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Noncommutative Geometry and Integrable Models

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Abstract. A construction of conservation laws for σ-models in two dimensions is generalized within the framework of noncommutative geometry of commutative algebras. This is done by replacing the ordinary calculus of differential forms with other differential calculi and introducing an analogue of the Hodge operator on the latter. The general method is illustrated with several examples.

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1. Introduction

In a recent work [1], we have shown that completely integrable discrete versions of two-dimensional σ-models (chiral models) are obtained via certain deformations [2] of the ordinary calculus of differential forms on $\mathbb{R}^2$. The procedure is based on a generalization of the construction of conserved currents presented for continuum σ-models in [3]. In the present work, we further generalize this method in several ways. We present rather weak conditions to be imposed on a differential calculus and on a generalized Hodge $*$-operator such that the classical field equation $\mathrm{d} * g^{-1}\,\mathrm{d}g = 0$ for a σ-model makes sense and a construction of an infinite sequence of conserved currents still works (Section 3). In Section 2, we introduce two-dimensional 'noncommutative geometries' with several examples to which we refer in the sequel. Section 3 presents our notion of a generalized σ-model, the construction of conserved currents for it, and a linear system of which the field equations are integrability conditions. This linear system is then further discussed in Section 4 from a slightly more general point of view, revealing a kind of 'duality' between the σ-model field equation and the zero curvature condition. Some integrable equations are derived from the examples of noncommutative geometries in Section 2. Section 5 contains our conclusions and further remarks.

2. Two-Dimensional Noncommutative Geometries

Let $\mathcal{A}$ be a commutative algebra of functions of two variables, t and x. Let $\Omega(\mathcal{A})$ be a differential calculus on $\mathcal{A}$ such that $\mathrm{d}t$ and $\mathrm{d}x$ constitute a left and right $\mathcal{A}$-module basis of the space $\Omega^1(\mathcal{A})$ of 1-forms. Though the algebra $\mathcal{A}$ itself is commutative (and can thus be realized as an algebra of functions on some topological space), the differential calculus may be such that functions and differentials do not commute. In that case, we speak of a 'noncommutative differential calculus' and geometric structures built on it inherit this noncommutativity. In this sense we obtain a 'noncommutative geometry' on a commutative algebra. Our basic geometric ingredient is a $\mathbb{C}$-linear operator $*\colon \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A})$ with the property

$$* (\omega f) = f(*\omega) \tag{1}$$

for all $f \in \mathcal{A}$ and $\omega \in \Omega^1(\mathcal{A})$. Then

$$* \,\mathrm{d}t = \gamma\,\mathrm{d}t + \alpha\,\mathrm{d}x, \qquad *\,\mathrm{d}x = \beta\,\mathrm{d}t + \delta\,\mathrm{d}x \tag{2}$$

and the operator $*$ is determined by the choice of $\alpha, \beta, \gamma, \delta \in \mathcal{A}$. The $*$-operator generalizes the Hodge operator of Riemannian geometry. In the following sections we shall need some additional properties for this operator. We require $*$ to be *symmetric* in the sense that

$$\omega * \omega' = \omega' * \omega \tag{3}$$

for all $\omega, \omega' \in \Omega^1(\mathcal{A})$. Depending on the choice of differential calculus, these conditions restrict the possibilities for the $*$ operator. A simple calculation shows that the symmetry condition is equivalent to*

$$\mathrm{d}t\, f\beta\,\mathrm{d}t + \mathrm{d}t\, f\,\delta\,\mathrm{d}x - \mathrm{d}x\, f\gamma\,\mathrm{d}t - \mathrm{d}x\, f\alpha\,\mathrm{d}x = 0 \tag{4}$$

for all $f \in \mathcal{A}$. We also require $*$ to be invertible. As a consequence of (1) its inverse then satisfies $*^{-1}(f\omega) = (*^{-1}\omega)f$. Moreover, we demand that

$$\mathrm{d}\,(* * \omega) = 0 \Leftrightarrow \mathrm{d}\,\omega = 0. \tag{5}$$

In Section 3, we also need the triviality of the first cohomology group of $\Omega(\mathcal{A})$, i.e., closed 1-forms have to be exact. This condition is fulfilled for all the following examples.

EXAMPLE 1. Let $\mathcal{A}$ be the algebra of C^∞-functions on $\mathbb{R}^2$ and $\Omega(\mathcal{A})$ the ordinary differential calculus (where functions commute with 1-forms). According to the Poincaré Lemma every closed form is exact, i.e., the cohomology is trivial. The symmetry condition (4) becomes $\delta = -\gamma$. The $*$-operator is invertible iff $\mathcal{D} := \alpha\beta + \gamma^2$ is everywhere different from zero. We find $** = \mathcal{D} id$. The condition (5) is satisfied iff $\mathcal{D}$ is constant.

* It is possible to extend a first-order differential calculus to higher orders by demanding that the product of any two differentials vanishes. Then the following condition is trivially satisfied, but also the field equation which we consider in Section 3. It is more natural and convenient, of course, to constrain the space of two-forms only by those conditions which are derived from the first-order calculus using the general rules of differential calculus (cf. [2], for example).

EXAMPLE 2. Let $\mathcal{A}$ be the set of all functions on the two-dimensional lattice $a\mathbb{Z} \times b\mathbb{Z}$ where a, b are positive real constants. x and t are the canonical coordinate functions. A differential calculus on $\mathcal{A}$ is then determined by the commutation relations*

$$[\mathrm{d}x, x] = a\,\mathrm{d}x, \qquad [\mathrm{d}x, t] = 0 = [\mathrm{d}t, x], \qquad [\mathrm{d}t, t] = b\,\mathrm{d}t. \tag{6}$$

More generally, we have

$$\mathrm{d}t\, f(x,t) = f(x, t+b)\,\mathrm{d}t, \qquad \mathrm{d}x\, f(x,t) = f(x+a, t)\,\mathrm{d}x \tag{7}$$

for $f \in \mathcal{A}$. Acting with the exterior derivative on (6) leads to

$$\mathrm{d}x\,\mathrm{d}x = 0 = \mathrm{d}t\,\mathrm{d}t, \quad \mathrm{d}x\,\mathrm{d}t = -\mathrm{d}t\,\mathrm{d}x. \tag{8}$$

In general, however, 1-forms do not anticommute in this calculus. For the differential of a function f we get

$$\mathrm{d}f = \partial_{+x} f\,\mathrm{d}x + \partial_{+t} f\,\mathrm{d}t \tag{9}$$

where ∂_{+t} and ∂_{+x} are discrete partial derivatives, i.e.,

$$\begin{aligned}(\partial_{+x} f)(x,t) &= \frac{1}{a}[f(x+a,t) - f(x,t)],\\ (\partial_{+t} f)(x,t) &= \frac{1}{b}[f(x,t+b) - f(x,t)].\end{aligned} \tag{10}$$

The symmetry condition (4) becomes $\gamma = \delta = 0$ and the $*$-operator is invertible iff $\alpha\beta$ nowhere vanishes on the lattice. Furthermore, one finds

$$\begin{aligned}&**[\mathrm{d}x\, f(x,t) + \mathrm{d}t\, h(x,t)]\\ &\quad = \alpha(x,t)\beta(x,t-b)\,\mathrm{d}x f(x-a, t-b) +\\ &\qquad + \alpha(x-a,t)\beta(x,t)\,\mathrm{d}t\, h(x-a,t-b).\end{aligned} \tag{11}$$

The condition (5) in particular requires $**\,\mathrm{d}t$ and $**\,\mathrm{d}x$ to be closed. This leads to

$$\partial_{+t}[\alpha(x,t)\beta(x,t-b)] = 0, \qquad \partial_{+x}[\alpha(x-a,t)\beta(x,t)] = 0. \tag{12}$$

Thus

$$\alpha(x,t) = \frac{C(x)}{B(t-b)}\alpha(x-a,t-b), \qquad \beta(x,t) = \frac{B(t)}{\alpha(x-a,t)}, \tag{13}$$

* More precisely, these relations determine a differential calculus on the algebra of polynomials in x and t which can then be extended to the algebra of arbitrary functions. See also [2].

where $C(x)$ and $B(t)$ are arbitrary (nowhere vanishing) functions. Taking (12) into account, (5) applied to the closed 1-form $\mathrm{d}x\, t + \mathrm{d}t\, x$ yields

$$\alpha(x,t)\beta(x,t-b) = \alpha(x-a,t)\beta(x,t). \tag{14}$$

Together with (13), this requires C and B to be constant and, moreover, $C = B$. We end up with

$$\alpha(x,t) = \alpha(x-a,t-b), \qquad \beta(x,t) = \frac{C}{\alpha(x-a,t)}. \tag{15}$$

With these restrictions on α and β we have

$$* * \omega(x,t) = C\omega(x-a,t-b) \tag{16}$$

for all $\omega \in \Omega^1(\mathcal{A})$ and (5) is satisfied. In the limit $a \to 0, b \to 0$ we obtain the ordinary differential calculus (on C^∞-functions of x and t). The corresponding limit of the $*$-operator, however, does not exhaust the possibilities which we have for $a = b = 0$ (cf. Example 1). On the other hand, the limit $b \to 0$, keeping a constant (and different from zero), does exhaust the possibilities which one finds by investigating the limit calculus.

EXAMPLE 3. Let $\mathcal{A}$ be the algebra of C^∞-functions on $\mathbb{R}^2$ and $\Omega(\mathcal{A})$ the differential calculus determined by

$$[\mathrm{d}x, x] = \eta\, \mathrm{d}t, \qquad [\mathrm{d}x, t] = [\mathrm{d}t, x] = [\mathrm{d}t, t] = 0, \tag{17}$$

with a constant η (see also [4]). More generally, we have

$$\mathrm{d}t f = f\, \mathrm{d}t, \qquad \mathrm{d}x f = f\, \mathrm{d}x + \eta f_x\, \mathrm{d}t \tag{18}$$

for $f \in \mathcal{A}$. Here f_x denotes the partial derivative with respect to x. Furthermore, one finds

$$\mathrm{d}f = \left(f_t + \frac{\eta}{2} f_{xx}\right) \mathrm{d}t + f_x\, \mathrm{d}x \tag{19}$$

and $\mathrm{d}x\, \mathrm{d}x = 0 = \mathrm{d}t\, \mathrm{d}t$, $\mathrm{d}x\, \mathrm{d}t = -\mathrm{d}t \mathrm{d}x$. For $\eta \neq 0$ the symmetry condition (4) becomes $\alpha = 0$ and $\delta = -\gamma$ so that

$$* \, \mathrm{d}t = \gamma\, \mathrm{d}t, \qquad * \, \mathrm{d}x = \beta\, \mathrm{d}t - \gamma\, \mathrm{d}x. \tag{20}$$

The $*$-operator is invertible iff $\gamma \neq 0$. The condition (5) applied to the differentials $\mathrm{d}t$ and $\mathrm{d}x$ requires γ to be constant. Since every 1-form ω can be written as $\omega = \mathrm{d}t f + \mathrm{d}x h$ with functions f and h, a direct calculation now leads to

$$* * \omega = \gamma^2 \omega \tag{21}$$

so that (5) is indeed satisfied. There is no restriction for the function β.

EXAMPLE 4. For $ab \neq 0$, the constants a and b in (6) can be absorbed via a rescaling of x and t. We may therefore set $a = b = 1$. In terms of the 'light cone coordinates'

$$u := \mu\,(t + x), \qquad v := \nu\,(t - x), \tag{22}$$

where μ, ν are constants, (6) becomes

$$[\mathrm{d}u, u] = \mu\,\mathrm{d}u, \qquad [\mathrm{d}u, v] = [\mathrm{d}v, u] = \mu\,\mathrm{d}v, \qquad [\mathrm{d}v, v] = \frac{\nu^2}{\mu}\,\mathrm{d}u. \tag{23}$$

Performing the limit $\mu \to 0$ in such a way that $\nu^2/\mu \to \eta$ with a constant η, the calculus of Example 3 is recovered. Another calculus, which will be discussed in the following, is obtained in the limit $\nu \to 0$. After a renaming of the coordinate functions, we get

$$[\mathrm{d}t, t] = 0, \qquad [\mathrm{d}t, x] = [\mathrm{d}x, t] = \mu\,\mathrm{d}t, \qquad [\mathrm{d}x, x] = \mu\,\mathrm{d}x. \tag{24}$$

For a function f, this generalizes to

$$\begin{aligned} \mathrm{d}t\,f(x,t) &= f(x+\mu,t)\,\mathrm{d}t, \\ \mathrm{d}x\,f(x,t) &= f(x+\mu,t)\,\mathrm{d}x + \mu \dot{f}(x+\mu,t)\,\mathrm{d}t \end{aligned} \tag{25}$$

where $\dot{f} = \partial f/\partial t$. Furthermore,

$$\mathrm{d}f = \dot{f}(x+\mu,t)\,\mathrm{d}t + (\partial_{+x} f)(x,t)\,\mathrm{d}x. \tag{26}$$

The algebra $\mathcal{A}$ should now consist of functions on $\mu\mathbb{Z} \times \mathbb{R}$ which are smooth in the variable t. Again, (8) holds. The $\nu \to 0$ limit of the $*$-operator for the calculus of Example 2 (in the form (23)) only leaves us with $\alpha = \beta = 0$ and $\delta = -\gamma$ in (2). But a closer inspection of the above (limit) calculus shows that an arbitrary function β is permitted. The condition (5) requires γ to be constant and β not to depend on x, i.e., $\beta = \beta(t)$. Then

$$**\,\omega(x,t) = \gamma^2 \omega(x - 2\mu, t). \tag{27}$$

The above examples by far do not exhaust the possibilities.[⋆] Even these examples can be considerably generalized by replacing the constants appearing in the defining relations of the differential calculi by suitable functions. The commutation relations for the differentials then no longer take the simple form (8). If two differential calculi are related by a (suitable) coordinate transformation, they should be identified. A

[⋆] Further examples of two-dimensional differential calculi can be found in [5].

complete classification of two-dimensional differential calculi has not yet been achieved (see [5] for partial results). As a consequence of our definitions, the action of the $*$-operator can be calculated on any basis of $\Omega^1(\mathcal{A})$ if we know its action on one basis.

3. Generalized σ-Models and Conservation Laws

In case of the ordinary differential calculus on $\mathbb{R}^2$, the following construction of conserved currents is due to Brezin *et al.* [3]. In the form presented below, it also works for the noncommutative geometries introduced in the previous section. Γ denotes an algebra of finite matrices with entries in $\mathcal{A}$ and Γ^* the group of invertible elements of Γ. For $g \in \Gamma$ and

$$A := g^{-1}\,\mathrm{d}g \tag{28}$$

we consider the field equations

$$\mathrm{d} * A = 0 \tag{29}$$

and refer to such a classical field theory as a *generalized σ-model*. Since A is a 'pure gauge' we have

$$F := \mathrm{d}A + AA = 0. \tag{30}$$

Let $\Psi \in \Gamma$ and $D\colon \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ the 'exterior covariant derivative' given by

$$D\Psi = \mathrm{d}\Psi + A\Psi. \tag{31}$$

Using (1), (29) and (3), we find

$$\mathrm{d} * (A^i_j \Psi^j_k) = \mathrm{d}(\Psi^j_k * A^i_j) = (\mathrm{d}\Psi^j_k) * A^i_j = A^i_j * \mathrm{d}\Psi^j_k \tag{32}$$

and thus

$$\mathrm{d} * D\Psi = D * \mathrm{d}\Psi. \tag{33}$$

If there is one conserved current for a generalized σ-model, then an infinite sequence of conserved currents is obtained as follows. Suppose $J^{(m)} \in \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is conserved, i.e.,

$$\mathrm{d} * J^{(m)} = 0. \tag{34}$$

If the first cohomology group of $\Omega(\mathcal{A})$ is trivial and provided that (5) holds, there exists $\chi^{(m)} \in \Gamma$ such that

$$J^{(m)} = *\,\mathrm{d}\chi^{(m)}. \tag{35}$$

Then

$$J^{(m+1)} := D\chi^{(m)} \tag{36}$$

is also conserved since

$$\begin{aligned} \mathrm{d} * J^{(m+1)} &= \mathrm{d} * D\chi^{(m)} = D * \mathrm{d}\chi^{(m)} = DJ^{(m)} \\ &= DD\chi^{(m-1)} = F\chi^{(m-1)} = 0. \end{aligned} \tag{37}$$

Starting with $\chi^{(0)} = I$, the unit matrix, this procedure indeed generates an infinite number of conserved currents. Let us introduce

$$\chi := \sum_{m=0}^{\infty} \kappa^m \chi^{(m)} \tag{38}$$

where κ is a parameter. From (35) and (36) we obtain

$$* \mathrm{d}\chi^{(m+1)} = D\chi^{(m)}. \tag{39}$$

Multiplying by κ^{m+1} and summing over m leads to

$$* \ \mathrm{d}\chi = \kappa \, D\chi. \tag{40}$$

The field equations (29) are integrability conditions of the linear system (40). In a slightly more general setting this will be shown in the following section.

4. Another Look at the Linear System

Let $A \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Gamma$. Here A is not assumed to have the form (28). We still use the definitions (30) and (31), however. Let us consider a linear system of the form (40), i.e., $* \, \mathrm{d}\chi = \kappa D\chi$. It implies

$$\begin{aligned} 0 &= \mathrm{d}(*D\chi)^i_j = \mathrm{d} * \, \mathrm{d}\chi^i_j + \mathrm{d}(\chi^k_j * A^i_k) \\ &= d * \, \mathrm{d}\chi^i_j + A^i_k * \, \mathrm{d}\chi^k_j + \chi^k_j \, \mathrm{d} * A^i_k \\ &= (D * \, \mathrm{d}\chi)^i_j + \chi^k_j \, \mathrm{d} * A^i_k. \end{aligned} \tag{41}$$

On the other hand, (40) also leads to

$$D * \mathrm{d}\chi = \kappa \, D^2\chi = \kappa \, F\chi. \tag{42}$$

Hence

$$\chi^k_j \, \mathrm{d} * A^i_k = -\kappa \, F^i_k \, \chi^k_j. \tag{43}$$

We can now achieve $F = 0$ with the ansatz (28), i.e., $A = g^{-1}\mathrm{d}g$, as we did in the previous section. Then (29) is the integrability condition of (40) which then depends on g. Alternatively, we can satisfy $\mathrm{d} * A = 0$ by setting $A = * \, \mathrm{d}g'$. Then $F = 0$ is the integrability condition for the above linear system which now depends on g'. We should stress that in the two cases we are dealing with different linear systems and one should not expect the equations resulting from the two integrability conditions to be equivalent. In the following two examples, this turns out to be the case, however.

EXAMPLE 1. Let us consider the differential calculus of Example 2 in Section 2 with $b = 0$ (so that elements of $\mathcal{A}$ should be C^∞-functions of t) and $* \, \mathrm{d}t = \alpha \, \mathrm{d}x$, $* \, \mathrm{d}x = \beta \, \mathrm{d}t$ where α, β are constants different from zero. For $v \in \mathcal{A}$ we write $v_n(t) = v(na, t)$ where $n \in \mathbb{Z}$. Then

$$\mathrm{d}v_n = \mathrm{d}t \, \dot{v}_n + \mathrm{d}x \frac{1}{a}(v_n - v_{n-1}). \tag{44}$$

The 1-form

$$A(na, t) := * \, \mathrm{d}v_n = \alpha \dot{v}_n \, \mathrm{d}x + \frac{\beta}{a}(v_n - v_{n-1}) \, \mathrm{d}t \tag{45}$$

has the ‘curvature’

$$F(na, t) = \left[\alpha \ddot{v}_n - \frac{\beta}{a^2}(1 + \alpha a \dot{v}_n)(v_{n+1} - 2v_n + v_{n-1})\right] \mathrm{d}t \, \mathrm{d}x. \tag{46}$$

The zero curvature condition $F = 0$ is then equivalent to

$$[\ln(1 + \alpha a \dot{v}_n)]^{\cdot} = \frac{\beta}{a}(v_{n+1} - 2v_n + v_{n-1}). \tag{47}$$

This equation is ‘dual’, in the sense of an exchange of the roles of particles and interactions, and mathematically equivalent to that of the nonlinear Toda lattice equation, see [6], p. 18. The latter is

$$\ddot{u}_n = \frac{\beta}{\alpha a^2} \left(e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}}\right) \tag{48}$$

which is recovered from $\mathrm{d} * A = 0$ where now $A = e^u \, \mathrm{d}e^{-u}$, i.e., (28) with $g = e^{-u}$. See also [1].

EXAMPLE 2. We choose the differential calculus of Example 3 in Section 2. For the 1-form

$$A := * \, \mathrm{d}v = \left[\gamma \left(v_t - \frac{\eta}{2} v_{xx}\right) + \beta v_x\right] \mathrm{d}t - \gamma v_x \, \mathrm{d}x, \tag{49}$$

where $v \in \mathcal{A}$, the zero curvature condition $F = 0$ is

$$w_t + \frac{1}{2\gamma}\,(\beta w)_x - \frac{\eta}{4}\,\gamma\,(w^2)_x = 0, \tag{50}$$

where $w = v_x$. On the other hand, from $\mathrm{d} * A = 0$, where now $A := e^u\,\mathrm{d}e^{-u}$, we obtain the same equation by setting $w := \gamma u_x$.

EXAMPLE 3. Let us consider the $\nu = 0$ calculus of Example 4 in Section 2. With $A = * \,\mathrm{d}v$ where $v \in \mathcal{A}$, the zero curvature condition is equivalent to*

$$\bar{\partial}\dot{v}_n = -\frac{\beta(t)}{2\gamma}\Delta v_n + \frac{\gamma}{2}[\dot{v}_{n-1}\partial_{+x}v_n + \dot{v}_{n+1}\partial_{+x}v_{n-1}] \tag{51}$$

where $v_n := v(n\mu, t)$ and

$$\bar{\partial}v_n := \frac{1}{2\mu}(v_{n+1} - v_{n-1}), \qquad \Delta v_n := \frac{1}{\mu^2}(v_{n+1} - 2v_n + v_{n-1}). \tag{52}$$

On the other hand, with $A = \mathrm{e}^u\,\mathrm{d}\mathrm{e}^{-u}$ the equation $\mathrm{d} * A = 0$ leads to

$$\dot{u}_{n+1}\,\mathrm{e}^{u_n - u_{n+1}} - \dot{u}_{n-1}\,\mathrm{e}^{u_{n-1} - u_n} = \frac{\beta(t)}{\gamma\mu}[\mathrm{e}^{u_n - u_{n+1}} - \mathrm{e}^{u_{n-1} - u_n}]. \tag{53}$$

These are just a few examples of integrable equations. The relevance of the last two is unclear. They are included here mainly to illustrate the general method. So far we have restricted our examples to $g \in \mathcal{A}$ for simplicity. Generalizations to models where g takes values in some matrix group are easily obtained, as in the next example.

EXAMPLE 4. We generalize our Example 3 in the sense just mentioned. With $A = g^{-1}\,\mathrm{d}g$ (where $g \in \Gamma^*$) the equation $\mathrm{d} * A = 0$ is equivalent to

$$g_n^{-1}\dot{g}_{n+1} + (g_{n-1}^{-1})^{\cdot}\,g_n = -\frac{\beta}{\mu\gamma}(g_n^{-1}g_{n+1} - g_{n-1}^{-1}g_n). \tag{54}$$

The linear system (40) can be expressed as follows (when $\kappa \neq 0$),

$$(g_{n+1}\chi_{n+1})^{\cdot} = \frac{1}{\kappa}g_n(\gamma\dot{\chi}_{n-1} + \beta\partial_{+x}\chi_{n-1}), \tag{55}$$

$$g_{n+1}\chi_{n+1} = g_n[(1 - \gamma/\kappa)\chi_n + (\gamma/\kappa)\chi_{n-1}]. \tag{56}$$

Introducing $\xi_n := (\kappa g_n \chi_n, \chi_{n-1})^T$ and

$$L_n := \frac{1}{\kappa}\begin{pmatrix} \kappa - \gamma & \gamma\kappa g_n \\ g_n^{-1} & 0 \end{pmatrix}, \tag{57}$$

* The function $\beta(t)$ can be absorbed by choosing a suitable 'time' coordinate.

$$M_n := \frac{1}{\kappa - \gamma} \begin{pmatrix} \beta/\mu & -\kappa[\gamma \dot{g}_n + (\beta/\mu) g_n] \\ -[g_{n-1}^{-1} \dot{g}_{n-1} + \beta/(\mu\gamma)] g_{n-1}^{-1} & \kappa\beta/(\mu\gamma) \end{pmatrix} \tag{58}$$

(assuming $\kappa \neq \gamma$), the above system of equations can be written as follows,

$$\xi_{n+1} = L_n \xi_n, \qquad \dot{\xi}_n = M_n \xi_n. \tag{59}$$

The integrability conditions, which are the σ-model field equations, now take the form $\dot{L}_n + L_n M_n - M_{n+1} L_n = 0$. We have derived a formulation of the complete integrability of (54) in terms of a Lax pair.

In the way described in this section, and furthermore by choosing different differential calculi, we get a plethora of models which are integrable in the sense of Section 3. These models need to be further investigated (in particular with respect to soliton solutions) and somehow classified.

5. Conclusions

We have introduced a generalization of σ-models in the framework of noncommutative geometry. Obviously our constructive method leads to a large set of new completely integrable models. An interesting question is which of the known integrable models which are of interest in physics fit into this framework. For example, it has been shown in [1] (see also Example 1 in Section 4) that the nonlinear Toda lattice is a generalized σ-model in the sense of Section 3. Via the linear system (40) there is an integrable zero curvature model associated with each generalized σ-model. This ‘duality’ turned out to coincide with a physical duality in case of the nonlinear Toda lattice.

Our definition of generalized σ-models (and their duals) also makes sense in more than two dimensions and the construction of conserved currents in Section 3 still works. The problem, however, is to find a $*$-operator satisfying (1), (3) and (5). It should also be noticed that, in more than two dimensions, our $*$-operator (which acts in the space of 1-forms) is no longer an analogue of the Hodge operator of Riemannian geometry.

EXAMPLE. Let us consider the ordinary differential calculus on $\mathbb{R}^n$. A $*$-operator is then determined by

$$* \, \mathrm{d}x^i = a^i_j \, \mathrm{d}x^j \tag{60}$$

(using the summation convention). The symmetry condition (3) takes the form

$$\omega'_k a^k{}_{[i} \omega_{j]} = \omega_k a^k{}_{[i} \omega'{}_{j]} \tag{61}$$

where $\omega = \mathrm{d}x^i \omega_i$ and the square brackets indicate antisymmetrization of indices. In more than two dimensions ($n > 2$), this condition is only satisfied for all 1-forms

ω, ω' if all the functions a^i_j vanish.* Hence, there is no (generalized) σ-model in this case.

The last example leaves us with a rather pessimistic impression concerning the possibilities of higher-dimensional generalized σ-models. However, the situation may be different in case of other (noncommutative) differential calculi. The corresponding possibilities have still to be explored.

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* Choose any three of the n indices, like $\{1, 2, 3\}$, and evaluate the symmetry condition for $\omega_i, \omega'_j \in \{\delta^1_k, \delta^2_k, \delta^3_k\}$. This leads to $a^k_i = 0$ for all i and for $k = 1, 2, 3$. But since $\{1, 2, 3\}$ could number any triple of coordinates, we have $a^i_j = 0$ where $i, j = 1, \ldots, n$.