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Approximations of Quantum-Graph Vertex Couplings by Singularly Scaled Rank-One Operators

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Abstract. We investigate approximations of the vertex coupling on a star-shaped graph by families of operators with singularly scaled rank-one interactions. We find a family of vertex couplings, generalizing the δ'-interaction on the line, and show that with a suitable choice of the parameters they can be approximated in this way in the norm-resolvent sense. We also analyze spectral properties of the involved operators and demonstrate the convergence of the corresponding on-shell scattering matrices.

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1. Introduction

Quantum graphs are a versatile model of many physical systems; we refer to the recent monograph [3] for an extensive bibliography. One of the central items of this theory are vertex coupling conditions used to match wave functions supported by graph edges. From general principles they have to be chosen to make the graph Hamiltonian self-adjoint. This is a simple task, but the result leaves a lot a freedom through parameters entering those conditions the values of which have to be fixed. The background of such a choice is the question about the physical meaning of the coupling, important without any doubt; one has to keep in mind that different vertex couplings give rise to different quantum dynamics on the graph.

A natural approach to the problem is to analyze various approximations of those couplings, either on the graph itself or using a tubular network shrinking to

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the graph 'skeleton'. Even in the latter case, however, one considers the approximations on the graph as an intermediate step—cf. [7] and references therein. The task simplifies due to the fact that the vertex couplings, the physically interesting ones at least, are of a local nature; it is thus sufficient to solve the problem for a star-type graph with n edges meeting in a single vertex. One usually begins with the most simple coupling, often called Kirchhoff, and investigates families of scaled interaction supported in the vicinity of the junction. The simplest example is potentials scaled with their mean preserved, which give rise to one-parameter family of the so-called δ-couplings [4]. This is, however, only a small subclass of the couplings allowed by the self-adjointness requirement. Using potentials scaled in a more singular way many other conditions can be obtained—without going into details we refer to [6] and the bibliography therein.

These scaled-potential approximations, however, do not yield all the admissible couplings; in particular, one cannot obtain in this way strongly singular matching conditions such as the so-called δ'-coupling and its modifications, which are of interest mainly because their properties contrast in a sense to those of more regular couplings; for instance, a δ' junction is opaque at high energies.

Note that various results are known in the simplest nontrivial case $n=2$ where the coupling is nothing else that a generalized point interaction on the line [1]. In particular, an approximation of the δ'-interaction on the line with the help of scaled rank-one operators was proposed longtime ago by Šeba [11]. The aim of the present paper is to propose and analyze a similar approximation of a class of singular vertex couplings, given by relations (5) below, by nonlocal potentials for a general star-shaped graph. Our main results are demonstration of the norm-resolvent convergence of such an approximation, Theorem 3.2 below, and convergence of the corresponding on-shell scattering matrices, Theorem 5.1. Let us add that as in the case $n=2$, the constructed approximation is non-generic, and furthermore, it represents a new generalization of the δ'-interaction on the line, different from the two known ones [5]—we will say more on that in the concluding remarks—in contrast to those it leans on the permutation asymmetry of the approximating operators.

2. Preliminaries

To begin with, we recall a few basic notions concerning metric graphs. In what follows, we focus on noncompact star-shaped graphs Γ consisting of $n \in \mathbb{N}$ semi-infinite edges $\gamma_1, \ldots, \gamma_n$ connected at a single vertex. A map $\psi : \Gamma \to \mathbb{C}$ is said to be a function on the graph and its restriction to the edge γ_i will be denoted by ψ_i. Each edge γ_i has a natural parametrization x_i given by the arc length of the curve representing the edge, hence without loss of generality we may identify each γ_i with the half-line $[0, \infty)$. A differentiation is always related to this natural length parameter. We denote by $\psi_i'(0)$, the limit value of the derivative at the graph vertex taken conventionally in the outward direction, i.e., away from the vertex. The integral $\int_\Gamma \psi\,\mathrm{d}x$ of ψ over Γ is the sum of integrals over all edges $\sum_{i=1}^n \int_0^\infty \psi_i\,\mathrm{d}x_i$, the measure being the natural Lebesgue measure.

As usual, $L^2(\Gamma)$ denotes the Hilbert space of (equivalence classes of) such functions with the scalar product $(\psi_1,\psi_2)=\int_\Gamma \psi_1\bar{\psi}_2\,\mathrm{d}x$. Next, we introduce the Sobolev space $H^2(\Gamma)$ as the Banach space with the norm $\|\psi\|_{H^2(\Gamma)}=(\|\psi\|_{L^2(\Gamma)}+\|\psi''\|_{L^2(\Gamma)})^{1/2}$, observing that neither the functions belonging to $H^2(\Gamma)$ nor their derivatives should be continuous at the graph vertex. Finally, we say that a function ψ satisfies the Kirchhoff conditions at the graph vertex and write $\psi\in K(\Gamma)$ if ψ is continuous at this vertex and satisfies the condition $\sum_{i=1}^n \psi_i'(0)=0$.

Given a star graph Γ described above, we introduce the following family of Schrödinger operators on $L^2(\Gamma)$ labeled by the parameter $\varepsilon\in(0,1]$,

$$-\Delta^\varepsilon := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\lambda(\varepsilon)}{\varepsilon^3}V_\varepsilon(x)\langle\cdot,V_\varepsilon\rangle_\Gamma, \qquad \mathfrak{D}(-\Delta^\varepsilon)=H^2(\Gamma)\cap K(\Gamma), \tag{1}$$

with $V_\varepsilon(\cdot):=V(\frac{\cdot}{\varepsilon})$, where the real-valued function V belongs to the class $L^1_{\mathrm{loc}}(\Gamma)$ and has a compact support and zero mean, i.e., $\int_\Gamma V\,\mathrm{d}x=0$. Without loss of generality, we may (and shall) suppose that the support of V is contained in the unit ball centered at the graph vertex. With respect to the edge indices, V may be regarded as an $n\times n$ matrix function on $[0,\infty)$; we stress that it need not be diagonal. In a similar way, the differential part of $-\Delta^\varepsilon$ is a shorthand for the operator which acts as the negative second derivative on each edge γ_i. The function $\lambda(\cdot)$ in the above expression is supposed to be real-valued for real ε and holomorphic in the vicinity of the origin. In addition, it satisfies the condition

$$\lambda(\varepsilon)=\lambda_0+\varepsilon\lambda_1+\mathcal{O}(\varepsilon^2)\,, \qquad \varepsilon\to 0\,, \tag{2}$$

where λ_0 and λ_1 are nonzero real numbers. Our main goal in this paper is to investigate convergence of the operators $-\Delta^\varepsilon$ as $\varepsilon\to 0$ in the norm-resolvent topology.

To describe the outcome of the limiting process, we need the following quantities:

$$\vartheta_i := \int_0^\infty x_i V(x_i)\,\mathrm{d}x_i, \quad i=1,2,\ldots,n, \tag{3}$$

and

$$A := -\sum_{i=1}^n \int_0^\infty\int_0^\infty \min\{x_i,y_i\}\,V(x_i)V(y_i)\,\mathrm{d}x_i\mathrm{d}y_i\,. \tag{4}$$

Using them, we define the *limit operator* $-\Delta_\beta$ as the one acting as

$$-\Delta_\beta\psi := -\psi''$$

on functions $\psi \in H^2(\Gamma)$ that obey the matching conditions

$$\frac{\psi_i(0)-\psi_j(0)}{\vartheta_i-\vartheta_j}=\beta\sum_{\ell=1}^{n}\vartheta_\ell\,\psi_\ell'(0)\,,\quad 1\le i<j\le n\,,\qquad \sum_{\ell=1}^{n}\psi_\ell'(0)=0\,, \tag{5}$$

where we adopt the convention that $\psi_i(0)=\psi_j(0)$ holds if $\vartheta_i=\vartheta_j$. Here

$$\beta=\begin{cases}\frac{1}{\lambda_1 A^2} & \text{if } \lambda_0 A=1,\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

Remark 2.1. (i) In the particular case $n=2$, one gets from (5) the well-known one-dimensional δ'-interaction with the coupling parameter β.

(ii) The boundary conditions that determine the domain of the limit operator $-\Delta_\beta$ can alternatively be written in the conventional form [8] as

$$\mathcal{A}\Psi(0)+\mathcal{B}\Psi'(0)=0\,, \tag{7}$$

where

$$\mathcal{A}=\begin{pmatrix} 0 & 0 & 0 & \dots & 0\\ \frac{1}{\vartheta_1-\vartheta_2} & \frac{1}{\vartheta_2-\vartheta_1} & 0 & \dots & 0\\ \frac{1}{\vartheta_1-\vartheta_3} & 0 & \frac{1}{\vartheta_3-\vartheta_1} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\vartheta_1-\vartheta_n} & 0 & 0 & \dots & \frac{1}{\vartheta_n-\vartheta_1}\end{pmatrix},\qquad \mathcal{B}=-\beta\begin{pmatrix}\frac{1}{\beta} & \frac{1}{\beta} & \dots & \frac{1}{\beta}\\ \vartheta_1 & \vartheta_2 & \dots & \vartheta_n\\ \vartheta_1 & \vartheta_2 & \dots & \vartheta_n\\ \vdots & \vdots & \ddots & \vdots\\ \vartheta_1 & \vartheta_2 & \dots & \vartheta_n\end{pmatrix}.$$

It is useful to adopt the convention that all entries of the first row of $\mathcal{B}$ are -1 even if $\beta=0$. As it is well known, to make a graph Hamiltonian self-adjoint, a graph vertex at which n edges meet should be characterized by boundary conditions (7) in which the $n\times n$ matrices $\mathcal{A}$, $\mathcal{B}$ are such that the product $\mathcal{A}\mathcal{B}^*$ is self-adjoint, while the $2n\times n$ matrix $(\mathcal{A}|\mathcal{B})$ has rank n; it is straightforward to check that this is the case here.

(iii) If $\vartheta_1=\vartheta_j$ holds for some $j=2,\dots,n$, then the jth row of the above matrix $(\mathcal{A}|\mathcal{B})$ should be replaced by the vector $(1,0,\dots,0,-1,0,\dots,0)$ of the length $2n$ with -1 at the jth place. In what follows, we assume without loss of generality that all the ϑ_i are different.

Having stated the problem, let us review briefly the following contents of the paper. In the next section, we describe properties of the limit operator $-\Delta_\beta$, in particular, we characterize its spectrum as well as its resolvent, and describe the corresponding scattering matrix—cf. Theorems 3.1–3.3. In Sect. 4, we first discuss the structure of the resolvent of the approximation operators (1) with the aim prove our first main result, namely its closeness to that of the limit operator, stated in Theorem 4.1, together with its spectral consequences, Theorem 4.2. Finally, Sect. 5 contains the other main results of the paper, Theorem 5.1, which says that the on-shell scattering matrices of the operators (1) approximate the corresponding on-shell S-matrix of the limit one.

3. Limit Operator

After the above preliminaries, let us turn to basic properties of $-\Delta_\beta$. Starting from the explicit expression of the resolvent $(-\Delta_\beta-k^2)^{-1}$, we are going to describe the structure of the spectrum and scattering amplitudes of the limit

THEOREM 3.1. *The resolvent* $(-\Delta_\beta-k^2)^{-1}$ *of the limit operator is an integral operator on* $L^2(\Gamma)$ *with the kernel*

$$\Xi_k(x_i,y_j)=G_k(x_i,y_j)+\Lambda_{ij}(k^2)\exp(\mathrm{i}k(x_i+y_j)), \quad i,j=1,2,\ldots,n, \tag{8}$$

with $k^2\in\rho(-\Delta_\beta)$, $\Im k>0$, *and with* Λ *of the form*

$$\Lambda_{ij}(k^2)=\frac{\beta\Pi_{ij}}{1+\mathrm{i}k\beta B},$$

where

$$B:=\frac{1}{n}\left(\sum_{i=1}^n\vartheta_i\right)^2-\sum_{i=1}^n\vartheta_i^2, \quad \Pi_{ij}:=\left(\frac{1}{n}\sum_{\ell=1}^n\vartheta_\ell-\vartheta_i\right)\left(\frac{1}{n}\sum_{\ell=1}^n\vartheta_\ell-\vartheta_j\right). \tag{9}$$

In the relation (8),

$$G_k(x_i,y_j)=\frac{\mathrm{i}}{2k}\left[\delta_{ij}\exp(\mathrm{i}k|x_i-y_j|)+\left(\frac{2}{n}-\delta_{ij}\right)\exp(\mathrm{i}k(x_i+y_j))\right] \tag{10}$$

is the integral kernel of the resolvent of the free Hamiltonian $-\Delta_0$.

Proof. Let us first note that the claim, or that of Theorem 3.3 below, can be obtained easily referring to results in the literature. In order to make the text self-contained, we present a full proof first and comment on the alternative at the end of the section. According to Krein's formula, the sought Green's function is given by (8) with the matrix Λ to be found. Suppose that ψ solves the equation $(-\Delta_\beta-k^2)\psi=\phi$, then $\psi=(-\Delta_\beta-k^2)^{-1}\phi$ and relation (8) allows us to write the function ψ explicitly,

$$\psi(x_i)=\sum_{j=1}^n\int_0^\infty(G_k(x_i,y_j)+\Lambda_{ij}(k^2)\exp(\mathrm{i}k(x_i+y_j)))\phi(y_j)\,\mathrm{d}y_j.$$

Since the resolvent maps the whole space $L^2(\Gamma)$ onto the domain $\mathfrak{D}(-\Delta_\beta)$ of our operator, the function ψ has to satisfy the boundary conditions (7) at the vertex. Using the explicit form of G_k, we find that

$$\psi|_{x_i=0}=\sum_{j=1}^n\left(\frac{\mathrm{i}}{kn}+\Lambda_{ij}(k^2)\right)\int_0^\infty\phi(y_j)\exp(\mathrm{i}ky_j)\,\mathrm{d}y_j,$$

$$\psi'|_{x_i=0}=\sum_{j=1}^n\left(\mathrm{i}k\Lambda_{ij}(k^2)+\delta_{ij}-\frac{1}{n}\right)\int_0^\infty\phi(y_j)\exp(\mathrm{i}ky_j)\,\mathrm{d}y_j.$$

Substituting the above relations into (7), we get a system of equations,

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{0}^{\infty}\Big[\mathcal{A}_{\ell i}\left(\frac{\mathrm{i}}{kn}+\Lambda_{ij}(k^2)\right) \\ +\mathcal{B}_{\ell i}\left(\mathrm{i}k\Lambda_{ij}(k^2)+\delta_{ij}-\frac{1}{n}\right)\Big]\phi(y_j)\exp(\mathrm{i}ky_j)\,\mathrm{d}y_j=0$$

for $\ell=1,2,\dots,n$. Next, we require that the left-hand side vanishes for any ϕ, which yields the condition $\mathcal{A}\tilde{\Lambda}+\mathrm{i}k\mathcal{B}\tilde{\Lambda}+\mathcal{B}=0$, where $\tilde{\Lambda}_{ij}(k^2):=\Lambda_{ij}(k^2)+\frac{\mathrm{i}}{kn}$. This leads in a straightforward way to the following representation of the matrix $\tilde{\Lambda}(k^2)$,

$$\tilde{\Lambda}(k^2)=-(\mathcal{A}+\mathrm{i}k\mathcal{B})^{-1}\mathcal{B}.$$

Next, we apply the Gauss elimination method to get the chain of equivalences

$$(-(\mathcal{A}+\mathrm{i}k\mathcal{B})|\mathcal{B})\sim\cdots\sim(I|\underbrace{-(\mathcal{A}+\mathrm{i}k\mathcal{B})^{-1}\mathcal{B}}_{\tilde{\Lambda}(k^2)});$$

then by equivalent-row manipulations we pass to the matrix $(\mathcal{C}|\mathcal{D})$, where

$$\mathcal{C}=\mathrm{i}k\begin{pmatrix}\prod_{\ell=1}^{n}c_\ell & 0 & \dots & 0 & 0\\ 0 & \prod_{\ell=2}^{n}c_\ell & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & c_{n-1}c_n & 0\\ 0 & 0 & \dots & 0 & c_n\end{pmatrix},$$

and

$$\mathcal{D}=\begin{pmatrix} d_{11}\prod_{\ell=1}^{n-1}c_\ell & d_{12}\prod_{\ell=1}^{n-1}c_\ell & \dots & d_{1n-1}\prod_{\ell=1}^{n-1}c_\ell & d_{1n}\prod_{\ell=1}^{n-1}c_\ell\\ d_{21}\prod_{\ell=2}^{n-1}c_\ell & d_{22}\prod_{\ell=2}^{n-1}c_\ell & \dots & d_{2n-1}\prod_{\ell=2}^{n-1}c_\ell & d_{2n}\prod_{\ell=2}^{n-1}c_\ell\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ d_{n-11}c_{n-1} & d_{n-12}c_{n-1} & \dots & d_{n-1n-1}c_{n-1} & d_{n-1n}c_{n-1}\\ d_{n1} & d_{n2} & \dots & d_{nn-1} & d_{nn}\end{pmatrix}$$

with $c_j:=j+\mathrm{i}k\beta((\sum_{\ell=1}^{j}\vartheta_\ell)^2-j\sum_{\ell=1}^{j}\vartheta_\ell^2)$ and $d_{ij}:=-1+\mathrm{i}k\beta(n\Pi_{ij}-B)$. Consequently, we can divide each row of $(\mathcal{C}|\mathcal{D})$ by the corresponding diagonal element of $\mathcal{C}$. This yields the relation $(I|\tilde{\Lambda})$, where the entries of $\tilde{\Lambda}$ are given by the formula $\tilde{\Lambda}_{ij}=\frac{d_{ij}}{\mathrm{i}kc_n}$. Thus,

$$\Lambda_{ij}=\tilde{\Lambda}_{ij}+\frac{1}{\mathrm{i}kn}=\frac{1+\mathrm{i}k\beta B+d_{ij}}{\mathrm{i}kn(1+\mathrm{i}k\beta B)}=\frac{\beta\Pi_{ij}}{1+\mathrm{i}k\beta B},\tag{11}$$

which therefore completes the proof of the theorem. □

THEOREM 3.2. *The essential spectrum of $-\Delta_\beta$ is purely absolutely continuous and covers the nonnegative real axis, while the singularly continuous spectrum is empty,*

$$\sigma_{\mathrm{ess}}(-\Delta_\beta)=\sigma_{\mathrm{ac}}(-\Delta_\beta)=[0,\infty), \quad \sigma_{\mathrm{sc}}(-\Delta_\beta)=\emptyset.$$

If $\beta<0$, the operator $-\Delta_\beta$ has precisely one negative eigenvalue, namely its point spectrum $\sigma_{\mathrm{p}}(-\Delta_\beta)$ is

$$\sigma_{\mathrm{p}}(-\Delta_\beta)=\left\{-\frac{1}{\beta^2 B^2}\right\}.$$

If $\beta>0$, the limit operator has no eigenvalues,

$$\sigma_{\mathrm{p}}(-\Delta_\beta)=\emptyset, \quad \beta\notin(-\infty,0).$$

Proof. Since $(-\Delta_\beta-k^2)^{-1}-(-\Delta_0-k^2)^{-1}$, $k^2\in\rho(-\Delta_\beta)$, is of finite rank in view of (8), Weyl's essential spectrum theorem [10, Theorem XIII.14] implies that the essential spectrum of $-\Delta_\beta$ is not affected by the perturbation, i.e., $\sigma_{\mathrm{ess}}(-\Delta_\beta)=\sigma_{\mathrm{ess}}(-\Delta_0)=[0,\infty)$. Using (8) in combination with Theorem XIII.20 of [10], one can check easily the absence of $\sigma_{\mathrm{sc}}(-\Delta_\beta)$. The structure of the point spectrum of $-\Delta_\beta$ for negative β and the absence of negative eigenvalues for nonnegative one follow from the explicit meromorphic structure of the resolvent (8). On the other hand, we note that the pole in the right-hand side of (8) for $\beta>0$ corresponds to a resonance (antibound state). Finally, a short computation shows that the equation $-\Delta_\beta\psi=k^2\psi$ has no square integrable solutions for nonnegative k, which, in turn, leads to the absence of nonnegative eigenvalues for all real β. □

THEOREM 3.3. *For any momentum $k>0$, the on-shell scattering matrix $\mathcal{S}(k)$ for the pair $(-\Delta_\beta,-\Delta_0)$ takes the form*

$$\mathcal{S}_{ij}(k)=\frac{2}{n}-\delta_{ij}-\frac{2\mathrm{i}k\beta\Pi_{ij}}{1+\mathrm{i}k\beta B}$$

with Π_{ij} and B given by relations (9).

Proof. The scattering matrix can easily be obtained by substituting the scattering solution $\psi(x_i)=\delta_{ij}\exp(-\mathrm{i}kx_j)+\mathcal{S}_{ij}\exp(\mathrm{i}kx_j)$ into the matching conditions (7) which according to [8] yields

$$\mathcal{S}(k)=-(\mathcal{A}+\mathrm{i}k\mathcal{B})^{-1}(\mathcal{A}-\mathrm{i}k\mathcal{B}).$$

Reasoning in a way similar to the proof of Theorem 3.1, we get the chain of equivalences

$$(-(\mathcal{A}+\mathrm{i}k\mathcal{B})|(\mathcal{A}-\mathrm{i}k\mathcal{B}))\sim\cdots\sim(I\,|\underbrace{-(\mathcal{A}+\mathrm{i}k\mathcal{B})^{-1}(\mathcal{A}-\mathrm{i}k\mathcal{B})}_{\mathcal{S}(k)}).$$

Let the numbers c_j and the matrix Π be the same as in the mentioned proof and set

$$e_{ij} := \frac{2}{n}(c_n - \mathrm{i}kn^2\beta\Pi_{ij});$$

using again row manipulations we to pass to the matrix $(\mathcal{C}|\mathcal{E})$, where

$$\mathcal{E} = \mathrm{i}k \begin{pmatrix} (e_{11}-c_n)\prod_{\ell=1}^{n-1} c_\ell & e_{12}\prod_{\ell=1}^{n-1} c_\ell & \dots & e_{1n-1}\prod_{\ell=1}^{n-1} c_\ell & e_{1n}\prod_{\ell=1}^{n-1} c_\ell \\ e_{21}\prod_{\ell=2}^{n-1} c_\ell & (e_{22}-c_n)\prod_{\ell=2}^{n-1} c_\ell & \dots & e_{2n-1}\prod_{\ell=2}^{n-1} c_\ell & e_{2n}\prod_{\ell=2}^{n-1} c_\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n-11}c_{n-1} & e_{n-12}c_{n-1} & \dots & (e_{n-1n-1}-c_n)c_{n-1} & e_{n-1n}c_{n-1} \\ e_{n1} & e_{n2} & \dots & e_{nn-1} & e_{nn}-c_n \end{pmatrix}.$$

Finally, we divide each row of $(\mathcal{C}|\mathcal{E})$ by the corresponding diagonal entry of $\mathcal{C}$, obtaining thus $(I|\mathcal{S})$, where $\mathcal{S}$ is the sought scattering matrix and its entries are given by $\mathcal{S}_{ij} = \frac{e_{ij}}{c_n} - \delta_{ij}$. To complete the proof it is sufficient to use the explicit formulae for e_{ij} and c_n. $\square$

Let us now return to the alternative argument mentioned above. It was shown in [12] that the Green's function describing the general coupling (7) can be written in terms of pure waves weighted by the scattering amplitudes, namely as

$$G_k^{\mathcal{A},\mathcal{B}}(x_i, y_j) = \frac{\mathrm{i}}{2k}[\delta_{ij}\exp(\mathrm{i}k|x_i - y_j|) + \mathcal{S}_{ij}^{\mathcal{A},\mathcal{B}}(k)\exp(\mathrm{i}k(x_i + y_j))], \tag{12}$$

see also [9] for a more general formula. It implies that the proof of either Theorem 3.1 or Theorem 3.3 may be skipped, since it is a direct consequence of the other one and formula (12). In turn, the two proofs provide us with an alternative way to derive formula (12). Indeed, according to Krein's formula the sought Green's function is given by (8), and we find that the matrix $\Lambda^{\mathcal{A},\mathcal{B}}$ is given by

$$\Lambda^{\mathcal{A},\mathcal{B}}(k) = \frac{\mathrm{i}}{2k}\left(\mathcal{S}^{\mathcal{A},\mathcal{B}} + \mathcal{I} - \frac{2}{n}\mathcal{J},\right)$$

where $\mathcal{I}$ is an $n \times n$ identity matrix, while $\mathcal{J}$ is the matrix, whose all entries are one. From the proofs of Theorems 3.1, 3.3 one infers that

$$\Lambda^{\mathcal{A},\mathcal{B}}(k) = \frac{1}{2\mathrm{i}k}\left(2\mathrm{i}k\tilde{\Lambda}^{\mathcal{A},\mathcal{B}}(k) + \frac{2}{n}\mathcal{J}\right) = \frac{\mathrm{i}}{2k}\left(-(\mathcal{A}+\mathrm{i}k\mathcal{B})^{-1}(\mathcal{A}-\mathrm{i}k\mathcal{B}) + \mathcal{I} - \frac{2}{n}\mathcal{J}\right)$$

and the claim follows.

4. Convergence of the Resolvents and Spectra

This section is devoted to proof of the fact that the operator family $(-\Delta^{\varepsilon}-k^2)^{-1}$ approximates $(-\Delta_{\beta}-k^2)^{-1}$ in the uniform operator topology. We will do that by demonstrating that the kernel of $(-\Delta^{\varepsilon}-k^2)^{-1}$ approaches the kernel Ξ_k of $(-\Delta_{\beta}-k^2)^{-1}$ in $L^2(\Gamma)$ given by Theorem 3.1, which, in turn, makes it possible to verify the convergence of the corresponding operators in the Hilbert–Schmidt norm, and thus, a fortiori, in the uniform norm. To this aim, we are going to construct the resolvent $(-\Delta^{\varepsilon}-k^2)^{-1}$ explicitly; this resolvent allows us to determine the structure of the spectrum of the perturbed operator, and we show that the spectrum of $-\Delta^{\varepsilon}$ is close to that of $-\Delta_{\beta}$ for small ε. Our first main result reads

THEOREM 4.1. *As $\varepsilon \to 0$, the family of Hamiltonians $-\Delta^{\varepsilon}$ converges to $-\Delta_{\beta}$ in the norm-resolvent sense.*

Proof. To compare the resolvents of Δ^{ε} and Δ_{β}, fix $k := \mathrm{i}\varkappa$ belonging to the resolvent sets of both operators; as we shall see later from the explicit structure of the resolvents this can be achieved, e.g., by choosing $\varkappa > 0$ large enough.

We observe that the resolvent $(-\Delta^{\varepsilon}+\varkappa^2)^{-1}$ is an integral operator in $L^2(\Gamma)$, which has the kernel of the following form,

$$\begin{aligned}(-\Delta^{\varepsilon}+\varkappa^2)^{-1}(x_i, y_j) &= G_{\mathrm{i}\varkappa}(x_i, y_j)\\ &\quad - \zeta_{\varepsilon}((-\Delta_0+\varkappa^2)^{-1}V_{\varepsilon})(x_i)((-\Delta_0+\varkappa^2)^{-1}V_{\varepsilon})(y_j), \qquad (13)\end{aligned}$$

with $G_{\mathrm{i}\varkappa}$ from (10) being the Green's function of the free Hamiltonian $-\Delta_0$, and with the constant ζ_{ε} of the form

$$\zeta_{\varepsilon} := \left(\frac{\varepsilon^3}{\lambda(\varepsilon)} + \langle(-\Delta_0+\varkappa^2)^{-1}V_{\varepsilon}, V_{\varepsilon}\rangle_{\Gamma}\right)^{-1}.$$

This expression is obtained in the same way as in the particular case $n=2$, i.e., for point interactions on the line—see e.g., [2].

We start with the asymptotic behavior of the expression ζ_{ε} as $\varepsilon \to 0$. Using the Taylor expansion of $\exp(-\varepsilon\varkappa(x_i+y_i))$ together with the fact that V has a compact support and zero mean, one derives the formula

$$\begin{aligned}\langle(-\Delta_0+\varkappa^2)^{-1}V_{\varepsilon}, V_{\varepsilon}\rangle_{\Gamma} = \frac{\varepsilon^2}{2\varkappa}\Bigg[&\sum_{i=1}^{n}\int_0^{\infty}\int_0^{\infty} V(x_i)V(y_i)\exp(-\varepsilon\varkappa|x_i-y_i|)\,\mathrm{d}x_i\mathrm{d}y_i\\ &-\sum_{i=1}^{n}\int_0^{\infty}\int_0^{\infty} V(x_i)V(y_i)\exp(-\varepsilon\varkappa(x_i+y_i))\,\mathrm{d}x_i\mathrm{d}y_i\end{aligned}$$

$$+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty\int_0^\infty V(x_i)V(y_j)\exp(-\varepsilon\varkappa(x_i+y_j))\,\mathrm{d}x_i\mathrm{d}y_i\Bigg]$$
$$=-A\varepsilon^3+\varkappa B\varepsilon^4+\mathcal{O}(\varepsilon^5)\quad\text{as }\varepsilon\to 0.$$

Recall that the constants A and B are defined via formulae (4) and (9), respectively. We, thus conclude that

$$\zeta_\varepsilon=\frac{1}{\varepsilon^3\left(\frac{1}{\lambda_0}-A+\varepsilon(\varkappa B-\frac{\lambda_1}{\lambda_0^2})\right)}+\mathcal{O}\left(\frac{1}{\varepsilon^2}\right),\quad \varepsilon\to 0,$$

and finally, since $\beta=\frac{\lambda_0^2}{\lambda_1}$ when $\lambda_0=\frac{1}{A}$, that

$$\zeta_\varepsilon=\frac{-\beta}{\varepsilon^4(1-\varkappa\beta B)}+\mathcal{O}\left(\frac{1}{\varepsilon^3}\right),\quad \varepsilon\to 0. \tag{14}$$

In the next step, we have to discuss the asymptotical behavior of the functions $((-\Delta_0+\varkappa^2)^{-1}V_\varepsilon)(x_i)((-\Delta_0+\varkappa^2)^{-1}V_\varepsilon)(y_j)$. In a similar manner as above we find that

$$((-\Delta_0+\varkappa^2)^{-1}V_\varepsilon)(x_i)=\sum_{j=1}^{n}\int_0^\infty G_{\mathrm{i}\varkappa}(x_i,y_j)V_\varepsilon(y_j)\,\mathrm{d}y_j$$
$$=-\varepsilon^2\exp(-\varkappa x_i)\left[\sum_{j=1}^{n}\left(\frac{1}{n}-\delta_{ij}\right)\vartheta_j+\mathcal{O}(\varepsilon)\right],\quad \varepsilon\to 0,$$

which, in turn, gives the desired relation,

$$\begin{aligned}&((-\Delta_0+\varkappa^2)^{-1}V_\varepsilon)(x_i)((-\Delta_0+\varkappa^2)^{-1}V_\varepsilon)(y_j)\\&\quad=\varepsilon^4\exp(-\varkappa(x_i+y_j))\left(\Pi_{ij}+\mathcal{O}(\varepsilon)\right),\quad \varepsilon\to 0.\end{aligned} \tag{15}$$

Combining (13)–(15) we find that

$$\begin{aligned}(-\Delta^\varepsilon+\varkappa^2)^{-1}(x_i,y_j)&=G_{\mathrm{i}\varkappa}(x_i,y_j)+\exp(-\varkappa(x_i+y_j))\left(\frac{\beta\Pi_{ij}}{1-\varkappa\beta B}+\mathcal{O}(\varepsilon)\right),\\&=G_{\mathrm{i}\varkappa}(x_i,y_j)+\exp(-\varkappa(x_i+y_j))(\Lambda_{ij}(-\varkappa^2)+\mathcal{O}(\varepsilon))\end{aligned}$$

holds as $\varepsilon\to 0$. This allows us to conclude that the kernel $(-\Delta^\varepsilon+\varkappa^2)^{-1}(x_i,y_j)$ of the approximating operator resolvent converges as $\varepsilon\to 0$ to the kernel $\Xi_{\mathrm{i}\varkappa}(x_i,y_j)$ pointwise. We further observe that the function $(-\Delta^\varepsilon+\varkappa^2)^{-1}(x_i,y_j)$ decays exponentially, hence the integral expressing its norm converges and the dominated convergence theorem implies that this function tends to the kernel $\Xi_{\mathrm{i}\varkappa}(x_i,y_j)$ in $L^2(\Gamma\times\Gamma)$. From this, the resolvent converges in the Hilbert–Schmidt norm follows, i.e.,

$$\lim_{\varepsilon\to 0}\|(-\Delta^\varepsilon+\varkappa^2)^{-1}-(-\Delta_\beta+\varkappa^2)^{-1}\|_2=0,$$

and thus, a fortiori, the family $\{-\Delta^\varepsilon\}_{\varepsilon\geq 0}$ approximates $-\Delta_\beta$ in the norm-resolvent topology. □

As an immediate corollary of Theorem 4.1 we get the following result.

THEOREM 4.2. (i) *The essential spectrum of $-\Delta^\varepsilon$ is purely absolutely continuous and covers the nonnegative real axis, while the singularly continuous spectrum is empty.*

(ii) *If $\beta<0$, the operator $-\Delta^\varepsilon$ has for all ε small enough exactly one negative eigenvalue $-\varkappa_\varepsilon^2$ with the asymptotic behavior*

$$-\varkappa_\varepsilon^2 = -\frac{1}{\beta^2 B^2} + \mathcal{O}(\varepsilon)\,, \quad \varepsilon\to 0\,,$$

which tends to the eigenvalue of the limit operator. If $\beta>0$, the perturbed operator has no eigenvalues.

(iii) *If $\beta=0$, then there are two possibilities. If $\lambda_0<\frac{1}{A}$, the approximating operator has for all ε small enough exactly one negative eigenvalue $-\varkappa_\varepsilon^2$ with the asymptotics*

$$-\varkappa_\varepsilon^2 = -\frac{(A-\frac{1}{\lambda_0})^2}{\varepsilon^2 B^2} + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \quad \varepsilon\to 0\,,$$

tending to $-\infty$. In the opposite case, the operator $-\Delta^\varepsilon$ has no eigenvalues.

Proof. The arguments used in the proof of Theorem 3.2 together with the precise structure of the resolvent $(-\Delta^\varepsilon-k^2)^{-1}$ give the first statement of the theorem. To obtain the remaining two claims, we only need to observe that the resolvent $(-\Delta^\varepsilon+\varkappa^2)^{-1}$ has only one pole at $\varkappa_\varepsilon$ admitting the asymptotics

$$\varkappa_\varepsilon = \frac{1}{B}\left(\frac{A-\frac{1}{\lambda_0}}{\varepsilon} + \frac{\lambda_1}{\lambda_0^2} + \mathcal{O}(\varepsilon)\right), \quad \varepsilon\to 0\,,$$

and that the operator Δ^ε can have only negative eigenvalues. □

5. Convergence of the Scattering Matrices

In the final section, we investigate stationary scattering for the pair $(-\Delta^\varepsilon, -\Delta_0)$. Our aim is to show that the corresponding scattering amplitudes are close to those for the pair $(-\Delta_\beta, -\Delta_0)$ in the limit $\varepsilon\to 0$. We consider the incoming monochromatic wave $\exp(-ikx_i)$ approaching the vertex along the edge γ_i. The corresponding scattering solution ψ_i^ε has to solve the problem

$$-\psi'' + \frac{\lambda(\varepsilon)}{\varepsilon^3} V_\varepsilon(x)\langle\psi, V_\varepsilon\rangle_\Gamma = k^2\psi \quad \text{on } \Gamma, \quad \psi\in K(\Gamma)\,, \tag{16}$$

and, by virtue of the compactness of the support of V, it takes the form

$$\psi_i^\varepsilon(x_j)=\delta_{ij}\exp(-\mathrm{i}kx_j)+\mathcal{S}_{ij}^\varepsilon(k)\exp(\mathrm{i}kx_j)\,,\quad x_j\ge 1.$$

Hence to solve the scattering problem for the Hamiltonian $-\Delta^\varepsilon$, we need to analyze the behavior of the amplitudes $\mathcal{S}_{ij}^\varepsilon(k)$ as the scaling parameter ε approaches zero.

The integro-differential equation (16) for the scattering solution ψ_i^ε can easily be reformulated as an integral equation using the variation-of-constants method,

$$\begin{aligned}\psi_i^\varepsilon(x_j)=&-\frac{\lambda(\varepsilon)\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma}{k\varepsilon^3}\int_{x_j}^{\varepsilon}V_\varepsilon(y_j)\sin k(x_j-y_j)\,\mathrm{d}y_j\\&+\delta_{ij}\exp(-\mathrm{i}kx_j)+\mathcal{S}_{ij}^\varepsilon(k)\exp(\mathrm{i}kx_j).\end{aligned}\tag{17}$$

Noting that

$$\psi_i|_{x_j=0}=\frac{\lambda(\varepsilon)\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma}{k\varepsilon^3}\int_0^\varepsilon V_\varepsilon(y_j)\sin ky_j\,\mathrm{d}y_j+\delta_{ij}+\mathcal{S}_{ij}^\varepsilon(k)\,,$$

$$\psi_i'|_{x_j=0}=-\frac{\lambda(\varepsilon)\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma}{\varepsilon^3}\int_0^\varepsilon V_\varepsilon(y_j)\cos ky_j\,\mathrm{d}y_j-\mathrm{i}k\delta_{ij}+\mathrm{i}k\mathcal{S}_{ij}^\varepsilon(k),$$

we substitute these relations into the Kirchhoff matching conditions to conclude that

$$\begin{aligned}\mathcal{S}_{ij}^\varepsilon(k)=&\frac{\lambda(\varepsilon)\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma}{k\varepsilon^3}\left[-\int_0^\varepsilon V_\varepsilon(y_j)\sin ky_j\,\mathrm{d}y_j\right.\\&\left.+\frac{1}{\mathrm{i}n}\sum_{\ell=1}^n\int_0^\varepsilon V_\varepsilon(y_\ell)\exp(\mathrm{i}ky_\ell)\,\mathrm{d}y_\ell\right]+\frac{2}{n}-\delta_{ij}\\=&\frac{\lambda(\varepsilon)\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma}{\varepsilon}\left[\sum_{\ell=1}^n\left(\frac{1}{n}-\delta_{\ell j}\right)\vartheta_\ell+\mathcal{O}(\varepsilon)\right]+\frac{2}{n}-\delta_{ij}.\end{aligned}\tag{18}$$

Comparing formulæ (17) and (18), we derive the following Fredholm integral equation (with a degenerate kernel) for the scattering solution,

$$\psi_i(x_j)=\langle\psi_i^\varepsilon,V_\varepsilon\rangle_\Gamma W(x_j)+F(x_j),$$

where

$$\begin{aligned}W(x_j)=\frac{\lambda(\varepsilon)}{2\mathrm{i}k\varepsilon^3}&\left[\int_{x_j}^\varepsilon V_\varepsilon(y_j)\exp(\mathrm{i}k(y_j-x_j))\,\mathrm{d}y_j+\int_0^{x_j}V_\varepsilon(y_j)\exp(\mathrm{i}k(x_j-y_j))\,\mathrm{d}y_j\right.\\&\left.+\sum_{\ell=1}^n\left(\frac{2}{n}-\delta_{\ell j}\right)\int_0^\varepsilon V_\varepsilon(y_\ell)\exp(\mathrm{i}k(x_j+y_\ell))\,\mathrm{d}y_\ell\right]\end{aligned}$$

and

$$F(x_j) = -2\mathrm{i}\delta_{ij}\sin kx_j + \frac{2}{n}\exp(\mathrm{i}kx_j);$$

then as an immediate consequence of this fact, we get

$$\langle \psi_i^{\varepsilon}, V_{\varepsilon}\rangle_{\Gamma} = \left[\sum_{j=1}^{n}\int_0^{\varepsilon} F(x_j)V_{\varepsilon}(x_j)\,\mathrm{d}x_j\right]\left[1-\sum_{j=1}^{n}\int_0^{\varepsilon} W(x_j)V_{\varepsilon}(x_j)\,\mathrm{d}x_j\right]^{-1}. \tag{19}$$

Next, we are going to find the asymptotic behavior of the quantity $\langle \psi_i^{\varepsilon}, V_{\varepsilon}\rangle_{\Gamma}$ defined by (19), which will be used further to get the asymptotics of the scattering amplitudes via (18). To this end, we first denote the numerator of (19) as N and analyze its asymptotic behavior,

$$\begin{aligned} N &= -2\mathrm{i}\int_0^{\varepsilon} V_{\varepsilon}(x_i)\sin kx_i\,\mathrm{d}x_i + \frac{2}{n}\sum_{j=1}^{n}\int_0^{\varepsilon} V_{\varepsilon}(x_j)\exp(\mathrm{i}kx_j)\,\mathrm{d}x_j \\ &= -2\mathrm{i}\varepsilon\left[\int_0^1 V(x_i)\sin k\varepsilon x_i\,\mathrm{d}x_i + \frac{\mathrm{i}}{n}\sum_{j=1}^{n}\int_0^1 V(x_j)\exp(\mathrm{i}k\varepsilon x_j)\,\mathrm{d}x_j\right] \\ &= 2\mathrm{i}k\varepsilon^2\sum_{j=1}^{n}\left(\frac{1}{n}-\delta_{ij}\right)\vartheta_j + \mathcal{O}(\varepsilon^3), \quad \varepsilon\to 0. \end{aligned}$$

Then the denominator of (19) can be written as $1-D$, where D behaves as follows,

$$\begin{aligned} D = &\frac{\lambda(\varepsilon)}{2\mathrm{i}k\varepsilon^3}\left[\sum_{j=1}^{n}\int_0^{\varepsilon}\int_0^{\varepsilon} V_{\varepsilon}(x_j)V_{\varepsilon}(y_j)\exp(\mathrm{i}k|x_j-y_j|)\,\mathrm{d}x_j\mathrm{d}y_j\right. \\ &\left.+\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(\frac{2}{n}-\delta_{\ell j}\right)\int_0^{\varepsilon}\int_0^{\varepsilon} V_{\varepsilon}(x_j)V_{\varepsilon}(y_\ell)\exp(\mathrm{i}k(x_j+y_\ell))\,\mathrm{d}x_j\mathrm{d}y_\ell\right] \\ = &\frac{\lambda(\varepsilon)}{2\mathrm{i}k\varepsilon}\left[\sum_{j=1}^{n}\int_0^1\int_0^1 V(x_j)V(y_j)(\exp(\mathrm{i}k\varepsilon|x_j-y_j|)-\exp(\mathrm{i}k\varepsilon(x_j+y_j)))\,\mathrm{d}x_j\mathrm{d}y_j\right. \\ &\left.+\frac{2}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\int_0^1\int_0^1 V(x_j)V(y_\ell)\exp(\mathrm{i}k\varepsilon(x_j+y_\ell))\,\mathrm{d}x_j\mathrm{d}y_\ell\right] \\ = &\lambda(\varepsilon)(A+\mathrm{i}k\varepsilon B+\mathcal{O}(\varepsilon^2)), \quad \varepsilon\to 0, \end{aligned}$$

with the usual definitions of the constants A and B. Combining the above asymptotic formulæ for N and D along with (19), we finally conclude that the scattering amplitude $\mathcal{S}_{ij}^{\varepsilon}(k)$ has the following asymptotic behavior,

$$\mathcal{S}_{ij}^{\varepsilon} = \frac{2}{n} - \delta_{ij} + \frac{2\mathrm{i}k\varepsilon\lambda(\varepsilon)}{1-\lambda(\varepsilon)(A+\mathrm{i}k\varepsilon B)} \times \left[\sum_{\ell=1}^{n}\left(\frac{1}{n}-\delta_{\ell i}\right)\vartheta_\ell\right]\left[\sum_{\ell=1}^{n}\left(\frac{1}{n}-\delta_{\ell j}\right)\vartheta_\ell\right] + \mathcal{O}(\varepsilon) = \mathcal{S}_{ij} + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,$$

where the limit values $\mathcal{S}_{ij}$ are defined in Theorem 3.3. In this way, we have shown

THEOREM 5.1. *For any momentum $k > 0$, the on-shell scattering matrix for the pair $(-\Delta^{\varepsilon}, -\Delta_0)$ converges as $\varepsilon \to 0$ to that of $(-\Delta_\beta, -\Delta_0)$, and moreover, there is a constant C such that*

$$\|\mathcal{S}^{\varepsilon}(k) - \mathcal{S}(k)\| \le C\varepsilon, \quad \varepsilon \in (0,1],$$

where $\|\cdot\|$ stands for the operator norm of the matrix.

6. Concluding Remarks

First of all, let us address a natural question inspired by the above remarks, namely whether the formula (12) remains valid generally for Schrödinger operator on the graph with rank-one perturbations,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda V(x)\langle\cdot, V\rangle_\Gamma. \tag{20}$$

If this was the case, Theorems 4.1 and 5.1 could imply each other each other. Let us suppose that $\lambda \in \mathbb{R}$ and $V \in L^1_{\mathrm{loc}}(\Gamma)$ is of compact support. In a similar manner as in the previous section, we find that the scattering matrix $\mathcal{S}^V$ corresponding to (20) can be expressed as

$$\mathcal{S}_{ij}^{V}(k) = \frac{\lambda N}{k(D-1)}\left[\int_0^\infty V(y_j)\sin ky_j\,\mathrm{d}y_j + \frac{\mathrm{i}}{n}\sum_{\ell=1}^{n}\int_0^\varepsilon V(y_\ell)\exp(\mathrm{i}ky_\ell)\,\mathrm{d}y_\ell\right] + \frac{2}{n} - \delta_{ij},$$

where

$$N := -2\mathrm{i}\int_0^\infty V(x_i)\sin kx_i\,\mathrm{d}x_i + \frac{2}{n}\sum_{j=1}^{n}\int_0^\infty V(x_j)\exp(\mathrm{i}kx_j)\,\mathrm{d}x_j$$

and

$$D := \frac{\lambda}{2\mathrm{i}k}\sum_{j=1}^{n}\int_0^\infty V(x_j)\left[\int_0^\infty V(y_j)(\exp(\mathrm{i}k|x_j-y_j|) - \exp(\mathrm{i}k(x_j+y_j)))\,\mathrm{d}y_j + \frac{2}{n}\sum_{\ell=1}^{n}\int_0^\infty V(y_\ell)\exp(\mathrm{i}k(x_j+y_\ell))\,\mathrm{d}y_\ell\right]\mathrm{d}x_j.$$

On the other hand, the Green's function G_k^V corresponding to (20) can be expressed as

$$G_k^V(x_i, y_j) = G_k(x_i, y_j) - \frac{\lambda((-\Delta_0 - k^2)^{-1}V)(x_i)((-\Delta_0 - k^2)^{-1}V)(y_j)}{1 + \lambda\langle(-\Delta_0 - k^2)^{-1}V, V\rangle_\Gamma}.$$

It follows from (10) that

$$((-\Delta_0 - k^2)^{-1}V)(x_i) = \frac{\mathrm{i}}{2k}\left[\int_0^\infty V(y_i)(\exp(\mathrm{i}k|x_i - y_i|) - \exp(\mathrm{i}k(x_i + y_i)))\,\mathrm{d}y_i + \frac{2}{n}\sum_{j=1}^n \int_0^\infty V(y_j)\exp(\mathrm{i}k(x_i + y_j))\,\mathrm{d}y_j\right],$$

and consequently, $\langle(-\Delta_0 - k^2)^{-1}V, V\rangle_\Gamma = -D/\lambda$. Should formula (12) hold in this case, the above results would yield the equality

$$\int_{x_i}^\infty V(y_i)\sin k(y_i - x_i)\,\mathrm{d}y_i = 0,$$

which in general does not hold. It holds, however, asymptotically, i.e., in the limit $\varepsilon \to 0$, in the special case of the potentials considered in the previous sections.

The second thing to mention is the meaning of the obtained δ'-interaction. We have mentioned that such interactions are interesting in view of their particular scattering properties, being most transparent at low energies. We do not know how to realize an exact δ' physically, but using approximation results we are able to simulate such a behavior over large intervals of energy. We have recalled also that a δ' on the line can have different generalizations to a graph. The most common is the 'symmetrized' one introduced in [5], characterized by the matching conditions

$$\psi_i'(0) = \psi_j'(0) =: \psi'(0), \quad 1 \le i < j \le n, \qquad \sum_{\ell=1}^n \psi_\ell(0) = \beta\psi'(0). \tag{21}$$

An attentive reader would notice, however, that there is another extension introduced in [5], namely, the one described by the conditions

$$\psi_i(0) - \psi_j(0) = \frac{\beta}{n}(\psi_i'(0) - \psi_j'(0)), \quad 1 \le i < j \le n, \qquad \sum_{\ell=1}^n \psi_\ell'(0) = 0, \tag{22}$$

with $\beta \in \mathbb{R}$; the result of this paper provides still another generalization.

Both the couplings (21) and (22), as well as (5), yield operators with the essential spectrum which is purely absolutely continuous and covers the nonnegative real axis, while the singularly continuous one is empty. For a negative β, the couplings (21) and (22) produce a single eigenvalue, namely $-\frac{n^2}{\beta^2}$, of a different multiplicity, equal one and $n-1$ in the two cases, respectively. In the former case, the corresponding eigenfunction is $\exp(nx_j/\beta)$ on the jth edge. In the case of (22), the jth eigenfunction is $\exp(nx_j/\beta)$ on the jth edge, $-\exp(nx_n/\beta)$ on the nth edge and vanishes elsewhere, for $j = 1, \dots, n-1$.

The new coupling (5) leads to a simple eigenvalue $-\frac{1}{\beta^2 B^2}$—cf. Theorem 3.2—the corresponding eigenfunction is $(\sum_{\ell=1}^n \vartheta_\ell - n\vartheta_j)\exp(-x_j/(\beta B))$ on the jth edge. Note that even when the system has some symmetries, i.e., when some of the parameters ϑ_j coincide, the structure of the spectrum is preserved, the only exception being the case when all the parameters ϑ_j are equal mutually; then the eigenvalue disappears, since then couplings (5) reduces then to the free (Kirchhoff) coupling.

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