REMARK ON THE DECAY OF A MIXED STATE

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The initial decay rate of a mixed state is discussed; is shown to be zero for finite energy states.

In the previous paper [1] we investigated a general scheme for description of unstable systems. One of our results concerned the initial decay rate. Generalizing the earlier results of HORWITZ and MARCHAND (see [2] and other references contained in [1]) we proved there, that the initial decay rate of any pure state of the unstable system equals to zero, if this state is so called finite energy state.

Here we shall be interested in the same problem, assuming now the state of the decaying system to be in general *mixed.* Such assumption seems to be reasonable: firstly, from the point of view of an experiment it is too restrictive to treat only pure states of unstable systems. In particular, mixed states are generally considered in the recent studies about the influence of measuring devices on the time evolution of the unstable system [3]. Moreover, for this kind of problems the behaviour of the unstable system immediately after its preparation, especially the initial decay rate, is of great importance (see also [4, 5]).

We shall prove in this paper that for finite energy states the initial decay rate is equal to zero, which generalizes the result contained in [1].

We deal with the description of unstable systems considered in Ref. [1]. Let us remind that we understood there as *unstable* any physical system which obeys the following conditions:

(i) one can ascribe to it a state Hilbert space $\mathscr{H}_u$,
(ii) $\mathscr{H}_u$ is a proper subspace of a Hilbert space $\mathscr{H}$,
(iii) a strongly continuous unitary representation $U(t)$ of one-parameter translation group is realized on $\mathscr{H}$ and $\mathscr{H}_u$ is not an invariant subspace of $U(t)$ on $\mathscr{H}$ if $t > 0$.

Here $\mathscr{H}$ means the state Hilbert space of the "whole" system, i.e. it contains also vectors corresponding to decay products etc. Further $U(t) = \exp(-\mathrm{i}Ht)$ is the evolution operator, and H is therefore the total Hamiltonian.

For any $\psi \in \mathscr{H}_u$, $\|\psi\| = 1$, we define the *decay law* (non-decay probability) of the unstable system prepared at $t = 0$ in the state ψ as the function

$$P_\psi(t) = \|V(t)\,\psi\|^2 \tag{1a}$$

where

$$V(t) = E_u\,U(t)\,E_u$$

and E_u is the projection on $\mathscr{H}_u$. Especially, if $\dim \mathscr{H}_u = 1$, we have

$$P_\psi(t) = p_\psi(t) \equiv |(\psi, U(t)\,\psi)|^2\,. \tag{1b}$$

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The initial state need not be pure but can be a mixture; then we have to describe it by a density matrix ϱ, Ran $\varrho \subset \mathscr{H}_u$. In this case we define the *decay law* by

(2) $$P_\varrho(t) = \mathrm{Tr}\,\{\varrho(t)\,E_u\} = \mathrm{Tr}\,\{U(t)\,\varrho\,U^+(t)\,E_u\} = \mathrm{Tr}\,\{V^+(t)\,V(t)\,\varrho\}$$

(the notion of trace is everywhere related to $\mathscr{H}$); this definition generalizes naturally (1a).

The evolution operator $U(t)$ and the Hamiltonian H can be expressed in the form of spectral decomposition

(3) $$U(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t}\,\mathrm{d}E_H(\lambda)\,, \quad H = \int_{-\infty}^{\infty} \lambda\,\mathrm{d}E_H(\lambda)$$

(see e.g. [6]). Now we shall generalize the concept of finite energy state used in [1]: a density matrix ϱ is said to describe a **finite energy state** if the quantity

(4) $$\langle H \rangle_\varrho = \int_{-\infty}^{\infty} \lambda\,\mathrm{d}\mu_\varrho(\lambda)\,, \quad \mu_\varrho(\lambda) = \mathrm{Tr}\,\{\varrho\,E_H(\lambda)\}\,,$$

is finite. In order to show consistency of this definition and its connection to that of finite energy pure state we shall prove the following two assertions:

Propositon 1: The function $\mu_\varrho(.)$ defined by (4) determines a measure on $\mathbb{R}$.

Proof: It is sufficient that the function $\mu_\varrho(.)$ is non-decreasing and continuous on the right (see e.g. [7]). As any compact operator the density matrix ϱ has a pure point spectrum (for properties of density matrices see e.g. [8]) so that we can write it in the form

(5) $$\varrho = \sum_k w_k E_k\,, \quad \sum_k w_k = 1\,,$$

where $w_k \geqq 0$ for all k and E_k is one-dimensional projection containing the normalized eigenvector φ_k of ϱ in its range. For any $\lambda \in \mathbb{R}$ the spectral projection $E_H(\lambda)$ is bounded so that $\mu_\varrho(\lambda)$ is defined and

(6) $$\mu_\varrho(\lambda) = \sum_k w_k\,\mu_k(\lambda)\,, \quad \mu_k(\lambda) = (\varphi_k, E_H(\lambda)\,\varphi_k)\,.$$

The function $\mu_k(.)$ is non-decreasing and continuous on the right for all k; $\mu_\varrho(.)$ is therefore obviously non-decreasing. Let us assume dim Ran $\varrho = \infty$; otherwise also the continuity can be seen trivially.

Consider some $\lambda_0 \in \mathbb{R}$, then to any $\varepsilon > 0$ there exists an integer n_0 such that for all $n > n_0$ it holds

(*) $$\sum_{k=n+1}^{\infty} w_k < \tfrac{1}{2}\varepsilon\,,$$

and on the other hand, to any such n one can find positive numbers δ_k, $k = 1, 2, \ldots, n$, so that for all λ, $\lambda_0 < \lambda < \lambda_0 + \delta_k$, we have

$$0 \leqq \mu_k(\lambda) - \mu_k(\lambda_0) < \tfrac{1}{2}\varepsilon . \tag{**}$$

Let us define $\delta = \min_{1 \leqq k \leqq n} \delta_k$, then combining (*) and (**) we obtain for all λ, $\lambda_0 < \lambda < \lambda_0 + \delta$:

$$0 \leqq \mu_\varrho(\lambda) - \mu_\varrho(\lambda_0) \leqq \sum_{k=1}^{n} w_k(\mu_k(\lambda) - \mu_k(\lambda_0)) + \\ + \sum_{k=n+1}^{\infty} w_k\, \mu_k(\lambda) < \sum_{k=1}^{n} w_k \cdot \tfrac{1}{2}\varepsilon + \sum_{k=n+1}^{\infty} w_k < \varepsilon ,$$

i.e. $\mu_\varrho(.)$ is continuous on the right at an (arbitrary) point $\lambda_0 \in \mathbb{R}$.

The present definition of the finite energy mixed state is essentially mathematical. Naturally, one has to ask what it means physically. It is reasonable to expect the "energy-finiteness" of a mixed state to be connected with the analogous property of pure states contained in this mixture.

Proposition 2: Let ϱ be a finite energy state. Then any φ_k "contained in the mixture", i.e. such to which corresponds non-zero w_k, describes also a finite energy state.

Proof: In order to prove this assertion we have to say what the convergence of integral (4) means. Let us denote by $H_\pm$ the operators

$$H_+ = \int_0^\infty \lambda\, \mathrm{d}E_H(\lambda) , \quad H_- = \int_{-\infty}^0 \lambda\, \mathrm{d}E_H(\lambda) , \tag{3a}$$

so that $H = H_+ + H_-$ and

$$|H| = H_+ - H_- . \tag{3b}$$

The integral (4) is understood to be convergent if both $\langle H_+ \rangle_\varrho$ and $\langle H_- \rangle_\varrho$ converge (are finite), which is equivalent to finiteness of $\langle |H| \rangle_\varrho$.

We denote for convenience

$$\langle H \rangle_k = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}\mu_k(\lambda)$$

(see (6)), and analogously $\langle H_\pm \rangle_k$, $\langle |H| \rangle_k$. The absolute convergence of the series (6) (we consider the more complicated case dim Ran $\varrho = \infty$) implies that for any measurable set $\mathrm{M} \subset \mathbb{R}$ the relation holds

$$\mu_\varrho(\mathrm{M}) = \sum_k w_k\, \mu_k(\mathrm{M}) ,$$

and consequently

(7a) $$\langle H_\pm \rangle_\varrho = \sum_k w_k \langle H_\pm \rangle_k \,,$$

i.e.

(7b) $$\langle H \rangle_\varrho = \sum_k w_k \langle H \rangle_k$$

and

(7c) $$\langle |H| \rangle_\varrho = \sum_k w_k \langle |H| \rangle_k \,.$$

The last relation tells us that infiniteness of any $\langle |H| \rangle_k$ (to which corresponds $w_k > 0$) implies that ϱ cannot be a finite energy state.

Thus we can formulate an *alternative definition*: a state ϱ is a **finite energy state** if it is a mixture of finite energy pure states φ_k such that the series (7) converge.

Now we are in position to prove the announced statement concerning the initial decay rate:

Theorem: Let ϱ be a finite energy state. Then the initial decay rate $\dot{P_\varrho}(0)$ equals to zero.

Proof: We express the density matrix ϱ through (5) and write the decay law (2) in the form

$$P_\varrho(t) = \sum_{jk} w_k(\varphi_j, U(t)\, E_k\, U^+(t)\, \varphi_j)$$

(order of summing is irrelevant due to absolute convergence); further we obtain

$$P_\varrho(t) = \sum_{jk} w_k \| E_k\, U^+(t)\, \varphi_j \|^2 = \sum_{jk} w_k |(\varphi_j, U(t)\, \varphi_k)|^2 = \sum_{jk} w_k |(\varphi_j, V(t)\, \varphi_k)|^2 \,.$$

In analogy with the notation used in [1] we write

(8a) $$P_k(t) = \| V(t)\, \varphi_k \|^2 \,, \quad p_k(t) = |(\varphi_k, V(t)\, \varphi_k)|^2 \,;$$

these function obey the following inequalities

(8b) $$0 \leqq p_k(t) \leqq P_k(t) \leqq P_k(0) = p_k(0) = 1 \,.$$

We express in this notation the decay law:

$$P_\varrho(t) = \sum_k w_k\, P_k(t) \,.$$

The inequalities (8b) show that there holds

(9a) $$0 \leqq p_\varrho(t) \equiv \sum_k w_k\, p_k(t) \leqq P_\varrho(t)$$

for all t, and

$$p_\varrho(0) = P_\varrho(0) = 1 \,. \tag{9b}$$

Combining the conditions (9a,b) one easily finds (see [1]) that $\dot p_\varrho(0) = 0$ implies $\dot P_\varrho(0) = 0$.

We are therefore interested in $\dot p_\varrho(t) = (\mathrm{d}/\mathrm{d}t) \sum_k w_k\, p_k(t)$. According to Proposition 2 all the φ_k correspond to finite energy states, i.e. all $\langle |H| \rangle_k$ are finite and looking for

$$\frac{\mathrm{d}}{\mathrm{d}t} (\varphi_k, U(t)\, \varphi_k) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}\mu_k(\lambda)$$

we can interchange the derivative with the integral and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} (\varphi_k, U(t)\, \varphi_k) = -\mathrm{i} \int_{-\infty}^{\infty} \lambda \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}\mu_k(\lambda) \,. \tag{10}$$

Let us prove now that for all k the derivatives $\dot p_k(t)$ are bounded and continuous functions of t: Eqs. (8), (10) imply

$$\begin{aligned} \dot p_k(t) &= 2\, \mathrm{Re}\, (U(t)\, \varphi_k, \varphi_k) \,.\, \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_k, U(t)\, \varphi_k) = \\ &= 2\, \mathrm{Im}\, (U(t)\, \varphi_k, \varphi_k) \,.\, \int_{-\infty}^{\infty} \lambda \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}\mu_k(\lambda) \,. \end{aligned} \tag{11}$$

One can easily see that

$$|\dot p_k(t)| \leqq 2 \int_{-\infty}^{\infty} |\lambda|\, \mathrm{d}\mu_k(\lambda) = 2\langle |H| \rangle_k$$

which proves the boundedness of $\dot p_k(t)$. Moreover, φ_k as a finite energy state belongs to the domain of $\sqrt{|H|} = \sqrt{H_+} + \sqrt{-H_-}$. Since $U(t)$ is strongly continuous, we obtain

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \lambda [\mathrm{e}^{-\mathrm{i}\lambda t} - \mathrm{e}^{-\mathrm{i}\lambda t_0}]\, \mathrm{d}\mu_k(\lambda) \right| &\leqq \sum_{\alpha=\pm} |(\sqrt{\alpha H_\alpha})\, \varphi_k,\ [(U(t) - U(t_0)]\, \sqrt{(\alpha H_\alpha)}\, \varphi_k)| \leqq \\ &\leqq \sum_{\alpha=\pm} \|\sqrt{(\alpha H_\alpha)}\, \varphi_k\| \,.\, \|[U(t) - U(t_0)]\, \sqrt{(\alpha H_\alpha)}\, \varphi_k\| \,. \end{aligned}$$

The function $\int_{-\infty}^{\infty} \lambda \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}\mu_k(\lambda)$ is therefore continuous, and consequently $\dot p_k(t)$ is continuous.

If dim Ran $\varrho < \infty$, we may now write

$$\dot p_k(t) = \sum_k w_k\, \dot p_k(t) \,. \tag{12}$$

In the case dim Ran $\varrho = \infty$ it is necessary to verify whether the series on the right-hand side of (12) converges uniformly with respect to t. However, this is not difficult, since the series can be majorized as follows (see (7c), (11))

$$\left|\sum_k w_k\, p_k^{\cdot}(t)\right| \leqq \sum_k w_k \left|p_k^{\cdot}(t)\right| \leqq 2 \sum_k w_k \langle |H| \rangle_k = 2\langle |H| \rangle_\varrho .$$

Using (11), (12) we obtain now

$$p_\varrho^{\cdot}(t) = 2 \sum_k w_k \operatorname{Im}\left(U(t)\, \varphi_k, \varphi_k\right) . \int_{-\infty}^{\infty} \lambda \mathrm{e}^{-i\lambda t}\, \mathrm{d}\mu_k(\lambda) ,$$

especially

$$p_\varrho^{\cdot}(0) = 2 \sum_k w_k \operatorname{Im} \langle H \rangle_k = 0$$

(each $\langle H \rangle_k$ is real), and consequently $P_\varrho^{\cdot}(0) = 0$. □

Let us now pass to discussion of the results. Notice first that in the case of a finite energy *pure* state (when the density matrix ϱ reduces to one-dimensional projection containing a unit vector ψ in its range), the decay law (2) gives (1a) and $\langle H \rangle_\varrho = \langle H \rangle_\psi < \infty$, so that the proved theorem really generalizes the result deduced in [1].

If $P_\varrho^{\cdot}(0)$ does not exist, the initial decay rate is defined as the derivative on the right $P_\varrho^{\cdot}(+0)$ in the case when it exists. Our theorem says that states with non-zero initial decay rate could be found among infinite energy states[1]). What concerns ***the physical realizability of infinite energy states*** the same is true as in the special case of a pure state[2]): infiniteness of the integral $\langle |H| \rangle_\varrho$ alone cannot serve as a criterion for exclusion of the state ϱ as physically unrealizable. The reason is that what one actually measures is a probability

$$\operatorname{Tr}\{\varrho\, E_H(\Delta)\} = \int_\Delta \mathrm{d}\mu_\varrho(\lambda)$$

that the measured quantity (especially energy) would be found in an interval Δ (more generally in a Borel set Δ on $\mathbb{R}$ – cf. Ref. [9]). This probability is defined for any state ϱ and it holds $\int_{-\infty}^{\infty} \mathrm{d}\mu_\varrho(\lambda) = 1$. Since any real experiment is restricted to bounded region of $\mathbb{R}$ (bounded scale of a measuring apparatus), convergence of quantities of the type (4) is a matter of our extrapolation.

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[1]) Such states obviously exist: for example let ϱ be such that $\mathrm{d}\mu_k(\lambda) = (1/2\pi)\,[\Gamma_k/(\lambda - \varepsilon_k)^2 + \frac{1}{4}\Gamma_k^2]\,\mathrm{d}\lambda$ for all k, $\Gamma_k > 0$, then $P_\varrho(t) = \sum_k w_k \exp(-\Gamma_k t)$, $t \geqq 0$, i.e. $P_\varrho^{\cdot}(+0) < 0$. Notice that ϱ is an *infinite energy state*, though all the integrals $\int_{-\infty}^{\infty} \lambda\, \mathrm{d}\mu_k(\lambda)$ converge in the sense of principal value (cf. proof of Proposition 2).

[2]) We are indebted to Prof. I. Todorov for discussion about this problem which has followed publication of the paper [1].

Let us point out that the initial decay rate *is not* a directly measurable quantity. According to postulates of the quantum theory any reduction of a state represents itself the immediate process. However, the duration of a real measurement (which also includes preparation of a state) is finite, so that the concept of two measurements arbitrarily closely following each other has its limitation[3]).

Nevertheless, an information about the initial decay rate can be understood as a useful starting point for estimating how the decay law behaves near the time origin. As we have mentioned, this is important namely for studies of repeated measurements on unstable system.

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