g \tag{3.65}$$

on $M \times [0, T]$. Therefore

$$\partial_t H_{ij} = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta H_{ij} + B_{ij}, \quad H(0) = 0, \tag{3.66}$$

where $B_{ij} := A_{ij}/\delta$ satisfying

$$-\left(\frac{8m\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}H|^2_{\tilde{g}}\right)\tilde{g} \le B_{ij} \le \left(\frac{8m\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}H|^2_{\tilde{g}}\right)\tilde{g} \tag{3.67}$$

on $M \times [0, T]$. As in [19], define

$$\psi := \phi - \tilde{\phi}, \quad \Psi := \delta\psi. \tag{3.68}$$

Then $\partial_t\psi = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\psi + C$, where

$$C := g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\phi} + \beta_1|\tilde{\nabla}\tilde{\phi}|^2_g + \beta_2\tilde{\phi} + \beta_1|\tilde{\nabla}\psi|^2_g + \beta_2\psi + 2\beta_1\langle\tilde{\nabla}\psi, \tilde{\nabla}\tilde{\phi}\rangle_g. \tag{3.69}$$

From $2\delta < 1$, (3.60) and the proof of Theorem 3.8, we have from (3.69) that $|C| \le 2m\sqrt{k_0} + 2\beta_1 k_1 + \beta_2\sqrt{k_1} + \beta_2|\psi| + 2\beta_1|\tilde{\nabla}\psi|^2_{\tilde{g}} \lesssim 1$, where $\lesssim$ depends on $m, k_0, k_1, \alpha_1, \beta_1, \beta_2$. Consequently, $\partial_t H_{ij} = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta H_{ij} + B_{ij}$, $\partial_t\Psi = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\Psi + D$, and $H(0) = \Psi(0) = 0$. Since

$$-\tilde{g} \le H(t) \quad \le \quad \tilde{g}, \quad \frac{1}{1+\delta}\tilde{g}^{-1} \quad \le \quad g^{-1} \quad \le \quad \frac{1}{1-\delta}\tilde{g}^{-1},$$
$$|\tilde{\nabla}g^{-1}(t)|^2_{\tilde{g}} \le \frac{\delta^2}{(1-\delta)^4}|\tilde{\nabla}H(t)|^2_{\tilde{g}}, \quad |\Psi(t)|^2 \quad \le \quad 1$$

By the argument in [26], we arrive at $\sup_{M\times[0,T]}(|\tilde{\nabla}H(t)|^2_{\tilde{g}} + |\tilde{\nabla}\Psi(t)|^2_{\tilde{g}}) \le 1$ where $\lesssim$ depends only on $m, k_0, k_1, k_1, \alpha_1, \beta_1, \beta_2$.

Theorem 3.9. *There exists a positive constant $T = T(m, k_0, k_1, \alpha_1, \beta_1, \beta_2)$ depending only on $m, k_0, k_1, \alpha_1, \beta_1, \beta_2$ such that*

$$\sup_{M\times[0,T]}|\tilde{\nabla}g(t)|^2_{\tilde{g}} \le C, \qquad \sup_{M\times[0,T]}|\tilde{\nabla}\phi|^2_{\tilde{g}} \le C$$

where C is a positive constant depending only on $m, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

3.4. Second order derivative estimates

In this subsection we derive the second order derivative estimates. Let $(M, \tilde{g})$ be a smooth complete Riemannian manifold of dimension m and $\tilde{\phi}$ a smooth function on M, satisfying

$$|\widetilde{\mathrm{Rm}}|_{\tilde{g}} \le k_0, \quad |\tilde{\phi}|^2 + |\tilde{\nabla}\tilde{\phi}|^2_{\tilde{g}} \le k_1, \quad |\tilde{\nabla}^2\tilde{\phi}|^2_{\tilde{g}} \le k_2. \tag{3.70}$$

From Theorem 3.8 and Theorem 3.9, there exists a positive constant T depending on $m, k_0, k_1, \alpha_1, \beta_1, \beta_2$ such that the flow

$$\partial_t g_{ij} = -2R_{ij} + 2\alpha_1\nabla_i\phi\nabla_j\phi + \nabla_i V_j + \nabla_j V_i,$$
$$\partial_t\phi = \Delta\phi + \beta_1|\nabla\phi|^2_g + \beta_2\phi + \langle V, \nabla\phi\rangle_g,$$
$$(g(0), \phi(0)) = (\tilde{g}, \tilde{\phi}),$$

has a smooth solution $(g(t), \phi(t))$ on $M \times [0, T]$ that satisfies the estimates

$$\frac{1}{2}\tilde{g} \le g(t) \le 2\tilde{g}, \quad |\tilde{\phi}|^2 \lesssim 1, \quad |\tilde{\nabla} g|^2_{\tilde{g}} \lesssim 1, \quad |\tilde{\nabla}\phi|^2_{\tilde{g}} \lesssim 1, \tag{3.71}$$

where $\lesssim$ (used throughout this subsection) depend only on $m, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

According to Lemma 2.5 and the proof of equation (34) in page 266 of [26], we have

$$\partial_t \mathrm{Rm} = \Delta \mathrm{Rm} + g^{*-2} * \mathrm{Rm}^{*2} + g^{-1} * V * \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V + \nabla^2\phi * \nabla^2\phi. \tag{3.72}$$

Recall the formula (28) in page 266 of [26], that $\partial_t V = \Delta V + g^{*-2} * \mathrm{Rm} * V + g^{-1} * V + * \partial_t g + g^{-1} * \tilde{\nabla} g * \partial_t g^{-1} * g$. Since $\partial_t g = g^{-1} * \mathrm{Rm} + \nabla V + \nabla\phi * \nabla\phi$, $\partial_t g^{-1} = g^{-1} * g^{-1} * \partial_t g$ and $V = g^{-1} * \tilde{\nabla} g$, it follows that

$$\begin{aligned} \partial_t V = {} & \Delta V + g^{*-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{*-2} * \tilde{\nabla} g * \nabla V + g^{*-4} * g * \tilde{\nabla} g * \mathrm{Rm} \\ & + g^{*-3} * g * \tilde{\nabla} g * \nabla V + g^{*-2} * \tilde{\nabla} g * \nabla\phi * \nabla\phi + g^{*-3} * g * \tilde{\nabla} g * \nabla\phi * \nabla\phi. \end{aligned} \tag{3.73}$$

Since $\partial_t \nabla V = \nabla(\partial_t V) + V * \partial_t \Gamma$ and $\partial_t \Gamma = g^{-1} * \nabla(\partial_t g)$, it follows that (as the proof of the equation (99) in page 278 of [26])

$$\begin{aligned} \partial_t \nabla V = {} & \Delta \nabla V + g^{*-3} * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{*-3} * \tilde{\nabla} g * \nabla \mathrm{Rm} + g^{*-2} * \nabla\tilde{\nabla} g * \nabla V \\ & + g^{*-2} * \tilde{\nabla} g * \nabla^2 V + g^{*-4} * g * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{*-4} * g * \tilde{\nabla} g * \nabla \mathrm{Rm} \\ & + g^{*-3} * g * \nabla\tilde{\nabla} g * \nabla V + g^{*-3} * g * \tilde{\nabla} g * \nabla^2 V \\ & + g^{*-2} * \nabla\tilde{\nabla} g * \nabla\phi * \nabla\phi + g^{*-2} * \tilde{\nabla} g * \nabla\phi * \nabla^2\phi \\ & + g^{*-3} * g * \nabla\tilde{\nabla} g * \nabla\phi * \nabla\phi + g^{*-3} * g * \tilde{\nabla} g * \nabla\phi * \nabla^2\phi. \end{aligned} \tag{3.74}$$

By the diffeomorphisms $(\Psi_t)_{t\in[0,T]}$ defined by (2.17), we have

$$\begin{aligned} \partial_t \hat{g}_{ij}(x,t) &= -2\hat{R}_{ij}(x,t) + 2\alpha_1 \hat{\nabla}_i \hat{\phi}(x,t) \hat{\nabla}_j \hat{\phi}(x,t), \\ \partial_t \hat{\phi}(x,t) &= \hat{\Delta}\hat{\phi}(x,t) + \beta_1 |\hat{\nabla}\hat{\phi}|^2_{\hat{g}}(x,t) + \beta_2 \hat{\phi}(x,t), \end{aligned}$$

where $\hat{g}(x,t)$ and $\hat{\phi}(x,t)$ are defined by (2.16). Then $\nabla_i \nabla_j \phi = y^\alpha{}_{,i} y^\beta{}_{,j} \hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{\phi}$ and $y^\alpha{}_{,i} := \frac{\partial}{\partial x^i} y^\alpha$, and hence

$$\begin{aligned} \partial_t \nabla_i \nabla_j \phi(x,t) = {} & y^\alpha{}_{,i} y^\beta{}_{,j} \partial_t \hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{\phi}(y,t) + y^\alpha{}_{,i} y^\beta{}_{,j} \partial_t y^\gamma \frac{\partial}{\partial y^\gamma} \hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{\phi}(y,t) \\ & + \partial_t \left(y^\alpha{}_{,i} y^\beta{}_{,j} \right) \hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{\phi}(y,t). \end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned} y^\alpha{}_{,i} y^\beta{}_{,j} \partial_t \hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{\phi}(y,t) \quad = \quad & \Delta \nabla_i \nabla_j \phi + g^{*-2} * \mathrm{Rm} * \nabla^2\phi + \beta_2 \nabla^2\phi \\ + g^{-1} * \nabla\phi * \nabla\phi * \nabla^2\phi + g^{-1} * \nabla\phi * \nabla^3\phi & + g^{-1} * \nabla^2\phi * \nabla^2\phi + g^{*-2} * \mathrm{Rm} * \nabla\phi * \nabla\phi. \end{aligned}$$

Using (17) and (18) in page 263 of [26], we can conclude that

$$I + J = g^{-1} * V * \nabla^3\phi + g^{-1} * \nabla V * \nabla^2\phi.$$

Combining those identities yields

$$\begin{aligned}\partial_t \nabla^2\phi &= \Delta\nabla^2\phi + g^{*-2} * \mathrm{Rm} * \nabla^2\phi + \beta_2\nabla^2\phi + g^{-1} * \nabla\phi * \nabla\phi * \nabla^2\phi \\ &\quad + g^{-1} * \nabla\phi * \nabla^3\phi + g^{-1} * \nabla^2\phi * \nabla^2\phi + g^{*-2} * \mathrm{Rm} * \nabla\phi * \nabla\phi \\ &\quad + g^{-1} * V * \nabla^3\phi + g^{-1} * \nabla V * \nabla^2\phi.\end{aligned} \tag{3.75}$$

The volume form $dV := dV_{g(t)} = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^m$ evolves by

$$\partial_t dV = \frac{1}{2} g^{ij}\partial_t g_{ij}\, dV = \left(-R + \alpha_1|\nabla\phi|^2_g + \mathrm{div}_g V\right) dV. \tag{3.76}$$

In particular, $dV = d\tilde{V}$. For any point $x_0 \in M$ and any $r > 0$ we denote by $B_{\tilde{g}}(x_0, r)$ the metric ball with respect to $\tilde{g}$. Recall the definition $\mathbf{\Theta} = (g, \phi)$.

Lemma 3.10. *We have*

$$\int_0^T \left(\int_{B_{\tilde{g}}(x_0,r)} |\tilde{\nabla}^2\mathbf{\Theta}|^2_{\tilde{g}} d\tilde{V} \right) dt \lesssim 1$$

where $\lesssim$ *depends on* $m, r, k_0, k_1, \alpha_1, \beta_1, \beta_2$.

Proof. As in the proof of Lemma 6.2 in [26], we chose a cutoff function $\xi(x) \in C_0^\infty(M)$ such that $|\tilde{\nabla}\xi|_{\tilde{g}} \le 8$ in M and

$$\xi = \begin{cases} 1, & B_{\tilde{g}}(x_0, r), \\ 0, & M \setminus B_{\tilde{g}}(x_0, r + 1/2), \end{cases} \qquad 0 \le \xi \le 1 \text{ in } M. \tag{3.77}$$

Since $m = g^{ij}g_{ij}$, it follows that the constant 1 can be replaced by $g^{-1} * g$. From (3.16) we have

$$\begin{aligned} I &:= \frac{d}{dt} \int_{B_{\tilde{g}}(x_0,r+1)} |\tilde{\nabla} g|^2_{\tilde{g}} \xi^2\, d\tilde{V} = 2 \int_{B_{\tilde{g}}(x_0,r+1)} \langle \tilde{\nabla} g, g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla} g\rangle_{\tilde{g}}\xi^2\, d\tilde{V} \\ &+ \int_{B_{\tilde{g}}(x_0,r+1)} g^{-1} * g * \tilde{\nabla} g * \tilde{\nabla}\widetilde{\mathrm{Rm}}\,\xi^2\, d\tilde{V} + \int_{B_{\tilde{g}}(x_0,r+1)} \Big[g^{*-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} \\ &+ g^{*-2} * \tilde{\nabla} g * \tilde{\nabla}^2 g + g^{*-3} * (\tilde{\nabla} g)^{*3} + \tilde{\nabla}\phi * \tilde{\nabla}^2\phi \Big] * \tilde{\nabla} g\, \xi^2\, d\tilde{V} := I_1 + I_2 + I_3. \end{aligned}$$

Now the following computations are similar to that given in [19]. For convenience, we give a self-contained proof. By the Bishop-Gromov volume comparison (see [9]), we have

$$\int_{B_{\tilde{g}}(x_0,r+1)} d\tilde{V} \lesssim 1. \tag{3.78}$$

By the estimate (3.78), using (3.70) and (3.71), the I_3-term can be rewritten as

$$I_3 \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} \left(|\tilde{\nabla}^2 g|_{\tilde{g}} + |\tilde{\nabla}^2\phi|_{\tilde{g}}\right)\xi^2\, d\tilde{V}. \tag{3.79}$$

The I_1-term and I_2-term were computed in [26] (see (54) and (58) in page 270)

$$I_1 \le -\frac{1}{2}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|^2_{\tilde g}\xi^2\,d\tilde V + C_1, \tag{3.80}$$

$$I_2 \lesssim 1 + \int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|_{\tilde g}\xi\,d\tilde V. \tag{3.81}$$

From (3.79), (3.80), and (3.81), we arrive at

$$I \le -\frac{1}{4}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|^2_{\tilde g}\xi^2\,d\tilde V + C_2\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 \phi|_{\tilde g}\xi\,d\tilde V + C_2. \tag{3.82}$$

From (3.24) we have

$$\begin{aligned} J &:= \frac{d}{dt}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla\phi|^2_{\tilde g}\xi^2\,d\tilde V = 2\int_{B_{\tilde g}(x_0,r+1)} \langle\tilde\nabla\phi, g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta\tilde\nabla\phi\rangle_{\tilde g}\xi^2\,d\tilde V \\ &+ \int_{B_{\tilde g}(x_0,r+1)} \xi^2\Big[g^{*-2} * \widetilde{\mathrm{Rm}} * \tilde\nabla\phi + g^{*-2} * \tilde\nabla g * \tilde\nabla^2\phi + g^{*-2} * \tilde\nabla g * (\tilde\nabla\phi)^{*2} \\ &\quad + g^{-1} * \tilde\nabla\phi * \tilde\nabla^2\phi + \tilde\nabla\phi\Big]\tilde\nabla\phi\,d\tilde V := J_1 + J_2. \end{aligned}$$

By the estimate (3.78), the J_2-term can be bounded by

$$J_2 \lesssim 1 + \int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\phi|_{\tilde g}\xi\,d\tilde V. \tag{3.83}$$

From the integration by parts, we obtain

$$J_1 \le -\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\phi|^2_{\tilde g}\xi^2\,d\tilde V + C_3\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\phi|_{\tilde g}\xi\,d\tilde V. \tag{3.84}$$

From (3.83) and (3.84),

$$J \le -\frac{1}{2}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\phi|^2_{\tilde g}\xi^2\,d\tilde V + C_4. \tag{3.85}$$

Together with (3.82), we arrive at

$$\frac{d}{dt}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla\mathbf{\Theta}|^2_{\tilde g}\xi^2\,d\tilde V \le -\frac{1}{4}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\mathbf{\Theta}|^2_{\tilde g}\xi^2\,d\tilde V + C_5. \tag{3.86}$$

Integrating (3.86) over $[0,T]$ implies $\int_0^T(\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2\mathbf{\Theta}|^2_{\tilde g}\xi^2\,d\tilde V)dt \lesssim 1$. Since $\xi = 1$ on $B_{\tilde g}(x_0,r)$, the above estimate yields the desired inequality. $\square$

Using (3.71) we have $|\tilde\nabla^2 g|^2_g \le 16|\tilde\nabla^2 g|^2_{\tilde g}$, $|\tilde\nabla^2\phi|^2_g \le 4|\tilde\nabla^2\phi|^2_{\tilde g}$, and $dV \le 2^{m/2}d\tilde V$ on $M\times[0,T]$ and then

$$\int_0^T\left(\int_{B_{\tilde g}(x_0,r)} |\tilde\nabla^2\mathbf{\Theta}|^2_g dV\right)dt \lesssim \int_0^T\left(\int_{B_{\tilde g}(x_0,r)} |\tilde\nabla^2\mathbf{\Theta}|^2_{\tilde g} d\tilde V\right)dt.$$

By (66) in page 272 of [26], we have $|\nabla\tilde{\nabla}g|_g^2 \leq 2|\tilde{\nabla}^2 g|_g^2 + C_6$; on the other hand, by $\nabla\tilde{\nabla}\phi = \tilde{\nabla}^2\phi + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}\phi$, we get $|\nabla\tilde{\nabla}\phi|_g^2 \leq 2|\tilde{\nabla}^2\phi|_g^2 + C_7$. Thus

$$\int_0^T \left(\int_{B_{\tilde{g}}(x_0,r)} |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2 dV \right) dt \lesssim 1 + \int_0^T \left(\int_{B_{\tilde{g}}(x_0,r)} |\tilde{\nabla}^2\mathbf{\Theta}|_g^2 dV \right) dt.$$

Therefore,

Lemma 3.11. *We have*

$$\int_0^T \left(\int_{B_{\tilde{g}}(x_0,r)} \left(|\tilde{\nabla}^2\mathbf{\Theta}|_g^2 + |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2 \right) dV \right) dt \lesssim 1,$$

where $\lesssim$ depends on $m, r, k_0, k_1, \alpha_1, \beta_1, \beta_2$.

We now prove the integral estimates for Rm, $\nabla^2\phi$, and ∇V. The similar results were proved by Shi [26] for the Ricci flow and List [19] for the Ricci flow coupled with the heat flow.

Lemma 3.12. *We have*

$$\int_{B_{\tilde{g}}(x_0,r)} \left(|\mathrm{Rm}|_g^2 + |\nabla^2\phi|_g^2 + |\nabla V|_g^2 \right) dV \lesssim 1$$

where $\lesssim$ depends on $m, r, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

Proof. Keep to use the same cutoff function $\xi(x)$ introduced in the proof of Lemma 3.10. From $|\mathrm{Rm}|_g^2 = g^{i\alpha}g^{j\beta}g^{k\gamma}g^{\ell\delta}R_{ijk\ell}R_{\alpha\beta\gamma\delta}$, we get $\partial_t|\mathrm{Rm}|_g^2 = 2\langle \mathrm{Rm}, \partial_t\mathrm{Rm}\rangle_g + \mathrm{Rm}^{*2} * g^{*-3} * \partial_t g^{-1}$ and

$$\begin{aligned} \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2\xi^2\, dV &= \int_{B_{\tilde{g}}(x_0,r+1)} |\widetilde{\mathrm{Rm}}|_{\tilde{g}}^2\xi^2\, d\tilde{V} + \int_0^t \left(\int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2\xi^2\partial_t dV \right) dt \\ &+ \int_0^t \left[\int_{B_{\tilde{g}}(x_0,r+1)} \left(2\langle \mathrm{Rm}, \partial_t\mathrm{Rm}\rangle_g + g^{*-3} * \mathrm{Rm}^{*2} * \partial_t g^{-1} \right) \xi^2\, dV \right] dt. \end{aligned} \tag{3.87}$$

By the estimate (3.78) we have

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\widetilde{\mathrm{Rm}}|_{\tilde{g}}^2\xi^2\, d\tilde{V} \lesssim 1. \tag{3.88}$$

Using (3.76) implies

$$\begin{aligned} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2\xi^2\partial_t dV dt &= \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} \\ &+ g^{*-6} * \mathrm{Rm}^{*3} + g^{*-5} * \mathrm{Rm}^{*2} * \nabla V \Big) \xi^2\, dV dt. \end{aligned} \tag{3.89}$$

By the evolution $\partial_t g^{-1} = g^{*-3} * \mathrm{Rm} + g^{*-2} * \nabla V + g^{*-2} * (\nabla\phi)^{*2}$ above (3.73), we get

$$\begin{aligned}&\int_0^t \int_{B_{\tilde g}(x_0,r+1)} g^{*-3} * \mathrm{Rm}^{*2} * \partial_t g^{-1} \xi^2 \, dV dt \\ &= \int_0^t \int_{B_{\tilde g}(x_0,r+1)} g^{*-5} * \mathrm{Rm}^{*2} * \left(g^{-1} * \mathrm{Rm} + \nabla V + (\nabla\phi)^{*2}\right) \xi^2 \, dV dt.\end{aligned} \tag{3.90}$$

According to (3.72),

$$\begin{aligned}&2\int_0^t \int_{B_{\tilde g}(x_0,r+1)} \langle \mathrm{Rm}, \partial_t \mathrm{Rm}\rangle_g \xi^2 \, dV_t dt \\ &= 2\int_0^t \int_{B_{\tilde g}(x_0,r+1)} \langle \mathrm{Rm}, \Delta \mathrm{Rm}\rangle_g \xi^2 \, dV dt + \int_0^t \int_{B_{\tilde g}(x_0,r+1)} \Big(g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \\ &\quad + g^{*-5} * \mathrm{Rm}^{*2} * \nabla V + g^{*-5} * V * \mathrm{Rm} * \nabla \mathrm{Rm} + g^{*-6} * \mathrm{Rm}^{*3} \Big) \xi^2 \, dV dt.\end{aligned} \tag{3.91}$$

Substituting (3.88), (3.89), (3.90), and (3.91), into (3.87), we arrive at

$$\begin{aligned}\int_{B_{\tilde g}(x_0,r+1)} |\mathrm{Rm}|_g^2 \xi^2 \, dV \quad &\le \quad C_1 + 2\int_0^t \int_{B_{\tilde g}(x_0,r+1)} \langle \mathrm{Rm}, \Delta \mathrm{Rm}\rangle_g \xi^2 \, dV dt \\ &+ \int_0^t \int_{B_{\tilde g}(x_0,r+1)} \Big(g^{*-6} * \mathrm{Rm}^{*3} + g^{*-5} * \mathrm{Rm}^{*2} * \nabla V + g^{*-5} * V * \mathrm{Rm} * \nabla \mathrm{Rm} \\ &\quad + g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} + g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \Big) \xi^2 \, dV dt,\end{aligned} \tag{3.92}$$

where C_1 is a uniform positive constant independent of t and x_0. The second term on the right-hand side of (3.92) was estimated in [26] (equation (78) in page 274):

$$\begin{aligned}&2\int_0^t \int_{B_{\tilde g}(x_0,r+1)} \langle \mathrm{Rm}, \Delta \mathrm{Rm}\rangle_g \xi^2 \, dV dt \\ &\le -\frac{3}{2}\int_0^t \int_{B_{\tilde g}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV_t dt + C_2 \int_0^t \int_{B_{\tilde g}(x_0,r+1)} |\mathrm{Rm}|_g^2 dV dt.\end{aligned} \tag{3.93}$$

Using $V = g^{-1} * \tilde\nabla g$ and (3.71), as showed in [26] (equations (80), (81), (88), pages 275–277), we obtain

$$\int_0^t \int_{B_{\tilde g}(x_0,r+1)} g^{*-5} * V * \mathrm{Rm} * \nabla \mathrm{Rm} \, \xi^2 \, dV dt$$

$$\leq \frac{1}{4}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV_t dt + C_3 \int_0^k \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 \, dV dt, \tag{3.94}$$

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-5} * \mathrm{Rm}^{*2} * \nabla V \, \xi^2 \, dV dt$$

$$\leq \frac{1}{8}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV dt + C_4 \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 dV dt, \tag{3.95}$$

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-6} * \mathrm{Rm}^{*3} \, \xi^2 \, dV dt$$

$$\leq \frac{1}{8}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV dt + C_5 \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 dV dt. \tag{3.96}$$

Plugging (3.93), (3.94), (3.95), (3.96) into (3.92), yields

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 \xi^2 \, dV$$

$$\leq -\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV dt + C_6 \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 dV dt + C_7$$

$$+ \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} + g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \Big) \xi^2 \, dV dt. \tag{3.97}$$

Using (3.71) and the Cauchy-Schwarz inequality, we can conclude that $g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} \lesssim |\mathrm{Rm}|_g^2 |\nabla\phi|_g^2$; but $|\nabla\phi|_g^2 = g^{ij}\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi \leq 2|\tilde{\nabla}\phi|_{\tilde{g}}^2 \lesssim 1$, the above quantity $g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2}$ is bounded from above by $|\mathrm{Rm}|_g^2$. From the equation (90) in page 277 of [26], we have $g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} \lesssim 1 + |\nabla\tilde{\nabla}g|_{\tilde{g}}^2 \lesssim 1 + |\nabla\tilde{\nabla}g|_g^2 \lesssim 1 + |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2$; this estimate together with Lemma 3.11 gives us

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} \, \xi^2 \, dV dt \lesssim 1. \tag{3.98}$$

To deal with the last term on the right-hand side of (3.97), we perform the integration by parts to obtain

$$\int_{B_{\tilde{g}}(x_0,r+1)} g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \, \xi^2 \, dV \leq \frac{1}{2}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV$$
$$+ \epsilon \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2 \, dV + C_{11} \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2 dV$$

using (3.71), $\nabla^2\phi = \nabla\tilde{\nabla}\phi$, and $|\mathrm{Rm}|_g^2 \lesssim 1 + |\nabla\tilde{\nabla}g|_g^2$. Note also that the second term on the right-hand side of (3.97) is uniformly bounded by the same estimate for $|\mathrm{Rm}|_g^2$ and Lemma 3.11. Consequently,

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\mathrm{Rm}|_g^2 \xi^2 \, dV \leq -\frac{1}{2}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV dt + \epsilon \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2 \, dV dt + C_{12}. \tag{3.99}$$

We next establish the similar inequality for $|\nabla^2\phi|_g^2 = g^{ik}g^{j\ell}\nabla_i\nabla_j\phi\nabla_k\nabla_\ell\phi$. Calculate $\partial_t|\nabla^2\phi|_g^2 = 2\langle\nabla^2\phi, \partial_t\nabla^2\phi\rangle_g + (\nabla^2\phi)^{*2} * g^{-1} * \partial_t g^{-1}$ and

$$\begin{aligned}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 \xi^2 \, dV &= \int_{B_{\tilde{g}}(x_0,r+1)} |\tilde{\nabla}^2\tilde{\phi}|_{\tilde{g}}^2 \xi^2 \, d\tilde{V} + \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 \xi^2 \partial_t dV dt \\ &+ \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(2\langle\nabla^2\phi, \partial_t\nabla^2\phi\rangle_g + (\nabla^2\phi)^{*2} * g^{-1} * \partial_t g^{-1}\Big)\xi^2 \, dV dt.\end{aligned} \tag{3.100}$$

Since $|\tilde{\nabla}^2\tilde{\phi}|_{\tilde{g}}^2 \leq k_2$ by the assumption (3.70), we have

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\tilde{\nabla}^2\tilde{\phi}|_{\tilde{g}}^2 \xi^2 \, d\tilde{V} \lesssim 1. \tag{3.101}$$

Using (3.76) implies

$$\begin{aligned}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 \xi^2 \partial_t dV dt &= \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \\ &+ g^{*-3} * (\nabla\phi)^{*2} * (\nabla^2\phi)^{*2} + g^{*-3} * \nabla V * (\nabla^2\phi)^{*2}\Big)\xi^2 \, dV dt.\end{aligned} \tag{3.102}$$

By the evolution equation of $\partial_t g^{-1}$ above (3.90), we get

$$\begin{aligned}&\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{-1} * (\nabla^2\phi)^{*2} * \partial_t g^{-1} \, \xi^2 \, dV dt \\ &= \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-3} * (\nabla^2\phi)^{*2} * \Big(g^{-1} * \mathrm{Rm} + \nabla V + (\nabla\phi)^{*2}\Big)\xi^2 \, dV dt.\end{aligned} \tag{3.103}$$

By (3.75) the third term on the right-hand side of (3.100) can be written as

$$\begin{aligned}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} 2\langle\nabla^2\phi, \partial_t\nabla^2\phi\rangle_g \xi^2 \, dV dt &= 2\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \langle\nabla^2\phi, \Delta\nabla^2\phi\rangle_g \xi^2 \, dV dt \\ + \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(g^{*-2} * (\nabla^2\phi)^{*2} &+ g^{*-4} * \mathrm{Rm} * (\nabla^{*2}\phi)^2 + g^{*-3} * (\nabla\phi)^{*2} * (\nabla^2\phi)^{*2}\end{aligned}$$

$$+ g^{*-3} * (\nabla^2\phi)^{*3} + g^{*-3} * \nabla\phi * \nabla^{*2}\phi * \nabla^3\phi \tag{3.104}$$
$$+ g^{*-4} * \mathrm{Rm} * (\nabla\phi)^{*2} * \nabla^2\phi + g^{*-3} * V * \nabla^2\phi * \nabla^3\phi + g^{*-3} * \nabla V * (\nabla^2\phi)^{*2}\Big)\xi^2\, dV dt.$$

Substituting (3.101), (3.102), (3.103), and (3.104) into (3.100), we arrive at

$$\begin{aligned}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2\xi^2\, dV \quad &\leq \quad C_{13} + 2\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \langle\nabla^2\phi, \Delta\nabla^2\phi\rangle_g \xi^2\, dV dt \\ &+ \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big(g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} + g^{*-3} * (\nabla\phi)^{*2} * (\nabla^2\phi)^{*2} \\ &+ g^{*-3} * (\nabla^2\phi)^{*3} + g^{*-3} * \nabla\phi * \nabla^2\phi * \nabla^3\phi + g^{*-4} * \mathrm{Rm} * (\nabla\phi)^{*2} * \nabla^2\phi \\ &+ g^{*-3} * V * \nabla^2\phi * \nabla^3\phi + g^{*-3} * \nabla V * (\nabla^2\phi)^{*2} + g^{*-2} * (\nabla^2\phi)^{*2}\Big)\xi^2\, dV dt.\end{aligned} \tag{3.105}$$

By integration by parts, we find that the first term on the right-hand side of (3.105) equals

$$\begin{aligned}&2\int_0^t B_{\tilde{g}(x_0,r+1)} \langle\xi^2\nabla^2\phi, g^{\alpha\beta}\nabla_\alpha\nabla_\beta\nabla^2\phi\rangle_g dV dt \\ &\leq -\frac{3}{2}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2\xi^2\, dV dt + C_{14}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 dV dt,\end{aligned}$$

by (3.71) and the fact that $|\nabla\xi|_g = |\tilde{\nabla}\xi|_g \leq \sqrt{2}|\tilde{\nabla}\xi|_{\tilde{g}} \leq 8\sqrt{2}$. We now estimate the rest terms on the right-hand side of (3.105). By the estimate below (3.98), we have

$$\begin{aligned}\int_{B_{\tilde{g}}(x_0,r+1)} g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2}\xi^2 dV \quad &\leq \quad \frac{1}{4}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla\mathrm{Rm}|_g^2\xi^2\, dV \\ &+ \ \epsilon \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2\xi^2\, dV + C_{15}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2 dV,\end{aligned}$$

where we replaced the coefficients 1/2 by 1/4 (however the proof is the same). Using (3.71), $|\mathrm{Rm}|_g^2 \lesssim 1 + |\nabla\tilde{\nabla}g|_g^2$ (see the equation (90) in page 277 of [26]), and Lemma 3.11, the integral of $g^{*-3} * (\nabla\phi)^{*2} * (\nabla^2\phi)^{*2} + g^{*-4} * \mathrm{Rm} * (\nabla\phi)^{*2} * \nabla^2\phi + g^{*-2} * (\nabla^2\phi)^{*2}$ is bounded by

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left[|\nabla^2\phi|_g^2 + |\mathrm{Rm}|_g^2\right] dV dt \lesssim \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left[1 + |\nabla\tilde{\nabla}\mathbf{\Theta}|_g^2\right] dV dt \lesssim 1$$

where we used the fact that T depends on the given constants and the volume estimate (3.78), from the second step to the third step. By (3.71), we get

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-3} * (\nabla^2\phi)^{*3}\, \xi^2\, dV dt$$

$$\leq \epsilon \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2\, dV dt + C_{16} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 dV dt;$$

similarly, according to the definition $V = g^{-1} * \tilde{\nabla} g$,

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-3} * \nabla\phi * \nabla^2\phi * \nabla^3\phi\, \xi^2\, dV dt$$

$$\leq \epsilon \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2\, dV dt + C_{18} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 dV dt.$$

Taking the integration by parts on ∇V yields

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} g^{*-3} * \nabla V * (\nabla^2\phi)^{*2} \xi^2\, dV dt$$

$$\leq \epsilon \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2\, dV dt + C_{19} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 dV dt.$$

Substituting the above estimates into (3.105) and using Lemma 3.11, we have

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2\phi|_g^2 \xi^2\, dV \leq C_{20} + \frac{1}{4} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|_g^2 \xi^2\, dV dt - \left(\frac{3}{2} - 5\epsilon\right) \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2\, dV dt, \tag{3.106}$$

where ϵ is a sufficiently small positive number that shall be determined later. Combining (3.100) with (3.106), we arrive at

$$\int_{B_{\tilde{g}}(x_0,r+1)} \left[|\mathrm{Rm}|_g^2 + |\nabla^2\phi|_g^2\right] \xi^2 dV \leq C_{21} - \frac{1}{4} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 dV dt - \left(\frac{3}{2} - 5\epsilon\right) \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^3\phi|_g^2 \xi^2 dV dt. \tag{3.107}$$

As a consequence of the above estimate (3.107), we can conclude that the integral of $|\mathrm{Rm}|_g^2 + |\nabla^2\phi|_g^2$ over the metric ball $B_{\tilde{g}}(x_0, r)$ is uniformly bounded from above. We here keep the minus terms on the right-hand side of (3.107) to deal with the integral of $|\nabla V|_g^2 \xi^2$ over $B_{\tilde{g}}(x_0, r+1)$, and therefore, we prove Lemma 3.12.

Since the metric g is equivalent to $\tilde{g}$, we may write $g^{*-k} * g^\ell = g^{*(\ell-k)}$. Under this convenience, the equation (3.74) can be written as

$$\begin{aligned} \partial_t \nabla V = \Delta \nabla V &+ g^{*-3} * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{*-3} * \tilde{\nabla} g * \nabla \mathrm{Rm} + g^{*-2} * \nabla\tilde{\nabla} g * \nabla V \\ &+ g^{*-2} * \tilde{\nabla} g * \nabla^2 V + g^{*-2} * \nabla\tilde{\nabla} g * (\nabla\phi)^{*2} + g^{*-2} * \tilde{\nabla} g * \nabla\phi * \nabla^2\phi. \end{aligned} \tag{3.108}$$

From $|\nabla V|_g^2 = g^{ik}g^{j\ell}\nabla_i V_j \nabla_k V_\ell$ we obtain $\partial_t|\nabla V|_g^2 = 2\langle\nabla V, \partial_t \nabla V\rangle_g + g^{*-4} * \mathrm{Rm} * (\nabla V)^{*2} + g^{*-3} * (\nabla V)^{*3}$; by the evolution equation of $\partial_t g^{-1}$ above (3.73), we arrive at

$$\partial_t|\nabla V|_g^2 = 2\langle\nabla V, \partial_t \nabla V\rangle_g + g^{*-4} * \mathrm{Rm} * (\nabla V)^{*2} + g^{*-3} * (\nabla V)^{*3} \\ + \ g^{*-3} * (\nabla V)^{*2} * (\nabla\phi)^{*2}. \tag{3.109}$$

Calculate

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 \xi^2 \, dV = \int_0^t \left(\frac{d}{dt}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 \xi^2 \, dV\right) dt \tag{3.110}$$
$$= \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|^2 \xi^2 \partial_t dV dt + \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \partial_t |\nabla V|_g^2 \, \xi^2 \, dV dt,$$

since $V = 0$ at $t = 0$. Plugging (3.108), (3.109) into (3.100), we get

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 \xi^2 \, dV = 2\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \langle \nabla V, \Delta \nabla V\rangle_g \xi^2 \, dV dt$$
$$+ \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \Big[g^{*-5} * \nabla\tilde{\nabla} g * \mathrm{Rm} * \nabla V + g^{*-5} * \tilde{\nabla} g * \nabla \mathrm{Rm} * \nabla V$$
$$+ \ g^{*-4} * \nabla\tilde{\nabla} g * (\nabla V)^{*2} + g^{*-4} * \tilde{\nabla} g * \nabla V * \nabla^2 V + g^{*-4} * \mathrm{Rm} * (\nabla V)^{*2} \tag{3.111}$$
$$+ \ g^{*-3} * (\nabla V)^{*3} + g^{*-4} * \nabla\tilde{\nabla} g * \nabla V * (\nabla\phi)^{*2}$$
$$+ \ g^{*-4} * \tilde{\nabla} g * \nabla V * \nabla\phi * \nabla^2\phi + g^{*-3} * (\nabla V)^{*2} * (\nabla\phi)^{*2} \Big] \xi^2 \, dV dt.$$

The first term on the right-hand side of (3.111) was computed in [26] (see the equation (104) in page 280):

$$2\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \langle \nabla V, \Delta\nabla V\rangle_g \xi^2 dV dt \le -\frac{15}{8}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2 V|_g^2 \xi^2 \, dV dt$$
$$+ \ C_{22}\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 dV dt. \tag{3.112}$$

Define

$$I_1 := g^{*-5} * \tilde{\nabla} g * \nabla \mathrm{Rm} * \nabla V + g^{*-4} * \tilde{\nabla} g * \nabla V * \nabla^2 V,$$
$$I_2 := g^{*-5} * \nabla\tilde{\nabla} g * \mathrm{Rm} * \nabla V,$$
$$I_3 := g^{*-4} * \nabla\tilde{\nabla} g * (\nabla V)^{*2} + g^{*-4} * \mathrm{Rm} * (\nabla V)^{*2} + g^{*-3} * (\nabla V)^{*3},$$
$$I_4 := g^{*-4} * \nabla\tilde{\nabla} g * \nabla V * (\nabla\phi)^{*2} + g^{*-4} * \tilde{\nabla} g * \nabla V * \nabla\phi * \nabla^2\phi \\ + \ g^{*-3} * (\nabla V)^{*2} * (\nabla\phi)^{*2}.$$

According to (106) in page 280 of [26], we have

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} I_1 \xi^2 \, dV dt \le \frac{1}{16} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left[|\nabla \mathrm{Rm}|_g^2 + |\nabla^2 V|_g^2 \right] \xi^2 \, dV dt + C_{23} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 dV dt; \tag{3.113}$$

according to (107) in page 280 and (112) in page 281 of [26], we have

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} I_2 \xi^2 \, dV dt \le \frac{1}{16} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left[|\nabla \mathrm{Rm}|_g^2 + |\nabla^2 V|_g^2 \right] \xi^2 \, dV dt + C_{24} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left[|\mathrm{Rm}|_g^2 + |\nabla V|_g^2 \right] dV dt, \tag{3.114}$$

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} I_3 \xi^2 \, dV dt \le \frac{1}{8} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2 V|_g^2 \xi^2 \, dV dt + C_{25} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 \xi^2 \, dV dt. \tag{3.115}$$

Using (3.71) implies

$$\int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} I_4 \xi^2 \, dV dt \lesssim \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} \left(|\nabla \tilde{\nabla} g|_g^2 + |\nabla V|_g^2 + |\nabla^2 \phi|_g^2 \right) \xi^2 \, dV dt. \tag{3.116}$$

Substituting (3.112), (3.113), (3.114), (3.115), (3.116) into (3.111), using the fact that $\nabla V = g^{-1} * \nabla \tilde{\nabla} g$, and using Lemma 3.11, we obtain

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla V|_g^2 \xi^2 \, dV_t \le -\frac{13}{8} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla^2 V|_g^2 \xi^2 \, dV dt + \frac{1}{8} \int_0^t \int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_g^2 \xi^2 \, dV dt + C_{26}. \tag{3.117}$$

Choosing $\epsilon = 11/40$ in (3.107) and combining with (3.117), we arrive at

$$\max_{t \in [0,T]} \int_{B_{\tilde{g}}(x_0,r)} \left[|\mathrm{Rm}|_g^2 + |\nabla^2 \phi|_g^2 + |\nabla V|_g^2 \right] dV \lesssim 1,$$

$$\int_0^T \int_{B_{\tilde{g}}(x_0,r)} \left[|\nabla \mathrm{Rm}|_g^2 + |\nabla^3 \phi|_g^2 + |\nabla^2 V|_g^2 \right] dV dt \lesssim 1$$

since $\xi \equiv 1$ on $B_{\tilde{g}}(x_0, r)$. $\square$

Recall that $\partial_t \text{Rm} = \Delta \text{Rm} + g^{*-2} * \text{Rm}^{*2} + g^{-1} * V * \nabla \text{Rm} + g^{-1} * \text{Rm} * \nabla V + (\nabla^2\phi)^{*2}$. Since $\nabla(g^{-1} * V * \text{Rm}) = g^{-1} * V * \nabla \text{Rm} + g^{-1} * \text{Rm} * \nabla V$, it follows that

$$\partial_t \text{Rm} = \Delta \text{Rm} + \nabla P_1 + Q_1, \tag{3.118}$$

where $P_1 := g^{-1} * V * \text{Rm}$ and $Q_1 := g^{-1} * \text{Rm} * \nabla V + g^{*-2} * \text{Rm}^{*2} + (\nabla^2\phi)^{*2}$. Recall the equations $\partial_t \nabla V = \nabla(\partial_t V) + V * \partial_t \Gamma$ and $\partial_t \Gamma = g^{-1} * \nabla(\partial_t g)$ after (3.73) and the equation $\partial_t g = g^{-1} * \text{Rm} + \nabla V + (\nabla\phi)^{*2}$. Hence

$$\begin{aligned}\partial_t \nabla V &= \nabla\left(\Delta V + g^{*-3} * \tilde{\nabla} g * \text{Rm} + g^{*-2} * \tilde{\nabla} g * \nabla V + g^{*-2} * \tilde{\nabla} g * (\nabla\phi)^{*2}\right)\\ &\quad + g^{*-2} * V * \nabla \text{Rm} + g^{-1} * V * \nabla^2 V + g^{-1} * V * \nabla\phi * \nabla^2\phi.\end{aligned}$$

From the Ricci identity $\nabla \Delta V = \Delta \nabla V + g^{*-2} * \text{Rm} * \nabla V + g^{*-2} * V * \nabla \text{Rm}$, it follows that

$$\begin{aligned}\partial_t \nabla V &= \Delta \nabla V + \nabla\left(g^{*-3} * \tilde{\nabla} g * \text{Rm} + g^{*-2} * \tilde{\nabla} g * \nabla V + g^{*-2} * \tilde{\nabla} g * (\nabla\phi)^{*2}\right)\\ &+ g^{*-2} * V * \nabla \text{Rm} + g^{-1} * V * \nabla^2 V + g^{*-2} * \text{Rm} * \nabla V + g^{-1} * V * \nabla\phi * \nabla^2\phi.\end{aligned}$$

Since $\nabla(g^{*-2} * V * \text{Rm}) = g^{*-2} * V * \nabla \text{Rm} + g^{*-2} * \text{Rm} * \nabla V$, $\nabla(g^{-1} * V * \nabla V) = g^{-1} * V * \nabla^2 V + g^{-1} * (\nabla V)^{*2}$ and $V = g^{-1} * \tilde{\nabla} g$, we obtain $g^{*-2} * V * \nabla \text{Rm} = \nabla(g^{*-3} * \tilde{\nabla} g * \text{Rm}) + g^{*-2} * \text{Rm} * \nabla V$, $g^{-1} * V * \nabla^2 V = \nabla(g^{*-2} * \tilde{\nabla} g * \nabla V) + g^{-1} * (\nabla V)^{*2}$ and hence

$$\partial_t \nabla V = \Delta \nabla V + \nabla P_2 + Q_2, \tag{3.119}$$

where

$$\begin{aligned}P_2 &= g^{*-3} * \tilde{\nabla} g * \text{Rm} + g^{*-2} * \tilde{\nabla} g * \nabla V + g^{*-2} * \tilde{\nabla} g * (\nabla\phi)^{*2},\\ Q_2 &= g^{*-2} * \text{Rm} * \nabla V + g^{-1} * (\nabla V)^{*2} + g^{-1} * V * \nabla\phi * \nabla^2\phi.\end{aligned}$$

Finally, according to (3.75), we have

$$\partial_t \nabla^2\phi = \Delta \nabla^2\phi + \nabla P_3 + Q_3, \tag{3.120}$$

where

$$\begin{aligned}P_3 &= g^{-1} * \nabla\phi * \nabla^2\phi + g^{-1} * V * \nabla^2\phi,\\ Q_3 &= g^{*-2} * \text{Rm} * \nabla^2\phi + g^{-1} * (\nabla\phi)^{*2} * \nabla^2\phi + g^{*-2} * \text{Rm} * (\nabla\phi)^{*2} + \beta_2 \nabla^2\phi.\end{aligned}$$

Lemma 3.13. *For any integer $n \geq 1$, we have*

$$\int_0^T \int_{B_{\tilde{g}}(x_0,r)} u^{n-1} |\nabla \tilde{\nabla} g|_g^2 dV dt, \quad \max_{t\in[0,T]} \int_{B_{\tilde{g}}(x_0,r)} u^n \, dV, \quad \int_0^T \int_{B_{\tilde{g}}(x_0,r)} u^{n-1} v \, dV dt \lesssim 1,$$

where

$$u := |\text{Rm}|_g^2 + |\nabla^2\phi|_g^2 + |\nabla V|_g^2, \quad v := |\nabla \text{Rm}|_g^2 + |\nabla^3\phi|_g^2 + |\nabla^2 V|_g^2,$$

and $\lesssim$ depends on $m, n, r, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

Proof. The case $n = 1$ follows from Lemma 3.11 and Lemma 3.12. We now prove by induction on n. Suppose that for $s = 1, \cdots, n-1$, we have

$$\int_0^T \int_{B_{\tilde{g}}(x_0,r)} u^{s-1} |\nabla\tilde{\nabla} g|_g^2 dV dt, \ \max_{t\in[0,T]} \int_{B_{\tilde{g}}(x_0,r)} u^s \, dV, \ \int_0^T \int_{B_{\tilde{g}}(x_0,r)} u^{s-1} v \, dV dt \lesssim 1.$$

For convenience, define $w := |\mathrm{Rm}|_{\tilde{g}}^2 + |\nabla^2\phi|_{\tilde{g}}^2 + |\nabla V|_{\tilde{g}}^2$. By (66) in page 272 of [26] and (3.71), we have $|\nabla\tilde{\nabla} g|_g^2 \le 2|\tilde{\nabla}^2 g|_g^2 + C_1 \le 32|\tilde{\nabla}^2 g|_{\tilde{g}}^2 + C_1$ and hence

$$\int_0^T \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-1} |\nabla\tilde{\nabla} g|_g^2 dV dt \le 32 \int_0^T \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-1} |\tilde{\nabla}^2 g|_{\tilde{g}}^2 dV dt + C_2$$

by (3.78) and (3.71). To estimate $\int_0^T \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-1} |\nabla\tilde{\nabla} g|_g^2 dV dt$, since $\frac{1}{16} w \le u \le 16w$, we suffice to estimate $\int_0^T \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1} |\tilde{\nabla}^2 g|_{\tilde{g}}^2 d\tilde{V} dt$ since $dV \le 2^{m/2} d\tilde{V}$. Consider the same cutoff function $\xi(x)$ used in the proof of Lemma 3.10. Calculate

$$\begin{aligned} K &:= \frac{d}{dt} \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1} |\tilde{\nabla} g|_{\tilde{g}}^2 \xi^2 \, d\tilde{V} = \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1} 2\langle \tilde{\nabla} g, \partial_t \tilde{\nabla} g\rangle_{\tilde{g}} \xi^2 \, d\tilde{V} \\ &+ \int_{B_{\tilde{g}}(x_0,r+1)} |\tilde{\nabla} g|_{\tilde{g}}^2 (n-1) w^{n-2} \left[\partial_t |\mathrm{Rm}|_{\tilde{g}}^2 + \partial_t |\nabla^2\phi|_{\tilde{g}}^2 + \partial_t |\nabla V|_{\tilde{g}}^2\right] \xi^2 \, d\tilde{V} := I + J. \end{aligned}$$

Using the evolution equation of $\tilde{\nabla} g$ after (3.77) yields

$$\begin{aligned} I &= 2 \int_{B_{\tilde{g}}(x_0,r+1)} \xi^2 w^{n-1} \Big\langle \tilde{\nabla} g, g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\nabla} g + g^{-1} * g * \tilde{\nabla}\widetilde{\mathrm{Rm}} + \tilde{\nabla}\phi * \tilde{\nabla}^2\phi \\ &\quad + g^{*-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} + g^{*-2} * \tilde{\nabla} g * \tilde{\nabla}^2 g + g^{*-3} * (\tilde{\nabla} g)^{*3} \Big\rangle_{\tilde{g}} d\tilde{V} \\ &\le C_3 \int_{B_{\tilde{g}}(x_0,r+1)} \left(1 + |\tilde{\nabla}^2 g|_{\tilde{g}}\right) w^{n-1} \xi^2 \, d\tilde{V} + I_1 + I_2 + I_3 \end{aligned}$$

by (3.71), where

$$\begin{aligned} I_1 &= 2 \int_{B_{\tilde{g}}(x_0,r+1)} \xi^2 w^{n-1} \langle \tilde{\nabla} g, g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\nabla} g\rangle_{\tilde{g}} d\tilde{V}, \\ I_2 &= \int_{B_{\tilde{g}}(x_0,r+1)} \xi^2 w^{n-1} * g^{-1} * g * \tilde{\nabla} g * \tilde{\nabla}\widetilde{\mathrm{Rm}} \, d\tilde{V}, \\ I_3 &= \int_{B_{\tilde{g}}(x_0,r+1)} \xi^2 w^{n-1} * \tilde{\nabla} g * \tilde{\nabla}\phi * \tilde{\nabla}^2\phi \, d\tilde{V}. \end{aligned}$$

By integration by parts, the term I_1 can be estimated by

$$I_1 \le \int_{B_{\tilde g}(x_0,r+1)} \Big[-|\tilde\nabla^2 g|^2_{\tilde g}\xi^2 w^{n-1} + C_4|\tilde\nabla^2 g|_{\tilde g}\xi w^{n-1} + \xi^2 w^{n-2} * g^{-1} * \tilde\nabla g$$
$$* \tilde\nabla^2 g * \left(\mathrm{Rm} * \tilde\nabla\mathrm{Rm} + \nabla V * \tilde\nabla\nabla V + \nabla^2\phi * \tilde\nabla\nabla^2\phi\right)\Big] d\tilde V$$

by (3.71). The Cauchy-Schwartz inequality implies

$$C_4 \int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|_{\tilde g}\xi w^{n-1}\, d\tilde V \le \frac{1}{8}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|^2_{\tilde g}\xi^2 w^{n-1}\, d\tilde V + C_5$$

by the inductive hypothesis and $\frac{1}{16}w \le u \le 16w$. Similarly,

$$\int_{B_{\tilde g}(x_0,r+1)} \xi^2 w^{n-2} * g^{-1} * \tilde\nabla g * \tilde\nabla^2 g * \left(\mathrm{Rm} * \tilde\nabla\mathrm{Rm} + \nabla V * \tilde\nabla\nabla V + \nabla^2\phi * \tilde\nabla\nabla^2\phi\right) d\tilde V$$
$$\le \frac{1}{8}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla^2 g|^2_{\tilde g} w^{n-1}\xi^2\, d\tilde V + C_7 \int_{B_{\tilde g}(x_0,r+1)} w^{n-2}\xi^2\left(|\tilde\nabla\mathrm{Rm}|^2_{\tilde g} + |\tilde\nabla\nabla V|^2_{\tilde g} + |\tilde\nabla\nabla^2\phi|^2_{\tilde g}\right) d\tilde V.$$

According to $\Gamma - \tilde\Gamma = g^{-1} * \tilde\nabla g$, we have $|\tilde\nabla\mathrm{Rm}|^2_{\tilde g} + |\tilde\nabla\nabla V|^2_{\tilde g} + |\tilde\nabla\nabla^2\phi|^2_{\tilde g} \lesssim u + v$ and then

$$\int_{B_{\tilde g}(x_0,r+1)} \xi^2 w^{n-2} * g^{-1} * \tilde\nabla g * \tilde\nabla^2 g * \left(\mathrm{Rm} * \tilde\nabla\mathrm{Rm} + \nabla V * \tilde\nabla\nabla V + |\tilde\nabla\nabla^2\phi|^2_{\tilde g}\right) d\tilde V$$
$$\le \frac{1}{8}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla g|^2_{\tilde g} w^{n-1}\xi^2\, d\tilde V + C_9 \int_{B_{\tilde g}(x_0,r+1)} u^{n-2} v\xi^2\, d\tilde V + C_{10}$$

by the inductive hypothesis. Consequently,

$$I_1 \le -\frac{3}{4}\int_{B_{\tilde g}(x_0,r+1)} |\tilde\nabla g|^2_{\tilde g} w^{n-1}\xi^2\, d\tilde V + C_{11}\int_{B_{\tilde g}(x_0,r+1)} u^{n-2} v\, d\tilde V + C_{12}. \tag{3.121}$$

If we directly use the inequalities (3.71), we can get the uniform upper bound for I_2 by the inductive hypothesis. However, in this case the bound shall depend on an upper bound of $|\widetilde{\tilde\nabla\mathrm{Rm}}|_{\tilde g}$. To fund the dependence of k_0, we will argue as follows. Again using the integration by parts, we get

$$I_2 \le \frac{1}{8}\int_{B_{\tilde g}(x_0,r+1)} w^{n-1}|\tilde\nabla^2 g|^2_{\tilde g}\xi^2\, d\tilde V + C_{15}\int_{B_{\tilde g}(x_0,r+1)} u^{n-2} v\, d\tilde V + C_{15} \tag{3.122}$$

by the inductive hypothesis, (3.71) and the previous estimates. From (3.71), we also have

$$I_3 \lesssim \int_{B_{\tilde g}(x_0,r+1)} w^{n-1}|\tilde\nabla^2\phi|_{\tilde g}\xi\, d\tilde V. \tag{3.123}$$

Since $C_3|\tilde\nabla^2 g|_{\tilde g} \le \frac{1}{8}|\tilde\nabla^2 g|^2_{\tilde g} + 2C_3^2$, we infer from (3.121), (3.122), and (3.123) that

$$I \le -\frac{1}{2}\int_{B_{\tilde{g}}(x_0,r+1)} |\tilde{\nabla}^2 g|^2_{\tilde{g}} w^{n-1}\xi^2\, d\tilde{V} + C_{16}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\, d\tilde{V} + C_{16}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}|\tilde{\nabla}^2\phi|_{\tilde{g}}\xi\, d\tilde{V} + C_{16} \tag{3.124}$$

by the inductive hypothesis. Note that the estimate (3.82) is a special case of (3.124). According to (3.71),

$$J \le C_{17}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\left[2\langle \mathrm{Rm}, \partial_t \mathrm{Rm}\rangle_{\tilde{g}} + 2\langle \nabla^2\phi, \partial_t\nabla^2\phi\rangle_{\tilde{g}} + 2\langle \nabla V, \partial_t \nabla V\rangle_{\tilde{g}}\right]\xi^2\, d\tilde{V} \quad := \quad C_{17}(J_1 + J_2 + J_3),$$

where

$$J_1 := \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2} 2\langle \mathrm{Rm}, \partial_t \mathrm{Rm}\rangle_{\tilde{g}}\xi^2\, d\tilde{V},$$

$$J_2 := \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2} 2\langle \nabla V, \partial_t \nabla V\rangle_{\tilde{g}}\xi^2\, d\tilde{V},$$

$$J_3 := \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2} 2\langle \nabla^2\phi, \partial_t \nabla^2\phi\rangle_{\tilde{g}}\xi^2\, d\tilde{V}.$$

Substituting (3.72) into J_1 we find that

$$J_1 = \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 2\Big\langle \mathrm{Rm}, \Delta \mathrm{Rm} + g^{*-2} * \mathrm{Rm}^{*2} + g^{-1} * V * \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V + (\nabla^2\phi)^{*2}\Big\rangle_{\tilde{g}} d\tilde{V}. \tag{3.125}$$

We now estimate each term in (3.125). Since $\Delta \mathrm{Rm} = g^{\alpha\beta}\nabla_\alpha\nabla_\beta \mathrm{Rm}$, the first term on the right-hand side of (3.125) is bounded by

$$\begin{aligned} -2\int_{B_{\tilde{g}}(x_0,r+1)} \langle \nabla_\beta \mathrm{Rm}, \nabla_\alpha(\xi^2 w^{n-2} g^{\alpha\beta}\mathrm{Rm})\rangle_{\tilde{g}} d\tilde{V} \quad &\le \quad -\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|^2_{\tilde{g}}\xi^2 w^{n-2}\, d\tilde{V} \\ + C_{18}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_{\tilde{g}}|\mathrm{Rm}|_{\tilde{g}}\xi w^{n-2}\, d\tilde{V} &+ C_{18}\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_{\tilde{g}}|\mathrm{Rm}|_{\tilde{g}}\xi^2 w^{n-3} \\ \Big(|\mathrm{Rm}|_{\tilde{g}}|\nabla \mathrm{Rm}|_{\tilde{g}} + |\nabla V|_{\tilde{g}}|\nabla^2 V|_{\tilde{g}} &+ |\nabla^2\phi|_{\tilde{g}}|\nabla^3\phi|_{\tilde{g}}\Big) d\tilde{V} \end{aligned}$$

since $\nabla g = \tilde{\nabla} g + g^{-1} * \tilde{\nabla} g * g \lesssim 1$. By the inductive hypothesis, we have

$$\int_{B_{\tilde{g}}(x_0,r+1)} |\nabla \mathrm{Rm}|_{\tilde{g}}|\mathrm{Rm}|_{\tilde{g}}\xi w^{n-2}\, d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\, dV,$$

and

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-3}|\nabla \mathrm{Rm}|_{\tilde{g}}|\mathrm{Rm}|_{\tilde{g}}\Big(|\mathrm{Rm}|_{\tilde{g}}|\nabla \mathrm{Rm}|_{\tilde{g}} + |\nabla V|_{\tilde{g}}|\nabla^2 V|_{\tilde{g}} + |\nabla^2\phi|_{\tilde{g}}|\nabla^3\phi|_{\tilde{g}}\Big)\, d\tilde{V}$$
$$\lesssim \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2} v\, dV;$$

hence the first term on the right-hand side of (3.125) is bounded from above by $1+\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV$ up to a uniform positive multiple. Since $|\mathrm{Rm}|^2_g \lesssim 1+|\nabla\tilde{\nabla}g|^2_{\tilde{g}}$ by the equation (90) in page 277 of [26], it follows that the sum of the third and forth terms of the right-hand side of (3.125) is bounded from above by

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 * 2\mathrm{Rm} * \left(g^{-1} * V * \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V\right) d\tilde{V}$$
$$\leq \frac{1}{8C_{17}} \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}}\, d\tilde{V} + C_{21}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{21},$$

because of the inductive hypothesis and $V = g^{-1} * \tilde{\nabla} g \lesssim 1$. Similarly, the second term on the right-hand side of (3.125) is bounded from above by (up to a uniform positive multiple)

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 * g^{*-2} * \mathrm{Rm}^{*3}\, d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV,$$

by the equation (84) in page 276 of [26], g is equivalent to $\tilde{g}$, and the inductive hypothesis. The last term on the right-hand side of (3.125) is bounded from above by (up to a uniform positive multiple)

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 * \mathrm{Rm} * \nabla^2\phi * \nabla^2\phi\, d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV.$$

Note that when we do the integration by parts, we may replace $\tilde{g}$ by g since g is equivalent $\tilde{g}$, so that we have no extra terms $\nabla\tilde{g}$ and $\nabla\tilde{g}^{-1}$. Therefore, substituting those estimates into (3.125) implies

$$C_{17}J_1 \leq \frac{1}{8}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}} d\tilde{V} + C_{22}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{22}. \tag{3.126}$$

By (3.125), we have

$$\begin{aligned} J_2 = \int_{B_{\tilde{g}}(x_0,r+1)} & w^{n-2}\xi^2 2\Big\langle \nabla V, \Delta\nabla V + g^{*-3} * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{*-3} * \tilde{\nabla} g * \nabla \mathrm{Rm} \\ & + g^{*-2} * \nabla\tilde{\nabla} g * \nabla V + g^{*-2} * \tilde{\nabla} g * \nabla^2 V \\ & + g^{*-2} * \nabla\tilde{\nabla} g * (\nabla\phi)^{*2} + g^{*-2} * \tilde{\nabla} g * \nabla\phi * \nabla^2\phi \Big\rangle_{\tilde{g}} d\tilde{V}. \end{aligned} \tag{3.127}$$

As before, the first term on the right-hand side of (3.127) is bounded from above by (up to a uniform positive multiple)

$$2\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2\langle \nabla V, \Delta\nabla V\rangle_{\tilde{g}} d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV$$

by the inductive hypothesis. By the Cauchy-Schwarz inequality, the sum of the second, third, forth, and fifth terms on the right-hand side of (3.127) is bounded from above by

$$\frac{1}{8C_{17}}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}}\,d\tilde{V} + C_{25}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{25}.$$

The rest terms on the right-hand side of (3.127) is bounded from above by (up to a uniform positive multiple)

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 * \nabla V * g^{*-2} * \left(\nabla\tilde{\nabla}g * (\nabla\phi)^{*2} + \tilde{\nabla}g * \nabla\phi * \nabla^2\phi\right) d\tilde{V} \lesssim 1$$

by the inductive hypothesis. Hence

$$C_{17}J_2 \leq \frac{1}{8}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}}\,d\tilde{V} + C_{26}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{26}. \tag{3.128}$$

According to (3.75),

$$\begin{aligned} J_3 = \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}\xi^2 2\Big\langle &\nabla^2\phi, \Delta\nabla^2\phi + g^{*-2} * \mathrm{Rm} * \nabla^2\phi + \beta_2\nabla^2\phi \\ &+ g^{-1} * (\nabla\phi)^{*2} * \nabla^2\phi + g^{-1} * \nabla\phi * \nabla^2\phi + g^{-1} * (\nabla^2\phi)^{*2} \\ &+ g^{*-2} * \mathrm{Rm} * (\nabla\phi)^{*2} + g^{-1} * V * \nabla^3\phi + g^{-1} * \nabla V * \nabla^2\phi\Big\rangle_{\tilde{g}} d\tilde{V}. \end{aligned} \tag{3.129}$$

By the integration by parts, the first term on the right-hand side of (3.129) equals

$$-2\int_{B_{\tilde{g}}(x_0,r+1)} \langle\nabla_\beta\nabla^2\phi, \nabla_\alpha(\xi^2 w^{n-2} g^{\alpha\beta}\nabla^2\phi)\rangle_{\tilde{g}} d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}v\,d\tilde{V}.$$

The rest terms on the right-hand side of (3.129) are bounded from above by (up to a uniform positive multiple)

$$\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-2}(w + w^{1/2} + v^{1/2})\,d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV$$

by the inductive hypothesis. Hence

$$J_3 \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV. \tag{3.130}$$

Combining (3.126), (3.128), and (3.130), we find that

$$J \leq \frac{1}{4}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}}\,d\tilde{V} + C_{27}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{27}. \tag{3.131}$$

Substituting (3.124) and (3.131) into the definition of K yields

$$\frac{d}{dt}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}|\tilde{\nabla}g|^2_{\tilde{g}}\xi^2\,d\tilde{V} \leq -\frac{1}{4}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2 g|^2_{\tilde{g}}\,d\tilde{V} + C_{28} + C_{28}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{28}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}|\tilde{\nabla}^2\phi|_{\tilde{g}}\xi\,d\tilde{V}. \tag{3.132}$$

$$\begin{aligned} L &:= \frac{d}{dt}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}|\tilde{\nabla}\phi|^2_{\tilde{g}}\xi^2\,d\tilde{V} = 2\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2\langle\tilde{\nabla}\phi, g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{\nabla}\phi\rangle_{\tilde{g}}d\tilde{V} \\ &+ \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2\tilde{\nabla}\phi * \Big(g^{*-2} * \widetilde{\mathrm{Rm}} * \tilde{\nabla}\phi + g^{*-2} * \tilde{\nabla}g * \tilde{\nabla}^2\phi \\ &+ g^{*-2} * \tilde{\nabla}g * (\tilde{\nabla}\phi)^{*2} + g^{-1} * \tilde{\nabla}\phi * \tilde{\nabla}^2\phi + \tilde{\nabla}\phi\Big)d\tilde{V} \quad := \quad L_1 + L_2. \end{aligned}$$

For L_2, we have

$$L_2 \lesssim \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi(1+|\tilde{\nabla}^2\phi|_{\tilde{g}})d\tilde{V} \lesssim 1 + \int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi|\tilde{\nabla}^2\phi|_{\tilde{g}}d\tilde{V}$$

by the inductive hypothesis. Taking the integration by parts on L_1 implies

$$L_1 \leq -\frac{3}{4}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2\phi|^2_{\tilde{g}}\,d\tilde{V} + C_{30}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi|\tilde{\nabla}^2\phi|_{\tilde{g}}\,d\tilde{V} + C_{30}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV,$$

because of $\tilde{\nabla}\mathrm{Rm}$, $\tilde{\nabla}\nabla V$, $\tilde{\nabla}\nabla^2\phi \lesssim v^{1/2} + w^{1/2}$. Therefore

$$L \leq -\frac{1}{2}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}^2\phi|^2_{\tilde{g}}\,d\tilde{V} + C_{31}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{31}. \tag{3.133}$$

From (3.132) and (3.133), we arrive at

$$\begin{aligned} \frac{d}{dt}\Big(\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2\left(|\tilde{\nabla}g|^2_{\tilde{g}} + |\tilde{\nabla}\phi|^2_{\tilde{g}}\right)d\tilde{V}\Big) &\leq C_{32}\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-2}v\,dV + C_{32} \\ &\leq -\frac{1}{4}\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2\left(|\tilde{\nabla}^2 g|^2_{\tilde{g}} + |\tilde{\nabla}^2\phi|^2_{\tilde{g}}\right)d\tilde{V}. \end{aligned}$$

Consequently, $\int_0^T\int_{B_{\tilde{g}}(x_0,r+1)} w^{n-1}\xi^2|\tilde{\nabla}g|^2_{\tilde{g}}\,d\tilde{V}dt \lesssim 1$; in particular,

$$\int_0^T\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-1}|\nabla\tilde{\nabla}g|^2_g\xi^2\,dV dt \lesssim 1. \tag{3.134}$$

By the equations (90) in page 277 and (108) in page 281 of [26], together with $\nabla^2\phi = \tilde{\nabla}^2 + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}\phi$, we have $|\mathrm{Rm}|^2_g + |\nabla V|^2_g + |\nabla^2\phi|^2_g \lesssim 1 + |\nabla\tilde{\nabla}g|^2_g$; using the estimate (3.134), we obtain $\int_0^T\int_{B_{\tilde{g}}(x_0,r+1)} u^n\,dV dt \lesssim 1$.

As in the proof of Lemma 3.12, we can show that $\int_{B_{\tilde{g}}(x_0,r+1)} u^n\xi^2\,dV + \frac{1}{8}\int_0^t\int_{B_{\tilde{g}}(x_0,r+1)} u^{n-1}v\xi^2\,dV dt \lesssim 1$ by the previous estimates and inductive hypothesis. Thus the lemma is also true for $s = n$. □

We now can prove the following theorem, as in [26] where use the equations (3.118), (3.119), (3.120), and Lemma 3.13.

Theorem 3.14. *We have*

$$\sup_{M\times[0,T]}|\mathrm{Rm}|^2_g \lesssim 1, \quad \sup_{M\times[0,T]}|\nabla V|^2_g \lesssim 1, \quad \sup_{M\times[0,T]}|\nabla^2\phi|^2_g \lesssim 1, \tag{3.135}$$

where $\lesssim$ depend on $n, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

By the same argument used in [19,26], we have

Theorem 3.15. *Let $(M, \tilde{g})$ be a complete noncompact Riemannian m-manifold with bounded Riemann curvature $|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \le k_0$, and $\tilde{\phi}$ a smooth function on M satisfying*

$$|\tilde{\phi}|^2 + |\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_1, \quad |\nabla^2_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_2.$$

Then there exists a positive constant $T = T(m, k_0, k_1, \alpha_1, \beta_1, \beta_2) > 0$ such that the regular-$(\alpha_1, 0, \beta_1, \beta_2)$-flow

$$\begin{aligned}
\partial_t\hat{g}(t) &= -2\mathrm{Ric}_{\hat{g}(t)} + 2\alpha_1\nabla_{\hat{g}(t)}\hat{\phi}(t)\otimes\nabla_{\hat{g}(t)}\hat{\phi}(t),\\
\partial_t\hat{\phi}(t) &= \Delta_{\hat{g}(t)}\hat{\phi}(t) + \beta_1|\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} + \beta_2\hat{\phi}(t),\\
(\hat{g}(0), \hat{\phi}(0)) &= (\tilde{g}, \tilde{\phi})
\end{aligned}$$

has a smooth solution $(\hat{g}(t), \hat{\phi}(t))$ on $M \times [0, T]$ and satisfies the following estimate

$$\frac{1}{C_1}\tilde{g} \le \hat{g}(t) \le C_1\tilde{g}, \quad |\mathrm{Rm}_{\hat{g}(t)}|^2_{\hat{g}(t)} + |\hat{\phi}(t)|^2 + |\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} + |\nabla^2_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} \le C_2$$

on $M \times [0, T]$, where C_1, C_2 are uniform positive constants depending only on $m, k_0, k_1, k_2, \alpha_1, \beta_1, \beta_2$.

Suppose that $(\hat{g}(t), \hat{\phi}(t))$ is a smooth solution to the regular-$(\alpha_1, 0, \beta_1 - \alpha_2, \beta_2)$-flow

$$\begin{aligned}
\partial_t\hat{g}(t) &= -2\mathrm{Ric}_{\hat{g}(t)} + 2\alpha_1\nabla_{\hat{g}(t)}\hat{\phi}(t)\otimes\nabla_{\hat{g}(t)}\hat{\phi}(t),\\
\partial_t\hat{\phi}(t) &= \Delta_{\hat{g}(t)}\hat{\phi}(t) + (\beta_1 - \alpha_2)|\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{g}(t)} + \beta_2\hat{\phi}(t),\\
(\hat{g}(0), \hat{\phi}(0)) &= (\tilde{g}, \tilde{\phi}).
\end{aligned}$$

Consider a 1-parameter family of diffeomorphisms $\Phi(t) : M \to M$ by

$$\frac{d}{dt}\Phi(t) = \alpha_2\nabla_{\hat{g}(t)}\hat{\phi}(t), \quad \Phi(0) = \mathrm{Id}_M. \tag{3.136}$$

If we define

$$g(t) := [\Phi(t)]^*\hat{g}(t), \quad \phi(t) := [\Phi(t)]^*\hat{\phi}(t), \tag{3.137}$$

then

$$
\begin{aligned}
\partial_t g(t) &= -2\mathrm{Rm}_{g(t)} + 2\alpha_1 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t) + 2\alpha_2 \nabla^2_{g(t)}\phi(t), \\
\partial_t \phi(t) &= \Delta_{g(t)}\phi(t) + \beta_1 |\nabla_{g(t)}\phi(t)|^2_{g(t)} + \beta_2\phi(t).
\end{aligned}
$$

If we furthermore have $|\hat{\phi}(t)|^2 \lesssim 1$ and $|\nabla_{\hat{g}(t)}\hat{\phi}(t)|^2_{\hat{\phi}} \lesssim 1$ on $M \times [0, T]$, using the standard theory of ordinary differential equations we have that the system (3.136) has a unique smooth solution $\Phi(t)$ on $M \times [0, T]$. Therefore $(g(t), \phi(t))$ defined in (3.137) are also smooth on $M \times [0, T]$ and satisfies the above system of equations.

Theorem 3.16. *Let $(M, \tilde{g})$ be a complete noncompact Riemannian m-manifold with bounded Riemann curvature $|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \le k_0$, and $\tilde{\phi}$ a smooth function on M satisfying*

$$
|\tilde{\phi}|^2 + |\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_1, \quad |\nabla^2_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_2.
$$

Then there exists a positive constant $T = T(m, k_0, k_1, \alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ such that the regular-$(\alpha_1, \alpha_2, \beta_1, \beta_2)$-flow

$$
\begin{aligned}
\partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \\
\partial_t \phi(t) &= \Delta_{g(t)}\phi(t) + \beta_1 |\nabla_{g(t)}\phi(t)|^2_{g(t)} + \beta_2\phi(t), \\
(g(0), \phi(0)) &= (\tilde{g}, \phi)
\end{aligned}
$$

has a smooth solution $(g(t), \phi(t))$ on $M \times [0, T]$ and satisfies the following estimate

$$
\frac{1}{C_1}\tilde{g} \le g(t) \le C_1\tilde{g}, \quad |\mathrm{Rm}_{g(t)}|^2_{g(t)} + |\phi(t)|^2 + |\nabla_{g(t)}\phi(t)|^2_{g(t)} + |\nabla^2_{g(t)}\phi(t)|^2_{g(t)} \le C_2
$$

on $M \times [0, T]$, where C_1, C_2 are uniform positive constants depending only on $m, k_0, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2$.

3.5. Higher order derivatives estimates

To complete the proof of Theorem 3.1, we need only to prove the higher order derivatives estimates (3.1). Suppose we have a smooth solution $(g(t), \phi(t))$ on $M \times [0, T]$ and satisfies

$$
\begin{aligned}
\partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 2\alpha_2 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t) + 2\alpha_2 \nabla^2_{g(t)}\phi(t), \\
\partial_t \phi(t) &= \Delta_{g(t)}\phi(t) + \beta_1 |\nabla_{g(t)}\phi(t)|^2_{g(t)} + \beta_2\phi(t), \\
(g(0), \phi(0)) &= (\tilde{g}, \tilde{\phi}),
\end{aligned}
\tag{3.138}
$$

where $(M, \tilde{g})$ is a complete noncompact Riemannian m-manifold with bounded curvature $|\mathrm{Rm}_{\tilde{g}}|^2_{\tilde{g}} \le k_0$ and $\tilde{\phi}$ is a smooth function on M satisfying $|\tilde{\phi}|^2 + |\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_1$ and $|\nabla^2_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}} \le k_2$, and

$$
g(t) \approx \tilde{g}, \quad |\mathrm{Rm}_{g(t)}|^2_{g(t)} + |\phi(t)|^2 + |\nabla_{g(t)}\phi(t)|^2_{g(t)} + |\nabla^2_{g(t)}\phi(t)|^2_{g(t)} \lesssim 1 \tag{3.139}
$$

on $M \times [0, T]$, where $\lesssim$ or $\approx$ depends only on $m, k_0, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2$.

Lemma 3.17. *For any nonnegative integer n, there exist uniform positive constants C_k depending only on $m, n, k_0, k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ such that*

$$
\left|\nabla^n_{g(t)}\mathrm{Rm}_{g(t)}\right|^2_{g(t)} + \left|\nabla^{n+2}_{g(t)}\phi(t)\right|^2_{g(t)} \le \frac{C_n}{t^n} \tag{3.140}
$$

on $M \times [0, T]$.

Proof. As before, we always write $\mathrm{Rm} := \mathrm{Rm}_{g(t)}$, $\phi := \phi(t)$, etc. From Lemma 2.5, we have

$$\partial_t \mathrm{Rm} = \Delta \mathrm{Rm} + g^{*-2} * \mathrm{Rm}^{*2} + (\nabla^2\phi)^{*2} + g^{-1} * \nabla \mathrm{Rm} * \nabla\phi + g^{-1} * \mathrm{Rm} * \nabla^2\phi. \tag{3.141}$$

Then the norm $|\mathrm{Rm}|^2$ of Riemann curvature evolves by $\partial_t|\mathrm{Rm}|^2 = 2\langle \mathrm{Rm}, \partial_t \mathrm{Rm}\rangle_g + g^{*-3} * \partial_t g^{-1} * \mathrm{Rm}^{*2}$; substituting (3.138) and (3.141) into above yields

$$\begin{aligned}\partial_t|\mathrm{Rm}|^2 \quad &= \quad \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + g^{*-6} * \mathrm{Rm}^{*3} + g^{*-4} * \mathrm{Rm} * (\nabla^2\phi)^{*2} \\ &+ g^{*-5} * \mathrm{Rm}^{*2} * (\nabla\phi)^{*2} + g^{*-5} * \mathrm{Rm}^{*2} * \nabla^2\phi + g^{*-5} * \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla\phi.\end{aligned} \tag{3.142}$$

Introduce a family of vector-valued tensor fields

$$\mathbf{\Lambda} := (\mathrm{Rm}, \nabla\phi), \tag{3.143}$$

and define $|\nabla^k \mathbf{\Lambda}|^2 := |\nabla^k \mathrm{Rm}|^2 + |\nabla^{k+1}\phi|^2$ for each nonnegative integer k. According to (3.139), (3.142) and the Cauchy-Schwartz inequality, we have

$$\partial_t|\mathrm{Rm}|^2 \le \Delta|\mathrm{Rm}|^2 - \frac{3}{2}|\nabla \mathrm{Rm}|^2 + C_1. \tag{3.144}$$

Since $\partial_t \nabla \mathrm{Rm} = \nabla \partial_t \mathrm{Rm} + \partial_t \Gamma * \mathrm{Rm}$ and $\partial_t \Gamma = g^{*-2} * \nabla \mathrm{Rm} + g^{-1} * \nabla\phi * \nabla^2\phi + g^{-1} * \nabla^3\phi$, it follows that

$$\begin{aligned}\partial_t|\nabla \mathrm{Rm}|^2 \le \Delta|\nabla \mathrm{Rm}|^2 - 2|\nabla^2 \mathrm{Rm}|^2 + C_2|\nabla \mathrm{Rm}||\nabla^2 \mathrm{Rm}| + C_2|\nabla \mathrm{Rm}||\nabla^3\phi| \\ + C_2|\nabla \mathrm{Rm}| + C_2|\nabla \mathrm{Rm}|^2.\end{aligned} \tag{3.145}$$

On the other hand, Lemma 2.7 yields

$$\begin{aligned}\partial_t \nabla^2\phi = \Delta\nabla^2\phi + g^{*-2} * \mathrm{Rm} * \nabla^2\phi + \beta_2\nabla^2\phi + |\nabla\phi|^2\nabla^2\phi \\ + \nabla\phi * \nabla^3\phi + g^{-1} * (\nabla^2\phi)^{*2} + g^{*-2} * \mathrm{Rm} * (\nabla\phi)^{*2}.\end{aligned} \tag{3.146}$$

Plugging (3.146) into $\partial_t|\nabla^2\phi|^2 = 2\langle \nabla^2\phi, \partial_t\nabla^2\phi\rangle + g^{*-3} * \partial_t g * (\nabla^2\phi)^{*2}$, we arrive at

$$\partial_t|\nabla^2\phi|^2 \le \Delta|\nabla^2\phi|^2 - 2|\nabla^3\phi|^2 + C_3|\nabla^3\phi| + C_3. \tag{3.147}$$

Combining (3.145) and (3.147), we have

$$\partial_t|\nabla\mathbf{\Lambda}|^2 \le \Delta|\nabla\mathbf{\Lambda}|^2 - \frac{3}{2}|\nabla^2\mathbf{\Lambda}|^2 + C_4|\nabla\mathbf{\Lambda}|^2 + C_4. \tag{3.148}$$

According to Lemma 2.6 and (3.144), for any given positive number a, we get

$$\partial_t\left(a + |\mathbf{\Lambda}|^2\right) \le \Delta\left(a + |\mathbf{\Lambda}|^2\right) - \frac{3}{2}|\nabla\mathbf{\Lambda}|^2 + C_5. \tag{3.149}$$

Therefore

$$\begin{aligned}\partial_t\left[\left(a + |\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2\right] \le \Delta\left[\left(a + |\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2\right] - 2\left\langle \nabla|\mathbf{\Lambda}|^2, \nabla|\nabla\mathbf{\Lambda}|^2\right\rangle \\ -\frac{3}{2}|\nabla\mathbf{\Lambda}|^4 + C_5|\nabla\mathbf{\Lambda}|^2 - \frac{3}{2}\left(a + |\mathbf{\Lambda}|^2\right)|\nabla^2\mathbf{\Lambda}|^2 + C_4\left(a + |\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2 + C_4\left(a + |\mathbf{\Lambda}|^2\right).\end{aligned} \tag{3.150}$$

By the definition, the second term on the right-hand side of (3.150) is bounded from above by $g^{pq}\nabla_p(|\mathrm{Rm}|^2+|\nabla\phi|^2)\nabla_q(|\nabla\mathrm{Rm}|^2+|\nabla^2\phi|^2) \le \frac{3}{2}a|\nabla^2\mathbf{\Lambda}|^2 + \frac{2C_6^2}{a}|\nabla\mathbf{\Lambda}|^4 + \frac{2C_6^2}{a}$ where we used the inequality $x^2 \le x^4+1$ for any $x \ge 0$. Substituting this inequality into (3.150) implies

$$\partial_t\left[\left(a+|\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2\right] \le \Delta\left[\left(a+|\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2\right] - \left(\frac{3}{2}-\frac{2C_6^2}{a}\right)|\nabla\mathbf{\Lambda}|^4 + C_5|\nabla\mathbf{\Lambda}|^2 + C_4\left(a+|\mathbf{\Lambda}|^2\right)|\nabla\mathbf{\Lambda}|^2 + C_4\left(a+|\mathbf{\Lambda}|^2\right) + \frac{2C_6^2}{a}. \tag{3.151}$$

By (3.139), we can choose a so that $a \ge 4C_6^2$ and $a \ge \max_{M\times[0,T]}(|\mathrm{Rm}|^2+|\nabla^2\phi|^2)$; then $3/2 - 2C_6^2/a \ge 1$ and $a \le a+|\mathbf{\Lambda}|^2 \le 2a$. Consequently, we can deduce from (3.151) that $\partial_t[(a+|\mathbf{\Lambda}|^2)|\nabla\mathbf{\Lambda}|^2] \le \Delta[(a+|\mathbf{\Lambda}|^2)|\nabla\mathbf{\Lambda}|^2] - \frac{1}{8a^2}[(a+|\mathbf{\Lambda}|^2)|\nabla\mathbf{\Lambda}|^2]^2 + C_7$. Consider the function $u := (a+|\mathbf{\Lambda}|^2)|\nabla\mathbf{\Lambda}|^2 t$ defined on $M\times[0,T]$. Then $u=0$ on $M\times\{0\}$ and

$$\partial_t u \le \Delta u - \frac{1}{8a^2 t}u^2 + C_7 t + \frac{1}{t}u \quad \text{on } M\times[0,T]. \tag{3.152}$$

Fix a point $x_0 \in M$, consider the cutoff function $\xi(x)$ on M, introduced in [26], such that

$$\xi = 1 \quad \text{on } B_{\tilde g}(x_0,1), \quad \xi = 0 \quad \text{on } M\setminus B_{\tilde g}(x_0,2), \quad 0\le\xi\le 1,$$
$$|\tilde\nabla\xi|^2_{\tilde g} \le 16\xi, \quad \tilde\nabla^2\xi \ge -c\tilde g, \quad \text{on } M,$$

where c depends on k_0. As in [26], we define $F := \xi u$ on $M\times[0,T]$. Then from the definition, we have

$$F = 0 \text{ on } M\times\{0\}, \quad F=0 \text{ on } (M\setminus B_{\tilde g}(x_0,2))\times[0,T], \quad F\ge 0 \text{ on } M\times[0,T].$$

Without loss of generality, we may assume that F is not identically zero on $M\times[0,T]$. In this case, we can find a point $(x_1,t_1)\in B_{\tilde g}(x_0,2)\times[0,T]$ such that $F(x_1,t_1) = \max_{M\times[0,T]}F(x,t) > 0$ which implies $t_1>0$ and $\partial_t F(x_1,t_1) > 0$, $\nabla F(x_1,t_1) = 0$, $\Delta F(x_1,t_1) \le 0$. If $u(x_1,t_1)\le 1$, then $F(x_1,t_1)\le 1$ by and hence $at|\nabla\mathbf{\Lambda}|^2 \le u = F \le 1$ on $B_{\tilde g}(x_0,1)\times[0,T]$; in particular, $|\nabla\mathrm{Rm}|^2+|\nabla^2\phi|^2 \lesssim \frac{1}{t}$ on $M\times[0,T]$.

In the following we assume that $u(x_1,t_1)\ge 1$. Under this assumption, $\xi\partial_t u \ge 0$ at (x_1,t_1), we arrive at, at the point (x_1,t_1), $0 \le \xi\Delta u + \frac{u}{t}(C_8 - C_9 u)\xi$; thus $\xi\Delta u + \frac{\xi u}{t}(C_8 - C_9 u) \ge 0$ at (x_1,t_1). By the same argument in [26] (equations (28)–(35) in page 293), we find that $\frac{\xi u}{t}(C_9 u - C_8) \le C_{10}u - u\Delta\xi$ at (x_1,t_1). According to the equation (38) in page 294 of [26], we get $-\Delta\xi \le C_{11} + g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \tilde\Gamma^\gamma_{\alpha\beta})\tilde\nabla_\gamma\xi$. Since $\partial_t\Gamma = g^{*-2}*\nabla\mathrm{Rm} + g^{-1}*\nabla^3\phi + g^{-1}*\nabla\phi*\nabla^2\phi$, it follows that $|\partial_t\Gamma| \le C_{11}|\nabla\mathbf{\Lambda}| + C_{12} \le \frac{C_{11}}{\sqrt{at}}u^{1/2} + C_{12}$. Since $\xi(x_1)u(x_1,t) = F(x_1,t) \le F(x_1,t_1)$ for $t\in[0,T]$, we obtain $|\partial_t(x_1,t)| \le C_{11}[F(x_1,t_1)/a\xi(x_1)]^{1/2}t^{-1/2} + C_{12}$ for any $t\in[0,T]$. As showed in [26] (the equation (45) in page 295), together with $|\tilde\nabla\xi|_{\tilde g} \le 4\xi^{1/2}$, we find that $g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \tilde\Gamma^\gamma_{\alpha\beta})\tilde\nabla_\gamma\xi \le C_{13}F(x_1,t_1)^{1/2} + C_{14}$ and then $-\Delta\xi \le C_{15} + C_{13}F(X_1,t_1)^{1/2}$ at (x_1,t_1). Consequently, we have the following inequality $\xi u(C_9 u - C_8) \le C_{10}tu + C_{15}tu + C_{13}tuF^{1/2} \le C_{16}u + C_{16}uF^{1/2}$ at (x_1,t_1); multiplying by $\xi(x_1)$ yields $C_9F^2 \le C_{17}F + C_{16}F^{3/2}$ at (x_1,t_1), from which we deduce that $F(x_1,t_1)\lesssim 1$ and therefore $\xi u \lesssim 1$ on $M\times[0,T]$. In particular, $u\lesssim 1$ on $M\times[0,T]$ since x_0 was arbitrary. From the definition of u, this tells us the estimate $|\nabla\mathbf{\Lambda}|^2 \lesssim 1/t$ on $M\times[0,T]$, where $\lesssim$ depends only on $m, k_0, k_1, k_2, \alpha_1, \alpha 2, \beta_1, \beta_2$. Hence the lemma holds for $n=1$.

By induction, suppose for $s = 1,\cdots,n-1$ we have $|\nabla^s\mathrm{Rm}|^2 + |\nabla^{s+2}\phi|^2 \lesssim \frac{1}{t^s}$ on $M\times[0,T]$. As in [26], we define a function $v := (a + t^{n-1}|\nabla^{n-1}\mathbf{\Lambda}|^2)|\nabla^n\mathbf{\Lambda}|^2 t^n$ and choose a sufficiently large. Similarly, we can show that $\partial_t v \le \Delta v - (C_{18}/a^2t)v^2 + C_{19} + (C_{20}/t)v$ on $M\times[0,T]$. Using the same cutoff function ξ and arguing in the same way, we obtain that $v\lesssim 1$ on $M\times[0,T]$. Hence the inequality (3.140) holds for $s=n$. □

References

[1] Simon Brendle, Uniqueness of gradient Ricci solitons, Math. Res. Lett. 18 (3) (2011) 531–538, MR2802586 (2012e: 53073).
[2] Simon Brendle, Rotational symmetry of self-similar solutions to the Ricci flow, Invent. Math. 194 (3) (2013) 731–764, MR3127066.
[3] Simon Brendel, Rotational symmetry of Ricci solitons in higher dimensions, J. Differ. Geom. 97 (2) (2014) 191–214, MR3231974.
[4] Huai-Dong Cao, Existence of gradient Kähler-Ricci solitons, in: Elliptic and Parabolic Methods in Geometry, Minneapolis, MN, 1994, AK Peters, Wellesley, MA, 1996, pp. 1–16, MR1417944 (98a: 53058).
[5] Huai-Dong Cao, Recent progress on Ricci solitons, in: Recent Advances in Geometry Analysis, in: Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 1–38, MR2648937 (2011d: 53061).
[6] Huai-dong Cao, Qiang Chen, On locally conformally flat gradient steady Ricci solitons, Trans. Am. Math. Soc. 364 (5) (2012) 2377–2391, MR2888210.
[7] Giovanni Catino, Carlo Mantegazza, Evolution of the Weyl tensor under the Ricci flow, Ann. Inst. Fourier (Grenoble) 61 (4) (2011) 1407–1435, MR2951497.
[8] Bing-Long Chen, Strong uniqueness of the Ricci flow, J. Differ. Geom. 82 (2) (2009) 363–382, MR25207960 (2019h: 53095).
[9] Bennett Chow, Peng Lu, Lei Ni, Hamilton's Ricci flow, in: Graduate Studies in Mathematics, vol. 77, American Mathematical Society/Science Press, Providence, RI/New York, 2006, xxxv+608 pp. MR2274812 (2008a: 53068).
[10] Bennett Chow, Peng Lu, Bo Yang, Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons, C. R. Math. Acad. Sci. Paris 349 (23–24) (2011) 1265–1267, MR2861997.
[11] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, Lei Ni, The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007, xxiv+536 pp. MR2302600 (2008 f: 53088).
[12] Manuel Fernández-López, Eduardo García-Río, A sharp lower bound for the scalar curvature of certain steady gradient Ricci solitons, Proc. Am. Math. Soc. 141 (6) (2013) 2145–2148, MR3034440.
[13] Richard S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differ. Geom. 17 (2) (1982) 255–306, MR0664497 (84a: 53050).
[14] Chun-Lei He, Hu Sen, De-Xing Kong, Kefeng Liu, Generalized Ricci flow I: local existence and uniqueness, in: Topology and Physics, in: Nankai Tracts Math., vol. 12, World Sci. Publ., Hackensack, NJ, 2008, pp. 151–171, MR2503395 (2010k: 53098).
[15] Richard S. Hamilton, The formation of singularities in the Ricci flow, in: Surveys in Differential Geometry, vol. II, Cambridge, MA, 1993, Int. Press, Cambridge, MA, 1995, pp. 7–136, MR1375255 (97e: 53075).
[16] Thomas Ivey, New examples of complete Ricci solitons, Proc. Am. Math. Soc. 122 (1) (1994) 241–245, MR1207538 (94k: 53057).
[17] Yi. Li, Generalized Ricci flow I: higher derivatives estimates for compact manifolds, Anal. Partial Differ. Equ. 5 (4) (2012) 747–775, MR3006641.
[18] Yi. Li, Eigenvalues and entropies under the harmonic-Ricci flow, Pac. J. Math. 267 (1) (2014) 141–184, MR3163480.
[19] B. List, Evolution of an Extended Ricci Flow System, PhD thesis, AEI Potsdam, 2005.
[20] Reto Müler, Monotone volume formulas for geometric flow, J. Reine Angew. Math. 643 (2010) 39–57, MR2658189 (2011k: 53086).
[21] Reto Müler, Ricci flow coupled with harmonic map flow, Ann. Sci. Éc. Norm. Supér. (4) 45 (1) (2012) 101–142, MR2961788.
[22] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.
[23] Grisha Perelman, Ricci flow with surgery on three-manifolds, arXiv:math/0303109.
[24] Grisha Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math/0307245.
[25] Peter Petersen, William Wylie, Rigidity of gradient Ricci solitons, Pac. J. Math. 241 (2) (2009) 329–345, MR2507581 (2010j: 53071).
[26] Wan-Xiong Shi, Deforming the metric on complete Riemannian manifolds, J. Differ. Geom. 30 (1) (1989) 223–301, MR1001277 (90i: 58202).
[27] Xue Hu, Yuguang Shi, Static flow on complete noncompact manifolds I: short-time existence and asymptotic expansions at conformal infinity, Sci. China Math. 55 (9) (2012) 1883–1900, MR2960867.
[28] Shing-Tung Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, Commun. Pure Appl. Math. 31 (3) (1978) 339–411, MR0480350 (81d: 53045).