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HIGHEST-WEIGHT REPRESENTATIONS OF THE sl(n + 1,C) ALGEBRAS: MAXIMAL REPRESENTATIONS

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1. Introduction

1.1 There are essentially two reasons which make the highest-weight representations of semisimple Lie algebras interesting. The first of them concerns their applications in quantum mechanics and elementary particle physics reviewed, e.g., in the introduction of Ref.1 . On the other hand, mathematically they are a generalization of the finite-dimensional irreducible representations conserving some of their properties.

1.2 The finite-dimensional irreducible representations with a highest weight Λ of a complex semisimple Lie algebra L are characterized by the condition that $\Lambda_i = 2(\Lambda,\omega_i)/(\omega_i,\omega_i)$, $i=1, 2,\dots,n$, are equal to non-negative integers (Refs.2,3) ; here ω_i and n are positive simple roots and rank of L , respectively. Properties of these representations are well-known/2-4/. The representations with ω_i arbitrary integers may be infinite-dimensional but remain integrable ;this case was studied by Harish-Chandra/5/.

The mentioned representations form, of course, only a small part among all the highest-weight representations of given L . Many results concerning the general case (with no restrictions on Λ) can be deduced from the theory of Verma modules/4/ ; an extensive treatment of this problem was presented by Gruber and Klimyk/1/. In their paper the so-called elementary representations were introduced and studied (cf.Sec.2.6 below ; essentially the same construction was used also by other authors for investigation of the highest-weight representations). The elementary representations are ex definitio representations with a highest weight ; generally they need not be irreducible, however, they are irreducible for a "great" subset in the set of all weights.

Since there is a one-to-one correspondence between the weights Λ and the irreducible highest-weight representations (cf. Theorem 2.4(b)), it might seem that no other highest-weight representations are needed, at least for those Λ for which the elementary representations are irreducible. However, representation spaces of the elementary representations are certain factor spaces (cf. Sec. 2.6 below). It makes their use extremely difficult even in the case of the lowest-dimensional algebras and represents itself a great practical disadvantage. This is why we suppose a search for other irreducible highest-weight representations to be meaningfull.

1.3 In this paper we shall give another set of irreducible highest-weight representations of $sl(n+1,\mathbb{C})$. A major part of them will be obtained in an explicit form in which matrix elements of generators can be easily calculated. In the subsequent paper /6/ we shall illustrate on an example $sl(3,\mathbb{C})\sim A_2$ that such explicit representations are given for all the weights Λ to which the irreducible elementary representations correspond. Moreover, we shall demonstrate that our method makes it possible to construct irreducible highest-weight representations also for some of the weights Λ such that the corresponding elementary representations are reducible.

1.4 The construction presented in the following sections is based on canonical realizations of $sl(n+1,\mathbb{C})$. Wide sets of these realizations are known for all the complex classical Lie algebras as well as for majority of their real forms /7-15/; a review of the subject was given in Ref.7. We treat here the algebras $sl(n+1,C) \sim A_n$ as the simplest case; in view of many common features of the mentioned realizations we believe that the method used below could be applicable also to other semisimple Lie algebras.

2. Preliminaries

2.1 The algebra $gl(n+1,\mathbb{C})$ is the $(n+1)^2$-dimensional complex Lie algebra with the standard basis $\{e_{ij} : i,j=1,2,\dots,n+1\}$ the elements of which obey

$$[e_{ij},e_{kl}] = \delta_{kj}e_{il} - \delta_{il}e_{kj} \quad . \tag{1}$$

This algebra is a direct sum of its one-dimensional centrum (generated by the element $e = \sum_{i=1}^{n+1} e_{ii}$) and the simple subalgebra $sl(n+1,\mathbb{C})\sim A_n$ whose generators are e_{ij}, $i \neq j$ and $a_i = e_{ii} - \frac{1}{n}e$, $i = 1,2,\dots,n$.

2.2 The standard Cartan subalgebra H in $L = sl(n+1,\mathbb{C})$ is generated by the "diagonal" elements a_i; its dimension, i.e., rank of L equals n. We choose the following Cartan-Weyl basis

$$h_i = a_{i+1} - a_i = e_{i+1,i+1} - e_{ii} \quad , \quad i = 1,2,\dots,n \quad , \tag{2a}$$

$$e_i = e_{i+1,i} \quad , \quad i = 1,2,\dots,n \quad , \tag{2b}$$

$$f_i = e_{-i} = e_{i,i+1} \quad , \quad i = 1,2,\dots,n \quad , \tag{2c}$$

$$e_{ij} \quad , \quad i > j+1 \quad , \tag{2d}$$

$$e_{ij} \quad , \quad i < j-1 \quad . \tag{2e}$$

The relations (1) imply that (2b-e) are the root vectors corresponding to the roots $\alpha_{ij} : \alpha_{ij}(\sum_{k=1}^{n} \lambda_k h_k) = \lambda_i - \lambda_j$. Among these roots $\omega_i \equiv \alpha_{i+1,i}$ and $\omega_{-i} = \alpha_{i,i+1}$ are simple, further α_{ij}, $i > j$, are positive. Following Ref.3 we call the elements (2a-c) canonical generators of L. They fulfil the relations

$$[h_i,h_j] = 0 \quad , \tag{3a}$$

$$[e_i,f_j] = \delta_{ij}h_i \quad , \tag{3b}$$

$$[h_i,e_j] = c_{ij}e_j \quad , \quad [h_i,f_j] = -c_{ij}f_j \quad , \tag{3c}$$

where c_{ij} are the Cartan numbers, $c_{ij} = 2, -1, 0$ for $i = j$, $|i-j| = 1$ and $|i-j| > 1$, respectively. Notice that the Cartan-Weyl basis (2) differs from the standard one /3,16/; they are connected by the automorphism generated by $e_{\pm i} \mapsto e_{\mp i}$, $h_i \mapsto -h_i$. We choose the basis (2) because it is suitable for our construction.

2.3 The universal enveloping algebra of L will be denoted conventionally as UL. Let ρ be a representation of L on a vector space V, by the same symbol we denote also the natural extension of ρ to UL. A representation $\rho : L \to \mathcal{L}(V)$ is called <u>representation with a highest weight</u> $\Lambda = (\Lambda_1,\dots,\Lambda_n)$ if there exists a vector $x_0 \in V$ (called <u>highest-weight vector</u>) such that the following three conditions are fulfilled :

(i) the linear form Λ on H, $\Lambda(h_i) = \Lambda_i$, is a weight of ρ: it holds $\rho(h)x_0 = \Lambda(h)x_0$ for all $h \in H$, or equivalently $\rho(h_i)x_0 = \Lambda_i x_0$, $i = 1,2,\dots,n$,

(ii) $\rho(e_i)x_0 = 0$, $i = 1,2,\dots,n$,

(iii) the vector x_0 is cyclic for ρ, i.e., $\rho(UL)x_0 \equiv = \{\rho(a)x_0 : a \in UL\} = V$.

Since a system of canonical generators exists in any semisimple Lie algebra, this definition applies not only to $L = sl(n+1,\mathbb{C})$ but to the other semisimple algebras as well. The lowest-weight representations are defined in the same way, the only change consists of replacement of $\rho(e_i)$ by $\rho(f_i)$ in (ii). Some important properties of the highest-weight representations are summed in the following assertions[1-3,17]:

2.4 <u>Theorem</u>: Let L be a complex semisimple Lie algebra and $\rho : L \to \mathcal{L}(V)$ its representation with a highest weight Λ. Then (a) the space V decomposes into a direct sum of finite-dimensional weight subspaces $V_M = \{x \in V : \rho(h)x = M(h)x$ for all $h \in H\}$, the subspace V_Λ being one-dimensional. Every weight M of ρ is of the form $M = \Lambda - \sum_{i=1}^{n} k_i\omega_i$, where ω_i are the positive simple roots of L and k_i are non-negative integers.

(b) To each linear form Λ on the Cartan subalgebra H of L there exists, up to equivalence, one and only one irreducible representation ρ of L with Λ as the highest weight.

2.5 <u>Theorem</u>: Let the assumptions of the previous theorem be valid. The representation ρ is finite-dimensional iff $\Lambda_i = \Lambda(h_i)$, $i = 1,2,\dots,n$, are non-negative integers.

2.6 Now we shall define the elementary representations of L. The algebra L decomposes into a direct sum $L = L_+ + H + L_-$ where L_- is the subalgebra generated by the elements f_i (cf. (2c); notice that each of the elements (2e) can be obtained from $f_1, f_2, \dots, f_n$ by Lie products). The universal enveloping algebra UL_- of L_- serves as a representation space. It can be identified with the free algebra of monomials

$$1 \quad , \quad f_{i_1}f_{i_2}\dots f_{i_m} \quad , \quad i_k = 1,2,\dots,n \quad , \quad m = 1,2,\dots$$

factorized by the ideal generated by the following elements:

$$[\dots[[f_{i_1},f_{i_2}],f_{i_3}],\dots f_{i_m}] \quad , \quad m = 2,3,\dots \quad ,$$

for those $(i_1,i_2,\dots,i_m)$ for which the sum of positive simple roots $\sum_{k=1}^{m}\omega_{i_k}$ is a root.

The <u>elementary representation</u> d_Λ corresponding to a linear form Λ on H is defined by the following relations *)

$$d_\Lambda(h)1 = \Lambda(h)1 \quad , \quad d_\Lambda(f_i)1 = f_i \quad , \quad d_\Lambda(e_i)1 = 0 \quad , \tag{4a}$$

$$d_\Lambda(h)f_{i_1}\dots f_{i_m} = (\Lambda - \omega_{i_1} - \dots - \omega_{i_m})(h)f_{i_1}\dots f_{i_m} \quad , \tag{4b}$$

$$d_\Lambda(f_i)f_{i_1}\dots f_{i_m} = f_i f_{i_1}\dots f_{i_m} \quad , \tag{4c}$$

$$d_\Lambda(e_i)f_{i_1}\dots f_{i_m} = f_{i_1}(d_\Lambda(e_i)f_{i_2}\dots f_{i_m}) + \delta_{ii_1}(\Lambda - \omega_{i_2} - \dots - \omega_{i_m})(h_i)f_{i_2}\dots f_{i_m} \; ; \tag{4d}$$

here ω_i are again the positive simple roots of L. The representation d_Λ is clearly a representation with the highest weight Λ; generally it is reducible but not completely reducible. Necessary and sufficient conditions for irreducibility of d_Λ can be found which employ action of the Weyl group W of L on the highest weight Λ (cf. theorems 5,6 of Ref.1).

2.7 The last introductory item concerns the canonical realizations which are the basic tool of our construction. The (complex) Weyl algebra W_{2N} is the associative algebra with unity 1 generated by the elements q_i, p_j, $i,j = 1,2,\dots,N$ which obey the standard CCR:

$$[p_i,p_j] = [q_i,q_j] = 0 \quad , \quad [p_i,q_j] = \delta_{ij}1 \quad .$$

Canonical realization of a Lie algebra L is a homomorphism $L \to W_{2N}$; it extends naturally to the homomorphism $UL \to W_{2N}$. In the following we shall deal with simple algebras; in this case any non-trivial realization is injective. Further notions concerning canonical realizations are given in Refs.7-15.

We shall use the canonical realizations of $gl(n+1,\mathbb{C})$ con-

*) The sign of the second term in (4d) differs from that of Ref.1 due to other choice of the canonical generators.

structed in Ref.9 . They are obtained recursively with the help of n canonical pairs, one complex parameter and a realization of $gl(n,\mathbb{C})$. The latter can be chosen in different ways : canonical realization of the same type, matrix representation or trivial realization ; in the first case the same possibilities appear after the next step in the choice of realization of $gl(n-1,\mathbb{C})$, etc. In what follows we employ mostly the first possibility when the reduction is performed to the end with canonical realizations of the same type *). The realization of e_{ij} of $gl(k+1,\mathbb{C})$ will be denoted as $\tau^{(k+1)}(e_{ij})$. It is convenient to numerate the canonical pairs in these realizations by two indices : q_i^{k+1} , p_j^{k+1} , $i,j = 1,2,\dots,k$, $k = 2,3,\dots$, then the following assertion is valid :

2.8 Proposition : To any complex numbers $\alpha_0,\alpha_1,\dots,\alpha_n$ there exists the realization of $gl(n+1,\mathbb{C})$ in W_{2N} , $N=(n+1)(n+2)/2$, given recursively by the formulae

$$\tau^{(n+1)}(e_{ij}) = q_i^{n+1}p_j^{n+1} + \tau^{(n)}(e_{ij}) + \tfrac{1}{2}\delta_{ij}1 \quad , \tag{5a}$$

$$\tau^{(n+1)}(e_{n+1,i}) = -p_i^{n+1} \quad , \tag{5b}$$

$$\tau^{(n+1)}(e_{i,n+1}) = q_i^{n+1}\Big(\sum_{j=1}^{n} q_j^{n+1}p_j^{n+1} + \frac{n+1}{2} - i\alpha_n\Big) + \sum_{j=1}^{n} q_j^{n+1}\tau^{(n)}(e_{ij}) \quad , \tag{5c}$$

$$\tau^{(n+1)}(e_{n+1,n+1}) = -\sum_{j=1}^{n} q_j^{n+1}p_j^{n+1} - \Big(\frac{n}{2} - i\alpha_n\Big)1 \quad , \tag{5d}$$

$i,j = 1,2,\dots,n$, where $\tau^{(1)}(e_{11}) = i\alpha_0$.

3. Maximal representations of $sl(n+1,\mathbb{C})$

3.1 Let B_{n+1} denote the set of all symbols (triangular matrices)

$$\left|\begin{array}{llll} m_{n1} & m_{n2} & \cdots\cdots\cdots & m_{nn} \\ \cdots & \cdots & \cdots & \\ m_{k1} & m_{k2} & \cdots\ m_{kk} & \\ \cdots & \cdots & & \\ m_{21} & m_{22} & & \\ m_{11} & & & \end{array}\right| \quad , \quad m_{kj} \in N_0 \quad , \tag{6}$$

where N_0 is the set of all non-negative integers. These symbols

*) The possibility of using matrix representations of some subalgebra will be employed in the subsequent paper (Ref.6).

will denote the basis vectors and the complex linear envelope $V_{n+1} = \mathbb{C}\{B_{n+1}\}$ will serve as the representation space. The vector $x_0^{n+1} = \left|\begin{array}{lll} 0 & \cdots & 0 \\ \cdots & & \\ 0 & & \end{array}\right|$ is called vacuum. We define the creation and annihilation operators $\bar{a}_j^{k+1}$, a_j^{k+1} , $j = 1,2,\dots,k$, $k = 1,2,\dots,n$ on V_{n+1} in the standard way :

$$\bar{a}_j^{k+1}\left|\begin{array}{l} m_{n1}\ \cdots\cdots\cdots\cdots\ m_{nn} \\ \cdots\cdots\cdots\cdots\cdots \\ m_{k1}\ \cdots\ m_{kj}\ \cdots\ m_{kk} \\ \cdots\cdots\cdots \\ m_{11} \end{array}\right| = \sqrt{m_{kj}+1}\left|\begin{array}{l} m_{n1}\ \cdots\cdots\cdots\cdots\ m_{nn} \\ \cdots\cdots\cdots\cdots\cdots \\ m_{k1}\ \cdots\ m_{kj}+1\ \cdots\ m_{kk} \\ \cdots\cdots\cdots \\ m_{11} \end{array}\right| \tag{7a}$$

$$a_j^{k+1}\left|\begin{array}{l} m_{n1}\ \cdots\cdots\cdots\cdots\ m_{nn} \\ \cdots\cdots\cdots\cdots\cdots \\ m_{k1}\ \cdots\ m_{kj}\ \cdots\ m_{kk} \\ \cdots\cdots\cdots \\ m_{11} \end{array}\right| = \sqrt{m_{kj}}\left|\begin{array}{l} m_{n1}\ \cdots\cdots\cdots\cdots\ m_{nn} \\ \cdots\cdots\cdots\cdots\cdots \\ m_{k1}\ \cdots\ m_{kj}-1\ \cdots\ m_{kk} \\ \cdots\cdots\cdots \\ m_{11} \end{array}\right| \tag{7b}$$

They obviously obey the canonical commutation relations

$$[\bar{a}_i^k,\bar{a}_j^l] = [a_i^k,a_j^l] = 0 \quad , \quad [a_i^k,\bar{a}_j^l] = \delta_{ij}\delta_{kl}\,I \quad ; \tag{8}$$

the same is true for the operators

$$Q_j^k(\beta) = \bar{a}_j^k\cos\beta + a_j^k\sin\beta \quad , \quad P_j^k(\beta) = -\bar{a}_j^k\sin\beta + a_j^k\cos\beta \tag{9}$$

Substituting now $Q_j^k(\beta)$, $P_j^k(\beta)$ into the formulae (5) for q_j^k , p_j^k , we obtain a representation of $gl(n+1,\mathbb{C})$ on V_{n+1} which depends on the parameters $\alpha_0,\alpha_1,\dots,\alpha_n$ and β . In the following we shall deal mostly with the case $\beta = 0$. The representation of $gl(k+1,\mathbb{C})$ obtained in this way will be denoted as $\rho^{(k+1)}$, $k = 1,2,\dots,n$. We shall use also E_{ij}^{k+1} as a shorthand for $\rho^{(k+1)}(e_{ij})$, $e_{ij} \in gl(k+1,\mathbb{C})$; in this notation the representations under consideration are given by the relations

$$E_{ij}^{n+1} = \bar{a}_i^{n+1}a_j^{n+1} + E_{ij}^n + \tfrac{1}{2}\delta_{ij}I \quad , \tag{10a}$$

$$E_{n+1,i}^{n+1} = -a_i^{n+1} \quad , \tag{10b}$$

$$E_{i,n+1}^{n+1} = \bar{a}_i^{n+1}\Big(\sum_{j=1}^{n}\bar{a}_j^{n+1}a_j^{n+1} + \frac{n+1}{2} - i\alpha_n\Big) + \sum_{j=1}^{n}\bar{a}_j^{n+1}E_{ij}^n \quad , \tag{10c}$$

$$E_{n+1,n+1}^{n+1} = -\sum_{j=1}^{n}\bar{a}_j^{n+1}a_j^{n+1} - \Big(\frac{n}{2} - i\alpha_n\Big)I \quad . \tag{10d}$$

Let us express further the corresponding representations of the subalgebra $sl(n+1,\mathbb{C})$ in terms of the basis (2). We obtain

$$(11a) = (10a)\ ,\quad (11b) = (10b)\ ,$$

$$E^{n+1}_{i,n+1} = \bar{a}^{n+1}_i\Big(\sum_{j=1}^{n}\bar{a}^{n+1}_j a^{n+1}_j - \sum_{k=1}^{i-1}\bar{a}^i_k a^i_k + \sum_{r=i+1}^{n}\bar{a}^r_i a^r_i - \sum_{s=i}^{n}\Lambda_s\Big) + \sum_{i\neq j=1}^{n}\bar{a}^{n+1}_j E^n_{ij}\ , \tag{11c}$$

$$H^{n+1}_j = E^{n+1}_{j+1,j+1} - E^{n+1}_{jj} = \sum_{r=j+2}^{n+1}(\bar{a}^r_{j+1}a^r_{j+1} - \bar{a}^r_j a^r_j) - 2\bar{a}^{j+1}_j a^{j+1}_j - \sum_{s=1}^{j-1}(\bar{a}^{j+1}_s a^{j+1}_s - \bar{a}^j_s a^j_s) + \Lambda_j I\ , \tag{11d}$$

where

$$\Lambda_j = i\alpha_j - i\alpha_{j-1} - 1\ ,\qquad j = 1,2,\ldots,n\ . \tag{12}$$

Appearance of these new parameters will be very important in the following. To any $\Lambda = (\Lambda_1,\Lambda_2,\ldots,\Lambda_n)$ the formulae (11) define a representation of $sl(n+1,\mathbb{C})$; we call it <u>maximal representation</u> and denote $\rho^{(n+1)}_\Lambda$ or simply ρ_Λ if there is no danger of misunderstanding.

3.2 <u>Proposition</u> : The restriction $\tilde{\rho}^{(n+1)}_\Lambda$ of $\rho^{(n+1)}_\Lambda$ to the subspace $V^\Lambda_{n+1} \equiv \rho^{(n+1)}_\Lambda(UL)x^{n+1}_0$ of V_{n+1} is a representation of $L = sl(n+1,\mathbb{C})$ with the highest weight Λ and the vacuum x^{n+1}_0 as its highest-weight vector.

<u>Proof</u> : The relations (11d) and (7) imply $\rho_\Lambda(h_j)x^{n+1}_0 = \Lambda_j x^{n+1}_0$, $j = 1,2,\ldots,n$. Further $\rho_\Lambda(e_i) = \rho_\Lambda(e_{i,i+1}) = \sum_{k=i+2}^{n+1}\bar{a}^k_{i+1}a^k_i - a^{i+1}_i$ due to (11a,b) so that $\rho_\Lambda(e_i)x^{n+1}_0 = 0$, $i = 1,2,\ldots,n$. The restriction $\tilde{\rho}^{(n+1)}_\Lambda$ is properly defined because $\rho_\Lambda(a)$ maps V^Λ_{n+1} into itself for any $a \in L$. The condition (iii) of Sec.2.3 is fulfilled automatically for $\tilde{\rho}_\Lambda$. ■

Thus we have constructed to any $\Lambda = (\Lambda_1,\ldots,\Lambda_n)$ the highest-weight representation $\tilde{\rho}^{(n+1)}_\Lambda$ of $sl(n+1,\mathbb{C})$. These representations are even irreducible as we shall show a little later. However, they are not yet suitable for practical use, because we do not now the representation space V^Λ_{n+1} explicitly. In the next section we shall find conditions under which $V^\Lambda_{n+1} = V_{n+1}$, i.e., $\tilde{\rho}^{(n+1)}_\Lambda = \rho^{(n+1)}_\Lambda$; they turn out to be irreducibility conditions for $\rho^{(n+1)}_\Lambda$.

4. <u>Irreducibility conditions for $\rho^{(n+1)}_\Lambda$</u>

4.1 We shall use the following simple fact :

<u>Proposition</u> : Every non-trivial invariant subspace $V' \subset V_{n+1}$ of $\rho^{(n+1)}_\Lambda$ contains the vacuum vector x^{n+1}_0 .

<u>Proof</u> : Since V' is assumed to be non-trivial it contains at least one non-zero vector $x \in V_{n+1}$. We can write

$$x = \sum_m \alpha_m \left|\begin{matrix} m_{n1} & \cdots & m_{nn} \\ \cdots\cdots & & \\ m_{11} & & \end{matrix}\right|\ ,$$

$$m = (m_{n1},\ldots,m_{nn},m_{n-1,1},\ldots,m_{n-1,n-1},m_{n-2,1},\ldots,m_{11})\ .$$

Let $\bar{m} = (\bar{m}_{n1},\ldots,\bar{m}_{11})$ be a "highest degree" of this sum understood in the following sense

$$\begin{aligned} \bar{m}_{n1} &= \max\{m_{n1} : \alpha_m \neq 0\}\ , \\ \bar{m}_{n2} &= \max\{m_{n2} : \alpha_{\bar{m}_{n1}m_{n2}\ldots m_{11}} \neq 0\}\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{m}_{11} &= \max\{m_{11} : \alpha_{\bar{m}_{n1}\ldots\bar{m}_{22}m_{11}} \neq 0\}\ . \end{aligned} \tag{$\ast$}$$

The relation (11a) implies

$$E^{n+1}_{ij} = \sum_{k=i+1}^{n+1}\bar{a}^k_i a^k_j - a^i_j\ ,\qquad i > j\ . \tag{11e}$$

Since V' is assumed to be an invariant subspace of ρ_Λ , the vector $E^{n+1}_{ij}y$ belongs to V' for any $y \in V'$. Consequently, the vector

$$\bar{x} = (E^{n+1}_{21})^{\bar{m}_{11}}(E^{n+1}_{32})^{\bar{m}_{22}}(E^{n+1}_{31})^{\bar{m}_{21}}(E^{n+1}_{43})^{\bar{m}_{33}}\ldots(E^{n+1}_{n1})^{\bar{m}_{n-1,1}}(E^{n+1}_{n+1,n})^{\bar{m}_{nn}}\ldots(E^{n+1}_{n+1,1})^{\bar{m}_{n1}}x$$

belongs to V' . The chosen order ensures that the sums from (11e) do not contribute. Further ($\ast$) together with (7b) imply $\bar{x} = c\alpha_{\bar{m}}x^{n+1}_0$, where c is some non-zero number (a product of powers of -1 and square roots of positive numbers) ; therefore $x^{n+1}_0 \in V'$. ■

4.2 <u>Corollary</u> : The representation $\tilde{\rho}_\Lambda^{(n+1)}$ from Proposition 3.2 is irreducible for any Λ.

<u>Proof</u> : Any non-trivial invariant subspace $V' \subset V_{n+1}$ of $\tilde{\rho}_\Lambda$ is in the same time invariant under ρ_Λ, thus it contains the vacuum vector x_0^{n+1}. It further implies $V^1 \equiv \rho_\Lambda(L)x_0^{n+1} \subset V'$, $V^2 \equiv \rho_\Lambda(L)V^1 \subset V'$, etc. We obtain therefore $\rho_\Lambda(UL)x_0^{n+1} \equiv V_{n+1} \subset V'$ so that $V' = V_{n+1}$. ■

4.3 Let us turn now to the problem mentioned at the end of the previous section. We shall prove the following important assertion :

<u>Theorem</u> : Let the conditions

$$(\Lambda_j + \Lambda_{j-1} + \dots + \Lambda_k + j - k) \notin N_0 \qquad (13)$$

be fulfilled for any pair of integers j,k, $1 \leq k \leq j = 1, 2, \dots, n$. Then the maximal representation $\rho_\Lambda^{(n+1)}$ of $sl(n+1,\mathbb{C})$ defined by the formulae (11) is irreducible ; it has the highest weight $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ and the highest-weight vector x_0^{n+1}.

<u>Proof</u> : In view of Proposition 3.2 and Corollary 4.2 it is sufficient to verify that under the stated assumptions $V_{n+1}^\Lambda = V_{n+1}$ holds. We shall do it in few steps :

4.4 We denote first as x_s^{n+1} the basis vectors (6) with all the indices equal to zero with exception of m_{n1} : $m_{kj} = s\delta_{kn}\delta_{j1}$, especially x_0^{n+1} is the vacuum vector as before. The following statement holds :

<u>Lemma</u> : If the conditions (13) are valid for $1 \leq k \leq j = n$, then the subspace V_{n+1}^Λ contains the vectors x_s^{n+1} for all $s \in N_0$.

<u>Proof</u> : Let us introduce the following finite sequences of operators (for convenience written as columns) :

$$R_1 = I \ , \quad R_{k+1} = \left| \begin{array}{c} R_1 E_{1,k+1}^{n+1} \\ R_2 E_{2,k+1}^{n+1} \\ \vdots \\ R_k E_{k,k+1}^{n+1} \end{array} \right| \ , \quad k = 1, 2, \dots, n \ . \qquad (*)$$

Let us further take an arbitrary $s \in N_0$ and denote

$$y_n = R_{n+1} x_s^{n+1} = \left| \begin{array}{c} R_1 E_{1,n+1}^{n+1} x_s^{n+1} \\ R_2 E_{2,n+1}^{n+1} x_s^{n+1} \\ \vdots \\ R_n E_{n,n+1}^{n+1} x_s^{n+1} \end{array} \right| .$$

This column has 2^{n-1} rows. We divide it into two parts with 2^{n-2} rows :

$$y_n^1 = \left| \begin{array}{c} R_1 E_{1,n+1}^{n+1} x_s^{n+1} \\ \vdots \\ R_{n-1} E_{n-1,n+1}^{n+1} x_s^{n+1} \end{array} \right| \quad \text{and} \quad y_n^2 = \left| R_n E_{n,n+1}^{n+1} x_s^{n+1} \right|$$

and introduce

$$y_{n-1} = (s - \Lambda_n) y_n^1 - y_n^2 \ .$$

Further we divide the column y_{n-1} into two parts : y_{n-1}^1 (consisting of the first 2^{n-3} rows) and y_{n-1}^2, then we define $y_{n-2} = (s - \Lambda_n - \Lambda_{n-1} - 1) y_{n-1}^1 - y_{n-1}^2$. Continuing this procedure, we put

$$y_{n-k} = (s - \Lambda_n - \Lambda_{n-1} - \dots - \Lambda_{n-k+1} - k + 1) y_{n-k+1}^1 - y_{n-k+1}^2$$

for all $1 \leq k \leq n-1$. Finally we obtain a one-row column, i.e., a vector y_1. We shall prove that the relation

$$y_1 = \sqrt{s+1} \prod_{j=1}^{n} (s - \Lambda_n - \Lambda_{n-1} - \dots - \Lambda_j - n + j) x_{s+1}^{n+1} \qquad (**)$$

holds. For this purpose we shall use the relations (11c) together with (7a,b). The latter imply

$$\bar{a}_j^{k+1} a_j^{k+1} x_s^{n+1} = s\delta_{kn}\delta_{j1} x_s^{n+1} \ , \quad a_j^{n+1} x_s^{n+1} = \sqrt{s}\, \delta_{kn}\delta_{j1} x_{s-1}^{n+1} \qquad (***)$$

so that we obtain for y_n the following expression :

$$\left|\begin{array}{l} R_1\bar{a}_1^{n+1}(s-\Lambda_n-\dots-\Lambda_1)x_s^{n+1} \qquad\qquad\qquad +R_1\sum_{j=2}^{n}\bar{a}_j^{n+1}E^n_{1j}x_s^{n+1} \\ R_2\bar{a}_2^{n+1}(s-\Lambda_n-\dots-\Lambda_2)x_s^{n+1}+R_2\bar{a}_1^{n+1}E^n_{21}x_s^{n+1} \qquad +R_2\sum_{j=3}^{n}\bar{a}_j^{n+1}E^n_{2j}x_s^{n+1} \\ \vdots \\ R_{n-1}\bar{a}_{n-1}^{n+1}(s-\Lambda_n-\Lambda_{n-1})x_s^{n+1}+R_{n-1}\sum_{j=1}^{n-2}\bar{a}_j^{n+1}E^n_{n-1,j}x_s^{n+1}+R_{n-1}\bar{a}_n^{n+1}E^n_{n-1,n}x_s^{n+1} \\ R_n\bar{a}_n^{n+1}(s-\Lambda_n)x_s^{n+1} \qquad +R_n\sum_{j=1}^{n-1}\bar{a}_j^{n+1}E^n_{nj}x_s^{n+1} \end{array}\right|$$

Here the terms containing $E^n_{ij}x_s^{n+1}$, $i > j$, are equal to zero because of (★★★) and the relation

$$E^n_{ij} = \sum_{k=i+1}^{n}\bar{a}_i^k a_j^k - a_j^i \quad , \quad i > j$$

which is obtained in the same way as (11e). Now we substitute for R_n from (★), then using further the relations (11a),(8) and (★★★) we get

$$\left|\begin{array}{l} R_1\bar{a}_1^{n+1}(s-\Lambda_n-\dots-\Lambda_1)x_s^{n+1}+R_1\sum_{j=2}^{n-1}\bar{a}_j^{n+1}E^n_{1j}x_s^{n+1}+R_1\bar{a}_n^{n+1}E^n_{1n}x_s^{n+1} \\ R_2\bar{a}_2^{n+1}(s-\Lambda_n-\dots-\Lambda_2)x_s^{n+1}+R_2\sum_{j=3}^{n-1}\bar{a}_j^{n+1}E^n_{2j}x_s^{n+1}+R_2\bar{a}_n^{n+1}E^n_{2n}x_s^{n+1} \\ \vdots \\ R_{n-1}\bar{a}_{n-1}^{n+1}(s-\Lambda_n-\Lambda_{n-1})x_s^{n+1} \qquad\qquad +R_{n-1}\bar{a}_n^{n+1}E^n_{n-1,n}x_s^{n+1} \\ R_1\bar{a}_1^{n+1}(s-\Lambda_n)x_s^{n+1} \qquad\qquad +(s-\Lambda_n)R_1\bar{a}_n^{n+1}E^n_{1n}x_s^{n+1} \\ \vdots \\ R_{n-1}\bar{a}_{n-1}^{n+1}(s-\Lambda_n)x_s^{n+1} \qquad\qquad +(s-\Lambda_n)R_{n-1}\bar{a}_n^{n+1}E^n_{n-1,n}x_s^{n+1} \end{array}\right|$$

Subtracting the lower half of this column from the upper one multiplied by $(s-\Lambda_n)$ we obtain the following expression for y_{n-1} :

$$\left|\begin{array}{l} R_1\bar{a}_1^{n+1}(s-\Lambda_n)(s-\Lambda_n-\dots-\Lambda_1-1)x_s^{n+1}+R_1(s-\Lambda_n)\sum_{j=2}^{n-1}\bar{a}_j^{n+1}E^n_{1j}x_s^{n+1} \\ R_2\bar{a}_2^{n+1}(s-\Lambda_n)(s-\Lambda_n-\dots-\Lambda_2-1)x_s^{n+1}+R_2(s-\Lambda_n)\sum_{j=3}^{n-1}\bar{a}_j^{n+1}E^n_{2j}x_s^{n+1} \\ \vdots \\ R_{n-1}\bar{a}_{n-1}^{n+1}(s-\Lambda_n)(s-\Lambda_n-\Lambda_{n-1}-1)x_s^{n+1} \end{array}\right|$$

In the next step we substitute for R_{n-1} from (★), then we use again the relations (11a),(8) and (★★★) and subtract the lower half from the upper one multiplied by $(s-\Lambda_n-\Lambda_{n-1})$:

$$y_{n-2} = (s-\Lambda_n)(s-\Lambda_n-\Lambda_{n-1}-1) \times$$
$$\times \left|\begin{array}{l} R_1\bar{a}_1^{n+1}(s-\Lambda_n-\dots-\Lambda_1-2)x_s^{n+1}+R_1\sum_{j=2}^{n-2}\bar{a}_j^{n+1}E^n_{1j}x_s^{n+1} \\ R_2\bar{a}_2^{n+1}(s-\Lambda_n-\dots-\Lambda_2-2)x_s^{n+1}+R_2\sum_{j=3}^{n-2}\bar{a}_j^{n+1}E^n_{2j}x_s^{n+1} \\ \vdots \\ R_{n-2}\bar{a}_{n-2}^{n+1}(s-\Lambda_n-\Lambda_{n-1}-\Lambda_{n-2}-2)x_s^{n+1} \end{array}\right|$$

Repeating this procedure we obtain finally

$$y_1 = \prod_{j=1}^{n}(s-\Lambda_n-\Lambda_{n-1}-\dots-\Lambda_j-n+j)\bar{a}_1^{n+1}x_s^{n+1} \quad ,$$

i.e., the formula (★★). The presented construction shows that there exists an element $p \in UL$ such that

$$\rho_\Lambda(p)x_s^{n+1} = \sqrt{s+1}\prod_{j=1}^{n}(s-\Lambda_n-\dots-\Lambda_j-n+j)x_{s+1}^{n+1} \quad .$$

This vector is non-zero due to the assumption, thus if x_s^{n+1} belongs to the subspace $V^\Lambda_{n+1} = \rho_\Lambda(UL)x_0^{n+1}$ then the same is true for x_{s+1}^{n+1} . Since x_0^{n+1} is contained in V^Λ_{n+1} , the proof is completed by induction. ■

4.5 Now we can continue the proof of Theorem 4.3. Since $V^\Lambda_{n+1} \subset$ $\subset V_{n+1}$, we have to prove the opposite inclusion ; it is clearly sufficient to verify that all the basis vectors (6) of V_{n+1} are contained in V^Λ_{n+1} . We decompose V_{n+1} in the following way : let D_{n+1} be the set of all symbols $l = (l_1,\dots,l_n)$, $l_i \in N_0$ and L_{n+1} be the complex vector space spanned by D_{n+1} . Then we can write

$$V_{n+1} = L_{n+1} \otimes V_n \quad , \quad x_0^{n+1} = l_0 \otimes x_0^n \tag{14}$$

where

$$(l_1,\dots,l_n) \otimes \left|\begin{array}{l} m_{n-1,1} \cdot\cdot\; m_{n-1,n-1} \\ \dots\dots\dots \\ m_{11} \end{array}\right| \equiv \left|\begin{array}{l} l_1 \;\dots\dots\dots\; l_{n-1} \; l_n \\ m_{n-1,1} \cdot\cdot\; m_{n-1,n-1} \\ \dots\dots\dots \\ m_{11} \end{array}\right|$$

and $l_0 \equiv (0,\dots,0)$. We shall prove first

$$L_{n+1} \otimes x_0^n \subset V^\Lambda_{n+1} \quad . \tag{15}$$

Let us take $l = (l_1,\dots,l_k,0,\dots,0)$, $1 \le k \le n-1$. Using (11e) we get

$$(E^{n+1}_{k+1,k})^r(1\otimes x^n_0) = \binom{l_k}{r}^{1/2} r!\,((l_1,\dots,l_{k-1},l_k-r,r,0,\dots,0)\otimes x^n_0)$$

According to Lemma 4.4 the vectors $x^{n+1}_s = (s,0,\dots,0)\otimes x^n_0$ belong to V^Λ_{n+1}, so acting on them by powers of $E^{n+1}_{k+1,k}$ we stay within V^Λ_{n+1}. Starting with s large enough we can obtain in this way every basis vector of $L_{n+1}\otimes x^n_0$, thus the relation (15) holds.

4.6 Further we shall show that V^Λ_{n+1} contains $L_{n+1}\otimes x^n_r$ for any $r\in N_0$. We know that this is true for $r=0$, let us assume the same for $r=1,\dots,s$. The proof is analogous to that of Lemma 4.4 : we start with an arbitrary $\tilde{x}_s = 1\otimes x^n_s$ and denote $\tilde{y}_{n-1} = R_n\tilde{x}_s$. Then we divide this column into two parts $\tilde{y}^i_{n-1}$ and define $\tilde{y}_{n-2} = (s-\Lambda_n)\tilde{y}^1_{n-1} - \tilde{y}^2_{n-1}$. Continuing the procedure with

$$\tilde{y}_{n-k} = (s-\Lambda_{n-1}-\dots-\Lambda_{n-k+1}-k+2)\tilde{y}^1_{n-k+1} - \tilde{y}^2_{n-k+1}$$

we arrive finally to the vector $\tilde{y}_1$. According to the construction it belongs to V^Λ_{n+1} ; we shall prove

$$\tilde{y}_1 = \sqrt{s+1}\prod_{j=1}^{n-1}(s-\Lambda_{n-1}-\dots-\Lambda_j-n+j-1)(1\otimes x^n_{s+1}) + \tilde{x}'_s \qquad (*)$$

where $\tilde{x}'_s$ is some vector from $L_{n+1}\otimes x^n_s$. We write $\tilde{y}_{n-1}$ using (11a), then we express E^n_{ij} with the help of relations (11c) and (7) obtaining thus for $\tilde{y}_{n-1}$:

$$\left|\begin{array}{l}
R_1\bar{a}^n_1(s-\Lambda_{n-1}-\dots-\Lambda_1)\tilde{x}_s \qquad\qquad + R_1\sum_{j=2}^{n-1}\bar{a}^n_jE^{n-1}_{1j}\tilde{x}_s + a^{n+1}_1a^{n+1}_n\tilde{x}_s \\
R_2\bar{a}^n_2(s-\Lambda_{n-1}-\dots-\Lambda_2)\tilde{x}_s + R_2\bar{a}^n_1E^{n-1}_{21}\tilde{x}_s + R_2\sum_{j=3}^{n-1}\bar{a}^n_jE^{n-1}_{2j}\tilde{x}_s + \bar{a}^{n+1}_2a^{n+1}_n\tilde{x}_s \\
\vdots \\
R_{n-1}\bar{a}^n_{n-1}(s-\Lambda_{n-1})\tilde{x}_s + R_{n-1}\sum_{j=1}^{n-2}\bar{a}^n_jE^{n-1}_{n-1,j}\tilde{x}_s \qquad\qquad + \bar{a}^{n+1}_{n-1}a^{n+1}_n\tilde{x}_s
\end{array}\right|$$

Now we can proceed further in the same way as in the proof of Lemma 4.4. The added vectors $\bar{a}^{n+1}_ja^{n+1}_n\tilde{x}_s$ belong to $L_{n+1}\otimes x^n_s$, thus the same holds for any linear combination of them. Finally we obtain

$$\tilde{y}_1 = \prod_{j=1}^{n-1}(s-\Lambda_{n-1}-\dots-\Lambda_j-n+j-1)\bar{a}^n_1\tilde{x}_s + \tilde{x}'_s \quad ,$$

$$\tilde{x}'_s = \prod_{k=2}^{n-1}(s-\Lambda_{n-1}-\dots-\Lambda_{n-k+1}-k+2)\bar{a}^{n+1}_1a^{n+1}_1\tilde{x}_s -$$

$$-\sum_{j=2}^{n-1}\prod_{k=2}^{n-j}(s-\Lambda_{n-1}-\dots-\Lambda_{n-k+1}-k+2)\bar{a}^{n+1}_ja^{n+1}_n\tilde{x}_s \quad ;$$

it proves the relation (*). The induction assumption implies $\tilde{x}'_s\in V^\Lambda_{n+1}$, then also the vector $\tilde{y}_1-\tilde{x}'_s$ belongs to V^Λ_{n+1}. The conditions (13) are assumed to be valid for $j=1,\dots,n$, especially for $j=n-1$, thus $\tilde{y}_1-\tilde{x}'_s$ is a non-zero multiple of $\tilde{x}_{s+1}$ which belongs therefore to V^Λ_{n+1}.

Further we decompose V_{n+1} into the tensor product $V_{n+1} = L_{n+1}\otimes L_n\otimes V_{n-1}$ in the analogy with (14). Let us take some $k=1,\dots,n-2$ and natural r and assume the vectors $x(m,l(k,s))\equiv m\otimes l(k,s)\otimes x^{n-1}_0$ to belong to V^Λ_{n+1} for any $m\in L_{n+1}$, $l(k,s))\equiv(l_1,\dots,l_k,s,0,\dots,0)$, l_i arbitrary elements of N_0 and $s=0,1,\dots,r-1$. The relations (7) and (11e) imply

$$(E^{n+1}_{k+1,k})^r x(m,l(k,0)) = \sum_{j=0}^{r} r!\binom{m_k}{r-j}^{1/2}\binom{m_{k+1}+r-j}{r-j}^{1/2}\binom{l_k}{j}^{1/2}(m_1,\dots$$

$$\dots,m_k-r,m_{k+1}+r,\dots,m_r)\otimes(l_1,\dots,l_k-j,j,0,\dots,0)\otimes x^{n-1}_0 \quad ,$$

thus by induction V^Λ_{n+1} contains the vectors $x(m,l(k,r))$ for all $r\in N_0$, i.e., if V^Λ_{n+1} contains the vectors $x(m,l(k,0))$ with arbitrary $l(k,0)$ and $m\in L_{n+1}$, then the same is true for $x(m,l(k+1,0))$. According to 4.6 the vectors $x(m,l(1,0))$ belong to V^Λ_{n+1} so applying once more the induction argument we obtain $x(m,l(n-1,0)) = x(m,l) = m\otimes l\otimes x^{n-1}_0\in V^\Lambda_{n+1}$ for any $m\in L_{n+1}$ and $l\in L_n$.

4.8 Now one has to repeat the considerations of Secs.4.6, 4.7 in order to "fill up" the third row. Continuing this procedure we arrive to the relation

$$V_{n+1} = \bigotimes_{k=1}^{n} L_{k+1}\subset V^\Lambda_{n+1}$$

which represents the desired result. ∎

5. Discussion

5.1 Let us assume all the components of the highest weight to be real, then coefficients of all polynomials used in the performed proofs are also real. It means that in this case the results obtained in the previous sections for $sl(n+1,\mathbb{C})$ apply to the real form $sl(n+1,\mathbb{R})$ as well.

5.2 The finite-dimensional irreducible representations of $sl(n+1,\mathbb{C})$ may be described completely in the framework of Gel'fand-Zetlin patterns (see Ref.18, sec.10.1). There exists a generalization of this method (Ref.18, sec.11.8, see also Ref.1, sec.2) which makes it possible to construct also some infinite-dimensional highest-weight representations. In the case of $sl(3,\mathbb{C})$, for example, one has to replace the Gel'fand-Zetlin patterns

$$\begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & \\ & m_{11} & \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} & m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & & \\ m_{11} & & & \end{pmatrix}$$

with $m_{12} \geqslant m_{13}+1$, $m_{13} \geqslant m_{22} \geqslant m_{23}$, $m_{12} \geqslant m_{11} \geqslant m_{22}$. Action of the standard Gel'fand-Zetlin formulae on these patterns defines an infinite-dimensional highest-weight representation of $sl(3,\mathbb{C})$ (determined by m_{13}, m_{23}, m_{33}). However, one can obtain in this way only representations with (possibly negative) integer components of the highest weight ; they correspond only to a small subset of the representations which we have studied here.

5.3 We have to compare our results first of all to the elementary representations introduced in Sec.2.6, because the latter are defined also for each weight Λ on H :

(a) every d_Λ is the highest-weight representation ; for our maximal representations this is true if the conditions (13) are satisfied, otherwise we obtain the highest-weight representation $\tilde{\rho}_\Lambda$ by restriction of ρ_Λ to the subspace V^Λ_{n+1}. On the other hand, the highest weight representations which we obtain are always irreducible. This difference is due to different incomplete reducibility of d_Λ and ρ_Λ : symbolically

$$d_\Lambda = \begin{pmatrix} \tilde{d}_\Lambda & 0 \\ \star & d'_\Lambda \end{pmatrix} \quad , \quad \rho_\Lambda = \begin{pmatrix} \tilde{\rho}_\Lambda & \star \\ 0 & \rho'_\Lambda \end{pmatrix}$$

where $\tilde{d}_\Lambda$ and $\tilde{\rho}_\Lambda$ are the irreducible components of d_Λ and ρ_Λ, respectively, and the star stands for non-zero blocks.

(b) Action of the operators $\rho_\Lambda(h_i)$, $\rho_\Lambda(e_{ij})$ on an arbitrary vector of V_{n+1} is obtained from the formulae (7) and (11). Especially, they allow us to calculate easily matrix elements of the generators. This is not true for the elementary representations for which the choice of a basis in the representation space is itself complicated. According to our opinion, this fact represents the main advantage of the maximal representations. We pay, of course, a price for it : the formulae (4) defining elementary representations are common for all the complex semisimple Lie algebras, while ours refer to the algebras A_n only. There exists, however, a hope of performing an analogous construction for the remaining classical semisimple Lie algebras.

(c) In the subsequent paper /6/ we shall illustrate the irreducibility conditions on the example of $sl(3,\mathbb{C})$. We shall prove that the conditions (13) are in this case necessary and sufficient for irreducibility of the maximal as well as the elementary representations. Further we shall show that starting from the canonical realizations (5) one can construct irreducible highest-weight representations also for some of the weights such that the corresponding elementary representations are reducible.

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