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Bound States of Infinite Curved Polymer Chains

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Abstract. We investigate an infinite array of point interactions of the same strength in $\mathbb{R}^d$, $d = 2, 3$, situated at vertices of a polygonal curve with a fixed edge length. We demonstrate that if the curve is not a line, but it is asymptotically straight in a suitable sense, the corresponding Hamiltonian has bound states. An example is given in which the number of these bound states can exceed any positive integer.

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1. Introduction

Methods of guiding particles or light quanta along a prescribed path are of interest from both the theoretical and practical points of view. One aspect of this problem is the relationship between the geometry of the 'channel' and the spectral and scattering properties of the corresponding Hamiltonian. The last decade has brought various results in this field – see, e.g., the paper [DE] and the recent books [Hu, LCM] for an extensive bibliography.

A particular question to be addressed in this Letter concerns the existence of curvature-induced bound states in infinite channels which are, in some sense, asymptotically straight. This effect was first demonstrated in [EŠ] and subsequently studied by numerous authors. A common feature of these studies, however, is that they assume a strict localization in the sense that the configuration space is a neighborhood of a given curve. This is not fully realistic, because in actual 'quantum waveguides', the confinement comes from a potential well of a finite depth, and it is not a-priori clear whether the binding effect will persist in the presence of a tunneling between different parts of the channel.

The aim of this Letter is to give an affirmative answer to the above question in a well-known model of a 'polymer chain', i.e., an array of point interactions in $\mathbb{R}^d$, $d = 2, 3$, with fixed coupling parameter, and the distance between the neighbors, which is certainly a very weak way to keep the particle 'within' the channel. The straight-polymer spectrum is thoroughly analyzed in [AGHH]: it is purely absolutely continuous and below bounded with at most one gap. We will show that making the chain curved – locally in a suitable sense – will lead to the emergence of isolated eigenvalues below the continuum threshold, and that making the curvature 'large' enough we can produce many such bound states.

2. Formulation of the Problem

Let $Y = \{y_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}^d$, $d = 2, 3$, with the following property: there is an $\ell > 0$ such that

$$|y_j - y_{j'}| \leqslant \ell\, |j - j'| \tag{2.1}$$

holds for any integers j, j'. In particular, the distance between neighboring points satisfies

$$|y_j - y_{j+1}| = \ell \tag{2.2}$$

for each $j \in \mathbb{Z}$. For simplicity, we shall use the symbol Y both for a map $\mathbb{Z} \to \mathbb{R}^d$ and the subset of $\mathbb{R}^d$ which is the range of this map. Furthermore, we shall assume that

(a1) there is $c_1 \in (0, 1)$ such that $|y_j - y_{j'}| \geqslant c_1\ell\, |j - j'|$. In particular, Y as a map is injective, and if it has asymptotes for $j \to \pm\infty$, they are not parallel.

(a2) Y is asymptotically straight in the following sense: there are positive c_2, μ, and $\omega \in (0, 1)$ such that the inequality

$$1 - \frac{|y_j - y_{j'}|}{|j - j'|} \leqslant c_2\big[1 + |j + j'|^{2\mu}\big]^{-1/2} \tag{2.3}$$

holds in the sector $S_\omega := \big\{(j, j')\colon j,\ j' \neq 0,\ \omega < j/j' < \omega^{-1}\big\}$ of $\mathbb{Z}^2$.

The operators we shall investigate are the point-interaction Hamiltonians $H_{\alpha,Y} \equiv -\Delta_{\alpha,Y}$ with the same interaction 'strength' at each point, which in the notation of [AGHH] means that α is a constant sequence. They are defined by means of the boundary conditions

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad j \in \mathbb{Z}\,, \tag{2.4}$$

expressed in terms of the generalized boundary values

$$L_0(\psi, y) := \lim_{|x-y|\to 0} \frac{\psi(x)}{\phi_d(x-y)}\,,$$
$$L_1(\psi, y) := \lim_{|x-y|\to 0} \big[\psi(x) - L_0(\psi, y)\,\phi_d(x - y)\big]\,,$$

where ϕ_d are the appropriate fundamental solutions,

$$\phi_2(x) = -\frac{1}{2\pi}\ln|x|\,, \qquad \phi_3(x) = \frac{1}{4\pi|x|}\,,$$

related to the free Green's functions

$$G_k(x-x') = \begin{cases} \dfrac{i}{4} H_0^{(1)}(k|x-x'|) \ldots d=2, \\ \dfrac{e^{ik|x-x'|}}{4\pi|x-x'|} \ldots d=3. \end{cases} \tag{2.5}$$

More exactly, one defines in this way the point-interaction Hamiltonian $H_{\alpha,\tilde{Y}}$ for any finite subset $\tilde{Y} \subset Y$, and $H_{\alpha,Y}$ is obtained as the strong resolvent limit over the filter of finite subsets [AGHH, §§ III.1, III.4]. The resolvent of $H_{\alpha,Y}$ is given by Krein's formula,

$$\begin{aligned}(-H_{\alpha,Y} - k^2)^{-1} \\ = G_k + \sum_{j,j'=-\infty}^{\infty} [\Gamma_{\alpha,Y}(k)]_{jj'}^{-1} \big(\overline{G_k(\cdot - y_{j'})}, \cdot\big) G_k(\cdot - y_j)\end{aligned} \tag{2.6}$$

for $k^2 \in \rho(H_{\alpha,Y})$ with $\operatorname{Im} k > 0$, where $\Gamma_{\alpha,Y}(k)$ is a closed operator (which is bounded in our case) on $\ell^2(\mathbb{Z})$ the matrix representation of which is

$$\Gamma_{\alpha,Y}(k) := \left[(\alpha - \xi_d^k)\delta_{jj'} - \tilde{g}_{jj'}^{Y,k} \right]_{j,j' \in \mathbb{Z}}, \tag{2.7}$$

where ξ_d^k is the regularized Green's function

$$\xi_2^k = -\frac{1}{2\pi}\left(\ln\frac{k}{2i} + \gamma\right), \qquad \xi_3^k = \frac{ik}{4\pi},$$

with $\gamma = -\psi(1)$ the Euler number, and

$$\tilde{g}_{jj'}^{Y,k} = \begin{cases} G_k(y_j - y_{j'}) \ \ldots\ j \neq j', \\ 0 \ \ldots\ j = j'. \end{cases}$$

Since α is independent of j, the map $k \mapsto \Gamma_{\alpha,Y}(k)$ is analytic in the open upper halfplane. Moreover, $\Gamma_{\alpha,Y}(k)$ is boundedly invertible for $\operatorname{Im} k > 0$ large enough, while for $k \in \mathbb{C}_+$ not too far from the real axis it may have a nontrivial null-space. By (2.6), the latter determines the spectrum of the original operator $H_{\alpha,Y}$ on the negative halfline. In particular, one easily checks the following result.

LEMMA 2.1. (i) *A point $-\kappa^2 < 0$ belongs to $\rho(H_{\alpha,Y})$ iff* $\operatorname{Ker}\Gamma_{\alpha,Y} = \{0\}$.

(ii) *If the operator-valued function $\kappa \mapsto \Gamma_{\alpha,Y}(i\kappa)^{-1}$ has bounded values in an open interval $I \subset \mathbb{R}_+$ with the exception of a point $\kappa_0 \in I$, where* $\dim\operatorname{Ker}\Gamma_{\alpha,Y}(i\kappa) = n$*, then $-\kappa_0^2$ is an isolated eigenvalue of $H_{\alpha,Y}$ of multiplicity n.*

3. The Essential Spectrum

Let us now ask how the geometry of the array Y is reflected in the spectral properties of the operator $H_{\alpha,Y}$. As a departure point, we recall some facts about straight polymers. If we denote by Y_0 the linearly arranged sequence, $|y_j - y_{j'}| = \ell\,|j - j'|$ for all $j, j' \in \mathbb{Z}$, the corresponding spectrum is purely absolutely continuous and consists of two bands – see [AGHH, §§ III.1.5. III.4] – which may overlap if α is not large enough negative, in particular, for $d = 3$ and $\alpha\ell \geqslant -(1/2\pi)\ln 2$. Its threshold $E_d^{\alpha,\ell} \equiv E_d^{\alpha,\ell}(0)$ is always negative. In the three-dimensional case, it is known explicitly,

$$E_3^{\alpha,\ell} = \frac{1}{\ell^2}\left[\ln\left(1 + \frac{1}{2}\,\mathrm{e}^{-4\pi\alpha\ell} + \mathrm{e}^{-2\pi\alpha\ell}\sqrt{1 + \frac{1}{4}\,\mathrm{e}^{-4\pi\alpha\ell}}\right)\right]^2, \tag{3.1}$$

while for $d = 2$, we have $E_2^{\alpha,\ell} = -\kappa_{\alpha,\ell}^2$, where $\kappa_{\alpha,\ell}$ solves the equation

$$\alpha + \frac{1}{2\pi}(\gamma - \ln 2) = g_{i\kappa}(0), \tag{3.2}$$

where $g_{i\kappa}(0)$ is given by (3.3) below – see [AGHH, § III.4] with an obvious correction. In both cases $E_d^{\alpha,\ell}$ is strictly monotonous with respect to α. The upper edge $E_d^{\alpha,\ell}(\pi/\ell)$ of the first spectral band is given by analogous expressions with $g_k(0)$ replaced by $g_k(\pi/\ell)$.

Notice, in addition, that the spectrum of $\Gamma_{\alpha,Y}(k)$ for a fixed value $k \in \{\zeta\colon \operatorname{Im}\zeta > 0\} \cup \mathbb{R}_+$ is absolutely continuous, because it is unitarily equivalent to an operator of multiplication by $\alpha + (1/2\pi)(\gamma - \ln 2)\delta_{d,2} - g_k(\theta)$ on the appropriate Brillouin zone, i.e. on $L^2(\mathcal{B}_\ell)$ with $\mathcal{B}_\ell := (-(\pi/\ell), (\pi/\ell))$, where

$$g_k(\theta) := \frac{1}{2\pi}\lim_{N\to\infty}\left\{\sum_{n=-N}^{N}\frac{1}{2}\left[\left(n + \frac{\theta\ell}{2\pi}\right)^2 - \left(\frac{k\ell}{2\pi}\right)^2\right]^{-1/2} - \ln N\right\}. \tag{3.3}$$

for $d = 2$ and

$$g_k(\theta) := -\frac{1}{4\pi\ell}\ln[2(\cos k\ell - \cos\theta\ell)] \tag{3.4}$$

for $d = 3$. It is easy to see that the functions (3.3) and (3.4) are nonconstant, in particular, for $k = i\kappa$ with $\kappa > 0$ they are decreasing w.r.t. $|\theta|$.

We want first to show that a deformation of the straight polymer which satisfies the above requirement of asymptotic straightness leaves the essential spectrum invariant.

PROPOSITION 3.1. *Let Y satisfy assumptions* (2.1), (2.2), (a1), (a2); *then* $\sigma_{\mathrm{ess}}(H_{\alpha,Y}) = \left[E_d^{\alpha,\ell}, E_d^{\alpha,\ell}(\pi/\ell)\right] \cup [0,\infty)$*, with the two bands overlapping for α large enough.*

Proof. Consider first the negative part of the spectrum. As we have said, the spectrum of $\Gamma_{\alpha,Y}(i\kappa)$ with $\kappa > 0$ for a straight polymer equals

$$\sigma\big(\Gamma_{\alpha,Y_0}(i\kappa)\big) = \left[\alpha+\frac{1}{2\pi}(\gamma-\ln 2)\delta_{d,2}-g_{i\kappa}(0),\, \alpha+\frac{1}{2\pi}(\gamma-\ln 2)\delta_{d,2}-g_{i\kappa}\Big(\frac{\pi}{\ell}\Big)\right]. \tag{3.5}$$

In view of Lemma 4.3 below, the same interval is contained in the spectrum of $\Gamma_{\alpha,Y}(i\kappa)$ and, thus by Lemma 2.1, no point of the interval

$$I_1^{(-)} := \left[E_d^{\alpha,\ell},\, \min\Big\{0,\, E_d^{\alpha,\ell}\Big(\frac{\pi}{\ell}\Big)\Big\}\right]$$

belongs to the resolvent set of the operator $H_{\alpha,Y}$, hence $I_1^{(-)} \subset \sigma_{\rm ess}(H_{\alpha,Y})$. By the same compact-perturbation argument, we find that apart of a discrete set corresponding to eigenvalues of a finite multiplicity, the points $-\kappa^2$ outside $I_1^{(-)}$ belong to $\rho\big(H_{\alpha,Y}\big)$, so the set $(-\infty, 0) \setminus I_1^{(-)}$ is not contained in the essential spectrum.

Let us turn to the positive halfline. Given a function $\phi \in C_0^\infty([0,2])$ with $0 \leqslant \phi(r) \leqslant 1$ and $\phi(r) = 1$ for $r \in [0,1]$, we define

$$\psi_n(x; p, x_n) := \phi(n^{-1}|x-x_n|)\, e^{ip.x}$$

for any $n \in \mathbb{Z}_0$ and $p, x_n \in \mathbb{R}^d$. After normalization, the functions ψ_n with a sequence $\{x_n\} \subset \mathbb{R}^2$ such that $|x_n| \to \infty$ are easily seen to form a Weyl sequence of the free Hamiltonian H_0 corresponding to the point $|p|^2$ of its essential spectrum.

Notice now that for any $N \in \mathbb{Z}_+$ there is a ball $B_N \subset \mathbb{R}^d$ of radius N which does not intersect with Y, for otherwise we might take a family of such balls centered, say, at the points $(3n_1N, 0, 0)$ and $(0, 3n_2N, 0)$ with $n_1, n_2 \in \mathbb{Z}$, and any array Y intersecting with all of them will violate the assumption (a2). This means that we can choose $\{x_n\}$ in such a way that the balls $B_{2n}(x_n)$ are mutually different and do not intersect with Y, in which case $H_{\alpha,Y}\psi_n(\cdot; p, x_n) = H_0\psi_n(\cdot; p, x_n)$. In this way, we have constructed a Weyl sequence to $H_{\alpha,Y}$ for any point of $[0,\infty)$ which completes the proof. □

4. Curvature-Induced Discrete Spectrum

Now we are ready to prove the main result of this paper showing that a nonstraight array Y of the class specified in Section 2 generates a nonempty discrete spectrum.

THEOREM 4.1. *In addition to the above assumptions, suppose that the inequality* (2.1) *is sharp for some $j, j' \in \mathbb{Z}$, then $H_{\alpha,Y}$ has at least one isolated eigenvalue below $E_d^{\alpha,\ell}$ for any $\alpha \in \mathbb{R}$.*

Proof. In view of Lemma 2.1, we have to look for solutions of the equation $\Gamma_{\alpha,Y}(i\kappa)\psi = \psi$, where the operator is given by (2.7). We will write it as a sum of $\Gamma_{\alpha,Y_0}(i\kappa)$ and the perturbation $\mathcal{D}_\kappa := \Gamma_{\alpha,Y}(i\kappa) - \Gamma_{\alpha,Y_0}(i\kappa)$ and investigate how the latter affects spectral properties of a straight polymer. Since $G_{i\kappa}(\cdot)$ is by (2.5) strictly decreasing in $\mathbb{R}_+$, the inequality (2.1) implies

$$[\mathcal{D}_\kappa]_{jj'} = \tilde{g}^{Y_0,i\kappa}_{jj'} - \tilde{g}^{Y,i\kappa}_{jj'} = G_{i\kappa}(\ell(j-j')) - G_{i\kappa}(y_j - y_{j'}) \leqslant 0 \tag{4.1}$$

for any pair of nonidentical $j, j' \in \mathbb{Z}$, while $[\mathcal{D}_\kappa]_{jj} = 0$. We use this negativity property to show that a sharp inequality in (2.1) moves the threshold of the entire spectrum.

LEMMA 4.2. $\inf \sigma\big(\Gamma_{\alpha,Y}(i\kappa)\big) < \alpha + (1/2\pi)(\gamma - \ln 2)\delta_{d,2} - g_{i\kappa}(0)$ *holds for any* $\kappa > 0$ *if the array* Y *is not straight.*

Proof. The claim will be justified if we find $\psi \in \ell^2(\mathbb{Z})$ such that

$$\big(\psi, \Gamma_{\alpha,Y}(i\kappa)\psi\big) < \left(\alpha + \frac{1}{2\pi}(\gamma - \ln 2)\delta_{d,2} - g_{i\kappa}(0)\right)\|\psi\|^2 ,$$

which, in view of (2.7), can be rewritten as

$$\begin{aligned} &\sum_{j\neq j'} [\mathcal{D}_\kappa]_{jj'} \bar{\psi}_j \psi_{j'} - \sum_{j\neq j'} G_{i\kappa}(\ell(j-j'))\, \bar{\psi}_j \psi_{j'} + \\ &\qquad + \left(\frac{\delta_{d,2}}{2\pi} \ln\kappa + \frac{\kappa}{4\pi}\delta_{d,3} + g_{i\kappa}(0)\right) \sum_j |\psi_j|^2 < 0\,. \end{aligned} \tag{4.2}$$

Consider first the last two terms which we shall treat by means of the Fourier representation, $\hat{\psi}(\theta) = \sum_{j\in\mathbb{Z}} \psi_j e_j(\theta)$, with respect to the standard trigonometric basis $\{e_j\}$ in $L^2(\mathcal{B}_\ell)$. By the Parseval relation, the last norm equals $\int_{\mathcal{B}_\ell} |\hat{\psi}(\theta)|^2 \mathrm{d}\theta$. Furthermore, for the second term we use the Poisson summation formula [AGHH] which can be written as

$$\sum_{j\neq 0} G_{i\kappa}(\ell j)\, \mathrm{e}^{-ij\ell\theta} = g_{i\kappa}(\theta) + \frac{\delta_{d,2}}{2\pi}\ln\kappa + \frac{\kappa}{4\pi}\delta_{d,3},$$

thus giving

$$\sum_{j\neq j'} G_{i\kappa}(\ell(j-j'))\, \bar{\psi}_j \psi_{j'} = \int_{\mathcal{B}_\ell} \left(g_{i\kappa}(\theta) + \frac{\delta_{d,2}}{2\pi}\ln\kappa + \frac{\kappa}{4\pi}\delta_{d,3}\right) |\hat{\psi}(\theta)|^2 \mathrm{d}\theta\,.$$

Consequently, the sum of the last two terms in (4.2) is given by the expression

$$\int_{\mathcal{B}_\ell} (g_{i\kappa}(0) - g_{i\kappa}(\theta)) |\hat{\psi}(\theta)|^2 \mathrm{d}\theta \tag{4.3}$$

which is obviously nonnegative. By (3.3) and (3.4), the difference contained in the integral is a finite, smooth, and even function on $\mathcal{B}_\ell$, so there is a $c_\kappa > 0$ such that

$$0 \leqslant g_{i\kappa}(0) - g_{i\kappa}(\theta) \leqslant c_\kappa \theta^2. \tag{4.4}$$

For instance, the inequality is valid with

$$c_\kappa = \frac{\ell}{16\pi}\left(\sinh\frac{\kappa}{2}\right)^{-2} \quad \text{if } d = 3$$

Let us now choose for ψ the unit vector given by

$$\psi_j = (\tanh\lambda)^{1/2} e^{-\lambda|j|}$$

with $\lambda > 0$, for which

$$\hat{\psi}(\theta) = \sqrt{\frac{\ell}{2\pi}}(\tanh\lambda)^{1/2}\frac{1-e^{-2\lambda}}{1+e^{-2\lambda}-e^{-\lambda}\cos\theta\ell}\,.$$

For small enough λ, we have $\tanh\lambda \leqslant 2\lambda$ and $\frac{1}{2} < e^{-\lambda} < 1-2^{-1/2}\lambda$; then we can estimate

$$\hat{\psi}(\theta)^2 < \frac{\ell}{2\pi}(2\lambda)^3\left(1+e^{-2\lambda}-e^{-\lambda}\cos\theta\ell\right)^{-2} < \frac{\ell}{2\pi}(2\lambda)^3\left[\frac{\lambda^2}{2}+\frac{2}{\pi^2}(\theta\ell)^2\right]^{-2},$$

so (4.3) has the following upper bound:

$$\frac{c_\kappa}{2\pi\ell^2}(2\lambda)^3\int_{\mathbb{R}} u^2\left[\frac{\lambda^2}{2}+\frac{2}{\pi^2}u^2\right]^{-2} du = c_\kappa\pi^3\ell^{-2}\lambda^2\,.$$

On the other hand, the inequality in (4.1) is, by assumption, sharp on a non-empty subset of $\mathbb{Z}\times\mathbb{Z}$, which means that the first term in (4.2) is

$$\sum_{j\neq j'}[\mathcal{D}_\kappa]_{jj'}\tanh\lambda\, e^{-\lambda(|j|+|j'|)} \leqslant -c\lambda$$

for some positive c and all λ small enough. Hence, this term dominates the left-hand side of (4.2) as $\lambda\to 0+$; this result yields the sought trial vector. □

Our next goal is to show that the perturbation (4.1) is a compact operator provided the array Y is sufficiently straight at large distances.

LEMMA 4.3. *Under the assumption* (a2), $\mathcal{D}_\kappa$ *is Hilbert–Schmidt if* $\mu > \frac{1}{2}$.

Proof. We have to estimate the right-hand side of (4.1). For brevity, we introduce the following notation for arguments appearing at this expression:

$$\varrho \equiv \varrho_{jj'} := |y_j - y_{j'}|\,, \quad \sigma \equiv \sigma_{jj'} := \ell|j-j'|\,.$$

We employ the fact that the free Green's functions (2.5) are convex for $k = i\kappa$. This yields

$$0 \leqslant G_{i\kappa}(\varrho) - G_{i\kappa}(\sigma) \leqslant -\varrho\, G'_{i\kappa}(\varrho)\frac{\sigma-\varrho}{\varrho}\,, \tag{4.5}$$

where the derivative at the right-hand side is equal to $-\kappa K_1(\kappa|\cdot|)$ for $d=2$, and $-e^{-\kappa|\cdot|}/4\pi|\cdot|^2(1+\kappa|\cdot|)$ for $d=3$. Notice that the function $\varrho\mapsto\varrho\, G'_{i\kappa}(\varrho)$ is bounded in $(0,\infty)$ for $d=2$; for $d=3$ it has a singularity at the origin but it is bounded in $[c_1\ell,\infty)$, i.e., for all values of ϱ making a nonvanishing contribution to $\mathcal{D}_\kappa$.

At the same time, we have

$$0 \leqslant \frac{\sigma-\varrho}{\varrho} \leqslant \frac{1-c_1}{c_1} \tag{4.6}$$

in view of $c_1\sigma \leqslant \varrho \leqslant \sigma$, hence the matrix elements of $\mathcal{D}_\kappa$ are bounded.

This is not enough, of course, we need to know their decay properties. Away of the sector S_ω, we employ the fact that there is a $c>0$ such that

$$-\varrho\, G'_{i\kappa}(\varrho) \leqslant c\,\mathrm{e}^{-\kappa\varrho/2} \leqslant c\,\mathrm{e}^{-c_1\kappa\sigma/2} \tag{4.7}$$

holds for all nonzero ϱ, σ, while in the said sector we have by (a2) the estimate

$$\frac{\sigma-\varrho}{\varrho} \leqslant \frac{\sigma-\varrho}{c_1\sigma} \leqslant \frac{c_2}{c_1}\left[1+|j+j'|^{2\mu}\right]^{-1/2}. \tag{4.8}$$

Combining the estimates (4.5)–(4.8) we get a bound to the Hilbert–Schmidt norm in question:

$$\sum_{j\neq j'}[\mathcal{D}_\kappa]^2_{jj'} \leqslant \left(\frac{1-c_1}{c_1}\right)^2 c^2 \sum_{\mathbb{Z}^2\setminus S_\omega} \mathrm{e}^{-c_1\kappa|j-j'|} + \left(\frac{cc_2}{c_1}\right)^2 \sum_{S_\omega} \frac{{}^{-c_1\kappa|j-j'|}}{1+|j+j'|^{2\mu}}\,. \tag{4.9}$$

Denoting $s=j'-j$, we can rewrite the first sum at the right-hand side as

$$2\sum_{s=1}^{\infty}\sum_{j=[-\frac{s}{1-\omega}]+1}^{[\frac{s\omega}{1-\omega}]} \mathrm{e}^{-c_1\kappa s} \leqslant 2\sum_{s=1}^{\infty}\left(1+s\,\frac{1+\omega}{1-\omega}\right)\mathrm{e}^{-c_1\kappa s}$$

with the last series obviously convergent. The second series at the right-hand side of (4.9) is not diminished if we sum over all the $\mathbb{Z}^2$. Passing then to $j\pm j'$ as the new summation variables, we see that it is finite for $\mu>\frac{1}{2}$. □

As the last ingredient, we need the following continuity result:

LEMMA 4.4. *Under the same assumptions as above, the map* $\kappa \mapsto \Gamma_{\alpha,Y}(i\kappa)$ *is operator-norm continuous and* $\inf\sigma\big(\Gamma_{\alpha,Y}(i\kappa)\big) \to \pm\infty$ *as* $\kappa\to\infty$ *and* $\kappa\to 0+$, *respectively.*

Proof. The claim holds for the 'free' operator $\Gamma_{\alpha,Y_0}(i\kappa)$, because the functions $\kappa\mapsto g_{i\kappa}(\theta)$ are continuous for any $\theta\in\mathcal{B}_\ell$, and $g_{i\kappa}(0)$ which determines the spectrum bottom has the needed limits. More specifically, $g_{i\kappa}(0)\to-\infty$ as $\kappa\to\infty$, its asymptotics being logarithmic for $d=2$ and linear for $d=3$, while $g_{i\kappa}(0)\to+\infty$ as $\kappa\to 0+$. It is thus sufficient to check that the map $\kappa\mapsto\mathcal{D}_\kappa$ is continuous and remains bounded as $\kappa\to\infty$. We employ the inequality

$$[\mathcal{D}_\kappa-\mathcal{D}_{\kappa'}]^2_{jj'} \leqslant 2\Big([\mathcal{D}_\kappa]^2_{jj'}+[\mathcal{D}_{\kappa'}]^2_{jj'}\Big) \leqslant 4[\mathcal{D}_{\kappa_0}]^2_{jj'},$$

which holds true for any $\kappa_0\leqslant\min(\kappa,\kappa')$. Hence, the series expressing the HS-norm of $\mathcal{D}_\kappa-\mathcal{D}_{\kappa'}$ can be uniformly majorized and the limit may be interchanged with the sum giving

$$\|\mathcal{D}_\kappa - \mathcal{D}_{\kappa'}\|_{\rm HS} \to 0 \quad \text{as } \kappa' \to \kappa\,.$$

At the same time, the estimate (4.10) shows that not only the norm remains bounded, but even $\|\mathcal{D}_\kappa\|_{\rm HS} \to 0$ as $\kappa \to \infty$ which concludes the proof. □

Proof of Theorem 4.1 (continued). We have $\inf \sigma(\Gamma_{\alpha,Y}(i\kappa)) < \inf \sigma(\Gamma_{\alpha,Y_0}(i\kappa))$ by Lemma 4.2 whenever the array Y is not straight. On the other hand, a deformation Y of Y_0 which is asymptotically straight in the sense of assumption (a2) leaves, through Lemma 4.3, the essential spectrum invariant, so the part of $\sigma(\Gamma_{\alpha,Y}(i\kappa))$ outside the interval 3.5 consists of isolated eigenvalues of a finite multiplicity at most. Putting the two results together, we find that for any $\kappa > 0$, there is at least one such eigenvalue $\lambda_{\alpha,Y}(\kappa)$ below $\inf \sigma(\Gamma_{\alpha,Y_0}(i\kappa))$. Finally, by Lemma 4.4, the function $\lambda_{\alpha,Y}(\cdot)$ is continuous and $\lambda_{\alpha,Y}(\kappa) \to \pm\infty$ as $\kappa \to \infty$ and $\kappa \to 0+$, respectively. This means that for any $\alpha \in \mathbb{R}$ there is a point κ_0 such that $\lambda_{\alpha,Y}(\kappa_0) = 0$, and the eigenvalue $-\kappa_0^2$ which corresponds to it, through Lemma 2.1 satisfies $-\kappa_0^2 < E_d^{\alpha,\ell}$. □

5. Concluding Remarks

Having established the existence of 'trapped modes' in locally curved polymers, we are naturally interested on the cardinality of $\sigma_{\rm disc}(H_{\alpha,Y})$ or, in other words, how rich the discrete part of the spectrum could be. We want to illustrate that the number of bound states can be any positive integer provided Y is curved 'enough'.

EXAMPLE 5.1. Consider the set $Y_{M,N}$ obtained by an appropriate ordering of

$$\{(j,j'): j \neq j', \max(j,j') \leqslant M\} \cup \{(j,j): |j| \leqslant N\},$$

to which we ascribe the operator $H_{M,N} \equiv H_{\alpha,Y_{M,N}}$, again with the same $\alpha \in \mathbb{R}$ at all points. Depending on α, it has at most $2M(2M+1) + 2N + 1$ eigenvalues naturally ordered as $\mu_{M,N}^{(1)} \leqslant \mu_{M,N}^{(2)} \leqslant \cdots < 0$.

Using the monotonicity of point-interaction Hamiltonians w.r.t. the coupling constants, as in [AGHH, § II.1.1], one checks that the above operator family is nonincreasing w.r.t. both M and N which means, in particular,

$$\mu_{M,N}^{(j)} \geqslant \max\left(\mu_{M+1,N}^{(j)}, \mu_{M,N+1}^{(j)}\right)$$

for each j, M, N. The sequence $\{H_{M,M}\}_{M=1}^\infty$ converges in the strong resolvent sense, respectively, to the Hamiltonian of the square crystal if $d = 2$, and of a monomolecular layer if $d = 3$ [AGHH, §§ III.1.6, III.4]. Both have absolutely continuous spectra with the threshold strictly below the number $E_d^{\alpha,\ell}$ described in Section 3. Hence, for any positive integer j_0, one can find $M \equiv M(j_0)$ such that $H_{M,M}$ has j_0 eigenvalues of less than $E_d^{\alpha,\ell}$.

On the other hand, the sequence $\{H_{M,N}\}_{N=M}^{\infty}$ converges in the strong resolvent sense to the operator $H_{\alpha,Y}$ corresponding to the array $Y \equiv Y_{M,\infty}$ which is straight except for the central part where it is 'tightly packed'. Using the monotonicity again, we find that $H_{\alpha,Y}$ has at least j_0 eigenvalues below the essential-spectrum threshold $E_d^{\alpha,\ell}$.

Let us conclude the Letter with several remarks. One may ask about the meaning of the asymptotic straightness requirement (a2). In a companion paper [EI] which treats an analogous problem for measure perturbations of two-dimensional Schrödinger operators supported by curves, we show that a sufficient condition for a planar curve to satisfy such a condition is that its signed curvature decay with respect to the arc length s is $\mathrm{o}\big(|s|^{-5/4-\varepsilon}\big)$.

Another natural question is what a curvature will do to the spectrum of a two-dimensional array under the influence of a constant magnetic field, where, in the straight case, we have absolutely continuous bands sandwiched between the Landau levels [EJK]. The argument in this Letter does not extend to this situation, because its crucial ingredient is the real-valuedness and monotonicity of the free Green's function which is no longer true for magnetic Schrödinger operators. Thus, the question deserves a separate treatment which we postpone to another paper. The same is true for the continuous-spectrum part of the nonmagnetic problem where the curvature leads in general to a nontrivial scattering.

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References

[AGHH] Albeverio, S., Gesztesy, F., Høegh-Krohn, R. and Holden, H.: *Solvable Models in Quantum Mechanics*, Springer, Heidelberg, 1988.

[DE] Duclos, P. and Exner, P.: Curvature-induced bound states in quantum waveguides in two and three dimensions, *Rev. Math. Phys.* **7** (1995), 73–102.

[EI] Exner, P. and Ichinose, T.: Geometrically induced spectrum in curved leaky wires, *J. Phys. A* **34**(7) (2001), 1439–1450.

[EJK] Exner, P., Joye, A. and Kovařík, H.: Edge currents in the absence of edges, *Phys. Lett. A* **264** (1999), 124–130.

[EŠ] Exner, P. and Šeba, P.: Bound states in curved quantum waveguides, *J. Math. Phys.* **30** (1989), 2574–2580.

[Hu] Hurt, N. E.: *Mathematical Physics of Quantum Wires and Devices*, Kluwer Acad. Publ., Dordrecht, 2000.

[LCM] Londergan, J. T., Carini, J. P. and Murdock, D. P.: *Binding and Scattering in Two-Dimensional Systems*, Lecture Notes in Math. 60, Springer, New York, 1999.