THE VON NEUMANN WAY TO TREAT SYSTEMS OF MIXED DIMENSIONALITY*

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We show how the von Neumann theory of self-adjoint extensions can be used to investigate quantum systems the configuration space of which can be decomposed into parts of different dimensionalities. The method can be applied in many situations; we illustrate it on examples including point contact spectroscopy, nanotube systems, microwave resonators, or spin conductance oscillations.

Keywords: singular coupling, self-adjoint extensions, spectral and scattering properties.

1. Introduction

Looking back it is usually not difficult to distinguish great ideas, in particular, because they have a long and twisted life, and usually make impact in areas of which their authors knew nothing. The work of John von Neumann, the centenary of whom we are commemorating, offers various examples. The most striking one is probably his ground breaking contribution to the theory and practice of digital computers; it is hardly necessary to explain numerous ways in which this concept changed the lifestyle of the whole society.

In this paper we want to discuss a "later life" of another von Neumann's great idea coming from his early work on foundations of quantum mechanics. To give a proper meaning to the heuristic concept of hermiticity he distinguished the class of self-adjoint operators as those which can play role of quantum mechanical observables. On the way he created—somehow *en passant*—the theory of self-adjoint extensions of symmetric operators, without paying much attention to its possible physical applications.

For three decades the theory remained mostly a nice piece of mathematics. Its first important physical application is due to Berezin and Faddeev [4] who used it to explain the concept of point interaction which can be traced back to

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Fermi [16]. The extension approach proved to be very useful and many works followed; a comprehensive review of the point-interaction theory can be found in the monograph [1]. In the second half of the eighties similar techniques based on self-adjoint extensions were applied to more complicated quantum systems in which different "parts" interacted through a singular coupling. It is more than a coincidence that at the same time experimental physicists learned how to produce various structures for which such models offered a reasonable if idealized theoretical description. In a decade and a half which passed since then a substantial progress was achieved on both the experimental and theoretical side. Our aim in this review is to describe some of these new developments.

2. Systems with a decomposable state space

In both classical and quantum mechanics there are systems with constraints which make the configuration space a nontrivial subset of $\mathbb{R}^n$. Sometimes it happens that one can idealize it as a *union of components coupled through sets of a lower dimension* as in the examples of Fig. 1. In classical mechanics it is not a big problem. It is not that one could not find examples of such a behaviour, rather the "local" character of classical motion allows always to "magnify" the junction region and to study trajectories there in more detail.

In contrast, quantum mechanics offers various examples of physical importance. To name just a few, recall for instance:

- *quantum graphs*, i.e. tiny graph-like structures built from various materials: metallic, semiconductor, carbon, etc.,
- the *point-contact spectroscopy*, in which material properties are deduced from current through a junction between a surface and a needle, or alternatively between two thin films separated by an insulating layer in which a small crack is made,
- *STEM-type microscopes*, which are again based on measuring the current between a tip and a surface,
- compositions of *nanotubes* with *fullerene* molecules, etc.

Moreover, some classical *electromagnetic systems*, such as flat microwave resonators with attached antenna, can be described by the same equations and thus discussed simultaneously—see Section 7 below.

How such a singular coupling can be constructed? Consider a general case in which quantum dynamics on $M_1 \cup M_2$ should interact through a point contact

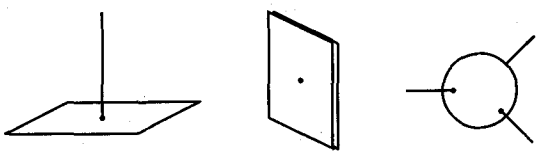

Fig. 1. Examples of systems with singularly coupled "parts".

$x_0 \in M_1 \cap M_2$. Take first Hamiltonians H_j on the *isolated* manifolds M_j and restrict them to functions vanishing in the vicinity of the point x_0. The operator $H_0 := H_{1,0} \oplus H_{2,0}$ obtained in this way is symmetric, but in general not self-adjoint. The idea is to seek Hamiltonian of the coupled system among *self-adjoint extensions* of H_0.

Naturally, it is not a priori ensured that such a construction will give a nontrivial result. In the nonrelativistic quantum mechanics which we consider here, where H_j is a *second-order differential operator*, the method works for $\dim M_j \leq 3$, or more generally, the *codimension* of the contact should not exceed three, since otherwise the restriction H_0 is *e.s.a.*, and as such it has just the trivial extension given by the closure. For Dirac operators the dimensional restriction is even more strict: the codimension have to be at most *one*. On the other hand, apart of the trivial case, the construction does not give a unique result because in general there are many self-adjoint extensions. A junction where n configuration-space components meet contributes typically by n to the deficiency indices of the restricted operator H_0, and thus—by von Neumann theory—it adds n^2 parameters to the resulting Hamiltonian class. The choice of a "true" extension is a nontrivial physical problem, to be discussed in each particular model separately—see Section 6 below.

The most common example of such system is represented by *quantum graphs* mentioned above where the components M_j can be identified with curve segments, finite or infinite. There are many works devoted to this subject and we restrict ourselves to quoting some recent work like [19] or [20] and other papers in the same issue for a bibliography. Our aim here is to discuss some less known systems including cases when the components M_j can be of different dimensions.

3. Coupling dimensions one and two

For simplicity we use "rational" units, in particular, the Hamiltonian acts at each configuration component as $-\Delta$, or more generally as Laplace–Beltrami operator if M_j has a nontrivial metric. An archetypal example of a system in which dimensions one and two are coupled corresponds to the first picture in Fig. 1. In other words, the state Hilbert space is $\mathcal{H} = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}^2)$, so the wave functions are pairs $\phi := \binom{\phi_1}{\phi_2}$ of square integrable functions on the halfline and the plane, respectively.

Let us apply the described construction. Restricting $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)_{\mathrm{D}} \oplus (-\Delta)$ to functions vanishing in the vicinity of the junction we get a symmetric operator with deficiency indices (2, 2). The corresponding four-parameter family of extensions can be obtained using von Neumann's general prescription [14]. It is practical to characterize it by means of *boundary conditions* expressed in terms of *generalized boundary values*[*]

$$L_0(\Phi) := \lim_{r\to 0} \frac{\Phi(\vec{x})}{\ln r}, \qquad L_1(\Phi) := \lim_{r\to 0} \left[\Phi(\vec{x}) - L_0(\Phi) \ln r\right].$$

[*]In a similar way one can couple dimensions one and three, then L_0 would be the coefficient at the pole singularity and the definition of L_1 would be appropriately modified.

Typical boundary conditions determining a self-adjoint extension are*

$$\phi_1'(0-) = A\phi_1(0-) + BL_0(\Phi_2),$$
$$L_1(\Phi_2) = C\phi_1(0-) + DL_0(\Phi_2),$$

where $A, D \in \mathbb{R}$ and $B = 2\pi\bar{C}$. The allowed values of the coefficients can be checked easily by computing the boundary form of H_0^*. Recall that the last named operator is given by the same differential expression as H_0 and that *only the s-wave part* of Φ in the plane, $\Phi_2(r,\varphi) = (2\pi)^{-1/2}\phi_2(r)$ can be coupled nontrivially to the halfline, because in other partial waves the restriction is e.s.a. Integration by parts gives

$$(\phi, H_0^*\psi) - (H_0^*\phi, \psi) = \bar{\phi}_1'(0)\psi_1(0) - \bar{\phi}_1(0)\psi_1'(0) + \lim_{\varepsilon\to 0+} \varepsilon\left(\bar{\phi}_2(\varepsilon)\psi_1'(\varepsilon) - \bar{\phi}_2'(\varepsilon)\psi_2(\varepsilon)\right),$$

and using the asymptotic behaviour

$$\phi_2(\varepsilon) = \sqrt{2\pi}\,\left[L_0(\Phi_2)\ln\varepsilon + L_1(\Phi_2) + \mathcal{O}(\varepsilon)\right],$$

we can express the above limit term as

$$2\pi\left[L_1(\Phi_2)L_0(\Psi_2) - L_0(\Phi_2)L_1(\Psi_2)\right],$$

so the form vanishes if the coefficients are as indicated above.

Consider now a *transport through the point contact*. Using the boundary conditions we can match the plane wave solution $e^{ikx} + r(k)e^{-ikx}$ on the halfline with $t(k)(\pi kr/2)^{1/2}H_0^{(1)}(kr)$ in the plane obtaining

$$r(k) = -\frac{\mathcal{D}_-}{\mathcal{D}_+}, \qquad t(k) = \frac{2iCk}{\mathcal{D}_+}$$

with

$$\mathcal{D}_\pm := (A \pm ik)\left[1 + \frac{2i}{\pi}\left(\gamma_E - D + \ln\frac{k}{2}\right)\right] + \frac{2i}{\pi}BC,$$

where $\gamma_E \approx 0.5772$ is Euler's number. The scattering is *nontrivial* if $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is not diagonal. With any choice of the self-adjoint extension, however, the on-shell S-matrix is *unitary*, in particular, we have $|r(k)|^2 + |t(k)|^2 = 1$. The most important feature of this system is that *reflection dominates at high energies*: notice that $|t(k)|^2 = \mathcal{O}((\ln k)^{-2})$ holds as $k \to \infty$.

4. Single-mode geometric scatterers

Consider a sphere with a pair of leads attached (Fig. 2). For the sake of simplicity we suppose that the coupling at both vertices is given by the same coefficient

*The full description of the coupling can be given in the form $\mathcal{A}\binom{\phi_1}{L_0} + \mathcal{B}\binom{\phi_1'}{L_1} = 0$, see [14] for details, in the generical case described here we may put $\mathcal{B} = -I$.

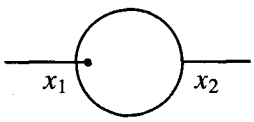

Fig. 2. A single-mode geometric scatterer.

matrix $\mathcal{A}$; three different one-parameter families of such $\mathcal{A}$ were investigated in [7, 13, 18].

It appears that scattering properties *en gross* are not very sensitive to the coupling: the system has numerous resonances and in the background reflection dominates as $k \to \infty$.

Let us describe the argument in detail: construction of generalized eigenfunctions means to couple plane-wave solution at the leads with

$$u(x) = a_1 G(x, x_1; k) + a_2 G(x, x_2; k),$$

where $G(\cdot, \cdot; k)$ is Green's function of Δ_{LB} on the sphere. The latter has a logarithmic singularity so $L_j(u)$ express in terms of $g := G(x_1, x_2; k)$ and

$$\xi_j \equiv \xi(x_j; k) := \lim_{x \to x_j} \left[G(x, x_j; k) + \frac{\ln|x - x_j|}{2\pi} \right].$$

Introduce $Z_j := \frac{D_j}{2\pi} + \xi_j$ and $\Delta := g^2 - Z_1 Z_2$, and consider, for instance, the coupling

$$\mathcal{A}_j = \begin{pmatrix} (2a)^{-1} & (2\pi/a)^{1/2} \\ (2\pi a)^{-1/2} & -\ln a \end{pmatrix}$$

with $a > 0$; we will explain below that it is in a sense a natural choice. Then the solution of the matching conditions yields the reflection and transmission amplitudes for this system,

$$r(k) = -\frac{\pi\Delta + Z_1 + Z_2 - \pi^{-1} + 2ika(Z_2 - Z_1) + 4\pi k^2 a^2 \Delta}{\pi\Delta + Z_1 + Z_2 - \pi^{-1} + 2ika(Z_1 + Z_2 + 2\pi\Delta) - 4\pi k^2 a^2 \Delta},$$

$$t(k) = -\frac{4ikag}{\pi\Delta + Z_1 + Z_2 - \pi^{-1} + 2ika(Z_1 + Z_2 + 2\pi\Delta) - 4\pi k^2 a^2 \Delta}.$$

So far the result is valid for any compact manifold G; to make use of it we need to know g, Z_1, Z_2, Δ. The spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of Δ_{LB} on G is purely discrete with eigenfunctions $\{\phi(x)_n\}_{n=1}^{\infty}$; then we find easily

$$g(k) = \sum_{n=1}^{\infty} \frac{\phi_n(x_1)\overline{\phi_n(x_2)}}{\lambda_n - k^2}$$

and

$$\xi(x_j, k) = \sum_{n=1}^{\infty} \left(\frac{|\phi_n(x_j)|^2}{\lambda_n - k^2} - \frac{1}{4\pi n} \right) + c(G),$$

where $c(G)$ depends on the manifold only*. Using these formulae for a sphere with the leads attached at poles, one can prove the following result.

THEOREM 4.1 ([18, 13]). *For any l large enough the interval $(l(l-1), l(l+1))$ contains a point μ_l such that $\Delta(\sqrt{\mu_l}) = 0$. Let $\varepsilon(\cdot)$ be a positive, strictly increasing function which tends to ∞ and obeys the inequality $|\varepsilon(x)| \leq x \ln x$ for $x > 1$. Furthermore, denote*

$$K_\varepsilon := \mathbb{R} \setminus \bigcup_{l=2}^{\infty} \left(\mu_l - \varepsilon(l)(\ln l)^{-2},\ \mu_l + \varepsilon(l)(\ln l)^{-2}\right).$$

Then there is a positive c such that the transmission probability satisfies

$$|t(k)|^2 \leq c\varepsilon(l)^{-2}$$

in the background, *i.e. for $k^2 \in K_\varepsilon \cap (l(l-1), l(l+1))$ and any l large enough. On the other hand, there are* resonance peaks *localized at K_ε with the property*

$$|t(\sqrt{\mu_l})|^2 = 1 + \mathcal{O}\left((\ln l)^{-1}\right) \quad \text{as} \quad l \to \infty.$$

This can be illustrated by calculating the transmission probability numerically. Fig. 3 shows its shape for a sphere of unit radius and $a = 10^{-2}$. Notice that the high-energy behaviour shares features with strongly singular interaction such as δ', for which $|t(k)|^2 = \mathcal{O}(k^{-2})$. *We conjecture* that *coarse-grained* transmission through our "bubble" has the same decay as $k \to \infty$; the comparison given in the lower graph supports this guess.

Notice also that while the general features mentioned above are expected to be the same if the angular distance of junctions is less than π, the detailed shape of the transmission plot changes—see [7] for examples. Let us also mention that in a similar way one can construct a *general scattering theory* on such "hedgehog" manifolds composed of compact scatterers, connecting edges and external leads [6].

5. Arrays of geometric scatterers

Consider next infinite periodic systems of such geometric scatterers which can be naturally treated by the usual Floquet–Bloch decomposition–see Fig. 4. Examples of the band spectrum worked out numerically for different parameter values can be found in [13].

One can naturally ask whether the scattering properties of such junctions are reflected in *gap behaviour* of periodic families of geometric scatterers *at high energies*? To explain, why such a question is of a non-negligible interest, recall the properties of *singular Wannier–Stark* systems with the Hamiltonian given formally by

$$-\frac{d^2}{dx^2} + \beta \sum_{n \in \mathbb{Z}} \delta'(x - na) - Fx,$$

*Notice that changing $c(G)$ is equivalent to a coupling constant renormalization.

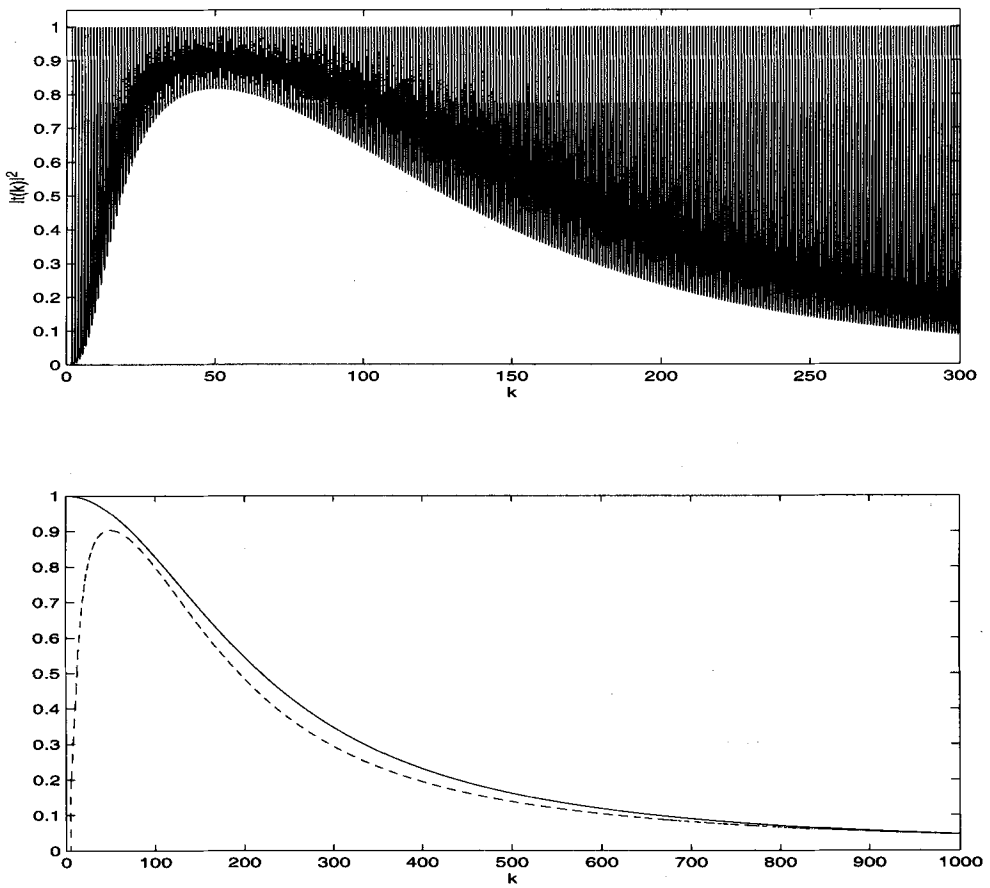

Fig. 3. The transition probability through a sphere scatterer and its value averaged over groups of ten resonance peaks, compared to δ' transition probability for an appropriate value of the parameter.

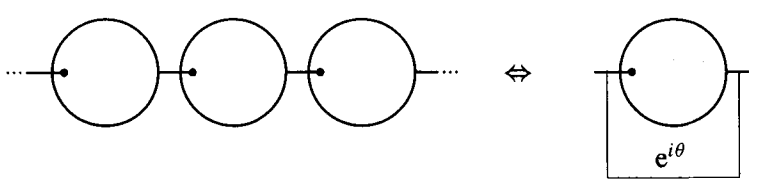

Fig. 4. Floquet treatment of geometric scatterer arrays.

where the δ' interaction is defined by means of appropriate boundary conditions [1]. They have counterintuitive properties: their absolute continuous spectrum is void [3, 9], and even more surprisingly, for "most" parameter values* the spectrum is pure point [2]. The reason behind are *large gaps* of δ' Kronig–Penney systems; this explains the above question.

We will thus consider *periodic combinations of spheres and segments* and adopt the following assumptions:

- periodicity in one or two directions (it is illustrative to speak about "loose bead arrays" and "loose bead carpets", respectively),
- angular distance between the contacts equals π or $\pi/2$,
- the simplest sphere-segment coupling,

$$\mathcal{A} = \begin{pmatrix} 0 & 2\pi\alpha^{-1} \\ \bar{\alpha}^{-1} & 0 \end{pmatrix}.$$

Moreover, we allow also a *tight coupling* when the spheres touch each other. The corresponding boundary conditions are then generically of the form

$$L_1(\Phi_1) = AL_0(\Phi_1) + CL_0(\Phi_2),$$
$$L_1(\Phi_2) = \bar{C}L_0(\Phi_1) + DL_0(\Phi_2)$$

with $A, D \in \mathbb{R}$, $C \in \mathbb{C}$, which ensures the self-adjointness; for the sake of simplicity we put $A = D = 0$. Denote by B_n, G_n the widths of the nth band and gap, respectively; then we have the following result.

THEOREM 5.1 ([5]). *There is a positive constant c such that*

$$\frac{B_n}{G_n} \leq c\, n^{-\epsilon}$$

holds as $n \to \infty$ for loosely connected *systems, where $\epsilon = \frac{1}{2}$ for arrays and $\epsilon = \frac{1}{4}$ for carpets. For* tightly coupled *systems to any $\epsilon \in (0, 1)$ there is a $\tilde{c} > 0$ such that the inequality $B_n/G_n \leq \tilde{c}\,(\ln n)^{-\epsilon}$ holds as $n \to \infty$.*

It is *conjectured* that similar results hold for other couplings and angular distances of the junctions. The problem is just technical; the dispersion curves determining the band spectra are less regular in general which makes the analysis rather complicated.

6. How to choose the self-adjoint extension

At a glance this question is simple: one should start from a more realistic model, for instance, one in which the manifolds involved have a finite thickness and the Hamiltonian is defined without any ambiguity, and to look what happens when the width tends to zero. Unfortunately this problem is more difficult than it seems.

*It is conjectured that this is true for any nonzero F and β.

It has been studied on graphs, approximated by a family of "fat graphs" which support Laplacian with *Neumann* boundary conditions [21–23], and also more general "sleeve-shaped" manifolds have been considered [12], however, the physically most important situation with Dirichlet boundary conditions remains an open problem.

For the graph-like manifolds of the first two classes mentioned above which are *compact* the shrinking limit can be worked out: one finds that eigenvalues at the bottom of the spectrum tend to those of the graph Laplacian with *Kirchhoff boundary conditions*, which mean continuity of the wave function at each vertex v_k of the graph together with

$$\sum_{\text{edges meeting at } v_k} \psi_j'(v_k) = 0;$$

in a sense these conditions describe a "free motion" on the graph. On can try to get other coupling by using a more general squeezing limit, for instance by supposing that the edge and vertex parts of the approximating manifolds shrink at different rates, say, ε versus ε^μ with $\mu < 1$, however, this again does not lead to a nontrivial vertex coupling [12]. It seems that a solution to the problem would require more than the geometry, for instance, to introduce additionally families of potentials, in general with a nonlinear scaling as recent approximation results on graphs suggest [8, 10].

Since for our problem with different dimension even such partial results are absent, we will try something else and describe a *heuristic way* to choose coupling in the plane-and-halfline system. To this aim we compare *low-energy scattering* in the problem of Section 3 to the situation when the halfline is replaced by a *tube of radius a* as sketched in Fig. 5; we disregard at that the effect of the sharp edge at interface of the two parts.

Rotational symmetry allows us again to treat each partial wave separately. Given orbital quantum number ℓ one has to match smoothly the corresponding radial parts of the solutions

$$\psi(x) := \begin{cases} e^{ikx} + r_a^{(\ell)}(t)e^{-ikx} & \text{for } x \le 0, \\ \sqrt{\dfrac{\pi kr}{2}}\, t_a^{(\ell)}(k) H_\ell^{(1)}(kr) & \text{for } r \ge a. \end{cases}$$

This yields

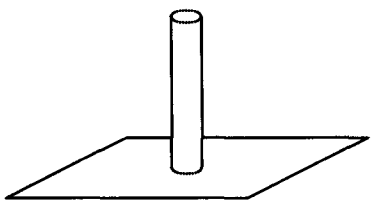

Fig. 5. A tube and plane system.

$$r_a^{(\ell)}(k) = -\frac{\mathcal{D}_-^a}{\mathcal{D}_+^a}, \qquad t_a^{(\ell)}(k) = 4i\sqrt{\frac{2ka}{\pi}}\,(\mathcal{D}_+^a)^{-1}$$

with

$$\mathcal{D}_\pm^a := (1 \pm 2ika)H_\ell^{(1)}(ka) + 2ka\left(H_\ell^{(1)}\right)'(ka).$$

The Wronskian relation $W(J_\nu(z), Y_\nu(z)) = 2/\pi z$ implies unitarity of the scattering, in particular, it shows that

$$|r_a^{(\ell)}(k)|^2 + |t_a^{(\ell)}(k)|^2 = 1.$$

Using asymptotic properties of Bessel functions for small values of the argument, $ka \ll 1$, we get

$$|t_a^{(\ell)}(k)|^2 \approx \frac{4\pi}{((\ell-1)!)^2}\left(\frac{ka}{2}\right)^{2\ell-1}$$

for $\ell \neq 0$, so the transmission probability vanishes fast as $k \to 0$ in the higher partial waves. The situation is different for $\ell = 0$ where

$$H_0^{(1)}(z) = 1 + \frac{2i}{\pi}\left(\gamma + \ln\frac{ka}{2}\right) + \mathcal{O}(z^2 \ln z).$$

A comparison shows that $t_a^{(0)}(k)$ coincides, in the leading order as $k \to 0$, with the corresponding expression for the plane-and-halfline system provided

$$A := \frac{1}{2a}, \qquad D := -\ln a, \qquad B = 2\pi C = \sqrt{\frac{2\pi}{a}};$$

this justifies a posteriori the coupling choice we made in Section 4. The parameter of this extension family has a natural meaning, namely the radius of the junction—similarly as the "coupling constant" of the two-dimensional δ interaction is intimately related to its scattering length [1].

7. Illustration on microwave experiments

As we have mentioned the models discussed here do not apply to quantum mechanical systems only. Consider an *electromagnetic resonator*. If it is *very flat*, the Maxwell equations simplify: TE modes effectively decouple from TM ones and we can describe them by Helmholz equation. Let the resonator be equipped with an *antenna* which serves as a source. Such a system has many resonances; it is natural to ask about distribution of their spacings.

The reflection amplitude for a compact manifold with one lead attached at x_0 is found as above: we have

$$r(k) = -\frac{\pi Z(k)(1-2ika) - 1}{\pi Z(k)(1+2ika) - 1},$$

where $Z(k) := \xi(\vec{x}_0; k) - \frac{1}{2\pi}\ln a$. Suppose that the resonator is *rectangular*. To evaluate regularized Green's function as in Section 4 we use eigenvalues and

eigenfunctions of Dirichlet Laplacian in $G = [0, c_1] \times [0, c_2]$, namely

$$\phi_{nm}(x, y) = \frac{2}{\sqrt{c_1 c_2}} \sin\left(n\frac{\pi}{c_1}x\right) \sin\left(m\frac{\pi}{c_2}y\right),$$

$$\lambda_{nm} = \frac{n^2\pi^2}{c_1^2} + \frac{m^2\pi^2}{c_2^2}.$$

Resonances are given by complex zeros of the denominator of $r(k)$, i.e. by solutions of the algebraic equation

$$\xi(\vec{x}_0, k) = \frac{\ln a}{2\pi} + \frac{1}{\pi(1 + ika)}.$$

Their distribution computed from the model can be compared with *experimental results* obtained at University of Marburg in a setting with $a = 1\,\text{mm}$ and different resonator sizes and antenna positions; we average over x_0 and $c_1, c_2 = 20 \sim 50\,\text{cm}$. The result adopted from [15] is shown in Fig. 6.

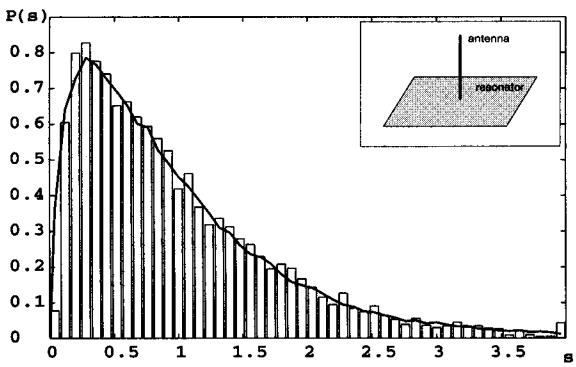

Fig. 6. The bins show experimental results, the dashed line is the theoretical result; both are averaged over different experiment geometries.

A comment is needed, however. The excellent agreement shown in the figure is achieved with the *lower third* of the actually measured frequencies; using all experimental data we get a mismatch. This is not a flaw; it *confirms the validity* of the above described approximation, since the shorter wavelengths in this experiment are already comparable with the antenna radius a and approximation condition $ka \ll 1$ is no longer valid.

8. Spin conductance oscillations

In our final example we want to illustrate that the manifolds we consider need not be separate spatial entities, but rather copies of the configuration space labelled by an internal quantum number. To this aim we employ a spin conductance problem. Recently the authors of [17] measured conductance of polarized electrons through an InAs sample; they came to a surprising conclusion that the results *depended*

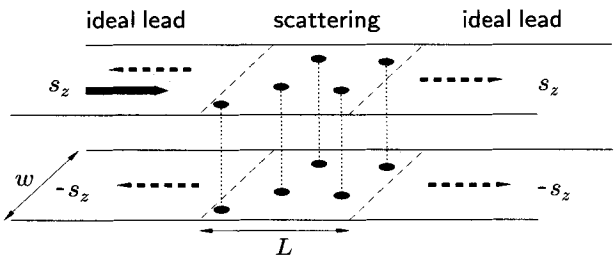

Fig. 7. A model of spin flip on impurity atoms.

on length L of the semiconductor "bar", an moreover, that the *spin-flip processes dominated* for some values of L.

The physical mechanism of the spin flip is clearly based on the *spin-orbit interaction with impurity atoms.* However, the corresponding equations are complicated and no realistic transport theory of that type is likely to be constructed soon. This inspired us to construct a *model* in which spin-flip interaction has a *point character.* Semiconductor bar is described as *two strips coupled at. the impurity sites* as sketched in Fig. 7; a point coupling between the strips (understood as two-dimensional manifolds with a boundary) is described by the boundary condition we have used in Section 5. Such a system can be treated by means of Krein's formula, in analogy with description of two-dimensional point interaction in a single strip—cf. [11].

In fact we can use results of the last mentioned paper directly if we adopt an additional assumption, namely that the impurities are randomly distributed, however, they all have the same coupling, $A = D$ and $C \in \mathbb{R}$. Then we can instead study a

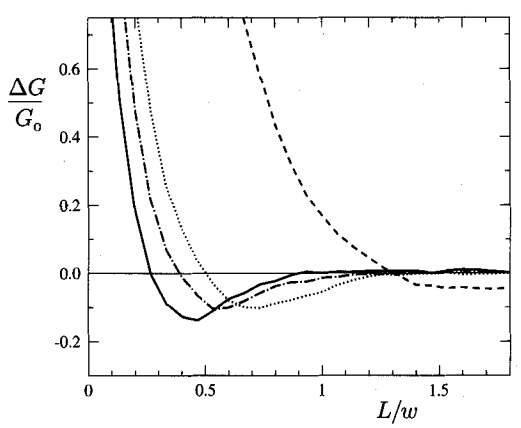

Fig. 8. Spin conductance vs. the sample length.

pair of decoupled strips described by the symmetric and antisymmetric combinations of the wave functions. The coupling becomes

$$L_1(\Phi_1 \pm \Phi_2) = (A \pm C)L_0(\Phi_1 \pm \Phi_2),$$

i.e. in each strip we have a set of point interactions, which have different effective coupling and naturally also different localizations lengths.

Returning to the original functions Φ_j, we expect that superposition of the above solutions may produce *spin conductance oscillations*. This is indeed the case; if we choose realistic values of the parameters and take also into account the coupling between the mesoscopic sample and the macroscopic leads we arrive at the result [24] shown in Fig. 8. In the vertical axis there is the difference of non-spin-flip and spin-flip conductances, normalized to the total conductance. The curves refer to several values of the coupling constants; they all show spin-flip dominance in a certain range of lengths.

9. Some open questions

The list of applications of the singular coupling technique we have discussed in this paper is by no means exhaustive. Moreover, the experimental solid state physics is developing rapidly and many new systems of complicated geometry are reported every year; this allows us to expect that there are many more applications of this technique to come.

We prefer to finish this survey with a short list of open questions. We believe that this is very much in the spirit of John von Neumann whom the centennial conference and this proceedings volume are commemorating.

- *General geometric scatterer systems:* we are interested in the asymptotic behaviour at high energies, in particular, localization of the resonances and background dominance. While these properties are expected to hold universally, the detailed resonance properties may depend crucially on the scatterer geometry.
- *Reduced Green's function on a compact manifold:* the expression for the function $\xi(x, k)$ is Section 4 contains the constant $c(G)$. In Section 7 we found a match with the experiment assuming that for a rectangle we have $c(G) = 0$. Is it true generally, and if not, how does the constant depend on the manifold G?
- *Wannier–Stark systems:* a comparison with δ' systems and Theorem 5.1 suggest the question how does the spectrum of sphere arrays look like when a *linear potential* is added? One can conjecture that it is *pure point*, but the problem is obviously difficult.
- *General periodic systems:* is the gap behaviour as $k \to \infty$ we have found in Section 5 a universal property for periodic systems of manifolds connected with elements of a lower dimension?
- And finally, the *coupling parameter choice:* it is natural to ask whether one can formulate the heuristic argument of Section 6 rigorously. More generally, it would be useful to know whether other junction couplings can be also obtained by approximations with additional potentials and a different contact geometry.

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