An isoperimetric problem for leaky loops and related mean-chord inequalities

Pavel Exner

Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague, Czechia, and Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czechia
exner@ujf.cas.cz

We consider a class of Hamiltonians in $L^2(\mathbb{R}^2)$ with attractive interaction supported by piecewise C^2 smooth loops Γ of a fixed length L, formally given by $-\Delta - \alpha\delta(x-\Gamma)$ with $\alpha > 0$. It is shown that the ground state of this operator is locally maximized by a circular Γ. We also conjecture that this property holds globally and show that the problem is related to an interesting family of geometric inequalities concerning mean values of chords of Γ.

1 Introduction

There is a small number of topics which can be regarded as trademark for mathematical physics. One of them without any doubt concerns relations between geometric properties of constraints and/or interaction and extremal values of a spectral quantity; classical examples are Faber-Krahn inequality [12, 14] or the PPW-conjecture proved by Ashbaugh and Benguria [3].

A common feature of these and analogous problems is that the extremum is reached by shapes having a rotational symmetry. At the same time, the nature of the extremum may be different. While a ball *minimizes* the principal eigenvalue of the Dirichlet Laplacian among regions of a fixed volume, for non-simply connected regions like annular strips or layers considered in [8, 13], built over a curve (surface) of a fixed length (area), the circular shape is on the contrary a *maximizer*. A natural topological way to understand this difference becomes smeared, however, when the particle is not localized by boundary conditions but by a potential, a regular or singular one.

In this paper we consider such a problem associated with a class of operators in $L^2(\mathbb{R}^2)$ which are given formally by the expression

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \tag{1.1}$$

where $\alpha > 0$ and Γ is a C^2 loop in the plane (see below for exact assumptions) having a fixed length $L > 0$. A motivation to study these operators comes from the theory of *leaky quantum graphs* – see [5, 9] and related papers, a bibliography can be found in [1] – aiming at a more realistic model of quantum wire structures which would take quantum tunneling into account.

Our aim is to show that the ground-state energy of $H_{\alpha,\Gamma}$ is (sharply) maximized when Γ is a circle. We will be able to prove that this property holds *locally* conjecturing its global validity. There are several reasons why one may expect this result to be valid. On one hand, we know from [11] that in the limit of strong coupling, $\alpha \to \infty$, the ground-state dependence on Γ is given in the leading order by the lowest eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2([0,L])$ with periodic boundary conditions where γ is the curvature of Γ, and the latter is easily seen to be globally sharply maximized when γ is constant along Γ. On the other hand, by [10] the operator $H_{\alpha,\Gamma}$ can be approximated in the strong resolvent sense by point interaction Hamiltonians with the point interactions equidistantly spaced along Γ and properly chosen coupling constants, and from [7] we know that the ground state of such an operator is locally maximized by a regular polygon.

Needless to say, neither of the above observations proves the desired result. The first one is global, but it holds only asymptotically and we do not know whether the error term will not spoil the inequality. The second argument holds for any $\alpha > 0$ suggesting the local validity, but the polygons approximating the circle do not have exactly the same lengths.

Our main tools in this paper are the generalized Birman-Schwinger principle in combination with the convexity of Green's function. They allow us to reformulate the problem in a purely geometric way, in terms of *mean value of chords* of arc segments of Γ. Since such geometric inequalities are of an independent interest, we discuss them in Sec. 4 separately in a broader context, including the discrete version which arose in connection with the polygon problem treated in [7]. Before doing that, we will formulate in the next section the problem and state our main result, Theorem 2.1, and provide the mentioned reformulation in Sec. 3. After the discussion of the inequalities we will finish the proof of Theorem 2.1 and present some concluding remarks.

2 Formulation and the main result

We will assume throughout that $\Gamma:\ [0,L]\to\mathbb{R}^2$ is *a closed curve,* $\Gamma(0)=\Gamma(L)$*, parametrized by its arc length, which is* C^1*-smooth, piecewise* C^2*, and has no cusps*[1]. Unless stated otherwise, we will mean by the curve Γ for simplicity both the above mentioned function and its image in the plane. Furthermore, we introduce the equivalence relation: Γ and Γ' belong to the same equivalence class if one can be obtained from the other by a Euclidean transformation of the plane. Spectral properties of the corresponding $H_{\alpha,\Gamma}$ and $H_{\alpha,\Gamma'}$ are obviously the same, and we will usually speak about a curve Γ having in mind the corresponding equivalence class. It is clear that the stated regularity assumptions are satisfied, in particular, by the circle, say $\mathcal{C}:=\{\,((L/2\pi)\cos s,(L/2\pi)\sin s):\ s\in[0,L]\,\}$, and its equivalence class.

First of all we have to give a rigorous meaning to the operator (1.1). Following [4, 5] we can do that in two ways. The more general one is to consider a positive Radon measure m on $\mathbb{R}^2$ and $\alpha>0$ such that

$$(1+\alpha)\int_{\mathbb{R}^2}|\psi(x)|^2\,\mathrm{d}m(x)\le a\int_{\mathbb{R}^2}|\nabla\psi(x)|^2\,\mathrm{d}x+b\int_{\mathbb{R}^2}|\psi(x)|^2\,\mathrm{d}x \tag{2.1}$$

holds for all ψ from the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ and some $a<1$ and b. The map I_m defined on $\mathcal{S}(\mathbb{R}^2)$ by $I_m\psi=\psi$ extends by density uniquely to

$$I_m:\ W^{1,2}(\mathbb{R}^2)\ \to\ L^2(m):=L^2(\mathbb{R}^2,\mathrm{d}m)\,; \tag{2.2}$$

abusing notation we employ the same symbol for a continuous function and the corresponding equivalence classes in both $L^2(\mathbb{R}^2)$ and $L^2(m)$. The inequality (2.1) extends to $W^{1,2}(\mathbb{R}^2)$ with ψ replaced by $I_m\psi$ at the left-hand

[1] There are, of course, no local cusps under the C^1 assumption, but we have not excluded self-intersections, so the last requirement means that the curve meets itself at such a point at a nonzero angle. In fact our main result can be pushed through under a slightly weaker regularity assumption, namely that $\dot\Gamma$ is absolutely continuous.

side. This makes it possible to introduce the following quadratic form,

$$\mathcal{E}_{-\alpha m}(\psi,\phi) := \int_{\mathbb{R}^2} \overline{\nabla\psi(x)}\nabla\phi(x)\,\mathrm{d}x - \alpha\int_{\mathbb{R}^2} (I_m\bar{\psi})(x)(I_m\phi)(x)\,\mathrm{d}m(x)\,, \quad (2.3)$$

with the domain $W^{1,2}(\mathbb{R}^2)$; it is straightforward to see that under the condition (2.1) it is closed and below bounded, with $C_0^\infty(\mathbb{R}^2)$ as a core, and thus associated with a unique self-adjoint operator. Furthermore, (2.1) is satisfied with any $a>0$ provided m belongs to the generalized Kato class,

$$\lim_{\epsilon\to 0}\sup_{x\in\mathbb{R}^2}\int_{B(x,\epsilon)} |\ln|x-y||\,\mathrm{d}m(y) = 0\,, \quad (2.4)$$

where $B(x,\epsilon)$ is the ball of radius ϵ and center x. Choosing now for m the Dirac measure supported by the curve one can check easily that the condition (2.4) is satisfied under our assumptions about Γ, hence we may identify the above mentioned self-adjoint operator with the formal one given by (1.1).

The described definition applies naturally to a much wider class of perturbations than we need here. Since Γ is supposed to be smooth, with the normal defined everywhere, we can define $H_{\alpha,\Gamma}$ alternatively through boundary conditions. Specifically, it acts as $-\Delta\psi$ on any ψ from the domain consisting of functions which belong to $W^{2,2}(\mathbb{R}^2\setminus\Gamma)$, they are continuous at the curve Γ and their normal derivatives have a jump there,

$$\frac{\partial\psi(x)}{\partial n_+} - \frac{\partial\psi(x)}{\partial n_-} = -\alpha\psi(x) \quad \text{for} \quad x=\Gamma(s)\,,\ \forall s\in[0,L]\,.$$

It is straightforward to check that such an operator is e.s.a. and its closure can be identified with (1.1) defined in the above described way [4]. The advantage of the second definition is that it has an illustrative meaning which corresponds well to the concept of a δ interaction in the cross cut of the curve.

Since the curve is finite, by [4, 5] we have $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0,\infty)$ while the discrete spectrum is nonempty and finite, so that

$$\epsilon_1 \equiv \epsilon_1(\alpha,\Gamma) := \inf\sigma\left(-\Delta_{\alpha,\Gamma}\right) < 0\,;$$

we ask for which Γ the principal eigenvalue is maximal. The main result of this paper is a partial answer to this question, namely:

Theorem 2.1 *Within the specified class of curves, $\epsilon_1(\alpha,\Gamma)$ is for any fixed $\alpha>0$ and $L>0$ locally sharply maximized by a circle.*

While we do not give a general answer here, we suggest what it should be.

Conjecture 2.2 *The circle is a sharp global maximizer, even under weaker regularity assumptions.*

3 Birman-Schwinger reformulation

For operators associated with the quadratic form (2.3) one can establish a generalized Birman-Schwinger principle – we refer to [4] for a detailed discussion. In particular, if k^2 belongs to the resolvent set of $H_{\alpha,\Gamma}$ we put $R^k_{\alpha,\Gamma} := (H_{\alpha,\Gamma} - k^2)^{-1}$. The free resolvent R^k_0 is defined for $\mathrm{Im}\, k > 0$ as an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

$$G_k(x-y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)\,.$$

Next we have to introduce embedding operators associated with R^k_0. Let μ, ν be arbitrary positive Radon measures on $\mathbb{R}^2$ with $\mu(x) = \nu(x) = 0$ for any $x \in \mathbb{R}^2$. By $R^k_{\nu,\mu}$ we denote the integral operator from $L^2(\mu) := L^2(\mathbb{R}^2, \mathrm{d}\mu)$ to $L^2(\nu)$ with the kernel G_k, in other words we suppose that

$$R^k_{\nu,\mu}\phi = G_k * \phi\mu$$

holds ν-a.e. for all $\phi \in D(R^k_{\nu,\mu}) \subset L^2(\mu)$. In our case the two measures will be the Dirac measure supported by Γ, denoted by m if necessary, and the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^2$, in different combinations. With this notation one can express the generalized BS principle as follows:

Proposition 3.1 *(i) There is a $\kappa_0 > 0$ such that the operator $I - \alpha R^{i\kappa}_{m,m}$ on $L^2(m)$ has a bounded inverse for any $\kappa \geq \kappa_0$.*
(ii) Let $\mathrm{Im}\, k > 0$. Suppose that $I - \alpha R^k_{m,m}$ is invertible and the operator

$$R^k := R^k_0 + \alpha R^k_{\mathrm{d}x,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,\mathrm{d}x}$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ is everywhere defined. Then k^2 belongs to $\rho(H_{\alpha,\Gamma})$ and $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$.
(iii) $\dim\ker(H_{\alpha,\Gamma} - k^2) = \dim\ker(I - \alpha R^k_{m,m})$ for any k with $\mathrm{Im}\, k > 0$.
(iv) an eigenfunction of $H_{\alpha,\Gamma}$ associated with such an eigenvalue k^2 can be written as

$$\psi(x) = \int_0^L R^k_{\mathrm{d}x,m}(x,s)\phi(s)\,\mathrm{d}s\,,$$

where ϕ is the corresponding eigenfunction of $\alpha R^k_{m,m}$ with the eigenvalue one.

Proof of (i)-(iii) is given in [4], for (iv) see [15]. ∎

Denoting conventionally $k = i\kappa$ with $\kappa > 0$ as corresponding to the bound-state energy $-\kappa^2$, we can thus rephrase our problem as a search for solutions to the integral-operator equation

$$\mathcal{R}^\kappa_{\alpha,\Gamma}\phi = \phi\,, \quad \mathcal{R}^\kappa_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,, \tag{3.1}$$

on $L^2([0,L])$, where K_0 is Macdonald function. Referring again to [4] and [15] we find that the operator-valued function $\kappa \mapsto \mathcal{R}^\kappa_{\alpha,\Gamma}$ is strictly decreasing in $(0,\infty)$ and $\|\mathcal{R}^\kappa_{\alpha,\Gamma}\| \to 0$ as $\kappa \to \infty$. In fact the two properties can be checked also directly. The first one follows from the one-to-one correspondence of the eigenvalue branches (as functions of κ) to those of $H_{\alpha,\Gamma}$ which are obviously strictly monotonous as functions of α; the second one in turn comes from the explicit form of the kernel together with the dominated convergence theorem.

Next we use the fact that the maximum eigenvalue of $\mathcal{R}^\kappa_{\alpha,\Gamma}$ is simple. This conclusion results from the following considerations: the kernel of the operator is by (3.1) strictly positive, so $\mathcal{R}^\kappa_{\alpha,\Gamma}$ is positivity improving. It further means that for any nonzero $\phi, \chi \geq 0$ the functions $\mathcal{R}^\kappa_{\alpha,\Gamma}\phi$, $\mathcal{R}^\kappa_{\alpha,\Gamma}\chi$ are also strictly positive. Hence $(\phi, (\mathcal{R}^\kappa_{\alpha,\Gamma})^2\chi) \neq 0$, and as a consequence, $\mathcal{R}^\kappa_{\alpha,\Gamma}$ is ergodic; then the claim follows from Thm. XIII.43 of [16]. In view of Proposition 3.1(iii) the ground state of $H_{\alpha,\Gamma}$ is, of course, also simple.

If Γ is a circle the operator $H_{\alpha,\mathcal{C}}$ has a full rotational symmetry, so the corresponding eigenspace supports a one-dimensional representation of the group $O(2)$. Let us denote the ground-state eigenfunction of $H_{\alpha,\mathcal{C}}$ as $-\tilde\kappa_1^2$ (we will use tilde to distinguish quantities referring to the circle). The correspondence between the eigenfunctions given by Proposition 3.1(iv) then requires that the respective eigenfunction of $\mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde\phi_1(s) = L^{-1/2}$. Then we have

$$\max\sigma(\mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}) = (\tilde\phi_1, \mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}\tilde\phi_1) = \frac{1}{L}\int_0^L\int_0^L \mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}(s,s')\,\mathrm{d}s\mathrm{d}s'\,,$$

and on the other hand, for the same quantity referring to a general Γ a simple variational estimate gives

$$\max\sigma(\mathcal{R}^{\tilde\kappa_1}_{\alpha,\Gamma}) \geq (\tilde\phi_1, \mathcal{R}^{\tilde\kappa_1}_{\alpha,\Gamma}\tilde\phi_1) = \frac{1}{L}\int_0^L\int_0^L \mathcal{R}^{\tilde\kappa_1}_{\alpha,\Gamma}(s,s')\,\mathrm{d}s\mathrm{d}s'\,.$$

Hence to check that the circle is a maximizer it sufficient to show that

$$\int_0^L \int_0^L K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,\mathrm{d}s\mathrm{d}s' \geq \int_0^L \int_0^L K_0(\kappa|\mathcal{C}(s)-\mathcal{C}(s')|)\,\mathrm{d}s\mathrm{d}s' \tag{3.2}$$

holds *for all* $\kappa > 0$ and Γ of the considered class, or at least for Γ in the vicinity of $\mathcal{C}$ to prove the local result in Theorem 2.1. Since the kernel is symmetric w.r.t. the two variables, we can replace the double integral by $2\int_0^L \mathrm{d}s \int_0^s \mathrm{d}s'$. By another simple change of variables we find that the above claim is equivalent to positivity of the functional

$$F_\kappa(\Gamma) := \int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s \Big[K_0(\kappa|\Gamma(s+u) - \Gamma(s)|) - K_0(\kappa|\mathcal{C}(s+u) - \mathcal{C}(s)|) \Big] ,$$

where the second term in the integrand is, of course, independent of s being equal to $K_0(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L})$. Now we employ the (strict) convexity of K_0 which yields by means of the Jensen inequality the following estimate,

$$\frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[K_0\left(\frac{\kappa}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)|\mathrm{d}s \right) - K_0\left(\frac{\kappa L}{\pi} \sin\frac{\pi u}{L} \right) \right] \mathrm{d}u\,,$$

where the inequality is sharp unless $\int_0^L |\Gamma(s+u) - \Gamma(s)|\mathrm{d}s$ is independent of s. Finally, we observe that K_0 is decreasing in $(0,\infty)$, hence it is sufficient to check the inequality

$$\int_0^L |\Gamma(s+u) - \Gamma(s)|\,\mathrm{d}s \,\leq\, \frac{L^2}{\pi} \sin\frac{\pi u}{L} \tag{3.3}$$

for all $u \in (0, \frac{1}{2}L]$ and to show that is sharp unless Γ is a circle.

4 Mean-chord inequalities

The inequality (3.3) to which we have reduced our problem can be regarded as an element of a wider family which we are now going to describe. Let $\Gamma : [0,L] \to \mathbb{R}^2$ be again a loop in the plane; for the moment we do not specify its regularity properties. Let us consider all the arcs of Γ having length $u \in (0, \frac{1}{2}L]$. The mentioned inequalities are the following

$$C_L^p(u): \qquad \int_0^L |\Gamma(s+u) - \Gamma(s)|^p\,\mathrm{d}s \,\leq\, \tfrac{L^{1+p}}{\pi^p} \sin^p \tfrac{\pi u}{L}\,, \quad p > 0\,, \tag{4.1}$$

$$C_L^{-p}(u): \qquad \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p}\,\mathrm{d}s \,\leq\, \tfrac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}\,, \qquad p > 0\,. \tag{4.2}$$

They have also a discrete counterpart for an equilateral polygon $\mathcal{P}_N$ of N vertices and side length $\ell > 0$. Let $\{y_n\}$ be the family of its vertices, where the index values are identified modulo N; then we introduce

$$D^p_{N,\ell}(m): \quad \textstyle\sum_{n=1}^N |y_{n+m} - y_n|^p \le \frac{N\ell^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}\,, \quad p > 0\,, \tag{4.3}$$

$$D^{-p}_{N,\ell}(m): \quad \textstyle\sum_{n=1}^N |y_{n+m} - y_n|^{-p} \le \frac{N \sin^p \frac{\pi}{N}}{\ell^p \sin^p \frac{\pi m}{N}}\,, \quad p > 0\,, \tag{4.4}$$

for any $m = 1, \dots, [\frac{1}{2}N]$, where $[\cdot]$ denotes as usual the entire part.

In all the cases the right-hand side corresponds, of course, to the case with maximal symmetry, i.e. to the circle and regular polygon $\tilde{\mathcal{P}}_N$, respectively. *We conjecture* that without regularity restrictions $C^{\pm p}_L(u)$ *holds for any* $p \le 2$ *and the same is true for* $D^{\pm p}_{N,\ell}(m)$, and furthermore, we expect the inequalities *to be sharp unless* $\Gamma = \mathcal{C}$ *or* $\mathcal{P}_N = \tilde{\mathcal{P}}_N$*, respectively.* In the polygon case it is clear that the claim may not be true for $p > 2$ as the example of a rhomboid shows: $D^p_{4,\ell}(2)$ is equivalent to $\sin^p \phi + \cos^p \phi \ge 1$ for $0 < \phi < \pi$. We are unable at this moment to demonstrate the inequalities (4.1)–(4.4) in full generality; below we will present a few particular cases.

It is obvious that the inequalities have a scaling property, so without loss of generality one can assume, e.g., $L = 1$ and $\ell = 1$; in such a case we drop the corresponding symbol from the label. If necessary we can include also the case $p = 0$ when the inequalities turn into trivial identities.

Proposition 4.1 $C^p_L(u) \Rightarrow C^{p'}_L(u)$ *and* $D^p_{N,\ell}(m) \Rightarrow D^{p'}_{N,\ell}(m)$ *if* $p > p' > 0$.

Proof: The claim follows from the convexity of $x \mapsto x^\alpha$ in $(0, \infty)$ for $\alpha > 1$,

$$\begin{aligned} \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} &\ge \int_0^L \left(|\Gamma(s+u) - \Gamma(s)|^{p'} \right)^{p/p'} \mathrm{d}s \\ &\ge L \left(\frac{1}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)|^{p'} \mathrm{d}s \right)^{p/p'} . \end{aligned}$$

It is now sufficient to take both sides to the power p'/p; in the same way one checks the second implication. ∎

Proposition 4.2 $C^p_L(u) \Rightarrow C^{-p}_L(u)$ *and* $D^p_{N,\ell}(m) \Rightarrow D^{-p}_{N,\ell}(m)$ *for any* $p > 0$.

Proof: The Schwarz inequality implies

$$\int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p}\,\mathrm{d}s \ge \frac{L^2}{\int_0^L |\Gamma(s+u)-\Gamma(s)|^p\,\mathrm{d}s} \ge \frac{L^2\pi^p}{L^{1+p}\sin^p\frac{\pi u}{L}}\,,$$

and similarly for the polygon case. ∎

These simple relations mean that to check the above stated conjecture one needs only to verify $C^2(u)$ and $D^2_N(m)$. We will address the continuous case in the next section, here we notice that the results of [7] in combination with the last two propositions leads to the following conclusions:

Theorem 4.3 *(a)* $D^1_{N,\ell}(m)$ *holds locally for any* N *and* $m=1,\dots,[\frac12 N]$, *i.e. in a vicinity of the regular polygon, and consequently,* $D^{\pm p}_{N,\ell}(m)$ *holds locally for any* $p\in(0,1]$.
(b) $D^1_{N,\ell}(2)$ *holds globally for any* N, *and so does* $D^{\pm p}_{N,\ell}(2)$ *for each* $p\in(0,1]$.

5 Proof of Theorem 2.1

After this interlude let us return to our main problem. Notice first that our regularity hypothesis allows us to characterize Γ by its (signed) curvature $\gamma:=\dot\Gamma_2\ddot\Gamma_1-\dot\Gamma_1\ddot\Gamma_2$ which is by assumption a piecewise continuous function in $[0,L]$. The advantage is that γ specifies uniquely the equivalence class related by Euclidean transformations which can be represented by

$$\Gamma(s)=\left(\int_0^s \cos\beta(s')\,\mathrm{d}s',\int_0^s \sin\beta(s')\,\mathrm{d}s'\right)\,, \tag{5.1}$$

where $\beta(s):=\int_0^s\gamma(s')\,\mathrm{d}s'$ is the bending angle relative to the tangent at the chosen initial point, $s=0$. To ensure that the curve is closed, the conditions

$$\int_0^L \cos\beta(s')\,\mathrm{d}s' = \int_0^L \sin\beta(s')\,\mathrm{d}s' = 0 \tag{5.2}$$

must be satisfied. Using this parametrization we can rewrite the left-hand side of the inequality (4.1) in the form

$$\int_0^L\left[\left(\int_s^{s+u}\cos\beta(s')\,\mathrm{d}s'\right)^2+\left(\int_s^{s+u}\sin\beta(s')\,\mathrm{d}s'\right)^2\right]^{p/2}\mathrm{d}s := c^p_\Gamma(u)\,,$$

or equivalently

$$c_\Gamma^p(u) = \int_0^L \mathrm{d}s \left[\int_s^{s+u} \mathrm{d}s' \int_s^{s+u} \mathrm{d}s'' \cos(\beta(s') - \beta(s'')) \right]^{p/2} .$$

By Proposition 4.1 it is sufficient to check that the quantity $c_\Gamma^2(u)$ is maximized by the circle, i.e. by $\beta(s) = \frac{2\pi s}{L}$. Rearranging the integrals we get

$$\begin{aligned} c_\Gamma^2(u) &= \int_0^L \mathrm{d}s' \int_{s'-u}^{s'+u} \mathrm{d}s'' \int_{\max\{s'-u,s''-u\}}^{\min\{s',s''\}} \mathrm{d}s \, \cos(\beta(s') - \beta(s'')) \\ &= \int_0^L \mathrm{d}s' \int_{s'-u}^{s'+u} \mathrm{d}s'' \, [\min\{s', s''\} - \max\{s'-u, s''-u\}] \cos(\beta(s') - \beta(s'')), \end{aligned}$$

or

$$c_\Gamma^2(u) = \int_0^L \mathrm{d}s' \int_{s'-u}^{s'+u} \mathrm{d}s'' \, [u - |s' - s''|] \, \cos(\beta(s') - \beta(s'')) \, .$$

Next we change the integration variables to $x := s' - s''$ and $z := \frac{1}{2}(s' + s'')$,

$$c_\Gamma^2(u) = \int_{-u}^u \mathrm{d}x \, (u - |x|) \int_0^L \mathrm{d}z \, \cos\left(\beta(z + \frac{1}{2}x) - \beta(z - \frac{1}{2}x) \right) ,$$

and since the functions involved are even w.r.t. x we finally get

$$c_\Gamma^2(u) = 2 \int_0^u \mathrm{d}x \, (u - x) \int_0^L \mathrm{d}z \, \cos\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} \gamma(s) \, \mathrm{d}s \right) . \tag{5.3}$$

As a certain analogy to Theorem 4.3(b) we can prove the sought global inequality in case when the curve arcs in question are sufficiently short and/or the tangent vector direction does change too fast.

Proposition 5.1 *Suppose that* Γ *has no self-intersections and the inequality* $\beta(z + \frac{1}{2}u) - \beta(z - \frac{1}{2}u) \le \frac{1}{2}\pi$ *is valid for all* $z \in [0, L]$, *then* $C_L^2(u)$ *holds.*

Proof: We employ concavity of cosine in $(0, \frac{1}{2}\pi)$ obtaining

$$\begin{aligned} c_\Gamma^2(u) &\le 2L \int_0^u \mathrm{d}x \, (u - x) \cos\left(\frac{1}{L} \int_0^L \mathrm{d}z \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} \gamma(s) \, \mathrm{d}s \right) \\ &= 2L \int_0^u \mathrm{d}x \, (u - x) \cos\left(\frac{1}{L} \int_0^L \mathrm{d}s \, \gamma(s) \int_{s-\frac{1}{2}x}^{s+\frac{1}{2}x} \mathrm{d}z \right) \\ &= 2L \int_0^u \mathrm{d}x \, (u - x) \cos \frac{2\pi x}{L} = \frac{L^3}{\pi^2} \sin^2 \frac{\pi u}{L} \, , \end{aligned}$$

since $\int_0^L \gamma(s)\,\mathrm{d}s = \pm 2\pi$ for a curve without self-intersections. Moreover, the function $z \mapsto \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} \gamma(s)\,\mathrm{d}s$ is constant for $x \in (0,u)$ *iff* $\gamma(\cdot)$ is constant, hence the circle corresponds to a sharp maximum. ∎

This result, however, does not help us with our main problem, because we need the inequality to be valid for all arc lengths. As indicated before, we can prove a local result which will imply Theorem 2.1.

Theorem 5.2 *Under the regularity assumptions of Sec. 3, the inequality $C_L^2(u)$ holds locally for any $L>0$ and $u \in (0, \frac{1}{2}L]$, and consequently, $C_L^{\pm p}(u)$ holds locally for any $p \in (0,2]$.*

Proof: Gentle deformations of a circle can be characterized by the curvature

$$\gamma(s) = \frac{2\pi}{L} + g(s)\,,$$

where g is a piecewise continuous functions which is small in the sense that $\|g\|_\infty \ll L^{-1}$ and satisfies the condition $\int_0^L g(s)\,\mathrm{d}s = 0$. The function in the last integral of (5.3) can be then expanded as

$$\cos\frac{2\pi x}{L} - \sin\frac{2\pi x}{L}\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s - \frac{1}{2}\cos\frac{2\pi x}{L}\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s\right)^2 + \mathcal{O}(g^3)\,,$$

where the error term is a shorthand for $\mathcal{O}(\|Lg\|_\infty^3)$. Substituting this expansion into (5.3) we find that the term linear in g vanishes, because

$$\int_0^L \mathrm{d}z \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s = \int_0^L \mathrm{d}s\, g(s) \int_{s-\frac{1}{2}x}^{s+\frac{1}{2}x} \mathrm{d}z = 0\,,$$

and thus

$$c_\Gamma^2(u) = \frac{L^3}{\pi^2}\sin^2\frac{\pi u}{L} - I_g(u) + \mathcal{O}(g^3)\,, \tag{5.4}$$

where

$$I_g(u) := \int_0^u \mathrm{d}x\,(u-x)\,\cos\frac{2\pi x}{L}\int_0^L \mathrm{d}z\,\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s\right)^2.$$

We need to show that $I_g(u) > 0$ unless $g=0$ identically. Notice that for $u \le \frac{1}{4}L$ this property holds trivially. For $u \in (\frac{1}{4}L, \frac{1}{2}L]$ we use the fact that g

is periodic and piecewise continuous, so we can write it through its Fourier series

$$g(s) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{2\pi n s}{L} + b_n \cos \frac{2\pi n s}{L} \right)$$

with the zero term missing, where $\sum_n (a_n^2 + b_n^2)$ is finite (and small). Using

$$\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\, \mathrm{d}s = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(a_n \sin \frac{2\pi n z}{L} + b_n \cos \frac{2\pi n z}{L} \right) \sin \frac{\pi n x}{L}$$

together with the orthogonality of the Fourier basis we find

$$I_g(u) = \int_0^u \mathrm{d}x\, (u-x)\, \cos \frac{2\pi x}{L} \sum_{n=1}^{\infty} \frac{L^3}{2\pi^2} \frac{a_n^2 + b_n^2}{n^2}\, \sin \frac{\pi n x}{L}\,.$$

Since the summation and integration can be obviously interchanged, we have

$$I_g(u) = \frac{L^5}{2\pi^4} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{n^2}\, F_n \left(\frac{\pi u}{L} \right) , \tag{5.5}$$

where

$$F_n(v) := \int_0^v (v-y)\, \cos 2y\, \sin ny\, \mathrm{d}y\,.$$

These integrals are equal to

$$\begin{aligned} F_1(v) &= \frac{1}{18}(\, 9 \sin v - \sin 3v - 6v\,)\,, \\ F_2(v) &= \frac{1}{32}(\, 4v - \sin 4v\,)\,, \\ F_n(v) &= \frac{nv}{n^2-4} - \frac{\sin(n-2)v}{2(n-2)^2} - \frac{\sin(n+2)v}{2(n+2)^2}\,, \qquad n \geq 3\,. \end{aligned}$$

Using the fact that $\sin x < x$ for $x > 0$ we see immediately that $F_n(v) > 0$ for $v > 0$ and $n \geq 2$. On the other hand, $F_1(v)$ has in the interval $(0, \frac{\pi}{2})$ a single positive maximum, at some $v > \frac{\pi}{4}$, from which it decreases to the value $F_1(\frac{\pi}{2}) = \frac{1}{18}(10 - 3\pi) > 0$. Summing up this argument, we have found that the quantity (5.5) is positive unless all the coefficients a_n, b_n are zero. ∎

Remark 5.3 One may wonder what happened with the closedness requirement (5.2). As the argument shows we were able to demonstrate the claim using only the weaker property that $\beta(0) = \beta(L)$. This is possible, of course, for small deformations only! As an illustration, consider Γ in the form of an "overgrown paperclip" which satisfies the condition $\beta(0) = \beta(L)$ but not (5.2), i.e. a line segment with two U-turns at the ends. Making the latter short one can get $c_\Gamma^2(\frac{1}{2}L)$ arbitrarily close to $\frac{1}{3}L^3$ which is larger than L^3/π^2.

6 Extensions and conclusions

To support our expectations that the result given in Theorem 2.1 holds globally and under weaker regularity assumptions, consider a simple example.

Example 6.1 Let Γ be a curve consisting of two circular segments of radius $R > \frac{L}{4\pi}$, i.e. it is given by the equations

$$\left(x \pm R\cos\frac{L}{2R}\right)^2 + y^2 = R^2 \qquad \text{for} \quad \pm x \geq 0\,. \tag{6.1}$$

For $R > \frac{L}{2\pi}$ it is "lens-shaped", for $\frac{L}{4\pi} < R < \frac{L}{2\pi}$ "apple-shaped"; it is not smooth except in the trivial case of a circle, $R = \frac{L}{2\pi}$. The curvature of this Γ equals

$$\gamma(s) = \frac{1}{R} + \left(\pi - \frac{L}{2R}\right)\left(\delta(s) + \delta(s - L/2)\right),$$

hence

$$c_\Gamma^2(u) = 2\int_0^u \mathrm{d}x\,(u-x)\left[(L-2x)\cos\frac{x}{R} - 2x\cos\frac{L-2x}{2R}\right]\mathrm{d}x\,,$$

and evaluating the integral, we arrive at

$$c_\Gamma^2(u) = 8R^3\left\{\frac{L}{2R}\sin^2\frac{u}{2R} + 4\left(\frac{u}{2R}\cos\frac{u}{2R} - \sin\frac{u}{2R}\right)\cos\frac{L}{4R}\,\cos\frac{L-2u}{4R}\right\}.$$

This function has for each $u \in (0, \frac{1}{2}L]$ a maximum at $R = \frac{L}{2\pi}$ and one can check directly that its value is smaller for any other R. In particular, in the limit $R \to \infty$ we have $c_\Gamma^2(u) \to Lu^2 - \frac{4}{3}u^3$ as one can find also directly with the "lens" degenerate into a double line segment; this value is less than $\frac{L^3}{\pi^2}\sin^2\frac{\pi u}{L}$ because $\sin^2 x > x^2 - \frac{4}{3\pi}x^3$ holds in $(0, \frac{\pi}{2})$.

To summarize our discussion, to prove Conjecture 2.2 it is sufficient to verify the inequality $C_L^p(u)$ for some $p \geq 1$ under appropriate regularity hypothesis. Naturally, one can ask also about ground-state maximizer in smaller families of curves Γ which do not contain the circle; examples could be polygonal loops with a fixed or limited number of vertices, or various prescribed compositions of arcs belonging to specific classes, circular, elliptic, parabolic, etc. Obviously a reasonable strategy is to look first for curves as close to the circle as possible within the given class. Sometimes one expects that the answer will be the curve with maximum symmetry as in the polygon case, in other situations it may not be true.

Another, and maybe more important extension of the present problem concerns a maximizer for the generalized Schrödinger operator in $\mathbb{R}^3$ with an attractive δ interaction supported by a closed surface of a fixed area A, and its generalization to closed hypersurfaces of codimension one in $\mathbb{R}^d$, $d > 3$. In the case of $d = 3$ we have a heuristic argument relying on [6, 8] similar to that used in the introduction which suggests that the problem is solved by the sphere provided the discrete spectrum is not empty, of course, which is a nontrivial assumption in this case – for properties of the corresponding operators see [2]. The Birman-Schwinger reduction of the problem similar to that of Sec. 3 can be performed again and the task is thus reduced to verification of a geometric inequality analogous to (4.1) which we can label as $C_A^{d,p}(u)$. We will discuss this problem in a following paper.

Acknowledgments

The research has been partially supported by the ASCR Grant Agency within the project A100480501.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, 2nd printing with an appendix by P. Exner, AMS Chelsea Publ., Providence, R.I., 2005.

[2] J.-P. Antoine, F. Gesztesy, J. Shabani: Exactly solvable models of spere interaction in quantum mechanics, *J. Phys. A: Math. Gen.* **20**, 3627–3712 (1987).

[3] M.S. Ashbaugh, R.D. Benguria: A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, *Ann. Math.* **135**, 601–628 (1992).

[4] J.F. Brasche, P. Exner, Yu.A. Kuperin, P. Šeba: Schrödinger operators with singular interactions, *J. Math. Anal. Appl.* **184**, 112-139 (1994).

[5] J.F. Brasche, A. Teta: Spectral analysis and scattering theory for Schrödinger operators with an interaction supported by a regular curve, in *Ideas and Methods in Quantum and Statistical Physics*, Cambridge Univ. Press 1992; pp. 197-211.

[6] P. Exner: Spectral properties of Schrödinger operators with a strongly attractive δ interaction supported by a surface, *Proceedings of the NSF Summer Research Conference (Mt. Holyoke 2002)*; AMS "Contemporary Mathematics" Series, vol. 339, Providence, R.I., 2003; pp. 25–36.

[7] P. Exner: An isoperimetric problem for point interactions, *J. Phys. A: Math. Gen.* **38**, to appear (2005); `math-ph/0406017`

[8] P. Exner, E.M. Harrell, M. Loss: Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature, in *Operator Theory : Advances and Applications*, vol. 108; Birkhäuser Verlag, Basel 1998; pp. 47–53.

[9] P. Exner, T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys. A: Math. Gen.* **34**, 1439–1450 (2001).

[10] P. Exner, K. Němcová: Leaky quantum graphs: approximations by point interaction Hamiltonians, *J. Phys. A: Math. Gen.* **36**, 10173–10193 (2003).

[11] P. Exner, K. Yoshitomi: Asymptotics of eigenvalues of the Schrödinger operator with a strong delta-interaction on a loop, *J. Geom. Phys.* **41**, 344–358 (2002).

[12] G. Faber: Beweiss das unter allen homogenen Membranen von Gleicher Fläche und gleicher Spannung die kreisförmige den Tiefsten Grundton gibt, *Sitzungber. der math.-phys. Klasse der Bayerische Akad. der Wiss. zu München*, 169–172 (1923).

[13] E.M. Harrell, P. Kröger, K. Kurata: On the placement of an obstacle or well so as to optimize the fundamental eigenvalue, *SIAM J. Math. Anal.* **33**, 240–259 (2001).

[14] E. Krahn: Über eine von Rayleigh formulierte minimal Eigenschaft des Kreises, *Ann. Math.* **94**, 97–100 (1925).

[15] A. Posilicano: Boundary triples and Weyl functions for singular perturbations of self-adjoint operator, *Meth. Funct. Anal. Topol.* **10**, 57–63 (2004).

[16] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York 1978.