Electron trapping by a current vortex

F. Bentosela,[a,b] P. Exner,[c,d] and V.A. Zagrebnov[a,b]

a) Centre de Physique Théorique, C.N.R.S., F–13288 Marseille Luminy
b) Université de la Mediterranée (Aix–Marseille II), F–13288 Luminy
c) Nuclear Physics Institute, Academy of Sciences, CZ–25068 Řež near Prague,
d) Doppler Institute, Czech Technical University, Břehová 7, CZ-11519 Prague,
bentosela@cpt.univ-mrs.fr, exner@ujf.cas.cz, zagrebnov@cpt.univ-mrs.fr

Abstract

We investigate an electron in the plane interacting with the magnetic field due to an electric current forming a localized rotationally symmetric vortex. We show that independently of the vortex profile an electron with spin antiparallel to the magnetic field can be trapped if the vortex current is strong enough. In addition, the electron scattering on the vortex exhibits resonances for any spin orientation. On the other hand, in distinction to models with a localized flux tube the present situation exhibits no bound states for weak vortices.

Interaction of charged particles with a localized magnetic field has been a subject of interest for a long time, both from the theoretical and experimental point of view — see, *e.g.*, [GBG, Št] and references therein. Such a field can have different sources, for instance, it may be induced by an electric current having one or more vortices. A lot of attention was payed in the last decade to vortex bound states in superconductors whose dynamics is governed by the Bogoliubov–de Gennes equation — *cf.* [HRD, SHDS, GS] and a bibliography given in [HIM].

Another, much simpler example involves a Pauli electron interacting with a flux tube modelling a vortex magnetic field — it is is appealing, in particular, since it has been observed that vortices appear often in the probability current associated with mesoscopic transport — see [EŠSF] and the literature there. The system of a tube and an electron has been investigated in a fresh paper by Cavalcanti et al. [CFC] who, however, seem to be unaware of another recent studies of the problem — *cf.* [Mo] and references therein. In these papers, the field is assumed to be constant within a circle and zero otherwise — the conclusion is then that in one spin state the electron can be always trapped by the vortex, independently of the magnetic flux value, as long as the effective gyromagnetic factor $g^* > 2$. This covers the physically important case of a free electron with $g^* = 2.0023$.

However, this claim depends substantially on the used magnetic field Ansatz. To illustrate this point we analyze in this letter the situation where the *vortex current distribution* represents the input. On one hand we are able to generalize the result of [Mo, CFC] by showing that vortices can trap electrons independently of their profile, and what is physically equally important, that they cause resonances in the electron–vortex scattering which get sharper as the vortex strength grows. On the other hand, the trapping needs in this more realistic setting certain minimum vortex strength (measured by the total circulating current or, say, the dipole moment of the field) to occur. To make things simpler we suppose that the vortex is *centrally symmetric*. We are convinced that the symmetry is not vital for the conclusions, but its absence makes the treatment technically much more complicated and we postpone the discussion of the general case to another paper.

The dynamics of a nonrelativistic electron in the plane exposed to a perpendicular but nonhomogeneous magnetic field is given by the respective Pauli equation, *i.e.*, by the Hamiltonian

$$H \,=\, \frac{1}{2m^*}\,\left(-i\hbar\vec\nabla \,+\, \frac{e}{c}\vec A(\vec x)\right)^2 +\, \frac12\, g\mu_B\sigma_3 B(\vec x)\,, \tag{1}$$

where m^* is the effective electron mass, g the gyromagnetic factor, $\mu_B = e\hbar/2m_e c$ the Bohr magneton, and the sign choice corresponds to the negative charge $-e$ of the electron. In the rational units, $2m^* = \hbar = c = 1$ this becomes

$$H \,=\, \left(-i\vec\nabla \,+\, e\vec A(\vec x)\right)^2 +\, \frac12\, g^* e\sigma_3 B(\vec x) \tag{2}$$

with the effective gyromagnetic factor $g^* \,=\, g\,\frac{m^*}{m_e}$. Since H is matrix–diagonal, each spin state can be treated separately. Moreover, we shall consider the situation when the magnetic field is rotationally symmetric. This allows us to perform the partial–wave decomposition and to replace H by the family of operators

$$H_\ell^{(\pm)} \,=\, -\,\frac{d^2}{dr^2} \,-\, \frac1r\,\frac{d}{dr} \,+\, V_\ell^{(\pm)}(r)\,, \quad V_\ell^{(\pm)}(r) := \left(eA(r) + \frac{\ell}{r}\right)^2 \pm\, \frac12\, g^* eB(r) \tag{3}$$

on $L^2(I\!R^+, r\,dr)$. To determine their spectral properties, we have to specify the involved functions, the angular component $A(r)$ of the vector potential and the related magnetic field, $B(r) = A'(r) + r^{-1}A(r)$.

We have said that they correspond a circulating electric current in the plane, which is supposed to be anticlockwise and having the angular component only, $\vec J(\vec x) \,=\, \lambda\delta(z)J(r)\vec e_\varphi$. Here r, φ are the polar coordinates in the plane; the total current is $\lambda\int_0^\infty J(r)\,dr$. The positive parameter λ can be, of course, absorbed into J; we introduce it as a tool to control the vortex "strength". The current density is supposed to obey the following modest requirements:

(i) J is C^2 smooth and non–negative, $J(r) \ge 0$,

(ii) at the origin $J(r) = ar^2 + \mathcal{O}(r^3)$,

(iii) at large distances $J(r) = \mathcal{O}(r^{-3-\epsilon})$ for some $\epsilon > 0$.

The vector potential in the plane $z=0$ is then also anticlockwise; its magnitude

$$A(r) \,=\, \lambda \int_0^{\infty} dr' r' J(r') \int_0^{2\pi} \frac{\cos\varphi'\, d\varphi'}{(r^2 + r'^2 - 2rr'\cos\varphi')^{1/2}} \tag{4}$$

is obtained by summing the contribution from all the circular current lines [Ja],

$$A(r) \,=\, \lambda \int_0^{\infty} J(r')\, \frac{4r}{r+r'}\, \frac{(2-\rho^2)K(\rho^2) - 2E(\rho^2)}{\rho^2}\, dr'\,, \tag{5}$$

where $\rho^2 := \frac{4rr'}{(r+r')^2}$ and $K,\,E$ are the full elliptic integrals of the first and the second kind, respectively. Using [AS, 17.3.29,30] (pay attention to a misprint there), one can also cast (5) into the form

$$A(r) \,=\, 4\lambda \int_0^{\infty} J(r')\, \frac{r'}{r_<} \left[K\left(\frac{r_<^2}{r_>^2}\right) - E\left(\frac{r_<^2}{r_>^2}\right)\right] dr'\,, \tag{6}$$

where we have used the usual shorthands, $r_< := \min(r,r')$ and $r_> := \max(r,r')$; the same result follows directly from (4) and [GR, 3.674.3]. In view of *(i)* the integral is finite for every r, because $E(\zeta)$ is regular at $\zeta=1$ and $K(\zeta)$ has a logarithmic singularity there.

Let us denote the Pauli Hamiltonian (2) with the vector potential (6) by $H(\lambda)$; the symbol $H_\ell^{(\pm)}(\lambda)$ will be used for its spin and orbital momentum components. Our main result is then the following:

Theorem. Under the stated assumptions, $\sigma(H(\lambda)) = [0,\infty)$ for $|\lambda|$ small enough. On the other hand, $H_0^{(-)}(\lambda)$ has a negative eigenvalue for a sufficiently large λ.

Proof: The first claim has to be checked for any $H_\ell^{(\pm)}(\lambda)$. By *(i)*, the potentials $V_\ell^{(\pm)}$ are C^1 smooth; they decay at infinity as

$$V_\ell^{(\pm)}(r) \,=\, \frac{\ell^2}{r^2} \,+\, \lambda e m\, \frac{2\ell \mp g^*}{r^3} \,+\, \mathcal{O}(r^{-3-\epsilon})\,, \tag{7}$$

where $m := \pi \int_0^\infty J(r')\, r'^2 dr'$ is the dipole moment of the current for $\lambda = 1$ — *cf.* [Ja]. Consequently, $\sigma_{ess}(H_\ell^{(\pm)}(\lambda)) = [0,\infty)$ following [RS, Sec. XIII.4]. We rewrite the potentials into the form

$$V_\ell^{(\pm)}(r) \,=\, \left(\lambda e A_1(r) + \frac{\ell}{r}\right)^2 \,\pm\, \frac{\lambda}{2}\, g^* e B_1(r)\,, \tag{8}$$

where the the indexed magnetic field refers to the value $\lambda = 1$. Since $H_\ell^{(\pm)}(\lambda)$ is just the s–wave part of the two–dimensional Schrödinger operator with the centrally symmetric potential (8), it is sufficient to check that the latter has no negative eigenvalues. If $\ell = 0$ the potential decay allows us to apply Thm. 3.4 of [Si] by which a negative eigenvalue exists for small positive λ if and only if $\int_0^\infty V_\ell^{(\pm)}(r)\, r\, dr \le 0$.

However, the flux through the circle of radius r is $2\pi\int_0^r B(r')\,r'dr' = 2\pi r A(r)$, so the second term in (8) does not contribute and the integral is determined by the first one which is positive for any nonzero λ. If $\ell\neq 0$ the decay is too slow, but this difficulty is easily overcome. We replace the first term, *e.g.*, by $\left(\lambda e A_1(r)+\frac{\ell}{r}\right)^2\Theta(r_0-r)$ with a positive r_0 and obtain the absence of a negative eigenvalue for a small λ; the same is by the minimax principle true for the original operator.

For the existence claim the behaviour of the potential around the origin is vital. We shall write the vector potential in the form

$$A(r)\,=\,\lambda\mu r+\alpha_0(r)\,,\qquad \mu\,:=\,\int_0^\infty J(r')\,\frac{dr'}{r'}\,. \tag{9}$$

The behaviour of α_0 follows from (6) and the following estimate on the difference of the elliptic integrals,

$$\frac{\pi\zeta}{4}\,\le\,K(\zeta)-E(\zeta)\,\le\,\frac{\pi}{2}\left(\frac{\zeta}{8}-\frac{3}{8}\ln(1-\zeta)+b\zeta^2\right) \tag{10}$$

for a sufficiently large $b>0$, which is a straightforward consequence of [AS, 17.3.11,12] and [GR, 8.123.2]. On the lower side we thus get

$$\alpha_0(r)\,\ge\,\frac{\pi}{r^2}\int_0^r J(r')\,r'^2dr'\,-\,\pi r\int_0^r J(r')\,\frac{dr'}{r'}\,\ge\,-\,\frac{3}{4}\,\pi a r^2+\mathcal{O}(r^3)\,.$$

The upper bound is similar. In view of *(i)–(iii)*, $J(r)\le\tilde a r$ for a suitable $\tilde a$; in the logarithmic term we employ the Taylor expansion which gives

$$\frac{3\pi\tilde a}{4r}\int_r^\infty\sum_{j=2}^\infty\frac{r^{2j}}{(r')^{2j-2}}\,dr'\,=\,\frac{3\pi\tilde a}{4}\,r^2\sum_{j=2}^\infty\frac{1}{j(2j-3)}\,.$$

Together we find $\alpha_0(r)=\mathcal{O}(r^2)$. This further implies

$$B(r)\,=\,2\lambda\mu+\beta_0(r)\,,\qquad \beta_0(r)\,:=\,\alpha_0'(r)+\frac{1}{r}\alpha_0(r)\,\in\mathcal{O}(r)\,. \tag{11}$$

The operator $H_0^{(-)}(\lambda)$ can be therefore written as

$$H_0^{(-)}(\lambda)\;=\;-\,\frac{d^2}{dr^2}\,-\,\frac{1}{r}\,\frac{d}{dr}\,+\,(\lambda e(\mu r+\alpha_0(r)))^2-\frac{1}{2}\,\lambda e g^*(2\mu+\beta_0(r))\,. \tag{12}$$

Using the rescaled variable $u:=r\sqrt{\lambda}$ we find it to be unitarily equivalent to the operator

$$\lambda A_\lambda\qquad\text{with}\qquad A_\lambda=A_0+W_\lambda \tag{13}$$

on $L^2(I\!\!R^+,u\,du)$, where

$$A_0\,:=\,-\,\frac{d^2}{du^2}\,-\,\frac{1}{u}\,\frac{d}{du}\,-g^*e\mu+e^2\mu^2u^2 \tag{14}$$

and

$$W_\lambda(u) \,:=\, 2\sqrt{\lambda}e^2\mu\, u\alpha_0\left(\frac{u}{\sqrt{\lambda}}\right) \,+\, \lambda e^2\,\alpha_0^2\left(\frac{u}{\sqrt{\lambda}}\right) \,-\, \frac{1}{2}\,g^*e\,\beta_0\left(\frac{u}{\sqrt{\lambda}}\right). \tag{15}$$

We have clearly $\sigma_{ess}(A_\lambda)=\sigma_{ess}(\lambda A_\lambda)=[0,\infty)$ for any $\lambda>0$, while A_0 as the s–wave part of the two–dimensional harmonic oscillator has a purely discrete spectrum. Despite the fact that the perturbation is large (the maximum of W_λ grows linearly with λ), one may attempt to employ the *asymptotic* perturbation theory.

Unfortunately, Thms. VIII.3.11,13 of [Ka] cannot be applied directly, because the family $\{W_\lambda\}$ is not monotonous. Instead we use the fact that W_λ tends to zero pointwise as $\lambda\to\infty$, since the above estimates yield

$$|W_\lambda(u)| \,\le\, \left(2e^2\mu\, c_\alpha u^3 + \frac{1}{2}\,g^*e\,c_\beta u\right)\lambda^{-1/2} \,+\, e^2c_\alpha^2u^4\lambda^{-1} \tag{16}$$

for some positive $c_\alpha,\, c_\beta$. The family $\{A_\lambda\}$ can be estimated from below because the potentials have in view of (15) a uniform lower bound. Hence one can choose $\xi_0<0$ which belongs to $\rho(A_\lambda)$ for all λ, and the resolvents form a bounded family of positive operators, $(A_\lambda-\xi)^{-1}\,\le\,(\xi_0-\xi)^{-1}$, for any $\xi\,<\,\xi_0$. Next we use a trick based on the resolvent identity: for a vector $f\,=\,(A_0-\xi)g$ with a fixed $g\in C_0^\infty(I\!\!R^+)$ we have

$$\left\|(A_\lambda-\xi)^{-1}f-(A_0-\xi)^{-1}f\right\| \,=\, \left\|(A_\lambda-\xi)^{-1}W_\lambda g\right\| \,\le\, (\xi_0-\xi)^{-1}\,\|W_\lambda g\| \,\to\, 0 \tag{17}$$

as $\lambda\to\infty$ in view of (16) and the compact support of g. However, the family of such f is dense in $L^2(I\!\!R^+,u\,du)$, so $A_\lambda\to A_0$ in the strong resolvent sense.

This allows us to employ Thm. VIII.1.14 of [Ka] by which to any $\nu_n\in\sigma_p(A_0)$ there is a family of $\nu_n(\lambda)\in\sigma(A_\lambda)$ such that $\nu_n(\lambda)\to\nu_n$ as $\lambda\to\infty$. It is therefore decisive that the unperturbed eigenvalue is stable in the sense of [Ka], which means negative in our case. The spectrum of A_0 is given explicitly by

$$\nu_n \,=\, e\mu\,(4n+2-g^*)\,, \qquad n\,=\,0,1,\dots\,, \tag{18}$$

so the condition is satisfied for $n=0$ if $g^*>2$ (as in [CFC], the next eigenvalue comes to play for $g^*>6$ which is physically not appealing). $\Box$

We finish the letter by remarks on extensions of the result and related topics:

(a) The difference in the *weak–coupling behaviour* comparing to [Mo, CFC] is not surprising. In their case $\int_0^\infty V_0^{(-)}(r)\,r\,dr$ is dominated for small $\lambda>0$ by the negative term due to the well. In reality, however, the magnetic field flux lines are closed in $I\!\!R^3$, so the well is compensated by a repulsive tail, small but extended, which prevents the trapping.

(b) The asymptotic perturbation theory yields also the ground state behaviour as $\lambda\to\infty$. By [Ka, Thm. VIII.2.6] the leading–order correction to ν_0 is

$$(\psi_0,W_\lambda\psi_0)=\frac{3e}{8}(g^*+2)\sqrt{\frac{\pi}{e\mu}}\,\alpha_0''(0)\,\lambda^{-1/2}+\mathcal{O}(\lambda^{-1})\,,$$

where ψ_0 is the ground–state eigenfuction of the two–dimensional harmonic oscillator. However, $\alpha_0''(0) = 0$, so the ground state of the original operator $H_0^{(-)}(\lambda)$ behaves as $-\lambda e\mu(g^* - 2) + \mathcal{O}(1)$.

(c) Large λ give rise to an *orbital series* of bound states: the above also argument works for $H_\ell^{(-)}(\lambda)$ with $\ell = -1, -2, \dots$. The potential in (14) is replaced at that by $e^2\mu^2u^2 + \ell^2r^{-2} + e\mu(2\ell - g^*)$, and one looks for negative eigenvalues among $\nu_{n,\ell} = e\mu\,(4n + 2(|\ell| + \ell) + 2 - g^*)$. The critical λ at which the eigenvalue emerges from the continuum is naturally ℓ–dependent.

(d) Positive eigenvalues of A_0 are unstable in the sense that they disappear in the continuum once the perturbation (15) is turned on. Following [Ka, Sec. VIII.5], however, they give rise to spectral concentration as $\lambda \to \infty$ which is manifested by *resonances* in electron scattering on the vortex. Knowing the shape of the potential barrier, one can compute their widths which vanish exponentially fast with λ. For a fixed ℓ, the number of resonances grows asymptotically linearly with λ, because the eigenvalues of the operator (14) are equally spaced and the top of the potential barrier in A_λ is asymptotically linear in λ. Due to the same reason, resonances exist at large λ for *both spin signs.* Notice also that the existence of resonances is not restricted by the value of the gyromagnetic factor, and therefore they may be observed in semiconductor systems where $|g^*|$ is typically less than one, in some cases even of order of 10^{-2}.

(e) The above mentioned resonance scattering is a *purely quantum effect.* A classical electron can be, of course, trapped in the current vortex if it is placed inside the potential barrier (3) with the energy less than its top (since there is no spin in this case, $g^* = 0$, the well bottom is at zero). On the other hand, an electron of the same energy outside the barrier gets scattered without the possibility of entering temporarily the interior of the vortex.

(f) In addition to smooth current distributions discussed above, the case $J(r) = \delta(r - R)$ is of a practical interest. As an illustration, imagine two adjacent thin films, one supporting a free electron gas while the other is equipped with a mesoscopic ring in which a persistent current circulates. The effective potential is now given explicitly by (2) and (6); in distinction to the smooth case it has a singularity of the type $(r - R)^{-1}$ and the operators $H_\ell^{(\pm)}(\lambda)$ need a regularization. Since the δ function above is an idealization of a sharply localized distribution, it is reasonable to choose for this purpose from the family of all admissible procedures a scheme based on the principal value of the singular potential [NZ, Ku].

(g) The reader may wonder whether the preferred spin orientation of the ground state does not conflict with the second theorem of Aharonov and Casher [AC]. The present model offers a good illustration that this claim should be taken *cum grano salis* indeed (in distinction to the first one which is sound — *cf.* [Th]). Recall that the Pauli Hamiltonian (2) can be factorized into the product $(\Pi_1 - i\sigma_3\Pi_2)(\Pi_1 + i\sigma_3\Pi_2)$, where $\vec{\Pi} := -i\vec{\nabla} + e\vec{A}$; if Ψ solves $H\Psi = E\Psi$

then $\tilde{\Psi} := (\Pi_1 + i\sigma_3\Pi_2)\Psi$ solves the equation with spin term sign switched. The nonzero component of the rotationally symmetric spinor Ψ turns at that into $-ie^{i\varphi}(\psi'(r) - eA(r)\psi(r))$. For the ground–state eigenfunction we have $\psi(0) \neq 0$ while $\psi'(0) = 0$ due to the smoothess of Ψ. However, the vector potential satisfies (9) with a positive coefficient. To belong to the domain of H, the nonzero component of $\tilde{\Psi}$ must be smooth at the origin. This requires $\psi''(0)\psi(0)^{-1} = \lambda e\mu$ but the *lhs* is negative; recall that ψ approaches the harmonic–oscillator ground state for large λ.

(h) Notice finally another difference. The model with a flux tube can have a natural "squeezing limit" in terms of Aharonov–Bohm Hamiltonians with a pointlike magnetic flux [Mo, AT, DŠ] because the vector potentials coincide outside the flux tube and the attractive part tends to a δ–well (with the exception of a single one, the resonances mentioned in (d) are lost at that as it is usual in such situations — *cf.* [AGHH]). On the other hand, the present case is more complicated being essentially three–dimensional as far as the magnetic field is concerned.

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