

Anomalous Pauli electron states for magnetic fields with tails

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We consider a two-dimensional electron with an anomalous magnetic moment, $g > 2$, interacting with a nonzero magnetic field B perpendicular to the plane which gives rise to a flux F . Recent results about the discrete spectrum of the Pauli operator are extended to fields with the $\mathcal{O}(r^{-2-\delta})$ decay at infinity: we show that if $|F|$ exceeds an integer N , there is at least $N + 1$ bound states. Furthermore, we prove that weakly coupled bound states exist under mild regularity assumptions also in the zero flux case.

1 Introduction

Several recent papers — see [3]–[6], and references therein — discussed the discrete spectrum of the two-dimensional Pauli operator with a localized magnetic field B , coming from an excess magnetic moment, $g > 2$. The most general result available concerns fields with a compact support [3]. In this situation the discrete spectrum is nonempty whenever B is nonzero, and its dimension is $1 + [F]$ where $[F]$ is the integer part of the related flux (in natural units).

The main aim of this letter is to extend this result to non-compactly supported fields which satisfy a mild regularity requirement and behave as $\mathcal{O}(|x|^{-2-\delta})$ for $|x| \rightarrow \infty$. As long as we consider a powerlike bound, this is an almost optimal condition, because B has to be integrable. We use a variational method to prove that if B is a nontrivial field with the stated decay and the absolute value of the flux exceeds

an integer N , then the Pauli operator with spin antiparallel to the flux has at least $N + 1$ bound states, counting multiplicity. The variational proof follows the same idea as in the compact-support case, but several modifications are needed.

Comparing to the mentioned theorem obtained in [3] the indicated result is slightly weaker giving one bound state less for integer values of the flux. The reason is that without the compact-support assumption we have less information about the asymptotic behaviour of the Aharonov-Casher states used in the construction, in particular, in case of integer flux the “last” one need not be bounded. On the other hand, we can replace the sophisticated mollifier of [3] by a simpler one.

The said difference is important in the case of zero flux when our main result, Theorem 3.1 below, becomes trivial. It was shown in [3] that the existence of a discrete spectrum can be then established for weak fields by the Birman-Schwinger technique (see also [4] for the strong field case), and moreover, that a bound state exists in this situation for both spin orientations. The drawback of this result was that it employed a (rather restrictive) assumption about the decay of the vector potential in the used gauge. We shall show that this condition can be relaxed and the existence of weakly coupled bound states can be proven under the mentioned assumptions on the magnetic field alone.

2 Preliminaries

We consider a two-dimensional electron interacting with a non-homogeneous magnetic field $B = \partial_1 A_2 - \partial_2 A_1$ perpendicular to the plane. For the sake of simplicity, we employ everywhere the natural units $2m = \hbar = c = e = 1$. The field corresponds to a vector potential $A = (A_1, A_2)$ for which we choose conventionally [12] the gauge $A_1 = -\partial_2 \phi$, $A_2 = \partial_1 \phi$, where

$$\phi(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} B(y) \ln |x - y| d^2 y, \quad (2.1)$$

Below we give conditions under which the vector potential components exist in the sense of distributions. The particle is described by the Pauli Hamiltonian

$$H_P^{(\pm)}(A) = (-i\nabla - A(x))^2 \pm \frac{g}{2} B(x) = D^* D + \frac{1}{2} (2 \pm g) B(x) \quad (2.2)$$

with $D := (p_1 - A_1) + i(p_2 - A_2)$, where the two signs correspond to the two possible spin orientations. We are particularly interested in the case when the electron has an excess magnetic moment, $g > 2$.

As in [3] we shall suppose that $B \in L^1(\mathbb{R}^2)$. This ensures the existence of a global quantity characterizing the field,

$$F := \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x) d^2 x, \quad (2.3)$$

i.e., the total flux measured in the natural units $(2\pi)^{-1}$. Without loss of generality we may assume $F \geq 0$; in that case we will be interested primarily in the operator $H_P^{(-)}(A)$ which describes an electron with its magnetic moment parallel to the flux.

The function (2.1) can be used to define the Aharonov-Casher states which satisfy $D\chi_j = 0$ and thus yield zero-energy solutions of the Pauli equation without the anomalous moment, $g = 2$. They are given by

$$\chi_j(x) = e^{-\phi(x)} (x_1 + ix_2)^j, \quad j = 0, 1, \dots \quad (2.4)$$

For fields with a compact support we have $\chi_j(x) = \mathcal{O}(|x|^{-F+j})$ as $|x| \rightarrow \infty$ – cf. [2], [12, Sec.7.2]. It means that if $F = N + \varepsilon$, $\varepsilon \in (0, 1]$ for a positive integer N , the operator $H_P^{(-)}(A)$ with $g = 2$ has N zero energy eigenvalues. Moreover, $\chi_{[F]}$ and possibly $\chi_{[F]-1}$ (in case that F is a positive integer; as usual, the symbol $[\cdot]$ denotes the integer part) are zero energy resonances, since they solve the equation $H_P^{(-)}(A)\chi_j = 0$ and remain bounded at large distances.

We shall assume the following:

(A.1) $B(x) = \mathcal{O}(|x|^{-2-\delta})$ for some $\delta > 0$,

(A.2) $B \in L_{\text{loc}}^{1+\epsilon}(\mathbb{R}^2)$ for some $\epsilon > 0$.

Remark 2.1 *If a positive number ϵ exists we can always choose it in such a way that $\delta(1 + \epsilon^{-1}) > 8$. Under the decay requirement of (A.1) the second assumption means that $B \in L^1(\mathbb{R}^2) \cap L^{1+\epsilon}(\mathbb{R}^2)$, in particular, that the flux (2.3) makes sense and the same is true for the integral (2.1) as we shall see in a while.*

The AC states now exist and their decay is given by the following result.

Proposition 2.2 *Assume (A.1) and (A.2). Then ϕ is locally bounded and to any $\varepsilon > 0$ there is a positive R such that*

$$|\phi(x) - F \ln |x|| < \varepsilon \ln |x| \quad (2.5)$$

holds for all $|x| > R$.

Proof: Given a positive c we denote $\langle y \rangle_c := \sqrt{c + y^2}$. Since $B \langle \cdot \rangle_c^{\delta/2} \in L^{1+\epsilon}$, and $\langle \cdot \rangle_c^{-\delta/2} \ln |x - \cdot| \in L^{1+\epsilon^{-1}}$ for $\delta(1 + \epsilon^{-1}) > 4$, the Hölder inequality yields a bound on $|\phi(x)|$. To prove the inequality (2.5), we denote $\mathcal{B}_R := \{x : |x| \leq R\}$ and $\bar{\mathcal{B}}_R := \mathbb{R}^2 \setminus \mathcal{B}_R$. Furthermore, we set

$$F_R := \frac{1}{2\pi} \int_{\mathcal{B}_R} B(x) d^2x, \quad (2.6)$$

and

$$\phi_R(x) := \frac{1}{2\pi} \int_{\mathcal{B}_R} B(y) \ln |x - y| d^2y, \quad \tilde{\phi}_R(x) := \phi(x) - \phi_R(x). \quad (2.7)$$

By assumption, to a given $\varepsilon > 0$ there is R_1 such that

$$\frac{1}{2\pi} \int_{\bar{\mathcal{B}}_{R_1}} |B(y)| \max\{1, \ln |y|\} d^2y < \frac{1}{4} \varepsilon. \quad (2.8)$$

It follows that

$$|F - F_{R_1}| < \frac{1}{4} \varepsilon. \quad (2.9)$$

For any $R > 0$ the quantity F_R is the flux of a cut-off field and ϕ_R is the corresponding “potential”. This allows us to employ the above mentioned estimate [12, Sec.7.2] by which

$$\phi_R(x) - F_R \ln |x| = \mathcal{O}(|x|^{-1}) \quad (2.10)$$

as $|x| \rightarrow \infty$. Finally, we shall prove that

$$|\tilde{\phi}_{R_1}(x)| < \frac{\varepsilon}{4} (\ln |x| + 1 + 2 \ln 2) + c|x|^{-2-\delta} \quad (2.11)$$

for some $c > 0$ and all $|x|$ large enough. To this end we decompose $\tilde{\phi}_{R_1} = \phi_1 + \phi_2$ corresponding to the integration over $|x - y| \leq R_1$ and $|x - y| > R_1$, respectively. The decay assumption yields

$$|\phi_1(x)| \leq \frac{1}{2\pi} \int_{|x-z| > R_1, |z| \leq R_1} c_1 |x - z|^{-2-\delta} \ln |z| d^2 z$$

for some $c_1 > 0$; we have used here the change of variable $x - y = z$. We have $|x - z|^{-2-\delta} \leq (|x| - R_1)^{-2-\delta} \leq |x|^{-2-\delta}$ for $|x| > R_1$, and therefore $\phi_1(x) = \mathcal{O}(|x|^{-2-\delta})$ as $|x| \rightarrow \infty$. Without loss of generality we may suppose that $R_1 > 1$ and $|x| \geq 1$. Since $|x - y| \leq |x| + |y| \leq (1 + |x|)(1 + |y|)$, the remaining part $\phi_2(x)$ is then in view of

$$0 \leq \ln |x - y| \leq \ln(1 + |x|) + \ln(1 + |y|) \leq 2 \ln 2 + \ln |x| + \ln |y|$$

and of (2.8) estimated by the first term at r.h.s. of (2.11). Putting now (2.9)-(2.11) together we find

$$|\phi(x) - F \ln |x|| < \frac{1}{2} \varepsilon \ln |x| + \frac{\varepsilon}{4} (1 + 2 \ln 2) + c_2 |x|^{-1}$$

with a suitable c_2 . There is an R_2 such that the sum of the last two terms is smaller than $\frac{1}{4} \varepsilon \ln |x|$ for $|x| > R_2$, so it is sufficient to set $R := \max\{1, R_1, R_2\}$. ■

The above assumptions allow us to prove a stronger claim about the regularity of ϕ . Let us first recall two definitions [1]. Given an open ball $\mathcal{B}^{(1)}$ centered at $x \in \mathbb{R}^n$ and an open ball $\mathcal{B}^{(2)}$ not containing x , the set $C_x = \mathcal{B}^{(1)} \cap \{x + \lambda(y - x); y \in \mathcal{B}^{(2)}, \lambda > 0\}$ is called a finite cone having vertex at x . An open domain $\Omega \subset \mathbb{R}^n$ has the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C . In particular, every non-empty open ball in \mathbb{R}^n has the cone property. We shall employ the Sobolev imbedding theorem (*cf.* the case C of the part 1 of Theorem 5.4 in [1]) for the sets

$$\begin{aligned} W^{m,p}(\Omega) &= \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}, \\ C_B^j(\Omega) &= \{u \in C^j(\Omega) : D^\alpha u \text{ is bounded on } \Omega \text{ for } |\alpha| \leq j\}. \end{aligned}$$

Lemma 2.3 *Let Ω be a domain in \mathbb{R}^n . Suppose that j and m are non-negative integers and $1 \leq p < \infty$, then the imbedding $W^{j+m,p}(\Omega) \rightarrow C_B^j(\Omega)$ exists provided $mp > n$.*

Lemma 2.4 (cf. [8, Thm. 9.9]) *Let Ω be a bounded domain in \mathbb{R}^2 and $f \in L^p(\Omega)$ with $1 < p < \infty$. Define $w(x) = \int_{\Omega} \Gamma(x-y)f(y) dy$, where $\Gamma(x) = \frac{1}{2\pi} \ln|x|$; then $w \in W^{2,p}(\Omega)$.*

Now we can state the indicated result:

Proposition 2.5 *Under the assumptions (A.1) and (A.2), ϕ is continuous in \mathbb{R}^2 .*

Proof: For arbitrary $x_0 \in \mathbb{R}^2$ and $R > 0$, we put $\mathcal{B}_R(x_0) = \{x \in \mathbb{R}^2; |x - x_0| < R\}$. We split ϕ as follows:

$$\phi(x) = \frac{1}{2\pi} \int_{\mathcal{B}_{2R}(x)} B(y) \ln|x-y| d^2y + \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \mathcal{B}_{2R}(x)} B(y) \ln|x-y| d^2y.$$

Since $\mathcal{B}_{2R}(x)$ has the cone property, the first term at the r.h.s. is in $W^{2,1+\epsilon}(\mathcal{B}_{2R}(x))$, and thus also in $C_B^0(\mathcal{B}_{2R}(x))$ by the preceding two lemmas. On the other hand, $\ln|x_1 - y| - \ln|x_2 - y| < \ln 3$ holds for any $x_1, x_2 \in \mathcal{B}_R(x)$ and any $y \in \mathbb{R}^2 \setminus \mathcal{B}_{2R}(x)$, so continuity of the second term follows by the Lebesgue dominated-convergence theorem. ■

Remark 2.6 *Proposition 2.5 can be proven in an alternative way. We define a probability measure $\mu(dx)$ on \mathbb{R}^2 by*

$$\mu(dx) := \frac{1}{2\pi\mathcal{N}} |B(x)| \langle x \rangle_c^{\delta/2} d^2x,$$

where $\mathcal{N} := \frac{1}{2\pi} \int_{\mathbb{R}^2} |B(y)| \langle y \rangle_c^{\delta/2} d^2y$ is the normalization factor, and a family of random variables $\{L_x\}_{x \in \mathcal{B}_\eta(x_0)}$ by

$$L_x(y) := \langle y \rangle_c^{-\delta/2} \ln|x-y|, \quad y \in \mathbb{R}^2, x \in \mathcal{B}_\eta(x_0).$$

of which we can check that it is uniformly integrable, i.e.,

$$\lim_{a \rightarrow \infty} \sup_{x \in \mathcal{B}_\eta(x_0)} \int_{\{y \mid |L_x(y)| \geq a\}} |L_x(y)| \mu(dy) = 0.$$

The argument leading to the last claim is based on simple estimates but it is lengthy and we skip the details. The relation

$$\lim_{x \rightarrow x_0} |\phi(x) - \phi(x_0)| = \mathcal{N} \lim_{x \rightarrow x_0} \int_{\mathbb{R}^2} |L_x(y) - L_{x_0}(y)| \mu(dy) = 0,$$

then follows from the abstract result given in [9, Theorem 3.7.4] or [10, Prop. II.5.4].

We will also need a bound on the vector potential, or equivalently, on the gradient of the potential (2.1). Its components are given by

$$(\partial_i \phi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x-z) \frac{z_i}{|z|^2} d^2 z, \quad (2.12)$$

at least for large enough $|x|$ where B is bounded. While in general they behave as $\mathcal{O}(|x|^{-1})$, in case of zero flux we have a stronger result.

Proposition 2.7 *In addition to the stated integrability and decay assumptions, suppose that $\int_{\mathbb{R}^2} B(y) d^2 y = 0$; then there is $\mu > 0$ such that $(\nabla \phi)(x) = \mathcal{O}(|x|^{-1-\mu})$ as $|x| \rightarrow \infty$.*

Proof: Consider $(\partial_1 \phi)(x)$; the argument for the other component is similar. We write it as $\sum_{j=1}^4 A_2^{(j)}(x)$, where the different contributions correspond to integration over the regions where $|x-z|$ and $|z|$ are respectively smaller and greater than R_3 . The last named number depends on $|x|$ and will be specified later.

Since $|z| \geq ||x| - |x-z||$, the term $A_2^{(1)}(x)$ with $|x-z| \leq R_3$ and $|z| \leq R_3$ is zero provided

$$|x| > 2R_3. \quad (2.13)$$

The term $A_2^{(2)}(x)$ obtained by changing the first inequality to $|x-z| > R_3$ is estimated easily as

$$\left| A_2^{(2)}(x) \right| \leq \frac{c_1}{2\pi} R_3^{-2-\delta} \int_{|z| \leq R_3} \frac{d^2 z}{|z|} = c_1 R_3^{-1-\delta}. \quad (2.14)$$

The third term corresponding to integration over $M_3 := \{z : |x-z| \leq R_3, |z| > R_3\}$ is the most complicated. Combining the decay and the zero-flux assumptions we get

$$\left| \int_{|y| \leq R_3} B(y) d^2 y \right| = \left| \int_{|y| > R_3} B(y) d^2 y \right| \leq \frac{2\pi c_1}{\delta} R_3^{-\delta}. \quad (2.15)$$

Next we split the field into the positive and negative part, $B = B_+ - B_-$, and write

$$A_2^{(3)}(x) = \frac{1}{2\pi} \int_{M_3} B_+(x-z) \frac{z_i}{|z|^2} d^2 z - \frac{1}{2\pi} \int_{M_3} B_-(x-z) \frac{z_i}{|z|^2} d^2 z.$$

It is straightforward to check that $|z_1| |z|^{-2} - |x|^{-1} \cos \theta \leq 5R_3 |x|^{-2}$ holds for $R_3 |x|^{-1}$ small enough, where θ is the angle corresponding to x in polar coordinates. We use this inequality to get an upper and lower bound to $z_i |z|^{-2}$ in the above integrals. Then we add and subtract $\frac{\cos \theta + 5R_3 |x|^{-1}}{2\pi |x|} \int_{M_3} B_-(x-z) d^2 z$ obtaining thus

$$\begin{aligned} A_2^{(3)}(x) &\leq \frac{\cos \theta + 5R_3 |x|^{-1}}{2\pi |x|} \int_{M_3} B(x-z) d^2 z + \frac{5R_3}{\pi |x|^2} \int_{M_3} B_-(x-z) d^2 z \\ &\leq \frac{2c_1}{\delta} R_3^{-\delta} \frac{\cos \theta}{|x|} + \frac{c_3}{|x|^2} R_3 \end{aligned} \quad (2.16)$$

and an analogous lower bound, where in the second step we have used (2.15) and the integrability of B . If we choose $R_3 = |x|^{1-\eta}$ for $\eta < 1$ we get

$$\left| A_2^{(3)}(x) \right| \leq c_4 \max\{|x|^{-1-\delta(1-\eta)}, |x|^{-1-\eta}\} \quad (2.17)$$

for some $c_4 > 0$ and large $|x|$; at the same time the condition (2.13) will be satisfied. The remaining term with $|x - z| > R_3$ and $|z| > R_3$ is estimated by

$$\left| A_2^{(4)}(x) \right| \leq \frac{1}{2\pi R_3} \int_{|x-z|>R_3} B(x-z) d^2z \leq \frac{c_1}{R_3} \int_{R_3}^{\infty} r^{-1-\delta} dr = \frac{c_1}{\delta} R_3^{-1-\delta}.$$

With our choice, $R_3 = |x|^{1-\eta}$, we get from here and (2.14)

$$\max \left\{ \left| A_2^{(2)}(x) \right|, \left| A_2^{(4)}(x) \right| \right\} \leq c_5 |x|^{-1-\delta+\eta(1+\delta)}, \quad (2.18)$$

so it is sufficient to set $\eta < \delta(1 + \delta)^{-1}$ to get a decay power smaller than -1 . ■

3 The main result

Now we are ready to extend the result of [3] about the existence and number of bound states to fields without a compact support.

Theorem 3.1 *Let B be nonzero, satisfying (A.1) and (A.2), and let the corresponding flux be $F = N + \eta$ for some $N \in \mathbb{N}_0$ and $\eta > 0$. Then the operator $H_P^{(-)}(A)$ has for $g > 2$ at least $N + 1$ isolated eigenvalues in $(-\infty, 0)$, multiplicity being counted.*

Proof: First we need to know that the essential spectrum covers the positive halfline. Since the last term in (2.2) can be viewed as a potential which is Δ -compact, it follows from [7, Thm. 6.1] and [11, Sec. XIII.4] that

$$\sigma_{\text{ess}}(H_P^{(\pm)}(A)) = [0, \infty). \quad (3.1)$$

In view of the minimax principle, it is then sufficient to find an $(N + 1)$ -dimensional subspace in $L^2(\mathbb{R}^2)$ on which the quadratic form

$$\psi \mapsto (\psi, H_P^{(-)}(A)\psi) = \int_{\mathbb{R}^2} |(D\psi)(x)|^2 d^2x - \frac{1}{2} (g - 2) \int_{\mathbb{R}^2} B(x) |\psi(x)|^2 d^2x$$

is negative. We will employ trial functions ψ_α of the following form

$$\psi_\alpha(x) = \sum_{j=0}^N \alpha_j (f_\varrho(r) \chi_j(x) + \varepsilon h_j(x)) \quad (3.2)$$

with $\alpha \in \mathbb{C}^{N+1}$; it is clearly sufficient to consider the unit sphere, $|\alpha| = 1$. Here f_ϱ is a mollifier which will be chosen as $f_\varrho(x) := f(|x|/\varrho)$ for a real-valued function

$f \in C_0^\infty(\mathbb{R}_+)$ such that $f(u) = 1$ for $u \leq 1$ and $f(u) = 0$ for $u \geq 2$. The functions $h_j \in C_0^\infty(\mathcal{B}_\varrho)$ will be specified later. By a direct computation,

$$\begin{aligned}
(\psi_\alpha, (D^*D + \mu B)\psi_\alpha) &= \sum_{j,k=0}^N \bar{\alpha}_j \alpha_k \left\{ \int_{\bar{\mathcal{B}}_\varrho} |f'_\varrho(r)|^2 (\bar{\chi}_j \chi_k)(x) d^2x \right. \\
&+ \varepsilon^2 \int_{\mathcal{B}_\varrho} (D\bar{h}_j)(x)(Dh_j)(x) d^2x + \mu \left[\int_{\mathbb{R}^2} (f_\varrho^2 B \bar{\chi}_j \chi_k)(x) d^2x \right. \\
&\left. \left. + \varepsilon \int_{\mathcal{B}_\varrho} ((\bar{h}_j \chi_k + \bar{\chi}_j h_k)B)(x) d^2x + \varepsilon^2 \int_{\mathcal{B}_\varrho} (B\bar{h}_j h_k)(x) d^2x \right] \right\} \quad (3.3)
\end{aligned}$$

where we have employed $D\chi_j = 0$ together with the fact that h_j and f'_ϱ have by construction disjoint supports: $D\Sigma_j \alpha_j f_\varrho \chi_j = 0$ holds inside \mathcal{B}_ϱ so $D\psi_\alpha = \varepsilon \Sigma_j \alpha_j Dh_j$ there, while outside we have instead $D\psi_\alpha = Df_\varrho \Sigma_j \alpha_j \chi_j = \psi_\alpha(-ix_1 + x_2)|x|^{-1} f'_\varrho$. We have to show that the r.h.s. is negative as long as $\mu < 0$, in particular, for $\mu = -\frac{1}{2}(g-2)$.

The mollifier is necessary since the sum (3.2) contains in general terms which are not L^2 . The corresponding contribution to the energy form, *i.e.*, the first term at the r.h.s. of (3.3) is positive and we have to make it small. Since f'_ϱ is supported in $\mathcal{B}_{2\varrho}$, it follows from Proposition 2.2 that

$$\frac{1}{\varrho^2} \int_{\bar{\mathcal{B}}_\varrho} \left| f' \left(\frac{|x|}{\varrho} \right) \right|^2 \left| \sum_{j=0}^N \alpha_j \chi_j(x) \right|^2 d^2x \leq \frac{4\pi \|f'\|_\infty^2}{1 + \varepsilon + N - F} (2\varrho)^{2(N-F+\varepsilon)} \quad (3.4)$$

provided $\varrho > R$. Without loss of generality we may assume $\eta \in (0, 1]$. Choosing then $\varepsilon \in (0, \eta)$, we obtain a bound which tends to zero as $\varrho \rightarrow \infty$, and therefore it allows us to handle the trial function tails.

The main part of the argument consists of checking that there exists a positive constant β such that

$$\int_{\mathbb{R}^2} B(x) \left| f_\varrho(x) \sum_j \alpha_j \chi_j(x) \right|^2 d^2x > \beta \quad (3.5)$$

holds for ϱ large enough and any α . We shall do it by *reductio ad absurdum* assuming the opposite. Now we have to specify the functions h_j . We set $h_j := h\chi_j$ for a real-valued $h \in C_0^\infty(\mathbb{R}_+)$, in which case the next term linear in ε acquires the form

$$2\varepsilon \int_{\mathbb{R}^2} \left| \sum_{j=0}^N \alpha_j \chi_j(x) \right|^2 h(x) B(x) d^2x.$$

Since B is nonzero by assumption, and $\sum_j \alpha_j \chi_j$ is a product of a positive function $e^{-\phi}$ and a polynomial having thus at most isolated zeros, one can choose h in such a way that the last expression is negative for any α . Moreover, as a continuous

function of α on the surface of a hypersphere it reaches a minimum there which is also negative. This implies that the sum of the second, fourth and fifth terms of Eq. (3.3), denoted as S , tends to 0 from below as ε tends to 0. Hence there is a number $\beta > 0$ such that for ϱ large enough and any α , one can find $h = h_{\alpha, \varrho}$ and $\varepsilon_{\alpha, \varrho}$ for which $S = -2\mu\beta$ holds. Suppose that

$$\int_{\mathbb{R}^2} B(x) \left| f_{\varrho}(x) \sum_j \alpha_j \chi_j(x) \right|^2 d^2x \leq \beta \quad (3.6)$$

holds true. Choosing then $h_{\alpha, \varrho}$ and $\varepsilon_{\alpha, \varrho}$ in the described way, we get

$$S + \text{the third term of Eq. (3.3)} \leq -2\mu\beta + \mu\beta = -\mu\beta < 0. \quad (3.7)$$

However, in view of (3.4) we have

$$\text{the first term of Eq. (3.3)} \leq \frac{4\pi \|f'\|_{\infty}^2}{1 + \varepsilon - \eta} (2\varrho)^{-2(\eta - \varepsilon)} \rightarrow 0 \quad (3.8)$$

as ϱ tends to ∞ , so the r.h.s. of (3.3) is negative for ϱ large enough. The argument can be carried over for any fixed value of μ , in particular, for $\mu = 2$. In that case, however, the supersymmetry property, $D^*D + 2B = DD^*$, applied to the l.h.s. of (3.3) leads to the absurd conclusion $\|D^*\psi_{\alpha}\|^2 < 0$, proving thus Eq. (3.5).

This means that the trial functions can be finally chosen in the form (3.2) with $\varepsilon = 0$. The energy form is then estimated by

$$(\psi_{\alpha}, H_P^{(-)}(A)\psi_{\alpha}) < \frac{4\pi \|f'\|_{\infty}^2}{1 + \varepsilon - \eta} (2\varrho)^{-2(\eta - \varepsilon)} - \frac{1}{2} (g - 2) \int_{\mathbb{R}^2} B(x) |\psi_{\alpha}(x)|^2 d^2x, \quad (3.9)$$

where the second term at the r.h.s. is smaller than $-\frac{1}{2}(g - 2)\beta$ and dominates for ϱ large enough. With our choice of the mollifier, ψ_{α} is within \mathcal{B}_{ϱ} just a linear combination of the Aharonov-Casher states (2.4). Since the latter are easily seen to be linearly independent we have accomplished the task of construction the sought $(N + 1)$ -dimensional subspace. ■

4 Zero flux case

In distinction to the analogous result in [3], Theorem 3.1 says nothing about the situation when $F = 0$. For radially symmetric strong and weak fields the bound state existence is established in [4] and [3], respectively. For weak fields without the rotational symmetry we can employ the method of Sec. 6 in [3], but without the assumption about the decay of $\nabla\phi$ used there. We need only a slightly stronger regularity requirement:

$$\text{(A.2')} \quad B \in L_{\text{loc}}^2(\mathbb{R}^2).$$

Recall that the said idea in [3] is based on the weak-coupling behaviour of two-dimensional Schrödinger operators with a potential depending on a coupling constant in a nonlinear way, specifically

$$H(\lambda) = -\Delta + \lambda V_1(x) + \lambda^2 V_2(x) \quad (4.1)$$

with $V_j \in L^{1+\delta}(\mathbb{R}^2) \cap L(\mathbb{R}^2, (1 + |x|^\delta) d^2x)$, $j = 1, 2$.

Lemma 4.1 [3, Sec. 4] *Suppose that $\int V_1(x) d^2x = 0$ and define*

$$\gamma_2 \equiv \gamma_2(V_1, V_2) := \frac{1}{2\pi} \int V_2(x) d^2x + \frac{1}{4\pi^2} \int V_1(x) \ln |x - y| V_1(y) d^2x d^2y. \quad (4.2)$$

The operator (4.1) has a weakly bound state for small nonzero λ iff the quantity (4.2) is negative. In that case the eigenvalue is $\epsilon(\lambda) = -e^{2/u(\lambda)}$ with $u(\lambda) = \gamma_2 \lambda^2 + \mathcal{O}(\lambda^3)$.

Lemma 4.2 *Under (A.1) and (A.2') the function $\phi \in W^{1,2}(\mathbb{R}^2)$.*

Proof: The function $\phi = \frac{1}{2\pi} B * \ln |\cdot|$ belongs to the first Sobolev space if the integral $\int (1 + |k|^2) |\hat{B}(k)|^2 |k|^{-2} d^2k$ is finite. The assumptions imply $B \in L^2$, and therefore also $\hat{B} \in L^2$; hence we have to check only its convergence around $k = 0$. We have $\hat{B}(0) = F = 0$, so

$$\hat{B}(k) = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(x) (e^{ikx} - 1) d^2x.$$

Further we decompose $\mathbb{R}^2 = \mathcal{B}_R \cup \tilde{\mathcal{B}}_R$ as in the proof of Proposition 2.2 with the circular boundary situated in the region where B is bounded. We estimate $|e^{ikx} - 1|$ by kR in the inner region and by $2|kx|^\eta$ with $\eta \in (0, 1)$ outside obtaining

$$|\hat{B}(k)| \leq c_1 |k| + \frac{1}{\pi} |k|^\eta \int_{\tilde{\mathcal{B}}_R} |x|^\eta |B(x)| d^2x$$

for some $c_1 > 0$. Choosing now η sufficiently small we can make the last integral finite; this yields $|\hat{B}(k)|^2 = \mathcal{O}(k^{2\eta})$ around the origin. ■

Now we can prove the following result.

Theorem 4.3 *Let a nonzero B with $F = 0$ satisfy (A.1) and (A.2'). Then each of the operators $H_P^{(\pm)}(\lambda A)$ with $g > 2$ has for small nonzero λ a bound state whose energy satisfies the bound*

$$\epsilon^{(\pm)}(\lambda) < - \exp \left\{ - \left(\frac{c\lambda^2}{16\pi} (g^2 - 4) \int_{\mathbb{R}^2} A(x)^2 d^2x \right)^{-1} \right\} \quad (4.3)$$

for any fixed $c \in (0, 1)$ and λ small enough.

Proof: It is established in [3] that the gradient term $2iA \cdot \nabla$ does not contribute to the energy form for real-valued functions, and therefore $H_p^{(\pm)}(\lambda A)$ can be estimated from above by the operators (4.1) with

$$V_1(x) = \pm \frac{g}{2} B(x), \quad V_2(x) = A(x)^2. \quad (4.4)$$

It remains to evaluate the coefficient (4.2). Since $|A|$ is square integrable by the preceding lemma and $|A(x)| = |(\nabla\phi)(x)|$, the first Green identity together with the equation $\Delta\phi = B$ and the Gauss theorem yield

$$\int_{\mathbb{R}^2} A(x)^2 d^2x = \lim_{R \rightarrow \infty} \oint_{\partial\mathcal{B}_R} \phi(x)(\nabla\phi)(x) \cdot d\vec{\sigma}(x) - \lim_{R \rightarrow \infty} \int_{\mathcal{B}_R} \phi(x)B(x) d^2x. \quad (4.5)$$

Substituting from (2.1) to the last term we see that it remains to establish that the first term at the r.h.s. vanishes as $R \rightarrow \infty$. However, this follows readily from Propositions 2.2 and 2.7. ■

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