Lieb-Thirring inequalities for geometrically induced bound states

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Abstract

We prove new inequalities of the Lieb-Thirring type on the eigenvalues of Schrödinger operators in wave guides with local perturbations. The estimates are optimal in the weak-coupling case. To illustrate their applications, we consider, in particular, a straight strip and a straight circular tube with either mixed boundary conditions or boundary deformations.

1 Introduction

Recent progress in experimental physics provides various examples of guided particles: electrons in semiconductor quantum wires or carbon nanotubes, atoms in hollow fibers, etc. Moreover, there is a close analogy between two-dimensional systems of this type and flat microwave resonators – see [4, 20, 24] for more details and bibliography. The most simple model of such quantum wave guides is a one-particle Schrödinger operator in a domain of a strip or tube form subject to various boundary conditions. If no external field is present, the stationary part of the problem, in particular the search for bound states, is then reduced to spectral analysis of the Laplace operator in such domains.

Consider the Dirichlet Laplacian on a straight tube $\mathbb{R} \times \omega_0$ with a rather general cross-section $\omega_0 \subset \mathbb{R}^{d-1}$. The spectrum of this operator is obviously purely absolutely continuous and it covers the interval $[\lambda_1(\omega_0), \infty)$, where $\lambda_1(\omega_0)$ is the lowest eigenvalue of the Dirichlet Laplacian on ω_0. If this ideal wave guide is perturbed, for example, by local deformations or by a local change of the boundary conditions, eigenvalues below the threshold λ_0 can appear. The corresponding bound states are sometimes called in the literature trapped modes; the corresponding electron wave functions are localized in the vicinity of the perturbation. This effect is well studied and, in particular, the asymptotic behavior of these eigenvalues for gentle deformations or small perturbations of the boundary conditions has been investigated in several papers, see e.g. [1, 2, 4, 5, 7, 9] and references therein.

On the other hand, only few quantitative results are known in the non-asymptotic regime. Here one looks for estimates on the discrete spectrum, such as the counting function [6, 9] of the trapped modes or their Riesz means [10]. In the last named paper it has been shown that due to the special geometry of mixed dimensionality of quantum wave guides, operator-valued Lieb-Thirring inequalities represent a suitable tool to tackle this problem. This was then applied to a straight wave guide with an attractive potential interaction. In the present work we are going to demonstrate how a similar approach can yield estimates for the case of locally deformed "quantum wires" or for bound states induced by a local modification of boundary conditions.

2 Preliminary about Lieb-Thirring inequalities

The aim of this section is to collect an auxiliary material on Lieb-Thirring estimates on $L^2(\mathbb{R}^d)$, which shall be of use in the following.

Let $\mathcal{G}$ be a separable Hilbert space and let W be a function on $\mathbb{R}^d$ which takes almost everywhere non-negative compact operators on $\mathcal{G}$ as its values. We consider eigenvalue moments of the Schrödinger type operator

$$H = 1_{\mathcal{G}} \otimes (-\Delta) - W(x) \quad \text{on} \quad \mathcal{G} \otimes L^2(\mathbb{R}^d)\,.$$

Suppose that $tr_{\mathcal{G}} W^{\sigma+\frac{d}{2}}(\cdot) \in L^{\sigma+\frac{d}{2}}(\mathbb{R}^d)$. Then for $\sigma \geq 1/2$ if $d = 1$, and for $\sigma > 0$ if $d \geq 2$, the following estimate holds true[1]:

$$tr_{\mathcal{G}\times L^2(\mathbb{R}^d)}\, H_-^\sigma \leq r(\sigma,d) L_{\sigma,d}^{cl} \int_{\mathbb{R}^d} tr_{\mathcal{G}}\, W^{\sigma+\frac{d}{2}}(x)\mathrm{d}x\,, \tag{1}$$

where

$$L_{\sigma,d}^{cl} := \frac{\Gamma(\sigma+1)}{2^d \pi^{d/2}\Gamma(\sigma+\frac{d}{2}+1)}\,.$$

[1] Throughout the paper, we use the notation $x_\pm := (|x| \pm x)/2$ for the positive and negative part of numbers, functions or self-adjoint operators, respectively.

Moreover, the constants $r(\sigma, d)$ in (1) satisfy the inequalities

$$r(\sigma, d) = 1 \quad \text{if} \quad \sigma \geq 3/2,\ d \in \mathbb{N}, \tag{2}$$
$$r(\sigma, d) \leq 2 \quad \text{if} \quad 1 \leq \sigma < 3/2,\ d \in \mathbb{N}, \tag{3}$$
$$r(\sigma, d) \leq 2 \quad \text{if} \quad 1/2 \leq \sigma < 1,\ d = 1, \tag{4}$$
$$r(\sigma, d) \leq 4 \quad \text{if} \quad 1/2 \leq \sigma < 1,\ d \in \mathbb{N},\ d \geq 2, \tag{5}$$

see [21, 17, 18, 16]. Usually these inequalities are stated for the scalar operator

$$H_\alpha = -\Delta - \alpha V \quad \text{on} \quad L^2(\mathbb{R}^d)\,,$$

i.e. for $\mathcal{G} = \mathbb{C}$, see [23, 3, 22, 27] and [29, 18]; then these bounds give estimates on spectral quantities in terms of the classical phase space volume.

The generalization (1) to operator-valued potentials has been the crucial step for the recent progress on the constants in Lieb-Thirring inequalities in higher dimensions. The idea of "lifting" in dimensions, given in [21], is also the base for the proof of the main result of this paper.

3 Statement of the result

Consider an open set $\Omega \subset \mathbb{R}^d$. Let $(x_1, \ldots, x_d)$ be the Cartesian coordinates in $\mathbb{R}^d$. For a vector $x \in \mathbb{R}^d$ we shall single out the first coordinate and write $x = (\xi, \eta)$ with $\eta = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$ and $\xi = x_1 \in \mathbb{R}$. For a given value of ξ let

$$\omega(\xi) = \{\eta \in \mathbb{R}^{d-1} |\, x = (\xi, \eta) \in \Omega\,\}$$

be the cross-section of Ω at the point ξ which is an open set in $\mathbb{R}^{d-1}$. We shall assume that the sets $\omega(\xi)$ are uniformly bounded and non-empty for any $\xi \in \mathbb{R}$, and that Ω is a straight tube with local perturbations, that is

$$\omega(\xi) = \omega_0 \quad \text{for all} \quad |\xi| > R\,.$$

for some open set ω_0 and a positive R. The local deformation of Ω is given by the shape of the cross-sections $\omega(\xi)$.

Consider further a set $\Gamma \subset \overline{\Omega}$, such that $\Omega \setminus \Gamma$ is open and that its projection onto the transverse plane,

$$P_\Gamma := \{\eta \in \mathbb{R}^{d-1} |\, \exists \xi \in \mathbb{R} \ \text{ such that } x = (\xi, \eta) \in \Gamma\}\,,$$

has zero Lebesgue measure in $\mathbb{R}^{d-1}$.

Let $-\Delta_\Gamma^\Omega$ be the self-adjoint realization of the Laplace operator on $L^2(\Omega \setminus \Gamma)$ with Dirichlet conditions on $\partial\Omega \setminus \Gamma$ and Neumann conditions on Γ. This means that the quadratic form

$$\int_{\Omega\setminus\Gamma} |\nabla u|^2 \mathrm{d}^d x$$

generating the operator $-\Delta_\Gamma^\Omega$ is defined on the closure (with respect to the $W^{1,2}$ Sobolev norm) of the set of all smooth functions in $\Omega \setminus \Gamma$, which vanish for large $|\xi|$ and in a vicinity of $\partial\Omega \setminus \Gamma$ and which are square integrable together with their first partial derivatives. For a fixed $\xi \in \mathbb{R}$ we define

$$\gamma(\xi) = \{\eta \in \mathbb{R}^{d-1} |\; x = (\xi, \eta) \in \Gamma\}\,.$$

As above let $-\Delta_\gamma^\omega$ be the self-adjoint realization of the Laplace operator on $\omega \setminus \gamma$ with Dirichlet conditions on $\partial\omega \setminus \gamma$ and Neumann conditions on γ, where $\omega = \omega(\xi)$ and $\gamma = \gamma(\xi)$. Under suitable conditions on γ the spectrum (or at least the lower portion of it) is discrete[2]. In this case the corresponding eigenvalues will be denoted by $\lambda_j(\omega, \gamma)\,,\ j = 1, 2, \ldots$; if $\gamma = \emptyset$ we shall simply write $\lambda_j(\omega)$ instead of $\lambda_j(\omega, \emptyset)$. Of particular importance is the "asymptotic" quantity $\lambda_1(\omega_0)$ with ω_0 from eq. (3). We will suppose that the functions $\xi \mapsto \lambda_j(\omega(\xi), \gamma(\xi))$ are measurable[3], or at least that this property is valid below $\lambda_1(\omega_0)$.

Assume now that the spectrum of $-\Delta_\Gamma^\Omega$ below $\lambda_1(\omega_0)$ is discrete. In general it may be empty, of course; we are interested in situations when it is not. Then the corresponding eigenvalues will be called $\Lambda_j(\Omega, \Gamma)\,,\ j = 1, 2, \ldots$, and in case of $\Gamma = \emptyset$ we write $\Lambda_j(\Omega)$ instead of $\Lambda_j(\Omega, \emptyset)$. In particular, if there is only one such eigenvalue we drop the index j. It is convenient to define the "shifted" operator

$$H := -\Delta_\Gamma^\Omega - \lambda_1(\omega_0)$$

on $L^2(\Omega \setminus \Gamma)$, the essential spectrum of which is by assumption and an elementary bracketing argument equal to

$$\sigma_{\mathrm{ess}}(H) = [0, \infty)\,,$$

while the perturbation can give rise to bound states of negative energy. The following estimate on the moments of these negative eigenvalues is the main result of this paper:

[2] In general, this is the case unless the set γ is too "wild" – see, e.g., [15, 28].

[3] This requirement imposes again a restriction on the geometry of Ω and Γ. For instance, in the pure Dirichlet case, $\Gamma = \emptyset$, this property is guaranteed provided that, apart of a discrete subset of $[-R, R]$, to each ξ and $\varepsilon > 0$ there is an open set $O \ni \xi$ such that for any $\xi' \in O$ the symmetric difference $\omega(\xi)\Delta\omega(\xi')$ is contained in the ε-neighborhood of the boundary $\partial\omega(\xi)$, because the eigenvalues are in this case piecewise continuous as functions of ξ – cf. [26].

Theorem 3.1 *Suppose that the spectrum of the operators* $-\Delta_\gamma^\omega$ *with* $\omega = \omega(\xi)$ *and* $\gamma = \gamma(\xi)$ *below* $\lambda_1(\omega_0)$ *is discrete and finite for almost all* $\xi \in \mathbb{R}$, *the eigenvalues are measurable w.r.t.* ξ, *and that*

$$I_{\Omega,\Gamma,\sigma} := \int_{\mathbb{R}} \operatorname{tr}\left(-\Delta_{\gamma(\xi)}^{\omega(\xi)} - \lambda_1(\omega_0)\right)_-^{\sigma+1/2} \mathrm{d}\xi = \int_{\mathbb{R}} \sum_j \left(\lambda_j(\omega(\xi),\gamma(\xi)) - \lambda_1(\omega_0)\right)_-^{\sigma+1/2} \mathrm{d}\xi$$

is finite for $\sigma \geq 1/2$. *Then the negative spectrum of* H *is discrete and the inequality*

$$\operatorname{tr} H_-^\sigma \leq r(\sigma,1)\, L_{\sigma,1}^{\mathrm{cl}} I_{\Omega,\Gamma,\sigma} \tag{6}$$

holds true.

We will prove Theorem 3.1 in Sec. 6. Before doing that we notice that it applies to a variety of particular cases, a selection of which is given in the following section.

4 Examples

4.1 Strip with a Neumann perturbation

Let $\Omega = \mathbb{R} \times (0,1)$ be a planar strip and $\Gamma = [0,\alpha] \times \{b\}$ with $\alpha > 0$ and $\frac{1}{2} < b \leq 1$ a line segment in the interior or on the boundary of Ω, away of the strip axis. Then the cross-section of the strip is $\omega(\xi) = \omega_0 = (0,1)$ while the cross-section of Γ is $\gamma(\xi) = \{b\}$ for $\xi \in [0,\alpha]$ and $\gamma(\xi) = \emptyset$ otherwise. The spectrum of the Laplacian $-\Delta_{\gamma(\xi)}^{\omega(\xi)}$ can be determined easily as its eigenfunctions are simple sine functions. The lowest eigenvalue of $-\Delta_\emptyset^{\omega_0}$ is $\lambda_1(\omega_0) = \pi^2$, which is therefore also the lower edge of the essential spectrum of $-\Delta_\Gamma^\Omega$. For $\xi \in [0,\alpha]$, the operator $\Delta_{\gamma(\xi)}^{\omega(\xi)}$ has a single eigenvalue $\frac{\pi^2}{4b^2}$ below π^2. Combining this information with (6) we obtain:

Corollary 4.1 *For* $H = -\Delta_\Gamma^\Omega - \pi^2$ *and* $\sigma \geq 1/2$ *the following inequality is valid,*

$$\operatorname{tr} H_-^\sigma \leq r(\sigma,1)\, L_{\sigma,1}^{\mathrm{cl}}\, \alpha \left(\pi^2 - \frac{\pi^2}{4b^2}\right)^{\sigma+1/2}. \tag{7}$$

The result remains valid, of course, for $b = \frac{1}{2}$ when it becomes trivial.

4.2 Strip with bulges

Suppose now that $\Gamma = \emptyset$ and $\Omega_f = \{(\xi,\eta) \in \mathbb{R}^2 |\ 0 < \eta < 1 + f(\xi)\}$ with a piecewise continuous and compactly supported function f such that $0 \leq f(\xi) < 1$. Then we get in a similar way as above the following bound:

Corollary 4.2 *For $H = -\Delta^{\Omega_f} - \pi^2$ and $\sigma \geq 1/2$ we have*

$$\operatorname{tr} H_-^\sigma \leq r(\sigma,1)\, L_{\sigma,1}^{\mathrm{cl}}\, \pi^{2\sigma+1} \int\limits_{-\infty}^{\infty} \left(1 - \frac{1}{(1+f(\xi))^2}\right)^{\sigma+1/2} \mathrm{d}\xi\,.$$

Note that the assumption $f(\xi) < 1$ is made here only for simplicity; it ensures that $-\Delta^{\omega(\xi)}$ has not more that one eigenvalue below π^2. It is straightforward to generalize the claim to a more general profile function replacing the integrand by $\sum_{j=1}^\infty \big(1 - j^2(1+f(\xi))^{-2}\big)_+^{\sigma+1/2}$, where the sum has, of course, only a finite number of nonzero terms for any fixed ξ.

4.3 Circular tube with bulges

As another particular case let us consider a tube in $\mathbb{R}^3$ with Dirichlet boundary which is circular outside a compact and has local bulges. The spectrum of the Laplace operator on a circular disk with unit radius is well known: it is purely discrete and expressed in terms of Bessel function zeros, in particular, the lowest eigenvalue is $j_{0,1}^2$, where $j_{0,1}$ is the first positive root of the function J_0. It is also known that among all domains of the same area, the first eigenvalue is minimized by the circular disk; this fact is expressed in the well-known Rayleigh-Faber-Krahn inequality [11, 19]

$$\lambda_1(\omega) \geq \frac{\pi j_{0,1}^2}{A(\omega)}\,, \tag{8}$$

where $A(\omega)$ is the area of the domain ω and $\lambda_1(\omega) = \inf \sigma\big(-\Delta^\omega\big)$.

We are again interested primarily in the situation when the bulge is not too big. Notice that the second eigenvalue $\lambda_2(\omega)$ can also be estimated with the help of (8): since $-\Delta^\omega$ commutes with the involution defined by complex conjugation, the eigenfunction Ψ_2^ω corresponding to $\lambda_2(\omega)$ can be chosen as real-valued; it vanishes on a smooth nodal line without endpoints in (the interior of) the cross section[4]. It follows that this curve divides ω into two parts, one of which must cover an area not exceeding $A(\omega)/2$; we call this part $\tilde\omega$. Then $\Psi_2(\omega)$ restricted to $\tilde\omega$ is also the ground-state eigenfunction of the Dirichlet Laplacian $-\Delta^{\tilde\omega}$, thus eq. (8) yields[5]

$$\lambda_2(\omega) \geq \frac{\pi j_{0,1}^2}{A(\tilde\omega)} \geq \frac{2\pi j_{0,1}^2}{A(\omega)}\,.$$

[4]The shape of this nodal line depends on the cross section geometry. If ω is simply connected the endpoints lie at the boundary, while for a non-simply connected ω it may be also a closed loop which does not touch the boundary [14, 12].

[5]The conclusion is not affected by the fact that $\omega \setminus \tilde\omega$ may have a larger area because the ground-state eigenvalue in the two parts must be the same, of course.

Consequently[6], if $A(\omega) \leq 2\pi$ then $\lambda_2(\omega) \geq j_{0,1}^2$ so that $\lambda_1(\omega)$ is the only eigenvalue which could be below $j_{0,1}^2$. It is indeed the case in the bulged part of the tube where $\omega \setminus \omega_0$ has a nonzero measure as it follows from the domain monotonicity of Dirichlet eigenvalues [13]. From the above remarks and Theorem 3.1 we make the following conclusion:

Corollary 4.3 *Define Ω as in Sec. 3 with ω_0 being a circular disk of unit radius and $\Gamma = \emptyset$ and $\omega(\xi) \supset \omega_0$ for all $\xi \in \mathbb{R}$. Moreover, suppose that the area $A(\omega(\xi))$ of $\omega(\xi)$ satisfies $A(\omega(\xi)) \leq 2\pi$. Then for $H = -\Delta^\Omega - j_{0,1}^2$ and $\sigma \geq 1/2$ the inequality*

$$\operatorname{tr} H_-^\sigma \leq r(\sigma, 1)\, L_{\sigma,1}^{cl}\, j_{0,1}^{2\sigma+1} \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \left(1 - \frac{\pi}{A(\omega(\xi))}\right)^{\sigma+1/2}$$

holds true.

5 Discussion of the results

Let us next compare the obtained results with those of earlier publications.

5.1 Strip with a small bulge

Consider the set $\Omega_{\alpha f}$ defined as in Corollary 4.2 with the function f replaced by αf to have a parameter which controls the deformation. For the sake of brevity we denote

$$F_n := \int_{-\infty}^{\infty} f(x)^n \,\mathrm{d}x \,.$$

It is known from [2] that for a sufficiently smooth f and small α the operator $-\Delta^{\Omega_{\alpha f}}$ has exactly one eigenvalue below π^2 and its asymptotic behavior is

$$\Lambda(\Omega_{\alpha f}) = \pi^2 - \pi^4 F_1^2 \alpha^2 + \mathcal{O}(\alpha^3)\,.$$

Expanding the estimate of Corollary 4.2 into powers of α, substituting $\pi^2 - \Lambda(\Omega_{\alpha f})$ for $\operatorname{tr} H_-$ and choosing $\sigma = \frac{1}{2}$, we obtain

$$\Lambda(\Omega_{\alpha f}) \geq \pi^2 - \pi^4 F_1^2\, \alpha^2 + 3\pi^4 F_1\, F_2\, \alpha^3 - \left(\frac{9}{4} F_2^2 + 4 F_1 F_3\right) \pi^4 \alpha^4 + \mathcal{O}(\alpha^5)\,,$$

which means that our Lieb-Thirring inequality reproduces the true weak-coupling asymptotics in this case.

[6]In particular cases one can do better. For instance, if the bulged tube is circular again, being described by a radius function r, then there is a single transverse eigenvalue below the threshold as long as $r(\xi) \leq j_{1,1}/j_{0,1} \approx 1.5933$ which means $A(\omega) \lesssim 2.5387\,\pi$.

5.2 Strip with Neumann perturbation on the boundary

The last claim need not be valid in general. Consider the set Ω of Corollary 4.1 with the perturbation at the boundary, i.e. take $\Gamma_\alpha = [0,\alpha]\times\{1\}$ with some $\alpha>0$. Then by [6] the operator $-\Delta^{\Omega}_{\Gamma_\alpha}$ has for small enough α exactly one eigenvalue below π^2. Choosing $\sigma=\frac{1}{2}$, Corollary 4.1 yields

$$\Lambda(\Omega,\Gamma_\alpha) \geq \pi^2 - \frac{9}{16}\pi^4\alpha^2.$$

On the other hand it is known from [7] that for small α there are positive c_1, c_2 such that[7]

$$\pi^2 - c_1\alpha^4 \leq \Lambda(\Omega,\Gamma_\alpha) \leq \pi^2 - c_2\alpha^4$$

holds, and consequently, our Lieb-Thirring inequality gives a too rough weak-coupling estimate in this case.

On the other hand, the estimate is of a correct order in α in the strong coupling case, i.e. for large α. To justify this claim, recall a simple bracketing bound used in [6]. The spectrum is estimated from above by means of adding extra Dirichlet conditions at $\xi = 0, a$ which yield the following orthogonal family of functions,

$$\Psi_n(\xi,\eta) := \begin{cases} \cos\frac{\pi}{2}\eta\,\sin\frac{n\pi}{\alpha}\xi & \text{for } \xi\in[0,\alpha]\,, \\ 0 & \text{for } \xi\notin[0,\alpha]\,. \end{cases}$$

This leads to a lower bound on $\operatorname{tr} H_-^\sigma$, namely

$$\begin{aligned} \operatorname{tr} \mathrm{H}_-^\sigma &\geq \sum_{n=1}^{\infty}\left(\frac{\pi^2}{4}+\frac{n^2\pi^2}{\alpha^2}-\pi^2\right)_-^\sigma \\ &= \pi^{2\sigma}\left(\frac{3}{4}\right)^{\sigma+1/2}\alpha\int_0^\infty (s^2-1)_-^\sigma\,\mathrm{d}s + o(\alpha) \\ &= L^{\mathrm{cl}}_{\sigma,1}\alpha\left(\frac{3\pi^2}{4}\right)^{\sigma+1/2} + o(\alpha)\,. \end{aligned}$$

In a similar way Neumann bracketing provides an upper bound on $\operatorname{tr} H_-^\sigma$ which differs from the lower one only by the summation range which now starts from $n=0$, and hence gives the same expression up to the error term. A comparison with eq. (7) for $b=1$ shows that our estimate exhibits the correct power of α, the only difference being the factor $r(\sigma,1)$ – cf. the relations (2)–(5).

[7] In fact, the eigenvalue has a Taylor expansion in α and the coefficient of the leading fourth-order term can be computed explicitly – see [25] and also [1].

5.3 General considerations

In the paper [10] a similar formula has been derived to estimate the moments of the binding energies in a straight wave guide with an attractive potential. The estimating expression differs from the r.h.s. of eq. (6): it consists of two terms reflecting the mixed dimensionality of the problem. One term describes the effect of a weak potential where the dominating behavior of the eigenfunctions is one-dimensional. The second one is important in the case of a strongly attractive potential where the influence of the boundary and the "leads" on the wave functions of the trapped particle in the lower part of the spectrum is negligible and the problem is essentially d-dimensional.

In the present work we have worked out estimates consisting of one term only, having on mind in the first place systems which have no more than one transverse eigenvalue below the threshold $\lambda_1(\omega_0)$. This can still yield a good estimate if the perturbation is rather "long" than "wide" as the previous example illustrates. Moreover, spectra of wave guides with large deformations can be well estimated by combination of bracketing and standard phase-space methods.

Our result exhibits the usual Lieb-Thirring features in the sense that it neglects repulsive components of the interaction, and the bound may become useless if the latter dominate. Consider, for instance, a deformed circular tube of Sec. 4.3 and suppose that the deformation is both squeezing and expanding the cross section. If the cross section in the deformed part deviates substantially from the circular shape, it may happen that the discrete spectrum is empty even if the deformation adds volume to Ω and the r.h.s. of the inequality in Corollary 4.3 is nonzero.

For sake of simplicity we have limited our considerations to wave guides which differ from a straight tube on a compact only. Some generalizations would not be difficult to derive. For example, the basic estimate (6) of Theorem 3.1 will also hold true for a wave guide the straight parts of which on both sides of the local perturbation are parallel but not in line with each other. In a similar way it is possible to generalize Theorem 3.1 to certain perturbations that are not compactly supported but still local in the sense that they fall off asymptotically fast enough. On the other hand, for instance, it is not possible to extend our results in a straightforward manner to the case of Neumann boundary conditions on a surface which is not parallel to the tube axis; the reason will become clear from the proof of Theorem 3.1 which we are now finally going to present.

6 Proof of Theorem 3.1

As usual the (shifted) Laplace operator on $L^2(\Omega)$ is associated with the closed quadratic form

$$h[\Psi,\Psi] = \int_\Omega \left(|\nabla\Psi|^2 - \lambda_1(\omega_0)|\Psi|^2\right)\, \mathrm{d}x\,, \tag{9}$$

where the boundary conditions are implemented by a proper choice of the domain $Q(h)$ of the form h. In our case, when we deal with $H = -\Delta_\Gamma^\Omega - \lambda_1(\omega_0)$, the form domain $Q(h)$ is given by the $|\cdot|_h$-closure[8] of the set $M(\Omega,\Gamma)$ of all functions $\Psi \in C^\infty(\Omega\setminus\Gamma)$, which vanish in the vicinity of $\partial\Omega\setminus\Gamma$ as well as for sufficiently large $|\xi|$, and for which the expression (9) is finite[9].

Now we define the smallest common envelope of the cross sections, which is bounded by assumption, and the corresponding cylindrical envelope of the tube by

$$\hat\omega := \bigcup_{\xi\in\mathbb{R}} \omega(\xi) \quad \text{and} \quad \hat\Omega := \mathbb{R}\times\hat\omega\,,$$

so we have $\Omega\subset\hat\Omega$. Consider the quadratic form on $L^2(\hat\Omega)$ given by

$$\hat h[\Psi,\Psi] := \int_\Omega \left(|\nabla\Psi|^2 - \lambda_1(\omega_0)|\Psi|^2\right)\, \mathrm{d}x + \int_{\hat\Omega\setminus\Omega} \left|\frac{\partial\Psi}{\partial\xi}\right|^2 \mathrm{d}x \tag{10}$$

with the form domain $Q(\hat h)$ equal to the $|\cdot|_{\hat h}$-closure of the set $\hat M(\Omega,\Gamma)$ of all functions $\Psi\in L^2(\hat\Omega)$ for which $\Psi|_\Omega \in M(\Omega,\Gamma)$ holds and the restriction $\Psi|_{\hat\Omega\setminus\Omega}$ is smooth and vanishes near $\partial\Omega$ and $\partial\hat\Omega$. Then $Q(\hat h) = Q(h)\oplus_{\hat h} Y$ where the set $Y\subset L^2(\hat\Omega\setminus\Omega)$ consists of all functions ϕ which are differentiable in the sense if distributions in the ξ-direction and satisfy $\frac{\partial\phi}{\partial\xi}\in L^2(\hat\Omega\setminus\Omega)$.

The closed quadratic form $\hat h$ is associated with the self-adjoint operator

$$\hat H = H \oplus \left(-\frac{\partial^2}{\partial\xi^2}\right) \quad \text{on} \quad L^2(\hat\Omega) = L^2(\Omega)\oplus L^2(\hat\Omega\setminus\Omega),$$

which is the direct sum of our original operator H on $L^2(\Omega)$ and the differential operator $-\frac{\partial^2}{\partial\xi^2}$ on $L^2(\hat\Omega\setminus\Omega)$ with Dirichlet condition on the part of $\partial\Omega$ which is not parallel to $\partial\hat\Omega$. The last named operator is positive by definition, and therefore $\hat H$ and H have the same negative spectrum.

[8] Here and in the following we use the symbol $|\cdot|_h$ for the slightly modified Sobolev norm defined by $|\cdot|_h^2 = h[\cdot,\cdot] + (\lambda_1(\omega_0)+1)||\cdot||^2$.

[9] In particular, such functions can have a "jump" on $\Gamma\cap\Omega$.

We can write the form $\hat{h}$ as a sum of parallel and transverse components,

$$\hat{h}[\Psi,\Psi] = \int_{\mathbb{R}} \mathrm{d}\xi \left(\int_{\hat{\omega}} \mathrm{d}\eta \left| \frac{\partial \Psi}{\partial \xi} \right|^2 + w(\xi)[\Psi(\xi,\cdot),\Psi(\xi,\cdot)] \right),$$

with the second term defined through the quadratic form

$$w(\xi)[\phi,\phi] := \int_{\omega(\xi)} \mathrm{d}\eta \left[|\nabla_\eta \phi(\eta)|^2 - \lambda_1(\omega_0)|\phi(\eta)|^2 \right] .$$

The domain of $w(\xi)$ can be chosen as

$$Q(w(\xi)) := \{\phi \in L^2(\hat{\omega}) : \phi|_{\omega(\xi)} \in Q(h(\xi))\},$$

where $Q(h(\xi))$ is the domain of the quadratic form $h(\xi)[\phi,\phi] = \int_{\omega(\xi)} |\nabla_\eta \phi|^2 \mathrm{d}\eta$ associated with $-\Delta_{\gamma(\xi)}^{\omega(\xi)}$ on $L^2(\omega(\xi))$. Indeed, with such a domain choice we have $\Psi(\xi,\cdot) \in Q(w(\xi))$ for any $\Psi \in Q(\hat{h})$ and almost every $\xi \in \mathbb{R}$.

It is straightforward to check that the form $w(\xi)$ is closed and associated with the operator

$$W(\xi) = \left[-\Delta_{\gamma(\xi)}^{\omega(\xi)} - \lambda_1(\omega_0) \right] \oplus 0_{\hat{\omega}\setminus\omega(\xi)} ,$$

where $0_{\hat{\omega}\setminus\omega(\xi)}$ is, of course, the zero operator on $L^2(\hat{\omega} \setminus \omega(\xi))$. It follows from the assumptions of Theorem 3.1 that the negative spectrum of $W(\xi)$ consists of at most finitely many negative eigenvalues. Let $W_-(\xi)$ be the negative part of the operator $W(\xi)$. Then $W_-(\xi)$ is an operator of finite rank on $L^2(\hat{\omega})$, and consequently, its quadratic form $w_-(\xi)$ is defined on $Q(w_-(\xi)) = L^2(\hat{\omega})$.

Next we introduce the quadratic form

$$\tilde{h}[\Psi,\Psi] := \int_{\mathbb{R}} \left(\int_{\hat{\omega}} \left| \frac{\partial \Psi}{\partial \xi} \right|^2 \mathrm{d}\eta - w_-(\xi)[\Psi(\xi,\cdot),\Psi(\xi,\cdot)] \right) d\xi ,$$

defined on the $|\cdot|_{\tilde{h}}$-closure of the set of all smooth functions in $L^2(\hat{\Omega})$. Making the closure explicit, we find that $Q(\tilde{h})$ consists of all functions $\Psi \in L^2(\hat{\Omega})$ for which the following conditions hold true:

(a) for a.e. $\eta \in \hat{\omega}$ the function $\Psi(\cdot,\eta)$ is differentiable in the sense of distributions in ξ-direction on $\mathbb{R}$ and $\frac{\partial \Psi}{\partial \xi} \in L^2(\hat{\Omega})$,

(b) for a.e. $\eta \in \hat{\omega}$ the function $\Psi(\cdot,\eta)$ satisfies the Dirichlet condition in the ξ-direction at points of $\partial\Omega \setminus \Gamma$.

Because the projection of the set Γ onto the η-coordinate plane has by assumption zero measure, the functions $\Psi(\cdot,\eta):\mathbb{R}\to\mathbb{C}$ given by some $\Psi\in\hat{M}(\Omega,\Gamma)$ are smooth in ξ-direction for a.e. $\eta\in\hat{\omega}$ and vanish at $\partial\Omega\setminus\Gamma$. Hence $Q(\tilde{h})$ contains the subset $\hat{M}(\Omega,\Gamma)$ which is a core[10] in the form domain of $\hat{h}$. Since $\hat{h}[\Psi,\Psi]\geq\tilde{h}[\Psi,\Psi]$ holds for all $\Psi\in\hat{M}(\Omega,\Gamma)$ and the norms $|\cdot|_{\hat{h}}$ and $|\cdot|_{\tilde{h}}$ are topologically compatible, it follows that $Q(\tilde{h})\supset Q(\hat{h})$. From this we infer that the inequality $\hat{h}\geq\tilde{h}$ is valid. This further means that the operator

$$\tilde{H}=-\Delta^{\mathbb{R}}\otimes 1_{L^2(\hat{\omega})}-W_-(\xi),\quad \xi\in\mathbb{R},$$

associated with $\tilde{h}$ is strictly bounded by that related to $\hat{h}$, i.e.

$$\tilde{H}<\hat{H}\,. \tag{11}$$

Now we are in position to apply the operator-valued Lieb-Thirring inequalities (1) for $d=1$ and $\sigma\geq 1/2$. In view of (11) and the observed fact that spectra of H and $\hat{H}$ coincide in the negative part we get

$$\begin{aligned}\mathrm{tr}\ H_-^\sigma &= \mathrm{tr}\ \hat{H}_-^\sigma\\ &\leq \mathrm{tr}\ \tilde{H}_-^\sigma\\ &\leq r(\sigma,1)L^{\mathrm{cl}}_{\sigma,1}\int_{\mathbb{R}}\mathrm{d}\xi\,\mathrm{tr}\ W_-^{\sigma+1/2}\\ &= r(\sigma,1)L^{\mathrm{cl}}_{\sigma,1}\int_{\mathbb{R}}\mathrm{d}\xi\,\mathrm{tr}\left(-\Delta_{\gamma(\xi)}^{\omega(\xi)}-\lambda_1(\omega_0)\right)_-^{\sigma+1/2}\ ;\end{aligned}$$

this completes the proof of Theorem 3.1.

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