K-THEORY, LOCAL COHOMOLOGY AND TANGENT SPACES TO HILBERT SCHEMES

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Abstract. By using K-theory, we construct a map from the tangent space to the Hilbert scheme at a point Y to the local cohomology group: $\pi : \mathrm{T_Y Hilb}^p(X) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}})$. We use this map π to answer (after slight modification) a question by Mark Green and Phillip Griffiths on constructing a map from the tangent space $\mathrm{T_Y Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$.

Contents

1. Introduction

Let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p. Considering Y as an element of $\mathrm{Hilb}^p(X)$, it is well known that the Zariski tangent space $\mathrm{T}_Y\mathrm{Hilb}^p(X)$ can be identified with $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf.

Y also defines an element of the cycle group $Z^p(X)$ and we are interested in defining the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$. In [8], Mark Green and Phillip Griffiths define $TZ^p(X)$ for $p = 1$(divisors) and $p = \dim(X)$(0-cycles) and leave the general case as an open question. Much of their theory was extended by Benjamin Dribus, Jerome W. Hoffman and the author in [6, 12]. In [12], we define $TZ^p(X)$ for any integer p satisfying $1 \leqslant p \leqslant \dim(X)$, generalizing

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Green and Griffiths' definitions. We recall the following fact from [12] for our purpose, and refer to [8, 12] for definition of $TZ^p(X)$.

Theorem 1.1 (Theorem 2.8 in [12])**.** *For X a smooth projective variety over a field k of characteristic 0, for any integer $p \geqslant 1$, the tangent space $TZ^p(X)$ is identified with $\mathrm{Ker}(\partial_1^{p,-p})$:*

$$TZ^p(X) \cong \mathrm{Ker}(\partial_1^{p,-p}),$$

where $\partial_1^{p,-p}$ is the differential of the Cousin complex [10] of $\Omega^{p-1}_{X/\mathbb{Q}}$ in position p:

$$0 \to \Omega^{p-1}_{k(X)/\mathbb{Q}} \to \cdots \to \bigoplus_{y \in X^{(p)}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{p,-p}} \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \cdots .$$

Now, we want to study the relation between $\mathrm{T_Y Hilb}^p(X)$ and $TZ^p(X)$. The following question is suggested by Mark Green and Phillip Griffiths in [8](see page 18 and page 87-89):

Question 1.2. [8] *For X a smooth projective variety over a field k of characteristic 0, for any integer $p \geqslant 1$, is it possible to define a map from the tangent space $\mathrm{T_Y Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$:*

$$\mathrm{T_Y Hilb}^p(X) \to TZ^p(X)?$$

For $p = \dim(X)$, this has been answered affirmatively by Green and Griffiths in [8], see Section 7.2 for details:

Theorem 1.3. [8] *For $p = d := \dim(X)$, there exists a map from the tangent space to the Hilbert scheme at a point Y to the tangent space to the cycle group*

$$\mathrm{T_Y Hilb}^d(X) \to TZ^d(X).$$

The main result of this short note is to construct a map in Definition 4.1

$$\pi : \mathrm{T_Y Hilb}^p(X) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}),$$

and use this map to study the above Question 1.2 by Mark Green and Phillip Griffiths.

In Example 4.4, we show, for general $Y \subset X$ of codimension p and $Y' \in \mathrm{T_Y Hilb}^p(X)$, $\pi(Y')$ may not lie in $TZ^p(X)$(the kernel of $\partial_1^{p,-p}$). However, we will show that, in Theorem 4.6, there exists $Z \subset X$ of codimension p and exists $Z' \in \mathrm{T_Z Hilb}^p(X)$ such that $\pi(Y') + \pi(Z') \in TZ^p(X)$.

As an application, we show how to find Milnor K-theoretic cycles in Theorem 4.7. In [13], we will apply these techniques to eliminate obstructions to deforming curves on a three-fold.

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Notations and conventions.

(1.) K-theory used in this note will be Thomason-Trobaugh non-connective K-theory, if not stated otherwise.

(2.) For any abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M\otimes_{\mathbb{Z}}\mathbb{Q}$.

(3.) $X[\varepsilon]$ denote the first order trivial deformation of X, i.e., $X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$, where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers.

2. **K-theory and tangent spaces to Hilbert schemes**

For X a smooth projective variety over a field k of characteristic 0 and $Y \subset X$ a subvariety of codimension p, let $i : Y \to X$ be the inclusion, then i_*O_Y is a coherent O_X-module and can be resolved by a bounded complex of vector bundles on X. Let $Y^{'}$ be a first order deformation of Y, that is, $Y^{'} \subset X[\varepsilon]$ such that $Y^{'}$ is flat over $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $Y^{'} \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong Y$. Then $i_*O_{Y'}$ can be resolved by a bounded complex of vector bundles on $X[\varepsilon]$, where $i : Y^{'} \to X[\varepsilon]$.

Let $D^{\mathrm{perf}}(X[\varepsilon])$ denote the derived category of perfect complexes of $O_X[\varepsilon]$-modules, and let $\mathcal{L}_{(i)}(X[\varepsilon]) \subset D^{\mathrm{perf}}(X[\varepsilon])$ be defined as

$$\mathcal{L}_{(i)}(X[\varepsilon]) := \{E \in D^{\mathrm{perf}}(X[\varepsilon]) \mid \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supph(E)}) \geq -i\},$$

where the closed subset $\mathrm{supph(E)} \subset X$ is the support of the total homology of the perfect complex E.

The resolution of $i_*O_{Y'}$, which is a perfect complex of $O_X[\varepsilon]$-module supported on Y, defines an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, denoted $[i_*O_{Y'}]$.

In general, the length of the perfect complex $[i_*O_{Y'}]$ may not be equal to p. Since $Y \subset X$ is of codimension p, for our purpose, we expect the perfect complex $[i_*O_{Y'}]$ is of length p. To achieve this, instead of considering $[i_*O_{Y'}]$ as an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, we consider its image in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^{\#}$, denoted $[i_*O_{Y'}]^{\#}$, where the idempotent completion is in the sense of Balmer-Schlichting [3]. And we have the following result:

Theorem 2.1. [2] *For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#} \simeq \coprod_{x[\varepsilon]\in X[\varepsilon]^{(-i)}} D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])$$

between the idempotent completion of the Verdier quotient $\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon])$ and the coproduct over $x[\varepsilon] \in X[\varepsilon]^{(-i)}$ of the derived category of perfect complexes of $O_{X[\varepsilon],x[\varepsilon]}$-modules with homology supported on the closed point $x[\varepsilon] \in \mathrm{Spec}(O_{X[\varepsilon],x[\varepsilon]})$. And consequently, one has

$$K_0((\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#}) \simeq \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(-i)}} K_0(D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])).$$

Let y be the generic point of Y and let $\mathcal{I}_Y$ be the ideal sheaf of Y. Then there exists the following short exact sequence:

$$0 \to \mathcal{I}_Y \to O_X \to i_*O_Y \to 0,$$

whose localization at y is the short exact sequence:

$$0 \to (\mathcal{I}_Y)_y \to O_{X,y} \to (i_*O_Y)_y \to 0.$$

We have $O_{Y,y} = O_{X,y}/(\mathcal{I}_Y)_y$. Since $O_{Y,y}$ is a field, $(\mathcal{I}_Y)_y$ is the maximal ideal of the regular local ring (of dimension p)$O_{X,y}$. So the maximal ideal $(\mathcal{I}_Y)_y$ is generated by a regular sequence of length p: $f_1, \cdots, f_p$.

Let $\mathcal{I}_{Y'}$ be the ideal sheaf of Y', then $\mathcal{I}_{Y'}/(\varepsilon)\mathcal{I}_{Y'} = \mathcal{I}_Y$ because of flatness. So we have $(\mathcal{I}_{Y'})_y/(\varepsilon)(\mathcal{I}_{Y'})_y = (\mathcal{I}_Y)_y$. Lift $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$ in $(\mathcal{I}_{Y'})_y$, where $g_1, \cdots, g_p \in O_{X,y}$, then $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$ generates $(\mathcal{I}_{Y'})_y$ because of Nakayama's lemma:

$$(\mathcal{I}_{Y'})_y = (f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p).$$

Moreover, $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$ is a regular sequence which can be checked directly.

We see that Y is generically defined by a regular sequence of length p: $f_1, \cdots, f_p$, where $f_1, \cdots, f_p \in O_{X,y}$. Y' is generically given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_1, \cdots, g_p \in O_{X,y}$. We use $F_{\bullet}(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$:

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow 0,$$

where each $F_i = \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p}$ and $A_i : \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p} \to \bigwedge^{i-1} (O_{X,y}[\varepsilon])^{\oplus p}$ are defined as usual.

Under the equivalence in Theorem 2.1, the localization at the generic point y sends $[i_*O_{Y'}]^{\#}$ to the Koszul complex $F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p)$:

$$[i_*O_{Y'}]^{\#} \to F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p).$$

Recall that Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [11].

Definition 2.2 (Definition 3.2 in [11]). *Let X be a finite equi-dimensional noetherian scheme and $x \in X^{(j)}$. For $m \in \mathbb{Z}$, Milnor K-group with support $K_m^M(O_{X,x}$ on $x)$ is rationally defined to be*

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace of $\psi^k = k^{m+j}$ and ψ^k are the Adams operations.

Theorem 2.3 (Prop 4.12 of [7]). *The Adams operations ψ^k defined on perfect complexes, defined by Gillet-Soulé in* [7]*, satisfy $\psi^k(F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p)) = k^p F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p)$.*

Hence, $F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p)$ is of eigenweight p and can be considered as an element of $K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$:

$$F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p) \in K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

Definition 2.4. *We define a map $\mu : \mathrm{T}_Y\mathrm{Hilb}^p(X) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ as follows:*

$$\begin{aligned} \mu :\ & \mathrm{T}_Y\mathrm{Hilb}^p(X) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\ & Y' \longrightarrow F_\bullet(f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p). \end{aligned}$$

3. Chern character

For any integer m, let $K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ denote the weight i eigenspace of the relative K-group, that is, the kernel of the natural projection

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} K_m^{(i)}(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

Recall that we have proved the following isomorphisms in [6, 11]:

Theorem 3.1 (Corollary 9.5 in [6], Corollary 3.11 in [11]). *Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. Chern character(from K-theory to negative cyclic homology) induces the following isomorphisms between relative K-groups and local cohomology groups:*

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H_y^p(\Omega_{X/\mathbb{Q}}^{\bullet,(i)}),$$

where

$$\begin{cases} \Omega^{\bullet,(i)}_{X/\mathbb{Q}} & = \Omega^{2i-(m+p)-1}_{X/\mathbb{Q}}, \text{for } \frac{m+p}{2} < \ i \leq m+p, \\ \Omega^{\bullet,(i)}_{X/\mathbb{Q}} & = 0, \text{else}. \end{cases}$$

The main tool for proving these isomorphisms is the space-level versions of Goodwillie's and Cathelineau's isomorphisms, proved in the appendix B of [5].

Let $K^M_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ denote the relative K-group, that is, the kernel of the natural projection

$$K^M_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\varepsilon=0} K^M_m(O_{X,y} \text{ on } y).$$

In other words, $K^M_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ is $K^{(m+p)}_m(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$. In particular, by taking $i = p$ and $m = 0$ in Theorem 3.1, we obtain the following formula:

Corollary 3.2.

$$K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Definition 3.3. *Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. There exists the following natural surjective map*

$$\mathrm{Ch} : K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}),$$

which is defined to be the composition of the natural projection

$$K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon),$$

and the following isomorphism

$$K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Now, We recall a beautiful construction of Angéniol and Lejeune-Jalabert which describes the map in Definition 3.3

$$\mathrm{Ch} : K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

An element $M \in K^M_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \subset K_0(O_{X,y}[\varepsilon] \textit{ on } y[\varepsilon])_{\mathbb{Q}}$ is represented by a strict perfect complex $L_\bullet$ supported at $y[\varepsilon]$:

$$0 \longrightarrow F_n \xrightarrow{M_n} F_{n-1} \xrightarrow{M_{n-1}} \dots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 \longrightarrow 0,$$

where each $F_i = O_{X,y}[\varepsilon]^{r_i}$ and M_i's are matrices with entries in $O_{X,y}[\varepsilon]$.

Definition 3.4 (Page 24 in [1])**.** *The local fundamental class attached to this perfect complex is defined to be the following collection:*

$$[L_\bullet]_{loc} = \{\frac{1}{p!} dM_i \circ dM_{i+1} \circ \cdots \circ dM_{i+p-1}\}, i = 0, 1, \cdots$$

where $d = d_{\mathbb{Q}}$ and each dM_i is the matrix of absolute differentials. In other words,

$$dM_i \in \mathrm{Hom}(F_i, F_{i-1} \otimes \Omega^1_{O_{X,y}[\varepsilon]/\mathbb{Q}}).$$

Theorem 3.5 (Lemme 3.1.1 on page 24, Def 3.4 on page 29 in [1])**.** *$[L_\bullet]_{loc}$ defined above is a cycle in $\mathcal{H}om(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet)$, and the image of $[L_\bullet]_{loc}$ in $H^p(\mathcal{H}om(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet))$ does not depend on the choice of the basis of $L_\bullet$.*

Since

$$H^p(\mathcal{H}om(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet)) = \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet),$$

the above local fundamental class $[L_\bullet]_{loc}$ defines an element in $\mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet)$:

$$[L_\bullet]_{loc} \in \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet).$$

Noting $L_\bullet$ is supported on y(same underlying space as $y[\varepsilon]$), there exists the following trace map, see page 98-99 in [1] for details,

$$\mathrm{Tr} : \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet) \longrightarrow H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}).$$

Definition 3.6 (Definition 2.3.2 on page 99 in [1])**.** *The image of $[L_\bullet]_{loc}$ under the above trace map, denoted $\mathcal{V}^p_{L_\bullet}$, is called Newton class.*

$K_0(O_{X,y}[\varepsilon]$ *on* $y[\varepsilon])$ is the Grothendieck group of the triangulated category $D^b(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$, which is the derived category of perfect complexes of $O_{X,y}[\varepsilon]$-modules with homology supported on the closed point $y[\varepsilon] \in \mathrm{Spec}(O_{X,y}[\varepsilon])$. Recall that the Grothendieck group of a triangulated category is the monoid of isomorphism objects modulo the submonoid formed from distinguished triangles.

Theorem 3.7 (Proposition 4.3.1 on page 113 in [1])**.** *The Newton class $\mathcal{V}^p_{L_\bullet}$ is well-defined on $K_0(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$.*

The truncation map $\rfloor \dfrac{\partial}{\partial \varepsilon} |_{\varepsilon=0}: \Omega^p_{X[\varepsilon]/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{Q}}$ induces a map,

$$\rfloor \frac{\partial}{\partial \varepsilon} |_{\varepsilon=0}: H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \longrightarrow H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Lemma 3.8. *The map(in Definition 3.3)*

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

can be described as a composition:

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet) \to H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

$$L_\bullet \longrightarrow [L_\bullet]_{loc} \longrightarrow \mathcal{V}^p_{L_\bullet} \longrightarrow \mathcal{V}^p_{L_\bullet} \rfloor \frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} .$$

In particular, for the Koszul complex $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$ in Definition 2.4, the Ch map can be described as follows. The following diagram,

$$\begin{cases} F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p) \longrightarrow O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p) \\ F_p(\cong O_{X,y}[\varepsilon]) \xrightarrow{[F_\bullet]_{loc}} F_0\otimes\Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}}(\cong\Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}}), \end{cases}$$

where $[F_\bullet]_{loc}$ is short for the local fundamental class attached to $F_\bullet(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p)$, gives an element in $Ext^p_{O_{X,y}[\varepsilon]}(O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p),\Omega^p_{X[\varepsilon]/\mathbb{Q}})$. This further gives an element in $H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}})$, denoted $\mathcal{V}^p_{F.}$.

We use $F_\bullet(f_1,\cdots,f_p)$ to denote the Koszul complex associated to the regular sequence $f_1,\cdots,f_p$, which is a resolution of $O_{X,y}/(f_1,\cdots,f_p)$. The truncation of $\mathcal{V}^p_{F.}$ in ε produces an element in $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$, which can be represented by the following diagram

$$\begin{cases} F_\bullet(f_1,\cdots,f_p) \longrightarrow O_{X,y}/(f_1,\cdots,f_p) \\ F_p(\cong O_{X,y}) \xrightarrow{[F_\bullet]_{loc}\rfloor\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}} F_0\otimes\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}(\cong\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}). \end{cases}$$

For simplicity, assuming $g_2=\cdots=g_p=0$, we see that $[F_\bullet]_{loc}\rfloor\dfrac{\partial}{\partial\varepsilon}|_{\varepsilon=0}=g_1df_2\wedge\cdots\wedge df_p$ and the truncation of $\mathcal{V}^p_{F.}$ in ε is represented by the following diagram

$$\begin{cases} F_\bullet(f_1,\cdots,f_p) \longrightarrow O_{X,y}/(f_1,\cdots,f_p) \\ F_p(\cong O_{X,y}) \xrightarrow{g_1df_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}(\cong\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}). \end{cases}$$

Further concrete examples can be found in Chapter 7 of [8](page 90-91).

4. **The map** π

Definition 4.1. *We define a map from* $\mathrm{T_Y Hilb}^p(X)$ *to* $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$ *by composing Ch in* Definition 3.3 *with* μ *in* Definition 2.4*:*

$$\pi:\mathrm{T_Y Hilb}^p(X)\xrightarrow{\mu}K_0^M(O_{X,y}[\varepsilon]\text{ on }y[\varepsilon])\xrightarrow{\mathrm{Ch}}H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Recall that the Cousin complex of $\Omega^{p-1}_{X/\mathbb{Q}}$ is of the form:

$$0\to\Omega^{p-1}_{k(X)/\mathbb{Q}}\to\cdots\to\bigoplus_{y\in X^{(p)}}H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})\xrightarrow{\partial_1^{p,-p}}\bigoplus_{x\in X^{(p+1)}}H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}})\to\cdots$$

and the tangent space $TZ^p(X)$ is identified with $\mathrm{Ker}(\partial_1^{p,-p})$, see Theorem 1.1.

For $p = d := \dim(X)$, $\partial_1^{d,-d} = 0$ because of dimensional reason. So $TZ^d(X) = \mathrm{Ker}(\partial_1^{d,-d}) = \bigoplus_{y \in X^{(d)}} H_y^d(\Omega^{d-1}_{X/\mathbb{Q}})$.

Corollary 4.2. *For $p = d := \dim(X)$, the map π defines a map from $\mathrm{T_Y Hilb}^d(X)$ to $TZ^d(X)$ and it agrees with the map by* Green *and* Griffiths *in* Theorem 1.3.

We want to know, for general p, whether this map π defines a map from $\mathrm{T_Y Hilb}^p(X)$ to $TZ^p(X)$, as Green and Griffiths asked in Question 1.2.

Remark 4.3. *In an email to the author,* Christophe Soulé *suggested that he consider the image of suitable Koszul complexes under the Ch map in Definition 3.3. This leads us to the following example, showing that π does not define a map from $\mathrm{T_Y Hilb}^p(X)$ to $TZ^p(X)$ in general. The Koszul technique is also used in* Theorem 4.6.

The author sincerely thanks Christophe Soulé *for very helpful suggestions.*

Example 4.4. *For X be a smooth projective three-fold over a field k of characteristic 0, let $Y \subset X$ be a curve with generic point y. We assume a point $x \in Y \subset X$ is defined by (f, g, h) and Y is generically defined by (f, g), then $O_{X,y} = (O_{X,x})_{(f,g)}$.*

We consider the infinitesimal deformation Y' of Y which is generically given by $(f + \varepsilon\frac{1}{h}, g)$, where $\frac{1}{h} \in O_{X,y} = (O_{X,x})_{(f,g)}$(Note $\frac{1}{h} \notin O_{X,x}$), whose Koszul complex is of the form

$$0 \to (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g,-f-\varepsilon\frac{1}{h})^{\mathrm{T}}} (O_{X,x})^{\oplus 2}_{(f,g)}[\varepsilon] \xrightarrow{(f+\varepsilon\frac{1}{h},g)} (O_{X,x})_{(f,g)}[\varepsilon] \to 0,$$

where $(-,-)^{\mathrm{T}}$ denotes transpose.

$\pi(Y') \in H_y^2(\Omega^1_{X/\mathbb{Q}})$ is represented by the following diagram,

$$\begin{cases} (O_{X,x})_{(f,g)} \longrightarrow (O_{X,x})^{\oplus 2}_{(f,g)} \longrightarrow (O_{X,x})_{(f,g)} \longrightarrow (O_{X,x})_{(f,g)}/(f,g) \longrightarrow 0 \\ (O_{X,x})_{(f,g)} \xrightarrow{\frac{1}{h}dg} \Omega^1_{(O_{X,x})_{(f,g)}/\mathbb{Q}}. \end{cases}$$

Let $F_\bullet(f, g, h)$ be the Koszul complex of f, g, h:

$$0 \to O_{X,x} \to O_{X,x}^{\oplus 3} \to O_{X,x}^{\oplus 3} \to O_{X,x} \to 0,$$

$\partial_1^{2,-2}(\pi(Y'))$ in $H_x^3(\Omega^1_{X/\mathbb{Q}})$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f,g,h) \longrightarrow O_{X,x}/(f,g,h) \\ O_{X,x} \xrightarrow{1dg} \Omega^1_{O_{X,x}/\mathbb{Q}}, \end{cases}$$

which is not zero.

This example shows that, in general, the image of π may not lie in $TZ^p(X)$(the kernel of $\partial_1^{p,-p}$). However, we will show that, in Theorem 4.6 below, given $Y \subset X$ of codimension p and $Y' \in \mathrm{T_Y Hilb}^p(X)$, there exists $Z \subset X$ of codimension p and $Z' \in \mathrm{T_Z Hilb}^p(X)$ such that $\pi(Y') + \pi(Z')$ is a nontrivial element of $TZ^p(X)$.

To fix notations, let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p, with generic point y. Let $W \subset Y$ be a subvariety of codimension 1 in Y, with generic point w. One assumes W is generically defined by $f_1, f_2, \cdots, f_p, f_{p+1}$ and Y is generically defined by $f_1, f_2, \cdots, f_p$. So one has $O_{X,y} = (O_{X,w})_P$, where P is the ideal $(f_1, f_2, \cdots, f_p) \subset O_{X,w}$.

Y' is generically given by $(f_1 + \varepsilon g_1, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p)$, where $g_1, \cdots, g_p \in O_{X,y}$. We assume $g_2 = \cdots = g_p = 0$ in the following. Since $O_{X,y} = (O_{X,w})_P$, we write $g_1 = \dfrac{a}{b}$, where $a, b \in O_{X,w}$ and $b \notin P$. b is either in or not in the maximal ideal $(f_1, f_2, \cdots, f_p, f_{p+1}) \subset O_{X,w}$.

Lemma 4.5. *If $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then $\partial_1^{p,-p}(\pi(Y')) = 0$.*

Proof. If $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then b is a unit in $O_{X,w}$, this says $g_1 = \dfrac{a}{b} \in O_{X,w}$. Then $\pi(Y')$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) & \longrightarrow & (O_{X,w})_P/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_P) & \xrightarrow{g_1 df_2 \wedge \cdots \wedge df_p} & F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \cdots, f_p)$ is of the form

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \ldots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i((O_{X,w})_P)^{\oplus p}$. Since $f_{p+1} \notin P$, f_{p+1}^{-1} exists in $(O_{X,w})_P$, we can write $g_1 df_2 \wedge \cdots \wedge df_p = \dfrac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p$.

$\partial_1^{p,-p}(\pi(Y'))$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1}) & \longrightarrow & O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}) \\ F_{p+1}(\cong O_{X,w}) & \xrightarrow{g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p} & F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

The complex $F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1})$ is of the form

$$0 \longrightarrow \bigwedge^{p+1}(O_{X,w})^{\oplus p+1} \xrightarrow{A_{p+1}} \bigwedge^{p}(O_{X,w})^{\oplus p+1} \longrightarrow \cdots.$$

Let $\{e_1, \cdots, e_{p+1}\}$ be a basis of $(O_{X,w})^{\oplus p+1}$, the map A_{p+1} is

$$e_1 \wedge \cdots \wedge e_{p+1} \to \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots e_{p+1},$$

where $\hat{e_j}$ means to omit the j^{th} term.

Since f_{p+1} appears in A_{p+1},

$$g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p \equiv 0 \in Ext^{p+1}_{O_{X,w}}(O_{X,w}/(f_1, \cdots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}),$$

$$\partial_1^{p,-p}(\pi(Y')) = 0.$$

□

This lemma doesn't contradict Example 4.4 above, where $h \in (f, g, h) \subset O_{X,x}$.

Theorem 4.6. *For $Y' \in \mathrm{T_Y Hilb}^p(X)$ which is generically defined by $(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$, where $g_1 = \dfrac{a}{b} \in O_{X,y} = (O_{X,w})_P$,*

- *Case 1: if $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then $\pi(Y') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y')) = 0$.*

- *Case 2: if $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$, there exists $Z \subset X$ of codimension p and exists $Z' \in \mathrm{T_Z Hilb}^p(X)$ such that $\pi(Y') + \pi(Z') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y') + \pi(Z')) = 0$.*

Proof. Case 1 is Lemma 4.5. Now, we consider the case $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$. Since $b \notin (f_1, f_2, \cdots, f_p)$, we can write $b = \sum_{i=1}^{p} a_i f_i^{n_i} + a_{p+1} f_{p+1}^{n_{p+1}}$, where a_{p+1} is a unit in $O_{X,w}$ and each n_j is some integer. For simplicity, we assume each $n_j = 1$ and $a_{p+1} = 1$.

Since Y' is generically given by $(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$, then $\pi(Y')$ is represented by the following diagram($g_1 = \dfrac{a}{b}$):

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \quad \longrightarrow \quad (O_{X,w})_P/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_P) \xrightarrow{\frac{a}{b} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \cdots, f_p)$ is of the form

$$0 \longrightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i ((O_{X,w})_P)^{\oplus p}$. Let $\{e_1, \cdots, e_p\}$ be a basis of $(O_{X,w})^{\oplus p}$, the map A_p is

$$e_1 \wedge \cdots \wedge e_p \to \sum_{j=1}^{p} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots e_p,$$

where $\hat{e_j}$ means to omit the j^{th} term.

Noting $\dfrac{1}{b}-\dfrac{1}{f_{p+1}}=\dfrac{-\sum_{i=1}^{p}a_if_i}{bf_{p+1}}$ and each $f_i(i=1,\cdots,p)$ appears in A_p, the above diagram representing $\pi(Y')$ can be replaced by the following one:

$$\begin{cases} F_\bullet(f_1,f_2,\cdots,f_p) \longrightarrow (O_{X,w})_P/(f_1,f_2,\cdots,f_p) \\ F_p(\cong (O_{X,w})_P) \xrightarrow{\frac{a}{f_{p+1}}df_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}(\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Then $\partial_1^{p,-p}(\pi(Y'))$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f_1,f_2,\cdots,f_p,f_{p+1}) \longrightarrow O_{X,w}/(f_1,f_2,\cdots,f_p,f_{p+1}) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{adf_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{O_{X,w}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Let P' denote the prime $(f_{p+1},f_2,\cdots,f_p)\subset O_{X,w}$, then P' defines a generic point $z\in X^{(p)}$ and one has $O_{X,z}=(O_{X,w})_{P'}$. We define the subscheme

$$Z:=\overline{\{z\}}.$$

Let Z' be a first order infinitesimal deformation of Z, which is generically given by $(f_{p+1}+\varepsilon\dfrac{a}{f_1},f_2,\cdots,f_p)$. $\pi(Z')$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f_{p+1},f_2,\cdots,f_p) \longrightarrow (O_{X,w})_{P'}/(f_{p+1},f_2,\cdots,f_p) \\ F_p(\cong (O_{X,w})_{P'}) \xrightarrow{\frac{a}{f_1}df_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}}(\cong \Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}}), \end{cases}$$

and $\partial_1^{p,-p}(\pi(Z'))$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f_{p+1},f_2,\cdots,f_p,f_1) \longrightarrow O_{X,w}/(f_{p+1},f_2,\cdots,f_p,f_1) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{adf_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{O_{X,w}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1,f_2,\cdots,f_p,f_{p+1})$ and $F_\bullet(f_{p+1},f_2,\cdots,f_p,f_1)$ are Koszul resolutions of $O_{X,w}/(f_1,f_2,\cdots,f_p,f_{p+1})$ and $O_{X,w}/(f_{p+1},f_2,\cdots,f_p,f_1)$ respectively.

These two Koszul complexes $F_\bullet(f_1,f_2,\cdots,f_p,f_{p+1})$ and $F_\bullet(f_{p+1},f_2,\cdots,f_p,f_1)$ are related by the following commutative diagram, see page 691 of [9],

$$\begin{tikzcd} O_{X,w} \arrow[r, "D_{p+1}"] \arrow[d, "\det A_1"'] & \wedge^p O_{X,w}^{\oplus p+1} \arrow[r, "D_p"] \arrow[d, "\wedge^p A_1"'] & \cdots \arrow[r] \arrow[d] & O_{X,w}^{\oplus p+1} \arrow[r, "D_1"] \arrow[d, "A_1"'] & O_{X,w} \arrow[d, "="'] \\ O_{X,w} \arrow[r, "E_{p+1}"] & \wedge^p O_{X,w}^{\oplus p+1} \arrow[r, "E_p"] & \cdots \arrow[r] & O_{X,w}^{\oplus p+1} \arrow[r, "E_1"] & O_{X,w}, \end{tikzcd}$$

where each D_i and E_i are defined as usual. In particular, $D_1 = (f_1, f_2, \cdots, f_p, f_{p+1})$, $E_1 = (f_{p+1}, f_2, \cdots, f_p, f_1)$, and A_1 is the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots 1 \\ 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots 0 \end{pmatrix}.$$

Since $\det A_1 = -1$, one has

$$\partial_1^{p,-p}(\pi(Z')) = -\partial_1^{p,-p}(\pi(Y')) \in Ext^{p+1}_{O_{X,w}}(O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}),$$

consequently, $\partial_1^{p,-p}(\pi(Z') + \pi(Y')) = 0 \in H^{p+1}_w(\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})$. In other words,

$$\pi(Z') + \pi(Y') \in TZ^p(X).$$

□

There exists the following commutative diagram, which is part of the commutative diagram of Theorem 3.14 in [11](taking j=1)

$$\begin{array}{ccc} \bigoplus_{x \in X^{(p)}} H^p_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \\ \partial_1^{p,-p} \downarrow & & d^{p,-p}_{1,X[\varepsilon]} \downarrow \\ \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) & \xleftarrow[\cong]{\mathrm{Ch}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]). \end{array}$$

For $Y' \in \mathrm{T_Y Hilb}^p(X)$ which is generically defined by $(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$, where $g_1 = \dfrac{a}{b} \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \cdots, f_p$. Theorem 4.6 implies the following.

Case 1: if $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, $\partial_1^{p,-p}(\pi(Y')) = 0$. The commutative diagram

$$\begin{array}{ccc} \pi(Y') & \xleftarrow{\mathrm{Ch}} & F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \\ \partial_1^{p,-p} \downarrow & & d^{p,-p}_{1,X[\varepsilon]} \downarrow \\ 0 & \xleftarrow[\cong]{\mathrm{Ch}} & d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)). \end{array}$$

says $d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)) = 0$.

Case 2: if $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exists $Z \subset X$ which is generically defined by

$(f_{p+1}, f_2, \cdots, f_p)$ and exists $Z' \in \mathrm{T_Z Hilb}^p(X)$ which is generically defined by $(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p)$ such that $\partial_1^{p,-p}(\pi(Y')+\pi(Z')) = 0$. We use $F_\bullet(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p)$ to denote the Koszul complex associated to $f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p$.

The commutative diagram

$$\begin{array}{ccc}
\pi(Y') + \pi(Z') & \xleftarrow{\mathrm{Ch}} & F_\bullet(f_1+\varepsilon\dfrac{a}{f_{p+1}}, f_2, \cdots, f_p) + F_\bullet(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p) \\
\partial_1^{p,-p}\big\downarrow & & \big\downarrow d^{p,-p}_{1,X[\varepsilon]} \\
0 & \underset{\cong}{\xleftarrow{\mathrm{Ch}}} & d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1+\varepsilon\dfrac{a}{f_{p+1}}, f_2, \cdots, f_p) + F_\bullet(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p)).
\end{array}$$

says $d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1+\varepsilon\dfrac{a}{f_{p+1}}, f_2, \cdots, f_p) + F_\bullet(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p)) = 0$.

Recall that, in Definition 3.4 and Corollary 3.15 in [11], the p-th Milnor K-theoretic cycles is defined as

$$Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])) := \mathrm{Ker}(d^{p,-p}_{1,X[\varepsilon]}).$$

The above can be summarized as:

Theorem 4.7. *For $Y' \in \mathrm{T_Y Hilb}^p(X)$ which is generically defined by $(f_1+\varepsilon g_1, f_2, \cdots, f_p)$, where $g_1 = \dfrac{a}{b} \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1+\varepsilon g_1, f_2, \cdots, f_p)$ to denote the Koszul complex associated to $f_1+\varepsilon g_1, f_2, \cdots, f_p$.*

- *Case 1: if $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then $F_\bullet(f_1+\varepsilon g_1, f_2, \cdots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon]))$.*
- *Case 2: if $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exists $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \cdots, f_p)$ and there exists $Z' \in \mathrm{T_Z Hilb}^p(X)$ which is generically defined by $(f_{p+1}+\varepsilon\dfrac{a}{f_1}, f_2, \cdots, f_p)$ such that*
$$F_\bullet(f_1+\varepsilon\frac{a}{f_{p+1}}, f_2, \cdots, f_p)+F_\bullet(f_{p+1}+\varepsilon\frac{a}{f_1}, f_2, \cdots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])).$$

The existence of Z and $Z' \in \mathrm{T_Z Hilb}^p(X)$ has applications in deformation of cycles, see [13] for a concrete example of eliminating obstructions to deforming curves on a three-fold.

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