A Model of Interband Radiative Transition

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We consider a simple model which is a caricature of a crystal interacting with a radiation field. The model has two bands of continuous spectrum and the particle can pass from the upper one to the lower by radiating a photon, the coupling between the excited and deexcited states being of a Friedrichs type. Under suitable regularity and analyticity assumptions we find the continued resolvent and show that for weak enough coupling it has a curve-type singularity in the lower halfplane which is a deformation of the upper-band spectral cut. We then find a formula for the decay amplitude and show that for a fixed energy it is approximately exponential at intermediate times, while the tail has a power-like behaviour.

I Introduction

A rigorous description of decay and resonance processes in quantum theory has a long history starting from the Friedrichs model presented in [1] and discussed later in numerous papers – see, e.g. [2], [3]. A systematic study of the problem started in the seventies. J. Howland and H. Baumgärtel with collaborators – see [4], [5], [6], [7] and the papers quoted there – used operator methods to establish the existence of resonance poles and to prove the Fermi rule for various systems with perturbed embedded eigenvalues. At the same time the seminal paper [8] by J. Aguilar and J.-M. Combes initiated the development of complex-scaling methods which are nowadays a very efficient tool to study resonances of Schrödinger operators.

In the eighties many papers dealing with quantum-field decay models appeared. A phenomenological models based on the Langevin equation were investigated in [9], [10], [11] and [12]. Moreover a generalization of them was given by A. Arai [13] within the Hamiltonian formalism. In a last few years the long-time behavior of canonical correlation functions for general Hamiltonians was investigated in [14] by applying the results of [13] and [15] via a quantum Langevin equation. From the point of view of virtual transitions, the long-time behavior of a correlation function was studied in [16]. It is also worth of noticing that, revisiting the decay problem, Bach, Fröhlich, and Sigal have developed a new manner to analyze the resonace problems for a class of models in quantum electrodynamics [17], [18].

In most of these models the unstable states come from perturbation of eigenvalues, either embedded in the continuous spectrum or isolated as in the case of Stark effect. Much less attention has been paid to the situation when the states which should decay belong to the continuous spectrum of the unperturbed Hamiltonian. An archetypal example of such a situation is a crystal in which an electron can radiate a photon and pass to a lower spectral band. A natural model in this case would be a Schrödinger operator with a periodic potential coupled to a quantized field. This is not easy, however. To start with a simpler case, we discuss in this paper a model of Friedrichs type with transitions between two bands of the absolutely continuous spectra which can be regarded as a one-photon approximation of the more realistic description.

While perturbed embedded eigenvalues typically give rise to resonance poles in the analytically continued resolvent, we are going to show that in the mentioned model the cut-like singularity corresponding to the "excited" spectral band gets deformed to the lower complex halfplane. Recall that a similar behavior has been observed in a completely different type of systems which involve a perturbation of a band spectrum, namely for scattering in finitely periodic systems [19]. Here we have a situation with a finite number of resonances which accumulate, however, along curves in the lower halfplane which are close to the spectral bands of the infinite system when the interaction is weak.

Let us describe briefly the contents of the paper. After formulating the model in the next section we shall compute in Section III the projection of the Hamiltonian resolvent onto the subspace of excited states corresponding to the upper spectral band of the "crystal". Under natural regularity assumptions we prove the mentioned claim about the change of the spectral singularity caused by a decay with the radiation of a "photon".

Then we turn to the time evolution of the undecayed state and show that its projection onto the upper-band subspace is – at least for a weak enough coupling – realized as multiplication by a function which we evaluate explicitly. The rest of the paper is devoted to properties of this decay amplitude. We show that in the weak-coupling case the latter is dominated at intermediate times by an exponential function. Hence the population of the excited spectral band changes in the course of the evolution: the wavefunction components supported in the regions where the deformed singularity is closer to the real axis survive longer. On the other hand, similarly to the usual decay theory, the deexcitation process cannot be purely exponential; we show that the decay amplitude has a power-like tail at long times.

II Description of the model

The "crystal part" of our model is assumed to have the simplest nontrivial spectrum consisting of a pair of disjoint absolutely continuous bands $I_0 = [\xi_0^{(-)}, \xi_0^{(+)}]$ and $I_1 = [\xi_1^{(-)}, \xi_1^{(+)}]$ with $-\infty < \xi_0^{(-)} < \xi_0^{(+)} < \xi_1^{(-)} < \xi_1^{(+)} < \infty$. Using the spectral representation [20] we can assume without loss of generality that the crystal state space is $L^2(I_1 \cup I_0, w(x)\, dx)$ with the Hamiltonian H_c acting as multiplication by the variable x; the weight function w is positive a.e., Lebesgue integrable, and satisfies

$$\int_{I_1 \cup I_0} w(x)\, dx = 1\,.$$

As we have said the “field part” is represented by the vacuum and one-photon (or phonon) states, which coexist with the upper and lower band of the “crystal”, respectively. The photon vacuum is by assumption a single state of zero energy, while the single-photon states belong to the space $L^2([\nu,\infty),\omega(z)\,dz)$, $\nu\ge 0$, on which the free Hamiltonian H_p acts as a multiplication by the variable z. The weight function ω is again Lebesgue integrable, non-negative a.e., and satisfies

$$\int_\nu^\infty \omega(z)\,dz = 1\,.$$

Putting the two components together we get the total state space of our model in the form

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 := L^2(I_1, w_1(x)\,dx) \oplus [L^2(I_0, w_0(y)\,dy) \otimes L^2(K, \omega(z)\,dz)]\,, \quad (2.1)$$

where $K = [\nu,\infty)$ and $w_\alpha := w\restriction I_\alpha$, $\alpha = 0,1$. The free Hamiltonian acts as

$$H_0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} H_c f \\ (H_c \otimes I + I \otimes H_p) g \end{pmatrix}$$

which means

$$\left(H_0 \begin{pmatrix} f \\ g \end{pmatrix} \right) \begin{pmatrix} x \\ y,z \end{pmatrix} = \begin{pmatrix} x f(x) \\ (y+z) g(y,z) \end{pmatrix} \quad (2.2)$$

with the arguments $x \in I_1$, $y \in I_0$, and $z \in K$.

Next we have to choose the interaction part of the Hamiltonian. Being inspired by the Friedrichs model we require

(i) the interaction includes necessarily a single photon emission/absorption, or in other words, the projections of $H_{\rm int}$ on $L^2(I_1, w_1(x)\,dx)$ and its orthogonal complement in $\mathcal{H}$ are zero,

(ii) the interaction is “minimal” in the sense that the action of $H_{\rm int}$ can be written in terms of multiplication by a ”formfactor”, integration, and possibly a change of variables.

It follows from *(i)* that $H_{\rm int} = \kappa L$ with an interaction constant κ and an “off-diagonal” operator L, where $L_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, i.e.

$$\left(H_{\rm int} \begin{pmatrix} f \\ g \end{pmatrix} \right) \begin{pmatrix} x \\ y,z \end{pmatrix} = \kappa \begin{pmatrix} (L_{01} g)(x) \\ (L_{10} f)(y,z) \end{pmatrix}\,. \quad (2.3)$$

Furthermore, in accordance with *(ii)* the operator L_{10} should be chosen in the form

$$(L_{10}f)(y,z) = \lambda(y,z)\, f(u(y,z))\,, \tag{2.4}$$

where $\lambda : I_0 \times K \to \mathbb{C}$ and $u : I_0 \times K \to I_1$ are functions containing the dynamical information about the system. This choice in turn restricts L_{01} because the full Hamiltonian (with a real coupling constant κ) must be symmmetric, which means

$$\int_{I_1} \overline{f(x)}(L_{01}g)(x)w_1(x)\,dx = \iint_{I_0\times K} \overline{\lambda(y,z)f(u(y,z))}g(y,z)w_0(y)\omega(z)\,dy\,dz \tag{2.5}$$

for all f and g from the operator domain. Suppose now that there are functions u, v such that $(y,z) \mapsto (u(y,z), v(y,z)) : I_0 \times K \to I_1 \times K$ is a bijective diffeomorphism which can be used as a substitution at the r.h.s. of (2.5) leading to

$$(L_{01}g)(x)w_1(x) = \int_K \overline{\lambda(y,z)}g(y,z) \left|\frac{D(y,z)}{D(u,v)}\right| w_0(y)\omega(z)\,dt\,, \tag{2.6}$$

the variables y, z being expressed as the inverse of $x = u(y,z)$ and $t = v(y,z)$ at the r.h.s.

Remarks 2.1 (a) For the sake of simplicity, assume that u depends on a single variable mapping I_0 onto I_1. This will reduce the dependence of the transition between a pair of states in I_1 and I_0, respectively, on the photonic component of the system.
(b) In the same vein we could suppose that

$$\lambda(y,z) = \lambda_0(y)\lambda_K(z) \tag{2.7}$$

which will turn $H_0 + H_{\mathrm{int}}$ – up to the isomorphism between I_1 and I_0 – into a direct integral of Friedrichs-type Hamiltonians. However, we choose a nontrivial setup and do not require that the dependence of the interaction strength on the energies of the excited state and the photon contained in the function λ factorizes. In other words, we will keep a general $\lambda : I_0 \times K \to \mathbb{C}$.

After this heuristic discussion, let us define the Hamiltonian which we shall consider in the following. We suppose that

(a1) $u : I_0 \to I_1$ is a bijective C^1-diffeomorphism,

then the interaction term $H_{\rm int}$ acts according to (2.3) with

$$\begin{aligned}(L_{10}f)(y,z) &:= \lambda(y,z)f(u(y))\,,\\ (L_{01}g)(x) &:= \frac{w_0(u^{-1}(x))}{|u'(u^{-1}(x))|w_1(x)}\int_K \overline{\lambda(u^{-1}(x),z)}g(u^{-1}(x),z)\omega(z)\,dz \end{aligned} \tag{2.8}$$

with $x\in I_1$, $y\in I_0$, and $z\in K$. The second expression makes sense because the two factors in the denominator are positive a.e. by assumption. The operator L defined in this way is formally symmetric and unbounded in general. To get a self-adjoint Hamiltonian we add a boundedness assumption. Specifically, we assume that

(a2) λ is Lebesgue measurable in $I_0\times K$ and there are positive C, C_1 such that

$$\int_K |\lambda(y,z)|^2\omega(z)\,dz \le C\,, \qquad w_0(y)\le C_1\,|u'(y)|w_1(u(y))$$

holds for every $y\in I_0$;

the last inequality means that the Radon-Nikodým derivative appearing as the first factor in $L_{01}g$ is bounded.

Proposition 2.2 *Under the assumptions (a1) and (a2),* $H_{\rm int}$ *is bounded and symmetric. Consequently,*

$$H=H(\kappa)=H_0+H_{\rm int}=H_0+\kappa L$$

is self-adjoint on the domain of H_0*.*

Proof: It remains to verify the boundedness of $H_{\rm int}$ which amounts to checking that the operators $L_{10}:\mathcal{H}_0\to\mathcal{H}_1$ and $L_{01}:\mathcal{H}_1\to\mathcal{H}_0$ are bounded. This is easily seen from the following estimates:

$$\begin{aligned}\|L_{10}f\|^2_{I_0\times K} &= \iint_{I_0\times K}|\lambda(y,z)|^2|f(u(y))|^2 w_0(y)\omega(z)\,dy\,dz\\ &\le C\int_{I_1}|f(x)|^2\frac{w_0(u^{-1}(x))}{|u'(u^{-1}(x))|}\,dx \,\le\, C_1C\|f\|^2_{I_1}\end{aligned}$$

and

$$
\begin{aligned}
\|L_{01}g\|^2_{I_1} &= \int_{I_1}\left[\frac{w_0(u^{-1}(x))}{u'(u^{-1}(x))w_1(x)}\right]^2\left[\int_K \lambda(u^{-1}(x),z)\overline{g(u^{-1}(x),z)}\omega(z)\,dz\right] \\
&\quad \times\left[\int_K \overline{\lambda(u^{-1}(x),t)}g(u^{-1}(x),t)\omega(t)\,dt\right]w_1(x)\,dx \\
&= \int_{I_0}\frac{w_0(y)^2}{|u'(y)|w_1(u(y))} \\
&\quad \times\left\{\iint_{K\times K}\left[\lambda(y,z)\overline{\lambda(y,t)}\right]\left[\overline{g(y,z)}g(y,t)\right]\omega(z)\omega(t)\,dz\,dt\right\}dy \\
&\le C_1\int_{I_0}\left[\int_K|\lambda(y,z)|^2\omega(z)\,dz\right]\left[\int_K|g(y,t)|^2\omega(t)\,dt\right]w_0(y)\,dy \\
&\le C_1C\|g\|^2_{I_0\times K},
\end{aligned}
$$

where we have used the Fubini theorem in combination with the Schwarz inequality for the scalar product in $L^2(K\times K,\omega(z)\omega(t)\,dz\,dt)$. ∎

Before proceeding further let us make a comment on the assumptions, part physical and part technical, which we will have to make in the following. Since the present model is rather a motivation study for a more realistic one, we do not strive for the maximal possible generality. On the other hand, we do not want to impose many unnecessary restrictions which would correspond to a fully specific system such as the one given below.

Example 2.3 Let $E_j(\cdot)$, $j=0,1$ be the lowest two dispersion curves of a one-dimensional crystal. Since we are discussing a caricature model, we neglect the multiplicity of the eigenvalues. In other words, we consider just a half of the Brillouin zone and regard E_j as maps $[0,\pi]\to I_j$ with E_0 strictly increasing and E_1 strictly decreasing. Moreover, both are restrictions to $[0,\pi]$ of real-analytic functions with the first derivatives vanishing at the endpoints of the interval but nonzero in its interior.

To rewrite the band projections of the crystal Hamiltonian in our formalism, we employ the operators $U_j : L^2([0,\pi])\to L^2(I_j,w_j(y)\,dy)$ defined by $(U_jf)(y):=f(E_j^{-1}(y))$; the definition makes sense since the inverse functions E_j^{-1} exist by assumption. The operators U_j are unitary provided we put

$$
w_j(y)=|E_j'(E_j^{-1}(y))|^{-1}\,. \tag{2.9}
$$

These functions are C^∞ in $(0,\pi)$ with singularities at the endpoints but the latter are integrable. In particular, if $E_j''(\vartheta) \neq 0$ at $\vartheta = 0,\pi$ we have $w_j(y) = O\left(|y - \xi_j^{(\pm)}|^{-1/2}\right)$ there.

One of the basic ingredients is, of course, the function u. Since the system of the crystal plus the radiation field is invariant w.r.t. the discrete group of translations on a multiple of the lattice constant, it is natural in the present example to suppose that the interaction does not couple states whose quasimomentum support in the upper and lower bands are disjoint. This is achieved if we choose

$$u(y) = E_1\left(E_0^{-1}(y)\right) \; ; \tag{2.10}$$

it is easy to see that it is a C^∞ function and

$$u'(y) = \frac{E_1'\left(E_0^{-1}(y)\right)}{E_0'\left(E_0^{-1}(y)\right)} \tag{2.11}$$

has finite limits at $\xi_0^{(\pm)}$ assuming that E_0 and E_1 have the first non-vanishing derivative at 0 resp. π of the same order. On the other hand we think of the radiation field as of the electromagnetic field in the rotating wave approximation. In this case we put the threshold energy $\nu = 0$ and $\omega(z) = \chi_{[0,\nu_{\max}]}(z)$ where $\nu_{\max}$ is a possible ultraviolet cut-off.

Under these model assumptions *(a1)* is satisfied automatically and the same is true for the second part of *(a2)*; it follows from (2.9) and (2.11) that it is valid for any $C_1 \geq 1$. The only remaining restriction is thus the boundedness condition $\int_0^{\nu_{\max}} |\lambda(y,z)|^2\, dz \leq C$ for the formfactor.

III The resolvent

As usual the spectral information is contained in the resolvent of the Hamiltonian. Under our assumptions, we can find it explicitly by solving the equation

$$(H-\zeta)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$$

for ζ in the resolvent set, in particular, for all $\zeta \in \mathbb{C}\setminus\mathbb{R}$. It is straightforward to check that

$$f(x) \;=\; r(x,\zeta) f_1(x) - \kappa r(x,\zeta)\, \frac{w_0(u^{-1}(x))}{|u'(u^{-1}(x))| w_1(x)}$$

$$\times \int_K \frac{\overline{\lambda(u^{-1}(x), z)}}{u^{-1}(x) + z - \zeta}\, g_1(u^{-1}(x), z)\, \omega(z)\, dz\,, \tag{3.1}$$

$$\begin{aligned} g(y,z) \;=\; & -\kappa \frac{\lambda(y,z)}{y+z-\zeta}\, r(u(y),\zeta)\, f_1(u(y)) \;+\; \frac{g_1(y,z)}{y+z-\zeta} + \kappa^2 \frac{\lambda(y,z)}{y+z-\zeta} \\ & \times r(u(y),\zeta)\, \frac{w_0(y)}{|u'(y)| w_1(u(y))} \int_K \frac{\overline{\lambda(y,r)}}{y+r-\zeta}\, g_1(y,r)\, \omega(r)\, dr\,, \end{aligned}$$

where

$$r(x,\zeta) := \left\{ x - \zeta - \kappa^2 \frac{w_0(u^{-1}(x))}{|u'(u^{-1}(x))| w_1(x)} \int_K \frac{|\lambda(u^{-1}(x), z)|^2}{u^{-1}(x) + z - \zeta}\, \omega(z)\, dz \right\}^{-1}\,.$$

Let P be the projection onto the subspace $\mathcal{H}_0 = L^2(I_1, w_1\, dx)$ of "undecayed" states in $\mathcal{H}$,

$$P \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}\,.$$

By (3.1), the reduced resolvent acts then as multiplication by the function r,

$$P(H-\zeta)^{-1}P = r(\cdot,\zeta)P\,. \tag{3.2}$$

For the sake of brevity we introduce the following notation,

$$v(y,z) \;:=\; |\lambda(y,z)|^2 \omega(z)\,, \tag{3.3}$$

$$\varrho(x) \;:=\; \frac{w_0(u^{-1}(x))}{|u'(u^{-1}(x))| w_1(x)}\,, \tag{3.4}$$

$$\mathcal{G}(y,\zeta) \;:=\; \int_K \frac{v(y,z)}{y+z-\zeta}\, dz\,, \tag{3.5}$$

so the function r can be written as

$$r(x,\zeta) = \left\{ x - \zeta - \kappa^2 \varrho(x)\, \mathcal{G}(u^{-1}(x),\zeta) \right\}^{-1} \tag{3.6}$$

for $\Im\zeta \neq 0$.

Remark 3.1 In the particular case of Example 2.3 it follows from (2.9) and (2.11) that $\varrho(x) = 1$, and moreover, $v(y,z) = |\lambda(y,z)|^2 \chi_{[0,\nu_{\max}]}(z)$.

To reveal the analytic properties of $r(x,\cdot)$ let us begin with those of $\mathcal{G}(y,\cdot)$.

Lemma 3.2 *Let $v(y,\cdot)$ have a locally bounded derivative in (ν,∞). Then for any $y \in I_0$ and a real $\zeta > y+\nu$ there exists finite principal value of the integral*

$$I(y,\zeta) := \mathcal{P}\int_\nu^\infty \frac{v(y,z)}{y+z-\zeta}\,dz\,. \tag{3.7}$$

Moreover, for any $k \in (0, \zeta-y-\nu)$,

$$\begin{aligned} I(y,\zeta) &= \int_\nu^{\zeta-y-k} \frac{v(y,z)}{y+z-\zeta}\,dz + \int_{\zeta-y-k}^{\zeta-y+k} \frac{v(y,z)-v(y,\zeta-y)}{y+z-\zeta}\,dz \\ &\quad + \int_{\zeta-y+k}^\infty \frac{v(y,z)}{y+z-\zeta}\,dz \end{aligned} \tag{3.8}$$

where all the three integrals are Lebesgue convergent.

Proof: Choose any $k \in (0,\zeta-y-\nu)$. As the integrals

$$\int_\nu^{\zeta-y-k} \frac{v(y,z)}{y+z-\zeta}\,dz \quad \text{and} \quad \int_{\zeta-y+k}^\infty \frac{v(y,z)}{y+z-\zeta}\,dz$$

exist due to the assumption *(a2)* it is sufficient to check the convergence of

$$I_k(y,\zeta) = \mathcal{P}\int_{\zeta-y-k}^{\zeta-y+k} \frac{v(y,z)}{y+z-\zeta}\,dz\ . \tag{3.9}$$

We employ the identity $v(y,z) = v(y,\zeta-y) + [v(y,z)-v(y,\zeta-y)]$ together with the estimate

$$|v(y,z)-v(y,\zeta-y)| \le c_1|y+z-\zeta|$$

with a finite c_1 independent of z. We see that finite

$$\int_{\zeta-y-k}^{\zeta-y+k} \frac{v(y,z)-v(y,\zeta-y)}{y+z-\zeta}\,dz$$

exists and it is sufficient to check $\mathcal{P}\int_{\zeta-y-k}^{\zeta-y+k}\frac{dz}{y+z-\zeta}$ which is easily seen to exist and to be equal to zero. ∎

As usual in similar situations to proceed one needs some analyticity assumption about the formfactor. In the present case we suppose that *(a3)* for all $y \in I_0$ the function $v(y,\cdot)$ can be holomorphically extended to an open set $\Omega_{v,y} \supset (\nu,\infty)$; we denote the extension again as $v(y,\cdot)$. Let us further assume that there is an open set Ω in $\mathbb{C}$ such that

$$(\xi_0^{(-)}+\nu,\infty) \subset \Omega \subset \cap_{y\in I_0}(y+\Omega_{v,y}) \quad .$$

Notice that the hypothesis of the previous lemma is satisfied under *(a3)*. Now we can make the following claim.

Lemma 3.3 *Let $y \in I_0$ and $\xi \in (y+\nu,\infty)$. Then*

$$\lim_{\pm\Im\zeta>0,\,\zeta\to\xi} \mathcal{G}(y,\zeta) = I(y,\xi) \pm i\pi v(y,\xi-y)\,.$$

Proof: Let us write again $\mathcal{G}(y,\zeta)$ defined by (3.5) as a sum of three integrals over the intervals $(\nu,\xi-y-k)$, $(\xi-y-k,\xi-y+k)$ and $(\xi-y+k,\infty)$ with $0<k<\xi-y-\nu$. The first and the third integral can be interchanged with limit by dominated convergence. The set $\Omega_{v,y}$ is open and contains (ν,∞), hence there is $k_1>0$ such that any $\vartheta \in \mathbb{C}$ satisfying $|\vartheta-\xi+y| \le k_1$ belongs to $\Omega_{v,y}$. Let us consider only ζ satisfying $|\zeta-\xi| \le k_1$ (so that $\zeta - y \in \Omega_{v,y}$) in the second integral and denote $\zeta_1 = \Re\zeta$, then we employ the identity $v(y,z) = v(y,\zeta_1-y) + [v(y,z)-v(y,\zeta_1-y)]$ and observe that

$$|v(y,z)-v(y,\zeta_1-y)| \le c_1(y,\xi,k,k_1)\,|y+z-\zeta_1|\,.$$

The contribution from the difference can be thus also handled by dominated convergence. In view of (3.8) we get

$$\lim_{\pm\Im\zeta>0,\,\zeta\to\xi} \mathcal{G}(y,\zeta) = I(y,\xi) + v(y,\xi-y) \lim_{\pm\Im\zeta>0,\,\zeta\to\xi} \int_{\xi-y-k}^{\xi-y+k} \frac{dz}{y+z-\zeta}$$

and the result follows by an easy calculation. ∎

Lemma 3.4 *Define the functions $\mathcal{G}_\Omega : I_0\times\Omega \to \mathbb{C}$ and $\mathcal{G}^\Omega : I_0\times\Omega\to\mathbb{C}$ by*

$$\mathcal{G}_\Omega(y,\zeta) = \begin{cases} \mathcal{G}(y,\zeta) & \dots \quad \Im\zeta>0 \\ I(y,\zeta)+i\pi v(y,\zeta-y) & \dots \quad \Im\zeta=0 \\ \mathcal{G}(y,\zeta)+2i\pi v(y,\zeta-y) & \dots \quad \Im\zeta<0 \end{cases} \tag{3.10}$$

$$\mathcal{G}^\Omega(y,\zeta) = \begin{cases} \mathcal{G}(y,\zeta)-2i\pi v(y,\zeta-y) & \dots \quad \Im\zeta>0 \\ I(y,\zeta)-i\pi v(y,\zeta-y) & \dots \quad \Im\zeta=0 \\ \mathcal{G}(y,\zeta) & \dots \quad \Im\zeta<0 \end{cases} \tag{3.11}$$

Under our assumptions (a1)–(a3)*, the functions $\mathcal{G}_\Omega(y,\cdot)$ and $\mathcal{G}^\Omega(y,\cdot)$ are holomorphic in $\Omega\setminus(-\infty,y+\nu]$ for any fixed $y\in I_0$.*

Proof: By Lemma 3.2 and assumption *(a3)*, $\mathcal{G}_\Omega$ is a finite function. Notice that $\zeta - y \in \Omega_{v,y}$ for $\zeta \in \Omega$ and $y \in I_0$. According to Lemma 3.3, the function $\mathcal{G}_\Omega(y,\cdot)$ is continuous in $\{\zeta \in \Omega | \Im\zeta \geq 0\} \setminus (-\infty, y+\nu]$ – see, e.g., Thm 146 in Ref. [21]. Alternatively, the continuity of $I(y,\cdot)$ in $(y+\nu,\infty)$ can be established directly from the dominated convergence used in the proof of Lemma 3.2. Similarly, the continuity in $\{\zeta \in \Omega \,|\, \Im\zeta \leq 0\} \setminus (-\infty, y+\nu]$ is seen and thus $\mathcal{G}_\Omega(y,\cdot)$ is continuous in $\Omega \setminus (-\infty, y+\nu]$. As it is holomorphic in $\{\zeta \in \Omega | \,\Im\zeta > 0\} \cup \{\zeta \in \Omega | \,\Im\zeta < 0\}$ it is also holomorphic in $\Omega \setminus (-\infty, y+\nu]$ due to a corollary (dubbed the *edge-of-wedge* theorem) of the Morera's theorem (stating that the continuous function is holomorphic iff the integrals over all rectangles with the edges parallel to the axes are zero – see, e.g., [22, Thm 168] or [23, Thm 10.17]). As to $\mathcal{G}^\Omega(y,\cdot)$, we can prove our statement in the same way as for $\mathcal{G}_\Omega(y,\cdot)$. ∎

Now we are in position to show what happens with the upper spectral band under influence of the perturbation. Let us formulate some further assumptions before.

(a4) The functions $\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\zeta)$ and $\varrho(x)\frac{\partial \mathcal{G}_\Omega(u^{-1}(x),\zeta)}{\partial\zeta}$ are continuous in the set $\{(x,\zeta) \in I_1 \times \Omega | \zeta \notin (-\infty, u^{-1}(x)+\nu]\}$.

(a5) For all $x \in I_1$,

$$x > u^{-1}(x) + \nu\,.$$

Remarks 3.5 (a) In the particular case of Example 2.3 the factor $\varrho(x) = 1$ can be dropped in *(a4)* and the assumption *(a5)* is satisfied.
(b) While most assumptions we make are of a technical nature, *(a5)* is a physical hypothesis saying that in no part of the excited spectral band the decay is prevented by energy conservation. It is satisfied, of course, if $\nu = 0$.

Let us denote

$$\Omega_x = \Omega \setminus (-\infty, u^{-1}(x)+\nu\,]\,.$$

Theorem 3.6 *Assume (a1)–(a5). Then the following statements hold.*
(a) There exist $\Delta > 0$, $\delta > 0$ and a unique function $\zeta : I_1 \times (-\delta,\delta) \to \mathbb{C}$ satisfying

$$\zeta(x,\kappa) \in (x-\Delta, x+\Delta) + i(-\Delta,\Delta) \subset \Omega_x\,, \tag{3.12}$$

$$x - \zeta(x,\kappa) - \kappa^2\varrho(x)\,\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa)) = 0\,. \tag{3.13}$$

The function ζ *is continuous in* $I_1 \times (-\delta, \delta)$ *and* $\zeta(x, \cdot) \in C^\infty(-\delta, \delta)$.
(b) The resolvent has no singularity in the upper complex half-plane, in particular

$$\Im\zeta(x, \kappa) \le 0 \tag{3.14}$$

holds for all $x \in I_1$, $\kappa \in (-\delta, \delta)$. *Moreover, if*

$$\varrho(x) v(u^{-1}(x), x - u^{-1}(x)) \neq 0 \tag{3.15}$$

for all x *from some compact* $I' \subset I_1$, *then there exists a* $\delta \in (0, \delta_1]$ *such that*

$$\Im\zeta(x, \kappa) < 0 \tag{3.16}$$

holds for all $0 < |\kappa| < \delta_1$ *and* $x \in I'$.

Proof: (a) Let us denote

$$D_+(x, \kappa, \zeta) := x - \zeta - \kappa^2 \varrho(x)\, \mathcal{G}_\Omega(u^{-1}(x), \zeta)\,. \tag{3.17}$$

The functions D_+ and $\frac{\partial D_+}{\partial \zeta}$ are continuous in $\{(x, \kappa, \zeta) | x \in I_1, \kappa \in \mathbb{R}, \zeta \in \Omega_x\}$ by assumption and $D_+(x, \cdot, \cdot) \in C^\infty(\mathbb{R} \times \Omega_x)$ by Lemma 3.4. Furthermore, $D_+(x, 0, x) = 0$ and

$$\frac{\partial D_+(x, 0, x)}{\partial \zeta} = -1 \neq 0\,.$$

By the implicit function theorem – see, e.g. [21, Thm 211] – to any $x_0 \in I_1$ there exist $d_{x_0} > 0$, $\delta_{x_0} > 0$ and $\Delta_{x_0} > 0$ such that for all $x \in (x_0 - d_{x_0}, x_0 + d_{x_0}) \cap I_1$ and $\kappa \in (-\delta_{x_0}, \delta_{x_0})$ there is just one $\zeta_{x_0}(x, \kappa) \in (x_0 - \Delta_{x_0}, x_0 + \Delta_{x_0}) + i(-\Delta_{x_0}, \Delta_{x_0}) \subset \Omega_x$ (recall *(a5)*) satisfying $D_+(x, \kappa, \zeta_{x_0}(x, \kappa)) = 0$, i.e. the relation (3.13). The function ζ_{x_0} is continuous in $((x_0 - d_{x_0}, x_0 + d_{x_0}) \cap I_1) \times (-\delta_{x_0}, \delta_{x_0})$ and $\zeta_{x_0}(x, \cdot) \in C^\infty(-\delta_{x_0}, \delta_{x_0})$ for any fixed $x \in (x_0 - d_{x_0}, x_0 + d_{x_0})$. We put

$$d'_{x_0} = \min(\Delta_{x_0}, d_{x_0})\,.$$

As I_1 is compact by assumption, the open covering of I_1 defined in this way has a finite subcovering, i.e. there exist a finite number of points $x_j \in I_1$, $j = 1, \dots, n$, such that

$$I_1 \subset \cup_{j=1}^n K_j\,;$$

we employ here the notation

$$K_j = (x_j - d'_{x_j}, x_j + d'_{x_j})\,, \quad J_j = K_j + i(-d'_{x_j}, d'_{x_j})$$

for $j = 1, \ldots, n$. Let us pick a point $x_{jk} \in K_j \cap K_k$ for given $j, k = 1, \ldots, n$; then there is $0 < \delta_{jk} \leq \min(\delta_{x_j}, \delta_{x_k})$ such that

$$\zeta_{x_j}(x_{jk}, \kappa) \in J_j \cap J_k \,, \quad \zeta_{x_k}(x_{jk}, \kappa) \in J_j \cap J_k \tag{3.18}$$

for $|\kappa| < \delta_{jk}$. Moreover, $\zeta_{x_j}(x, \kappa) = \zeta_{x_k}(x, \kappa)$ for all $x \in K_j \cap K_k$ and $|\kappa| < \delta_{jk}$; otherwise the uniqueness of ζ_{x_j} and ζ_{x_k} would be violated near at least one of the points

$$\sup\{x \in K_j \cap K_k \,|\, x \geq x_{jk}, \zeta_{x_j}(y, \kappa) = \zeta_{x_k}(y, \kappa) \,\text{for}\, x_{jk} \leq y \leq x, |\kappa| < \delta_{jk}\}\,,$$

$$\inf\{x \in K_j \cap K_k \,|\, x \leq x_{jk}, \zeta_{x_j}(y, \kappa) = \zeta_{x_k}(y, \kappa) \,\text{for}\, x \leq y \leq x_{jk}, |\kappa| < \delta_{jk}\}\,.$$

Choosing a number $\delta' > 0$ with $\delta' \leq \min_{1 \leq j \leq n} \delta_{x_j}$ and $\delta' \leq \min_{K_j \cap K_k \neq \emptyset} \delta_{jk}$, we conclude that there exists a unique $\zeta : I_1 \times (-\delta', \delta') \to \mathbb{C}$ such that $\zeta(x, \kappa) \in J_j$ for $x \in K_j$ and $D_+(x, \kappa, \zeta(x, \kappa)) = 0$. The function ζ is continuous in $I_1 \times (-\delta', \delta')$ and $\zeta(x, \cdot) \in C^\infty((-\delta', \delta'))$ for any fixed $x \in I_1$. Put

$$h_j(x) = \min(x - x_j + d'_{x_j}, x_j + d'_{x_j} - x)\,.$$

The function $h_j : I_1 \to \mathbb{R}$ defined in this way is continuous and $x \in K_j$ if and only if $h_j(x) > 0$. Then

$$h(x) := \max_{1 \leq j \leq n} h_j(x)$$

specifies a positive continuous function h on I_1. Let us denote

$$D = \min_{x \in I_1} h(x) > 0\,, \quad \Delta = \min\left(D, \min_{1 \leq j \leq n} d'_{x_j}\right) > 0\,.$$

As ζ is uniformly continuous on compact subsets of $I_1 \times (-\delta', \delta')$ there exists $0 < \delta \leq \delta'$ such that

$$\zeta(x, \kappa) \in (x - \Delta, x + \Delta) + i(-\Delta, \Delta)$$

for $x \in I_1$ and $|\kappa| < \delta$; hence the existence of the numbers δ, Δ and the function ζ is demonstrated.

Finally, to check the uniqueness of ζ let us assume that $\tilde{\zeta}$ is another function satisfying

$$\tilde{\zeta}(x, \kappa) \in (x - \Delta, x + \Delta) + i(-\Delta, \Delta)\,, \quad D_+(x, \kappa, \tilde{\zeta}(x, \kappa)) = 0$$

for $x \in I_1$, $\kappa \in (-\delta, \delta)$. Suppose that $x \in I_1$ and $|\kappa| < \delta$ are given. There exists an index $j = 1, \ldots, n$ such that

$$h_j(x) = h(x) \geq D\,, \quad x \in K_j\,, \quad \zeta(x,\kappa) \in J_j\,.$$

As the inequalities

$$x - x_j + d'_{x_j} \geq D\,, \quad x_j + d'_{x_j} - x \geq D\,, \quad -D < y - x < D$$

hold, where $y := \Re\tilde{\zeta}(x,\kappa)$, we have also

$$y - x_j + d'_{x_j} > 0\,, \quad x_j + d'_{x_j} - y > 0$$

and $y \in K_j$. Furthermore, $|\Im\tilde{\zeta}(x,\kappa)| < \Delta \leq d'_{x_j} \leq \Delta_{x_j}$. Then $\zeta(x,\kappa) = \tilde{\zeta}(x,\kappa)$ and the uniqueness is proven.
(b) Assume first that $\Im\zeta > 0$, then $\Im\mathcal{G}_\Omega(u^{-1}(x),\zeta) = \Im G(u^{-1}(x),\zeta) \geq 0$ by Eqs. (3.5) and (3.10), so the r.h.s of Eq. (3.17) has negative imaginary part. Consequently, there are no solutions $\zeta(x,\kappa)$ with positive imaginary parts, in other words (3.14) holds. We have checked here only that the open upper half-plane is a part of the resolvent set for the Hamiltonian. In the lower half-plane, the function $D_+(x,\kappa,\cdot)^{-1}$ is a meromorphic continuation of $r(x,\cdot)$ and may have singularities.

Suppose now that (3.15) holds. The expression $\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}$ is continuous in $(x,\kappa) \in I_1 \times (-\delta,\delta)$. It follows that

$$M := \max_{(x,\kappa)\in I'\times[-\frac{\delta}{2},\frac{\delta}{2}]} \left|\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}\right| < \infty\,.$$

Differentiating the equation defining $\zeta(x,\kappa)$ with respect to κ^2 we get

$$\frac{\partial\zeta(x,\kappa)}{\partial(\kappa^2)} + \varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa)) + \kappa^2\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}\frac{\partial\zeta(x,\kappa)}{\partial(\kappa^2)} = 0\,. \tag{3.19}$$

In combination with the previous inequality we conclude that

$$\frac{\partial\zeta(x,\kappa)}{\partial(\kappa^2)} = -\frac{\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{1+\kappa^2\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}} \tag{3.20}$$

is continuous in $(x,\kappa) \in I' \times \Big(-\min(\frac{\delta}{2}, M^{-\frac{1}{2}}), \min(\frac{\delta}{2}, M^{-\frac{1}{2}}) \Big)$ defining $M^{-\frac{1}{2}} = \infty$ for $M=0$. Furthermore, the assumption (3.15) together with (3.10) implies $\Im \frac{\partial \zeta(x,0)}{\partial(\kappa^2)} < 0$, hence there is $0 < \delta_1 \le \min(\frac{\delta}{2}, M^{-\frac{1}{2}})$ such that

$$\Im \frac{\partial \zeta(x,\kappa)}{\partial(\kappa^2)} < 0$$

for $(x,\kappa) \in I' \times (0,\delta_1)$ and (3.16) holds. ∎

Remarks 3.7 (a) Putting $\kappa = 0$ in (3.19) we obtain

$$\frac{\partial \zeta(x,0)}{\partial(\kappa^2)} = -\varrho(x) \mathcal{G}_\Omega(u^{-1}(x), x)\,, \tag{3.21}$$

where right-hand side is given by Lemma 3.4. This relation can be regarded as an analogue of the Fermi golden rule in the present situation.
(b) Notice that for the factorization (2.7) the term $|\lambda_0(u^{-1}(x))|^2$ factorizes from $\mathcal{G}_\Omega(u^{-1}(x),\zeta)$ and $\zeta(x,\kappa) = x$ holds whenever $\lambda_0(u^{-1}(x)) = 0$.

IV Decay of excited states

In accordance with the physical motivation, we are interested in transitions from a given state supported in I_1 into those in I_0. To find the time profile of the de-excitation probability it is sufficient to know the reduced evolution operator $PU(t)P = Pe^{-iHt}P$. Suppose that the initial state is of the form

$$\Psi_0 = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix},$$

for some $\psi_0 \in L^2(I_1, w_1(x)\,dx)$ with $\|\Psi_0\|^2 = \int_{I_1} |\psi_0(x)|^2\, w_1(x)\, dx = 1$. Its time evolution is given by the Stone formula,

$$U(t)\Psi_0 = \lim_{\eta\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \left[(H - \xi - i\eta)^{-1} - (H - \xi + i\eta)^{-1} \right] e^{-i\xi t} \Psi_0 \, d\xi\,,$$

according to [20, Thm VIII.5], and the projection P can be interchanged with the limit and the integral being a bounded operator. This yields the reduced evolution operator,

$$PU(t)\Psi_0 = \begin{pmatrix} \psi(t,\cdot) \\ 0 \end{pmatrix},$$

where

$$\psi(t,\cdot) = \lim_{\eta\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} [r(\cdot,\xi+i\eta) - r(\cdot,\xi-i\eta)]\, e^{-i\xi t}\psi_0(\cdot)\, d\xi \tag{4.1}$$

and r is given by (3.6). The integral and the limit refer to functions with values in $\mathcal{H}_0 = L^2(I_1, w_1(x)\,dx)$; they are known to be convergent as the Hamiltonian H is self-adjoint.

Let us now look for conditions under which the interchange of the limit and the integral in (4.1) is possible. To this end, we need more assumptions.

(a6) $v(y,z) \le C_2$ and $\left|\frac{\partial v(y,z)}{\partial z}\right| \le C_3$ holds for some positive constants C_2, C_3 and all $y \in I_0$, $z \in K$.

(a7) $v(y,\nu) = 0$ for all $y \in I_0$.

(a8) There exists a zero-measure set $N \subset I_1$ and a number

$$\nu_1 > d_1 := \sup_{x\in I_1} [x - u^{-1}(x) - \nu] > 0$$

such that

$$\varrho(x)v(u^{-1}(x),\xi) > 0$$

for all $x \in I_1 \setminus N$ and $\xi \in (\nu, \nu+\nu_1)$.

Lemma 4.1 *Assume (a1)–(a7). Then there exists a number C_4 such that*

$$|\mathcal{G}(y,\xi\pm i\eta)| \le C_4 \tag{4.2}$$

holds for all $y \in I_0$, $\xi \in \mathbb{R}$, and $0 \ne \eta \in \mathbb{R}$.

Proof: Recall the definition

$$\mathcal{G}(y,\xi\pm i\eta) = \int_\nu^\infty \frac{v(y,z)}{y+z-\xi\mp i\eta}\, dz = \int_\nu^\infty \frac{y+z-\xi\pm i\eta}{(y+z-\xi)^2+\eta^2} v(y,z)\, dz\,.$$

Using the first part of *(a6)*, we get

$$|\Im\mathcal{G}(y,\xi\pm i\eta)| = \int_\nu^\infty \frac{|\eta| v(y,z)}{(y+z-\xi)^2+\eta^2}\, dz \le C_2 \int_{-\infty}^\infty \frac{|\eta|}{z^2+\eta^2}\, dz = \pi C_2\,.$$

We fix $\alpha > 0$ and distinguish several cases.

(i) $\xi - y \geq \nu + \alpha$. Then

$$\Re\mathcal{G}(y, \xi \pm i\eta) = \left(\int_{\nu+y-\xi}^{-\alpha} + \int_{-\alpha}^{\alpha} + \int_{\alpha}^{\infty} \right) \frac{zv(y, z - y + \xi)}{z^2 + \eta^2} \, dz$$

where by *(a2)* we have

$$\left| \left(\int_{\nu+y-\xi}^{-\alpha} + \int_{\alpha}^{\infty} \right) \frac{zv(y, z - y + \xi)}{z^2 + \eta^2} \, dz \right| \leq \frac{1}{\alpha} \int_{\nu}^{\infty} v(y, z) \, dz \leq \frac{C}{\alpha} \, .$$

Using the mean value theorem,

$$\begin{aligned} J_2 : \; &= \int_{-\alpha}^{\alpha} \frac{zv(y, z - y + \xi)}{z^2 + \eta^2} \, dz \\ &= \int_{-\alpha}^{\alpha} \frac{z}{z^2 + \eta^2} \left[v(y, \xi - y) + z \partial_2 v(y, \vartheta(y, \xi, z)) \right] dz \end{aligned}$$

with $\vartheta(y, \xi, z)$ between $\xi - y$ and $\xi - y + z$. The integral of the first term is zero due to the antisymmetry in z while the second term can be estimated by *(a6)* giving $|J_2| \leq 2C_3\alpha$ and

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| \leq C\alpha^{-1} + 2C_3\alpha \, .$$

(ii) $\nu \leq \xi - y < \nu + \alpha$. Then

$$\Re\mathcal{G}(y, \xi \pm i\eta) = \left(\int_{\nu+y-\xi}^{\xi-\nu-y} + \int_{\xi-\nu-y}^{\alpha} + \int_{\alpha}^{\infty} \right) \frac{zv(y, z - y + \xi)}{z^2 + \eta^2} \, dz \, ,$$

where

$$\left| \int_{\nu+y-\xi}^{\xi-\nu-y} \frac{z}{z^2 + \eta^2} v(y, z - y + \xi) \, dz \right| \leq 2C_3(\xi - \nu - y) \leq 2C_3\alpha$$

follows by the same procedure as for the integral J_2 in case (i) and

$$\left| \int_{\alpha}^{\infty} \frac{z}{z^2 + \eta^2} v(y, z - y + \xi) \, dz \right| \leq C\alpha^{-1}$$

due to *(a2)*. In the remaining integral,

$$|v(y, z-y+\xi)| \le C_3(z-y+\xi-\nu)$$

by *(a6)* and *(a7)*. Denoting for a while $A = \xi-\nu-y$, we have now

$$\begin{aligned}
\left|\int_{\xi-\nu-y}^{\alpha} \frac{z}{z^2+\eta^2} v(y, z-y+\xi)\, dz\right| &\le C_3 \int_A^{\alpha} \frac{z^2+Az}{z^2+\eta^2}\, dz \\
&= C_3\left[\alpha - A + \frac{A}{2}\ln\frac{\alpha^2+\eta^2}{A^2+\eta^2} - |\eta|\left(\arctan\frac{\alpha}{|\eta|} - \arctan\frac{A}{|\eta|}\right)\right] \\
&\le C_3\left[\alpha + \frac{A}{2}\ln\frac{\alpha^2+\eta^2}{A^2+\eta^2}\right]
\end{aligned}$$

taking into account that $0 \le A \le \alpha$ in the last inequality. Let us estimate the maximum of function

$$f(A) = \frac{A}{2}\ln\frac{\alpha^2+\eta^2}{A^2+\eta^2}$$

in the mentioned interval of A. Clearly $f(0) = f(\alpha) = 0$ and $f(A) > 0$ for $0 < A < \alpha$. Hence f has a maximum at some point $A_0 \in (0, \alpha)$ satisfying

$$f'(A_0) = \frac{1}{2}\ln\frac{\alpha^2+\eta^2}{A_0^2+\eta^2} - \frac{A_0^2}{A_0^2+\eta^2} = 0\,.$$

From the last equation,

$$f(A_0) = \frac{A_0^3}{A_0^2+\eta^2} \le A_0 \le \alpha\,.$$

As a result,

$$\left|\int_{\xi-\nu-y}^{\alpha} \frac{z}{z^2+\eta^2}\, v(y, z-y+\xi)\, dz\right| \le 2C_3\alpha$$

and

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| \le 4C_3\alpha + C\alpha^{-1}$$

(iii) $\nu-\alpha \le \xi-y < \nu$. Then

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| = \left|\left(\int_{\nu+y-\xi}^{\alpha} + \int_{\alpha}^{\infty}\right) \frac{z}{z^2+\eta^2}\, v(y, z-y+\xi)\, dz\right|\,.$$

Here the second integral is bounded by $C\alpha^{-1}$ and the first one we estimate similarly as in the case (ii). Denoting here $B = \nu + y - \xi \in (0, \alpha]$, we obtain

$$\left| \int_{\nu+y-\xi}^{\alpha} \frac{z}{z^2+\eta^2} v(y, z-y+\xi)\, dz \right| \le C_3 \int_B^{\alpha} \frac{z(z-B)}{z^2+\eta^2}\, dz \le C_3 \alpha$$

and

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| \le C_3\alpha + C\alpha^{-1} .$$

(iv) $\xi - y < \nu - \alpha$. Then

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| = \left| \int_{\nu+y-\xi}^{\infty} \frac{z}{z^2+\eta^2} v(y, z-y+\xi)\, dz \right| \le C\alpha^{-1} .$$

Summing up the discussion, we have found that in all the cases the inequality

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| \le 4C_3\alpha + C\alpha^{-1}$$

holds. Minimizing the right-hand side with respect to $\alpha > 0$, we get

$$|\Re\mathcal{G}(y, \xi \pm i\eta)| \le 4\sqrt{CC_3}$$

and

$$|\mathcal{G}(y, \xi \pm i\eta)| \le \sqrt{16CC_3 + \pi^2 C_2^2}\, , \tag{4.3}$$

what we set out to prove. ∎

Theorem 4.2 *Assume (a1)–(a8). Then there exists $\delta_2 > 0$ such that for all $0 < |\kappa| < \delta_2$ and $t \in \mathbb{R}$*

$$\psi(t, x) = \mathcal{U}(t, x)\psi_0(x) \tag{4.4}$$

holds for almost every $x \in I_1$, where

$$\mathcal{U}(t, x) = \int_{\nu+u^{-1}(x)}^{\infty} W(x, \xi) e^{-i\xi t}\, d\xi\, , \tag{4.5}$$

$$W(x, \xi) = \frac{\kappa^2 \varrho(x) v(u^{-1}(x), \xi - u^{-1}(x))}{\left[x - \xi - \kappa^2 \varrho(x) I(u^{-1}(x), \xi)\right]^2 + \pi^2 \kappa^4 \varrho(x)^2 v(u^{-1}(x), \xi - u^{-1}(x))^2} . \tag{4.6}$$

Proof: Let δ, Δ and $\zeta(x,\kappa) = \zeta_1(x,\kappa) - i\zeta_2(x,\kappa)$ be as in Theorem 3.6. We first verify that $\zeta_2(x,\kappa) > 0$ for $x \in I_1 \setminus N$ and

$$0 < |\kappa| < \delta_2' := \min\left(\delta, \sqrt{\frac{\nu_1 - d_1}{C_1 C_4}}, \sqrt{\frac{d}{C_1 C_4}}\right) ,$$

where

$$d := \min_{x \in I_1} [x - \nu - u^{-1}(x)] > 0$$

by assumption *(a5)*. It is sufficient to show that $\zeta(x,\kappa)$ is not real as we know that $\zeta_2(x,\kappa) \geq 0$. By assumption *(a2)* and Lemma 4.1,

$$|D_+(x,\kappa,\xi)| \geq |\xi - x| - \kappa^2 C_1 C_4$$

for real ξ and there is no solution in $(-\infty, x - \kappa^2 C_1 C_4) \cup (x + \kappa^2 C_1 C_4, \infty)$. If $\zeta(x,\kappa) = \xi$ then the imaginary part of the Eq. (3.13) reads

$$\kappa^2 \pi \varrho(x) v(u^{-1}(x), \xi - u^{-1}(x)) = 0 \quad .$$

Thus there are no real solutions in $(\nu + u^{-1}(x), \nu + \nu_1 + u^{-1}(x)) \supset (x - d, \nu + \nu_1 + u^{-1}(x))$ by assumption *(a8)*. For the considered values of κ the intervals without real solutions ξ cover the whole real axis.

To any natural number n there exists an open set $N_n \subset \mathbb{R}$ of Lebesgue measure smaller then $\frac{1}{n}$ such that $N \subset N_{n+1} \subset N_n$. Let us denote $I_n' = I_1 \setminus N_n$. Let φ be an arbitrary vector from $\mathcal{H}_0$ and $\varphi_n = \varphi \chi_{I_n'}$. The scalar product

$$(\varphi_n, \psi(t,\cdot)) = \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{I_1 \times \mathbb{R}} \overline{\varphi_n(x)} \left(\Im r(x, \xi + i\eta)\right) e^{-i\xi t} \psi_0(x) w_1(x) \, dx \, d\xi \tag{4.7}$$

with $r(x, \xi + i\eta)$ given by (3.6). Fubini theorem can be used here as

$$|\mathcal{G}(u^{-1}(x), \xi + i\eta)| \leq \frac{C}{\eta} , \qquad |r(x, \xi + i\eta)| \leq \frac{1}{\eta} ,$$
$$|\Im r(x, \xi + i\eta)| \;=\; \mathcal{O}(\xi^{-2}) \quad \text{as} \quad \xi \to \pm\infty$$

for $\eta > 0$ using *(a2)*, (3.5) and (3.6) only. Let us next choose $0 < \Delta_1 < \Delta$, $0 < \eta_1 < \Delta$, denote

$$\delta_2 = \min\left(\delta_2', \sqrt{\frac{\Delta_1}{2 C_1 C_4}}\right) , \tag{4.8}$$

and consider further only $0 < |\kappa| < \delta_2$, $0 < \eta \le \eta_1$. Now

$$|\kappa^2 \varrho(x) \mathcal{G}(u^{-1}(x), \xi + i\eta)| \le \frac{1}{2}\Delta_1$$

by Lemma 4.1. We divide the integration range $I_1 \times \mathbb{R}$ of (4.7) into the parts where $|\xi - x| \ge \Delta_1$ and $|\xi - x| \le \Delta_1$, respectively, and construct the integrable majorant allowing us to use the dominated convergence in (4.7).

For $|\xi - x| \ge \Delta_1$, clearly

$$|\Im r(x, \xi + i\eta)| \le \frac{\eta_1 + \frac{1}{2}\Delta_1}{\left(|\xi - x| - \frac{1}{2}\Delta_1\right)^2} \le \frac{4\eta_1 + 2\Delta_1}{\Delta_1^2}\,.$$

Let us define function $g : \mathbb{R} \to \mathbb{R}$ as (recall that $I_1 = [\xi_1^{(-)}, \xi_1^{(+)}]$)

$$g(\xi) = \begin{cases} (\xi - \xi_1^{(-)} + \frac{\Delta_1}{2})^{-2}(\eta_1 + \frac{\Delta_1}{2}) & \ldots \quad \xi < \xi_1^{(-)} - \Delta_1 \\ \frac{4\eta_1 + 2\Delta_1}{\Delta_1^2} & \ldots \quad \xi_1^{(-)} - \Delta_1 \le \xi \le \xi_1^{(+)} + \Delta_1 \\ (\xi - \xi_1^{(+)} - \frac{\Delta_1}{2})^{-2}(\eta_1 + \frac{\Delta_1}{2}) & \ldots \quad \xi > \xi_1^{(+)} + \Delta_1 \end{cases} \tag{4.9}$$

Then $(x, \xi) \mapsto g(\xi)|\varphi(x)|\,|\psi_0(x)|\,w_1(x)$ is the sought majorant.

For $|\xi - x| \le \Delta_1$ we can consider only $x \in I_n'$ as $\varphi_n(x) = 0$ elsewhere. By Theorem 3.6,

$$m_{\kappa,n} := \min |D_+(x, \kappa, \xi + i\eta)| > 0$$

where D_+ is defined in (3.17) and the minimum is taken over the considered set of variables $x \in I_n'$, $\xi \in [x - \Delta_1, x + \Delta_1]$, $\eta \in [0, \eta_1]$ and a fixed value of $0 < |\kappa| < \delta_2$ (notice that $\xi + i\eta \in \Omega_x$ due to our choice of Δ_1, η_1 and the inclusion in (3.12)). The majorant can be now chosen as

$$m_{\kappa,n}^{-1}|\varphi(x)|\,|\psi_0(x)|\,w_1(x)\,.$$

Interchanging the limit with the integral in (4.7), using Lemma 3.3 and realizing that the integrand limit vanishes for $\xi < \nu + u^{-1}(x)$, we obtain

$$(\varphi_n, \psi(t, \cdot))_{I_n'} = (\varphi_n, \mathcal{U}(t, \cdot)\psi_0)_{I_n'} \tag{4.10}$$

with $\mathcal{U}$ given by (4.5)-(4.6) and scalar products in the space $L^2(I_n', w_1(x)\,dx)$. Here $\mathcal{U}(t, \cdot)\psi_0 \in L^2(I_n', w_1(x)\,dx)$ as $\mathcal{U}$ is bounded in $\mathbb{R} \times I_n'$ which can be seen using the majorant constructed above. As $\varphi_n \in L^2(I_n', w_1(x)\,dx)$ may be arbitrary, Eq. (4.4) follows for a.e. $x \in I_n'$. Now we see (4.4) for a.e. $x \in I_1$ in the limit $n \to \infty$. ∎

V Exponential decay at intermediate times

Recall that decays of unstable quantum systems are nonexponential at very short and very long times, however, they are usually exponential in a very good approximation over a wide range of intermediate times. Our aim here is to show that the present models exhibits a similar behaviour in the sense that the function $\mathcal{U}(\cdot, x)$ appearing in the restricted time evolution operator (4.4) can be approximated by an exponential for a.e. fixed $x \in I_1$.

The way to prove that is inspired by [3]. We employ the fact that the continued resolvent is for any fixed x a meromorphic function and show that for a sufficiently weak coupling the time evolution is dominated by the contribution from the residue term in (4.5).

In addition to the hypotheses made above, let us assume that there exist a constant C_5 such that

(a9) $\left| \frac{\partial^2 v(y,z)}{\partial z^2} \right| \le C_5$ holds for all $y \in I_0$ and $z \in K$.

Lemma 5.1 *For any $\alpha > -\nu$, $x \in I_1$, and $\xi > u^{-1}(x) + \nu$ the following estimates hold:*

$$\left| \Re \frac{\partial \mathcal{G}_\Omega(u^{-1}(x), \xi)}{\partial \xi} \right| \le \left(\frac{C_2}{\alpha + \nu} + 4C_3 \right) + C_3 \left| \ln \frac{\xi - u^{-1}(x) - \nu}{\alpha + \nu} \right| + C_5(\xi - u^{-1}(x) - \nu)\,, \tag{5.1}$$

$$\left| \Im \frac{\partial \mathcal{G}_\Omega(u^{-1}(x), \xi)}{\partial \xi} \right| \le \pi C_3\,. \tag{5.2}$$

Proof: Let us estimate

$$\frac{\partial \mathcal{G}(u^{-1}(x), \zeta)}{\partial \zeta} = \int_\nu^\infty \frac{v(u^{-1}(x), z)}{(u^{-1}(x) + z - \zeta)^2}\, dz \tag{5.3}$$

for $\zeta = \xi + i\eta$, $\eta > 0$, and get the result on the real axis by taking the limit $\eta \to 0^+$ using Lemma 3.4. We rewrite the derivative as

$$\frac{\partial \mathcal{G}_\Omega(u^{-1}(x), \zeta)}{\partial \zeta} = \int_{\nu + u^{-1}(x) - \xi}^\infty \frac{z^2 - \eta^2 + 2i\eta z}{(z^2 + \eta^2)^2}\, v(u^{-1}(x), z + \xi - u^{-1}(x))\, dz \tag{5.4}$$

and denote for a moment

$$\beta = \xi - u^{-1}(x) - \nu\,, \quad \gamma = \xi - u^{-1}(x) + \alpha\,; \tag{5.5}$$

by assumption we have $0 < \beta < \gamma$.

In the expression for the imaginary part of (5.4) we separate the integrals over $(-\beta, \beta)$ and (β, ∞). In the second integral the limit $\eta \to 0$ gives zero as can be seen easily by the dominated convergence. In the integral over $(-\beta, \beta)$, we insert the Taylor expansion

$$\begin{aligned} v(u^{-1}(x), z + \xi - u^{-1}(x)) &= v(u^{-1}(x), \xi - u^{-1}(x)) + \\ &\frac{\partial v(u^{-1}(x), \xi - u^{-1}(x))}{\partial z}\, z + \frac{1}{2}\frac{\partial^2 v(u^{-1}(x), \xi - u^{-1}(x) + \theta)}{\partial z^2}\, z^2 \end{aligned} \tag{5.6}$$

where θ in the error term lies between 0 and z. The contribution of the z^0 term to the integral vanishes because it gives rise to an odd function. The contribution of the second term is bounded by πC_3 in the limit $\eta \to 0$ as it follows from assumption *(a6)* and an explicit calculation. The z^2 term again does not contribute in view of assumption *(a9)* and an explicit calculation. In this way, inequality (5.2) is proved.

As for the real part of Eq. (5.4), we proceed similarly. Inserting the expansion (5.6) into the integral over $(-\beta, \beta)$ we obtain from the z^0 term

$$\left| -\frac{2}{\beta}\, v(u^{-1}(x), \xi - u^{-1}(x)) \right| \le 2C_3\,,$$

where the assumptions *(a6)* and *(a7)* were used in the last inequality. The term with z does not contribute and the term with z^2 is estimated by $C_5\beta$ in the limit $\eta \to 0$. The integral over (β, γ) in (5.4) can be handled by means of *(a6)* and *(a7)*,

$$\begin{aligned} &\left| \int_\beta^\gamma \frac{z^2 - \eta^2}{(z^2 + \eta^2)^2}\, v(u^{-1}(x), z + \xi - u^{-1}(x))\, dz \right| \\ &\quad = \left| \int_\beta^\gamma \frac{z^2 - \eta^2}{(z^2 + \eta^2)^2}\, \frac{\partial v(u^{-1}(x), \theta_1)}{\partial z} (z + \xi - u^{-1}(x) - \nu)\, dz \right| \\ &\quad \le C_3 \int_\beta^\gamma \frac{z + \xi - u^{-1}(x) - \nu}{z^2}\, dz = C_3 \left[\ln\left(1 + \frac{\alpha + \nu}{\beta}\right) + \frac{\alpha + \nu}{\gamma} \right] \\ &\quad \le C_3 \left[2 + \left| \ln \frac{\xi - u^{-1}(x) - \nu}{\alpha + \nu} \right| \right]\,, \end{aligned}$$

where we have employed $\nu < \theta_1 < z+\xi-u^{-1}(x)$ and the inequality $\ln(1+x) \le 1 + |\ln x|$. Finally, we have

$$\left| \int_\gamma^\infty \frac{z^2-\eta^2}{(z^2+\eta^2)^2}\, v(u^{-1}(x), z+\xi-u^{-1}(x))\, dz \right| \le C_2 \int_\gamma^\infty \frac{dz}{z^2} \le \frac{C_2}{\alpha+\nu}\,;$$

putting all these estimates together, we arrive at (5.1). ∎

Lemma 5.2 *There is $\delta_3 > 0$ such that for all $0 < |\kappa| < \delta_3$ and almost every $x \in I_1$, the function $W(x,\cdot)$ defined by formula (4.6) for $\xi > \nu + u^{-1}(x)$ and extended by zero to the rest of the real axis, $W(x,\xi) = 0$ for $\xi \le \nu + u^{-1}(x)$, is absolutely continuous in any compact subinterval of $\mathbb{R}$.*

Proof: From the proof of Theorem 4.2 we know that

$$W(x,\xi) = \frac{1}{\pi} \lim_{\eta\to 0^+} \Im r(x,\xi+i\eta)\,,$$

and therefore

$$W(x,\xi) \;=\; \frac{1}{\pi}\Im r_\Omega(x,\xi)\,, \tag{5.7}$$

$$r_\Omega(x,\xi) \;:=\; \frac{1}{x-\xi-\kappa^2\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\xi)} = [D_+(x,\kappa,\xi)]^{-1} \tag{5.8}$$

for $\xi > \nu + u^{-1}(x)$. Let δ and Δ be the numbers from Theorem 3.6. For $\kappa^2 < \Delta C_1^{-1} C_4^{-1}$, D_+ has no zeros if $|\xi - x| \ge \Delta$ (see Lemmas 3.4 and 4.1 and assumption *(a2)*). On the other hand, for $|\xi - x| < \Delta$ and $0 < |\kappa| < \delta_2$ real zeros can exist for at most zero-measure set of x which we neglect (see the proof of Theorem 4.2). Apart of it $W(x,\cdot)$ has a continuous derivative in $(\nu + u^{-1}(x), \infty)$ and therefore it is absolutely continuous in any compact subinterval. Let us denote

$$d := \min_{x\in I_1} [x - \nu - u^{-1}(x)] > 0\,,$$

where the positivity follows from assumption *(a5)*. Then $|D_+(x,\kappa,\xi)| > \frac{d}{3}$ for $\nu + u^{-1}(x) < \xi < \nu + u^{-1}(x) + \frac{d}{3}$, $\kappa^2 < \frac{d}{3C_1C_4}$, and $\frac{\partial W(x,\xi)}{\partial \xi}$ is bounded by an expression similar to the r.h.s. of (5.1) in the considered interval of ξ. Due to the integrability of $|\ln(\xi - \nu - u^{-1}(x))|$ and the estimate

$$|W(x,\xi)| \le \frac{3\pi C_3}{C_4 d}\, (\xi - \nu - u^{-1}(x))$$

(see assumptions *(a6)–(a7)* and (3.10)) $W(x,\cdot)$ is absolutely continuous in $[\nu+u^{-1}(x), \nu+u^{-1}(x)+\frac{d}{3}]$. Consequently, it is absolutely continuous in any compact subinterval of $\mathbb{R}$. Choosing

$$\delta_3 = \min\left(\delta, \delta_2, \sqrt{\frac{\Delta}{C_1C_4}}, \sqrt{\frac{d}{3C_1C_4}}\right).$$

we get the desired result. ∎

Lemma 5.3 *There exists $\delta_4 > 0$ such that*

$$M_1 := \max_{x\in I_1, |\kappa|\le\delta_4} \left|\varrho(x)\mathcal{G}_\Omega(u^{-1}(x), \zeta(x,\kappa))\right| < \infty, \tag{5.9}$$

$$M_2 := \max_{x\in I_1, |\kappa|\le\delta_4} \left|\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x), \zeta(x,\kappa))}{\partial\zeta}\right| < \infty, \tag{5.10}$$

and for all $|\kappa| < \delta_5 := \min(\delta_4, (2M_2)^{-\frac{1}{2}})$, $x \in I_1$, we have

$$|\zeta_1(x,\kappa) - x| \le 2M_1\kappa^2, \qquad 0 \le \zeta_2(x,\kappa) \le 2M_1\kappa^2,$$
$$|\zeta(x,\kappa) - x| \le 2M_1\kappa^2, \qquad \left|\frac{\partial\zeta(x,\kappa)}{\partial\kappa^2}\right| \le 2M_1,$$

where $\zeta(x,\kappa) = \zeta_1(x,\kappa) - i\zeta_2(x,\kappa)$ is the function from Theorem 3.6.

Proof: By Theorem 3.6 ζ is uniformly continuous in $I_1\times[-\frac{\delta}{2}, \frac{\delta}{2}]$. Hence there is $0 < \delta_4 \le \frac{\delta}{2}$ such that for $|\kappa| \le \delta_4$ and all $x \in I_1$ we have $|\zeta(x,\kappa) - x| < d$. Then $\zeta_1(x,\kappa) > \nu+u^{-1}(x)$ and the functions in the r.h.s of Eqs. (5.9), (5.10) are continuous. Consequently, M_1, M_2 are finite. For $|\kappa| < \delta_5$ we now have

$$\left|\frac{\partial\zeta(x,\kappa)}{\partial\kappa^2}\right| = \left|\frac{\varrho(x)\mathcal{G}_\Omega(u^{-1}(x), \zeta(x,\kappa))}{1+\kappa^2\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}}\right| \le \frac{M_1}{1-\kappa^2M_2} \le 2M_1$$

and the sought estimates on $\zeta(x,\kappa) - x$ follow. ∎

Lemma 5.4 *Let α be a number such that*

$$0 < \alpha < d := \inf_{x\in I_1}(x - u^{-1}(x) - \nu), \quad \alpha < \operatorname{dist}(I_1, \mathbb{C}\setminus\Omega),$$

and let us denote

$$N_{\alpha,x} := \{\vartheta \in \mathbb{C} \mid |\vartheta - x| \le \alpha\},$$
$$N_\alpha := \{(x,\vartheta) \in I_1\times\mathbb{C} \mid x \in I_1,\ |\vartheta - x| \le \alpha\}.$$

Then

(i) $N_\alpha \subset \{(x,\vartheta) \in I_1 \times \mathbb{C} \,|\, \vartheta \in \Omega \setminus (-\infty, u^{-1}(x) + \nu]\}$,

(ii) if $x \in I_1$ *and* $\vartheta \in N_{\alpha,x}$ *then* $(x,\vartheta) \in N_\alpha$,

(iii) N_α *is closed in* $\mathbb{R} \times \mathbb{C}$,

(iv) the numbers

$$M_3(\alpha) := \max_{(x,\vartheta)\in N_\alpha} \left|\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\vartheta)\right| < \infty\,, \tag{5.11}$$

$$M_4(\alpha) := \max_{(x,\vartheta)\in N_\alpha} \left|\varrho(x)\frac{\partial \mathcal{G}_\Omega(u^{-1}(x),\vartheta)}{\partial\vartheta}\right| < \infty \tag{5.12}$$

are finite,

(v) there exists an $\alpha' > \alpha$ *such that for any* $x \in I_1$, $\vartheta \in N_{\alpha',x}$, *and* $|\kappa| < \delta_6(\alpha) := \min\left(\delta_5, \sqrt{\frac{\alpha}{4M_1}}\right)$ *(see Lemma 5.3) we have*

$$\mathcal{G}_\Omega(u^{-1}(x),\vartheta) = \mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa)) + \frac{\partial \mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}(\vartheta - \zeta(x,\kappa)) + \mathcal{F}(x,\vartheta)(\vartheta-\zeta(x,\kappa))^2 \tag{5.13}$$

where $\mathcal{F}(x,\cdot)$ *is a function holomorphic in the interior of* $N_{\alpha',x}$ *and*

$$|\varrho(x)\mathcal{F}(x,\vartheta)| \leq \frac{8M_3(\alpha)}{\alpha^2} =: m_3(\alpha)\,, \tag{5.14}$$

$$\left|\varrho(x)\frac{\partial\mathcal{F}(x,\vartheta)}{\partial\vartheta}\right| \leq \frac{16M_3(\alpha)}{\alpha^3} =: m_4(\alpha) \tag{5.15}$$

holds for ϑ *in the interior of* $N_{\alpha/2,x}$.

Proof: The claims (i)–(iii) trivially follow from the definitions, the claim (iv) follows from the assumption *(a4)* and the claims (i), (iii). Under our assumptions there exists $\alpha' > \alpha$ satisfying all the assumptions of the lemma. Then for any $x \in I_1$, the function $\mathcal{G}_\Omega(u^{-1}(x),\cdot)$ is holomorphic in the interior of $N_{\alpha',x}$, the function $\mathcal{F}$ defined by Eq. (5.13) exists and $\mathcal{F}(x,\cdot)$ is holomorphic in the interior of $N_{\alpha',x}$. For $|\kappa| < \delta_6(\alpha)$ now $\zeta(x,\kappa)$ is in the interior of $N_{\alpha,x}$ and for all ϑ in the interior of $N_{\alpha,x}$ we have

$$\varrho(x)\mathcal{F}(x,\vartheta) = \frac{1}{2\pi i}\int_{\partial N_{\alpha,x}} \frac{\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),z)}{(z-\zeta(x,\kappa))^2(z-\vartheta)}\,dz\,,$$

$$\varrho(x)\frac{\partial\mathcal{F}(x,\vartheta)}{\partial\vartheta} = \frac{1}{2\pi i}\int_{\partial N_{\alpha,x}} \frac{\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),z)}{(z-\zeta(x,\kappa))^2(z-\vartheta)^2}\,dz\,.$$

If $|\kappa| < \delta_6(\alpha)$ and $\vartheta \in N_{\frac{\alpha}{2},x}$, then

$$|z-\zeta(x,\kappa)| \;\geq\; |z-x| - |x-\zeta(x,\kappa)| \geq \alpha - 2M_1\kappa^2 > \frac{\alpha}{2}\,, \quad (5.16)$$

$$|z-\vartheta| \;\geq\; |z-x| - |x-\vartheta| \geq \frac{\alpha}{2} \quad (5.17)$$

by Lemma 5.3, and the inequalities (5.14), (5.15) follow immediately. ∎

Theorem 5.5 *Assume (a1)-(a9). Then there exist finite constants $\delta' > 0$ and $C_6 > 0$ such that for all $|\kappa| < \delta'$ and $t > 0$ we have*

$$\left|\mathcal{U}(t,x) - A(x,\kappa)e^{-i\zeta_1(x,\kappa)t-\zeta_2(x,\kappa)t}\right| \leq \frac{C_6\kappa^2}{t} \quad (5.18)$$

for a.e. $x \in I_1$ where $\zeta(x,\kappa) = \zeta_1(x,\kappa) - i\zeta_2(x,\kappa)$ is the singularity location (with ζ_1 real, $\zeta_2 \geq 0$ – cf. Theorem 3.6) and

$$A(x,\kappa) := \left[1 + \kappa^2\varrho(x)\frac{\partial\mathcal{G}_\Omega(u^{-1}(x),\zeta(x,\kappa))}{\partial\zeta}\right]^{-1}.$$

Proof: If $\kappa = 0$ we have $\zeta(x,0) = x$ by (3.13) and $\mathcal{U}(t,x) = e^{-ixt}$ (see (2.2)) so the theorem holds with any C_6. Let us further suppose that $\kappa \neq 0$. By Theorem 3.6 and assumption *(a8)*, $\zeta_2(x,\kappa) > 0$ for a.e. $x \in I_1$ if $|\kappa| < \delta_2$. Let us exclude the remaining zero-measure set of x's from our considerations. Then the integral

$$\int_{-\infty}^{\infty} e^{-i\xi t}V(x,\xi)\,d\xi = \lim_{R\to\infty}\int_{-R}^{R} e^{-i\xi t}V(x,\xi)\,d\xi\,, \quad (5.19)$$

where

$$V(x,\xi) = \frac{1}{\pi}\,\Im\,\frac{A(x,\kappa)}{\zeta(x,\kappa)-\xi}\,, \quad (5.20)$$

exists in the generalized sense (5.19). While the Lebesgue integral does not exist due to the behavior at large $|\xi|$, the existence of generalized integral is well known and will be in fact seen from our calculations below. We shall estimate the difference between $\mathcal{U}(t,x)$ in Eq. (4.5) and the integral (5.19).

Let us recall from the proof of Theorem 4.2 that

$$W(x,\xi) = \lim_{\eta\to 0^+}\frac{1}{\pi}\,\Im\,r(x,\xi+i\eta) = \frac{1}{\pi}\,\Im\,\frac{1}{x-\xi-\kappa^2\varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\xi)}\,, \quad (5.21)$$

where the last equality should be used for $\xi > \nu + u^{-1}(x)$ only. Combining this with (4.6), assumptions *(a2)*, *(a6)* and Lemma 4.1 we arrive at the estimate

$$|W(x,\xi)| \le \frac{\kappa^2 C_1 C_2}{(\xi - x - \kappa^2 C_1 C_4)^2} \tag{5.22}$$

for $\xi > x + \kappa^2 C_1 C_4$. Due to Lemma 5.2 we can integrate by parts for $|\kappa| < \delta_3$,

$$\int_{-\infty}^{\infty} e^{-i\xi t}[W(x,\xi) - V(x,\xi)]\, d\xi = -\frac{i}{t}\int_{-\infty}^{\infty} e^{-i\xi t}\frac{\partial}{\partial \xi}[W(x,\xi) - V(x,\xi)]\, d\xi\,.$$

Let us choose an $\alpha > 0$ satisfying the assumptions of Lemma 5.4 and consider only the values of the coupling constants such that

$$0 < |\kappa| < \min\left(\delta_3, \delta_6(\alpha), \frac{1}{2}\sqrt{\frac{\alpha}{C_1 C_4}}\right)\,. \tag{5.23}$$

To calculate $\frac{\partial}{\partial \xi}W(x,\xi)$ let us denote for a while

$$\begin{aligned}
D_+ = D_+(x,\kappa,\xi) &= x - \xi - \kappa^2 \varrho(x)\mathcal{G}_\Omega(u^{-1}(x),\xi)\,,\\
D_1 = \Re D_+ &= x - \xi - \kappa^2 \varrho(x)\Re\mathcal{G}_\Omega(u^{-1}(x),\xi)\,,\\
D_2 = \Im D_+ &= -\kappa^2 \varrho(x)\Im\mathcal{G}_\Omega(u^{-1}(x),\xi)\,,\\
D_1' = \frac{\partial D_1}{\partial \xi} &= -1 - \kappa^2 \varrho(x)\Re\frac{\partial}{\partial \xi}\mathcal{G}_\Omega(u^{-1}(x),\xi)\,,\\
D_2' = \frac{\partial D_2}{\partial \xi} &= -\kappa^2 \varrho(x)\Im\frac{\partial}{\partial \xi}\mathcal{G}_\Omega(u^{-1}(x),\xi)\,.
\end{aligned}$$

Then

$$\frac{\partial}{\partial \xi}W(x,\xi) = -\pi^{-1}|D_+|^{-4}\left[(D_1^2 - D_2^2)D_2' - 2D_1 D_1' D_2\right]\,.$$

If now $|\xi - x| \ge \frac{\alpha}{2}$ the assumption *(a2)* together with Lemmas 4.1 and 5.1 (where we denote the constant as C_3') imply

$$\begin{aligned}
|D_+| &\ge |\xi - x| - \kappa^2 C_1 C_4 > \frac{\alpha}{4} > 0\,,\\
|D_+| &\ge \frac{1}{2}|\xi - x|\,,\\
|D_1| &\le |\xi - x| + \kappa^2 C_1 C_4 < 2|\xi - x|\,,\\
|D_2| &\le \kappa^2 C_1 C_4 < \frac{\alpha}{4}\,,\\
|D_1'| &\le C_3' + \kappa^2 C_1 C_3 |\ln(\xi - u^{-1}(x) - \nu)| + \kappa^2 C_1 C_5 (\xi - u^{-1}(x) - \nu)\,,\\
|D_2'| &\le \kappa^2 \pi C_1 C_3\,.
\end{aligned}$$

From here we get

$$\int_{(\nu+u^{-1}(x),x-\frac{\alpha}{2})\cup(x+\frac{\alpha}{2},\infty)} |\frac{\partial}{\partial\xi}W(x,\xi)|\,d\xi \le C_7\kappa^2\,,$$

where the explicit value of the constant C_7 can be expressed from the above estimates if necessary. What is important is that C_7 can be chosen independent of κ in the considered range.

Let us consider the term $V(x,\xi)$ now. We have the bounds

$$\frac{1}{1+\kappa^2M_2} \le |A(x,\kappa)| \le \frac{1}{1-\kappa^2M_2}$$

by Lemma 5.3, so

$$\frac{2}{3} \le |A(x,\kappa)| \le 2 \tag{5.24}$$

holds for

$$|\kappa| < \min\left(\delta_4,(2M_2)^{-1/2}\right)\,. \tag{5.25}$$

Denoting for a while $A_1 = \Re A(x,\kappa)$, $A_2 = \Im A(x,\kappa)$, we have

$$|A_1| \le |A(x,\xi)| \le 2\,, \quad |A_2| \le \kappa^2M_2|A(x,\kappa)|^2 \le 4\kappa^2M_2$$

and

$$\frac{\partial}{\partial\xi}V(x,\xi) = \frac{1}{\pi}\,\frac{A_2[(\xi-\zeta_1(x,\kappa))^2-\zeta_2(x,\kappa)^2]-2A_1\zeta_2(x,\kappa)(\xi-\zeta_1(x,\kappa))}{[(\xi-\zeta_1(x,\kappa))^2+\zeta_2(x,\kappa)^2]^2}\,.$$

If $|\xi-x| \ge \frac{\alpha}{2}$ and

$$|\kappa| < \min\left(\delta_5,\sqrt{\frac{\alpha}{8M_1}}\right)\,, \tag{5.26}$$

we have $|\xi-\zeta_1(x,\kappa)| \ge \frac{\alpha}{4}$ by Lemma 5.3, and therefore

$$\int_{(-\infty,x-\frac{\alpha}{2})\cup(x+\frac{\alpha}{2},\infty)} \left|\frac{\partial}{\partial\xi}V(x,\xi)\right|\,d\xi \le C_8\kappa^2$$

with a κ-independent finite constant C_8 which can be given explicitly if necessary.

Let us now turn to $\xi \in (x - \frac{\alpha}{2}, x + \frac{\alpha}{2})$. Using the expansion (5.13),

$$\begin{aligned} W(x,\xi) - V(x,\xi) &= \frac{1}{\pi}\,\Im \frac{\kappa^2 A(x,\kappa)^2 \varrho(x)\mathcal{F}(x,\xi)}{1+\kappa^2 A(x,\kappa)\varrho(x)(\xi-\zeta(x,\kappa))\mathcal{F}(x,\xi)}\,, \\ \frac{\partial}{\partial \xi}[W(x,\xi) - V(x,\xi)] &= \frac{1}{\pi}\,\kappa^2 \Im \Bigg\{ A(x,\kappa)^2 \\ &\quad \times \frac{\varrho(x)\frac{\partial \mathcal{F}(x,\xi)}{\partial \xi} - \kappa^2 A(x,\kappa)\varrho(x)^2 \mathcal{F}(x,\xi)^2}{[1+\kappa^2 A(x,\kappa)\varrho(x)(\xi-\zeta(x,\kappa))\mathcal{F}(x,\xi)]^2} \Bigg\}\,. \end{aligned}$$

Using (5.24), (5.14), (5.15) together with Lemma 5.3, and assuming that

$$|\kappa| < \min\left(\delta_6(\alpha), \frac{1}{2}\sqrt{\frac{\alpha}{M_1}}\,, \frac{1}{2\sqrt{m_3(\alpha)\alpha}}\right)\,, \tag{5.27}$$

we obtain

$$\left|\frac{\partial}{\partial \xi}[W(x,\xi) - V(x,\xi)]\right| \le C_9 \kappa^2\,,$$

where

$$C_9 := \frac{16}{\pi}\left(m_4(\alpha) + 2\delta_6(\alpha)^2 m_3(\alpha)^2\right)\,.$$

Putting all the estimates together, we get

$$\left|\int_{-\infty}^{\infty} e^{-i\xi t}\,\frac{\partial}{\partial \xi}[W(x,\xi) - V(x,\xi)]\,d\xi\right| \le C_6 \kappa^2\,,$$

where

$$C_6 = C_7 + C_8 + C_9\alpha$$

and

$$|\kappa| < \delta'\,;$$

δ' being the minimum of δ, δ_2 and the r.h.s. in (5.23),(5.25)–(5.27). Evaluating the generalized integral

$$\int_{-\infty}^{\infty} e^{-i\xi t}\,V(x,\xi)\,d\xi = A(x,\kappa)e^{-i\zeta(x,\kappa)t}$$

by closing the integration contour in the lower half-plane for $t > 0$ the inequality (5.18) is obtained. ∎

The theorem is apparently useless for very short and very large times when the error estimate $O(\kappa^2 t^{-1})$ is much larger then the amplitude value $\approx \exp(-\zeta_2(x,\kappa)t)$. On the other hand, we get a nontrivial bound for the times when

$$\frac{C_6\kappa^2}{t} \ll e^{-\zeta_2(x,\kappa)t} \tag{5.28}$$

where we take into account that $A(x,\kappa) \approx 1$. Let us write

$$\begin{aligned} \zeta_2(x,\kappa) &= \kappa^2\eta_2(x,\kappa)\,, \\ \eta_2(x,\kappa) &= \pi\varrho(x)v(u^{-1}(x), x-u^{-1}(x)) + O(\kappa^2) \end{aligned} \tag{5.29}$$

for small coupling κ. In the subsequent formulas we do not write the arguments of η_2, however, its x-dependence should be kept in mind in general. The relation (5.28) is valid for $T_1 \ll t \ll T_2$ where T_1, T_2 are two solutions of the equation

$$\kappa^2\eta_2 T_i e^{-\kappa^2\eta_2 T_i} = C_6\kappa^4\eta_2\,, \quad i=1,2. \tag{5.30}$$

If $\kappa^2 T_1$ is small we can approximate the equation by replacing the exponential with one obtaining

$$T_1 \approx C_6\kappa^2\,.$$

On the other hand, if $\kappa^2\eta_2 T_2 \gg 1$ we do not enlarge the range (T_1, T_2) by dropping the linear factor in (5.30). Then we obtain

$$T_2 \approx -\frac{1}{\kappa^2\eta_2}\ln(C_6\kappa^4\eta_2)\,.$$

The r.h.s. here is an decreasing function of η_2 in the interval $(0, C_6^{-1}\kappa^{-4})$. By (5.29) and assumptions *(a2), (a6)* we have

$$0 \le \eta_2 \le \pi C_1 C_2$$

in the κ^0 approximation. Restricting ourselves then to the coupling constant values with

$$|\kappa| \ll (\pi C_1 C_2 C_6)^{-1/4}\,,$$

we can safely use

$$T_2 \approx -\frac{1}{\pi C_1 C_2 \kappa^2}\ln(\pi C_1 C_2 C_6 \kappa^4)\,.$$

Hence we see that the announced approximately exponential behaviour of $\mathcal{U}(\cdot, x)$ holds in the weak-coupling regime over wide time range, roughly speaking from $C_1^{-\frac{1}{4}} C_2^{-\frac{1}{4}} C_6^{\frac{3}{4}}\kappa$ to $C_1^{-1}C_2^{-1}\kappa^{-2}$.

VI Long time behavior

The fact that the bound given by Theorem 5.5 becomes useless at very large times is not coincidental, because the decay rate is indeed slower there. To illustrate this claim, for instance, let $x \in I_1$ be such that by Lemma 3.4, Theorem 3.6(b), Theorem 4.2 and assumptions *(a6)–(a8)*, we have

$$W(x,\xi) \text{ is finite and continuous w.r.t. } \xi \in \left[\nu + u^{-1}(x), \infty\right) \tag{6.1}$$

for $0 < |\kappa| < \delta_2$, where δ_2 is the number from Theorem 4.2. This holds for a.e. $x \in I_1$.

By *(a6)* and (4.6), we get

$$|W(x,\xi)| \leq \frac{\kappa^2 \varrho(x) C_2}{\left[x - \xi - \kappa^2 \varrho(x) I(u^{-1}(x), \xi)\right]^2} =: T(\xi).$$

Since $\lim_{\xi\to\infty} I(u^{-1}(x),\xi) = 0$ by (3.8), we get

$$T(\xi) \underset{\xi\to\infty}{\sim} \frac{\kappa^2 \varrho(x) C_2}{(x-\xi)^2}. \tag{6.2}$$

Thus, $g(\xi) := \chi_{[\nu+u^{-1}(x),\infty)}(\xi) W(x,\xi)$ is in $L^2(\mathbb{R})$, but its support is not the whole $\mathbb{R}$, and

$$\mathcal{U}(t,x) = \int_{\nu+u^{-1}(x)}^{\infty} g(\xi) e^{-i\xi t} d\xi$$

by (4.5). Applying now [13, Corollary C2], we find that for a.e. $x \in I_1$ and $|\kappa| < \delta_2$

$$\mathcal{U}(t,x) \text{ does not decay exponentially as } |t| \to \infty. \tag{6.3}$$

To learn more about the long-time asymptotic behavior of $\mathcal{U}(t,x)$, we adopt the conditions *(a10)–(a13)* below, and employ the results of [13] and [15] in the same way as in [14, Thm 3.2(ii)].

Given $\nu \geq 0$ and $\theta \in (0, \pi/2)$, we define $\mathbf{D}_{\nu,\theta}$ by

$$\mathbf{D}_{\nu,\theta} := \left\{\zeta \in \mathbb{C} \,|\, \Re\zeta > \nu,\, -\theta < \arg\zeta < 0\right\}. \tag{6.4}$$

If $\nu < \nu'$, we have therefore

$$\mathbf{D}_{\nu',\theta} \subset \mathbf{D}_{\nu,\theta}. \tag{6.5}$$

Let us denote

$$\Omega(v) := \cap_{y\in I_0} (\Omega - y). \tag{6.6}$$

Notice that $\Omega(v) \subset \cap_{y\in I_0} \Omega_{v,y}$ by *(a3)*. We shall assume:

(a10) There exists $\theta_0 \in (0, \pi/4)$ such that $\overline{\mathbf{D}_{\nu,\theta_0}} \subset \Omega(v)$.

(a11) $v(y, \xi) > 0$ holds for each $y \in I_0$ and $\xi > \nu$.

(a12) Given $y \in I_0$, there exists $C_y > 0$ and $q_y > 0$ such that

$$|v(y, \zeta)| < C_y |\zeta|^{-q_y}$$

holds for any $\zeta \in \Omega(v)$.

Notice that for v which is continuous by *(a3)*, the assumption *(a11)* implies, in particular, that for each $x \in I_1$ and $\alpha, \beta \in (\nu, \infty)$ we have

$$\mu_{x,\alpha,\beta} := \inf_{\alpha \leq \xi \leq \beta} v\left(u^{-1}(x), \xi\right) > 0 \,. \tag{6.7}$$

For fixed $x \in I_1$ and $\zeta \in \Omega \setminus (-\infty, u^{-1}(x) + \nu]$, we have defined $D_+(x, \kappa, \zeta)$ by (3.17). In a similar way, we define three other functions, $D_-(x, \kappa, \zeta)$, $W(x, \zeta)$, and $g_x(\zeta)$ by

$$D_-(x, \kappa, \zeta) \;:=\; x - \zeta - \kappa^2 \varrho(x) \mathcal{G}^{\Omega}(u^{-1}(x), \zeta)\,, \tag{6.8}$$

$$W(x, \zeta) \;:=\; \frac{\kappa^2 \varrho(x) v\left(u^{-1}(x), \zeta - u^{-1}(x)\right)}{D_+(x, \kappa, \zeta) D_-(x, \kappa, \zeta)}\,, \tag{6.9}$$

$$\begin{aligned} g_x(\zeta) \;&:=\; W\left(x, \zeta + u^{-1}(x)\right) \\ &=\; \frac{\kappa^2 \varrho(x) v\left(u^{-1}(x), \zeta\right)}{D_+\left(x, \kappa, \zeta + u^{-1}(x)\right) D_-\left(x, \kappa, \zeta + u^{-1}(x)\right)}\,; \end{aligned} \tag{6.10}$$

in the last case $\zeta \in \Omega(v) \setminus (-\infty, \nu]$. Then, for a.e. $x \in I_1$ and $\kappa \in \mathbb{R}$ with $0 < |\kappa| < \delta_2$,

$$g_x \text{ can be regarded as measurable with } g_x \in L^1\left((\nu, \infty), d\xi\right) \tag{6.11}$$

by (6.1) and (6.2), and we can write the time evolution as follows,

$$\mathcal{U}(t, x) = e^{-iu^{-1}(x)t} \int_{\nu}^{\infty} g_x(\xi) e^{-i\xi t} d\xi\,, \tag{6.12}$$

by (4.5) and (4.6).

Next we need several lemmas. The first of them follows from *(a3), (a10),* and Lemma 3.4:

Lemma 6.1 *$g_x(\zeta)$ is meromorphic in $\mathbf{D}_{\nu,\theta_0}$ for every $x \in I_1$ and $\kappa \in \mathbb{R}$.*

Lemma 6.2 *For every $x \in I_1$ with $\varrho(x) \neq 0$, $\xi \in \mathbb{R}$ with $\xi > \nu$, and $\kappa \in \mathbb{R}$ with $0 < |\kappa| < \delta_2$,*

$$\lim_{\varepsilon \to 0^+} g_x(\xi - i\varepsilon) = g_x(\xi). \tag{6.13}$$

Proof: Let $\xi' \equiv \xi + u^{-1}(x)$. By Lemma 3.4 we have

$$\begin{aligned} D_+(x,\kappa,\xi') &:= \lim_{\varepsilon \to 0^+} D_+(x,\kappa,\xi' - i\varepsilon) && (6.14)\\ &= x - \xi' - \kappa^2 \varrho(x) \Big\{ I\left(u^{-1}(x), \xi'\right) + i\pi v\left(u^{-1}(x), \xi\right) \Big\}, \\ D_-(x,\kappa,\xi') &:= \lim_{\varepsilon \to 0^+} D_-(x,\kappa,\xi' - i\varepsilon) && (6.15)\\ &= x - \xi' - \kappa^2 \varrho(x) \Big\{ I\left(u^{-1}(x), \xi'\right) - i\pi v\left(u^{-1}(x), \xi\right) \Big\} \end{aligned}$$

for $\xi' > u^{-1}(x) + \nu$ $(\xi > \nu)$, which implies that

$$\begin{aligned} &D_+(x,\kappa,\xi') D_-(x,\kappa,\xi') && (6.16)\\ &\quad = \left[x - \xi' - \kappa^2 \varrho(x) I\left(u^{-1}(x), \xi'\right) \right]^2 + \pi^2 \kappa^4 \varrho(x)^2 v\left(u^{-1}(x), \xi\right)^2. \end{aligned}$$

Then $\lim_{\varepsilon \to 0^+} W(x, \xi' - i\varepsilon) = W(x, \xi')$ follows from (4.6) giving (6.13). ∎

Lemma 6.3 *For every $x \in I_1$, with $\varrho(x) \neq 0$, all sufficiently small $\varepsilon > 0$, every $\alpha, \beta \in (\nu, \infty)$ with $\alpha < \beta$, and every $\kappa \in \mathbb{R}$ with $0 < |\kappa| < \delta_2$, there exists a constant $C_{x,\alpha,\beta} > 0$ independent of ε such that*

$$\sup_{\alpha < \xi < \beta} |g_x(\xi - i\varepsilon)| \leq C_{x,\alpha,\beta}. \tag{6.17}$$

Proof: Set $S_{p,q} := \{\zeta \in \mathbb{C} \,|\, p \leq \Re\zeta \leq q,\ -\nu \tan\theta_0 \leq \Im\zeta \leq 0\}$. Fix $\varepsilon' \in \mathbb{R}$ with $0 < \varepsilon' < 1$ arbitrarily. $v\left(u^{-1}(x), \cdot\right)$ is uniformly continuous in $S_{\alpha,\beta}$ by *(a3)* and *(a10)* since $S_{\alpha,\beta} \subset \overline{\mathbf{D}_{\nu,\theta_0}}$. So there exists a constant $\varepsilon_1 \equiv \varepsilon_1(x, \varepsilon') > 0$ such that

$$\Big| v\left(u^{-1}(x), \xi - i\varepsilon\right) - v\left(u^{-1}(x), \xi\right) \Big| \leq \varepsilon' \Big| v\left(u^{-1}(x), \xi\right) \Big|$$

for $\alpha \leq \xi \leq \beta$ and $0 < \varepsilon < \varepsilon_1$ and we have

$$\Big| v\left(u^{-1}(x), \xi - i\varepsilon\right) \Big| \leq (1 + \varepsilon') \Big| v\left(u^{-1}(x), \xi\right) \Big| \tag{6.18}$$

for $\alpha \leq \xi \leq \beta$ and $0 < \varepsilon < \varepsilon_1$. Since $D_\pm(x,\kappa,\cdot)$ is holomorphic in $\Omega \setminus (-\infty, u^{-1}(x)+\nu]$ by Lemma 3.4, $D_\pm(x,\kappa,\cdot)$ is uniformly continuous in $S_{\alpha,\beta}+u^{-1}(x)$. In view of (6.14) and (6.15) there exists $\varepsilon_2 \equiv \varepsilon_2(x,\varepsilon') > 0$ such that

$$|D_\pm(x,\kappa,\xi'-i\varepsilon) - D_\pm(x,\kappa,\xi')| \leq \varepsilon' |D_\pm(x,\kappa,\xi')|$$

for $\xi' \equiv \xi + u^{-1}(x)$ with $\alpha \leq \xi \leq \beta$ and $0 < \varepsilon < \varepsilon_2$. Hence we have

$$(1-\varepsilon')\left|D_\pm\left(x,\kappa,\xi+u^{-1}(x)\right)\right| \leq \left|D_\pm\left(x,\kappa,\xi+u^{-1}(x)-i\varepsilon\right)\right| \qquad (6.19)$$

if $\alpha \leq \xi \leq \beta$ and $0 < \varepsilon < \varepsilon_2$. Using further (6.7), (6.10), (6.16), (6.18), and (6.19), we get

$$\begin{aligned} |g_x(\xi - i\varepsilon)| &\leq \frac{(1+\varepsilon')\kappa^2\varrho(x)|v\left(u^{-1}(x),\xi\right)|}{(1-\varepsilon')^2|D_+\left(x,\kappa,\xi+u^{-1}(x)\right)D_-\left(x,\kappa,\xi+u^{-1}(x)\right)|} \\ &\leq \frac{(1+\varepsilon')\kappa^2\varrho(x)|v\left(u^{-1}(x),\xi\right)|}{(1-\varepsilon')^2\pi^2\kappa^4\varrho(x)^2|v\left(u^{-1}(x),\xi\right)|^2} \\ &\leq \frac{(1+\varepsilon')}{(1-\varepsilon')^2\mu_{x,\alpha,\beta}\pi^2\kappa^2\varrho(x)} \end{aligned}$$

for $\alpha \leq \xi \leq \beta$ and $0 < \varepsilon < \varepsilon_0 \equiv \min\{\varepsilon_1,\varepsilon_2\}$, which implies the desired result. ∎

Lemma 6.4 *For every $x \in I_1$, all sufficiently large $|\zeta|$ with $\zeta \in \mathbf{D}_{\nu,\theta_0}$, and every $\kappa \in \mathbb{R}$ satisfying $0 < |\kappa| < \delta_2$,*

$$|g_x(\zeta)| \leq \frac{C_{10}}{|\zeta|^{2+q_y}}$$

with a constant $C_{10} > 0$ independent of $\zeta \in \mathbf{D}_{\nu,\theta_0}$.

Proof: In this proof, we set $y = u^{-1}(x)$, $\xi' \equiv \xi + u^{-1}(x)$, and let $\xi > \nu \geq 0$. Since

$$D_-(x,\kappa,\xi'-i\varepsilon) = x - (\xi'-i\varepsilon) - \kappa^2\varrho(x)\mathcal{G}(y,\xi'-i\varepsilon)$$

for every $\varepsilon > 0$, we get

$$|D_-\left(x,\kappa,\xi+u^{-1}(x)-i\varepsilon\right)|^2 \geq \left(\xi + A_{\varepsilon,x}(\xi)\right)^2,$$

where
$$A_{\varepsilon,x}(\xi) := \kappa^2 \varrho(x) \Re \mathcal{G}(y, \xi' - i\varepsilon) + u^{-1}(x) - x\,.$$
Set
$$B_x := \kappa^2 \varrho(x) C_4 + |u^{-1}(x)| + |x| > 0\,.$$
Then we get $|A_{\varepsilon,x}(\xi)| \le B_x$ by Lemma 4.1. Since we now take $\xi > 0$, we get for every C_- with $0 < C_- < 1$,
$$\begin{aligned}(\xi + A_{\varepsilon,x}(\xi))^2 - C_-^2 \xi^2 &\ge \xi^2 - 2B_x \xi - C_-^2 \xi^2 \\ &= (1 - C_-^2)\left(\xi - \frac{B_x}{1 - C_-^2}\right)^2 - \frac{B_x^2}{1 - C_-^2}\,.\end{aligned}$$
Thus there exists C_- with $0 < C_- < 1$ and $\xi_- \equiv \xi_-(x) > 0$ independent of $\varepsilon > 0$ such that
$$\left| D_-\left(x, \kappa, \xi + u^{-1}(x) - i\varepsilon\right) \right| > C_- \xi \tag{6.20}$$
for every $\xi > \xi_-$. As for $D_+(x, \kappa, \xi' - i\varepsilon)$, we have
$$D_+(x, \kappa, \xi' - i\varepsilon) = D_-(x, \kappa, \xi' - i\varepsilon) - 2i\kappa^2 \varrho(x) \pi v(y, \xi - i\varepsilon)$$
for any $\varepsilon > 0$. Moreover, by *(a12)* we get
$$|v(y, \xi - i\varepsilon)| \le \frac{C_y}{\{\xi^2 + \varepsilon^2\}^{q_y/2}} \le \frac{C_y}{\xi^{q_y}}$$
for $\xi - i\varepsilon \in \Omega(v)$. Thus there exists $\xi'_+ > 0$ independent of $\varepsilon > 0$ such that if $\xi > \xi'_+$, then
$$|v(y, \xi - i\varepsilon)| < \frac{C_-}{4\pi\kappa^2 \varrho(x)}\,.$$
Together we get
$$\begin{aligned}\left| D_+\left(x, \kappa, \xi + u^{-1}(x) - i\varepsilon\right) \right| &\ge \left| D_-\left(x, \kappa, \xi + u^{-1}(x) - i\varepsilon\right) \right| - \frac{C_-}{2} \\ &\ge C_- \xi - \frac{C_-}{2} > 0\end{aligned}$$
for $\xi \ge \max\left\{\xi_-, \xi'_+, 1\right\} =: \xi_+$ by (6.20); notice that ξ_+ is independent of $\varepsilon > 0$. On the other hand, we get
$$C_- \xi - \frac{C_-}{2} \ge \frac{C_-}{2}\,\xi$$

for $\xi > \xi_+$. Now we set $C_+ := C_-/2$; then $0 < C_+ < 1$ and

$$\left| D_+ \left(x, \kappa, \xi + u^{-1}(x) - i\varepsilon \right) \right| > C_+ \xi \tag{6.21}$$

for every $\xi > \xi_+$. Put $\xi \equiv \Re\zeta$ and $-\eta \equiv \Im\zeta$ so that $\eta > 0$. Then, having $\xi \geq \eta$, we get $2\xi^2 - (\xi^2 + \eta^2) = \xi^2 - \eta^2 \geq 0$. Hence by (6.20) and (6.21) we obtain

$$\frac{C_\pm}{\sqrt{2}} \leq \frac{C_\pm \xi}{\sqrt{\xi^2 + \eta^2}} \leq \frac{|D_\pm (x, \kappa, \zeta + u^{-1}(x))|}{|\zeta|}$$

for $\zeta = \xi - i\eta$ with $\xi \geq \max(\xi_\pm, \eta)$. If $\zeta \in \mathbf{D}_{y,\theta_0}$ with $\Re\zeta > \max(\xi_\pm, |\Im\zeta|)$, we have

$$\frac{C_\pm}{\sqrt{2}} |\zeta| \leq \left| D_\pm \left(x, \kappa, \zeta + u^{-1}(x) \right) \right| . \tag{6.22}$$

Using then (6.10), *(a12)*, and (6.22), we arrive at

$$|g_x(\zeta)| \leq 2 \frac{\kappa^2 \varrho(x)}{C_+ C_-} C_y |\zeta|^{-(2+q_y)}$$

for sufficiently large $|\zeta|$ with $\zeta \in \mathbf{D}_{\nu,\theta_0}$. ∎

Next we set for any $x \in I_1$

$$d_\nu^x \equiv x - \left(\nu + u^{-1}(x) \right) - \kappa^2 \varrho(x) \int_\nu^\infty \frac{v\left(u^{-1}(x), z\right)}{z - \nu} \, dz . \tag{6.23}$$

Remark 6.5 Recall that by *(a5)* d_ν^x is positive for sufficiently small $|\kappa|$.

Let us finally state the last assumption:

(a13) Given $x \in I_1$, there are constants $A_{\nu,x} \neq 0$ and $p_{\nu,x} \geq 0$ such that

$$\lim_{\substack{\zeta \to 0 \\ \zeta \in \mathbf{D}_{0,\theta_0}}} \frac{v\left(u^{-1}(x), \zeta + \nu\right)}{\zeta^{p_{\nu,x}}} = A_{\nu,x} .$$

We set

$$\begin{aligned} \kappa_{\nu,x}^2 &:= \frac{x - \nu - u^{-1}(x)}{\varrho(x) I\left(u^{-1}(x), \nu + u^{-1}(x)\right)} \\ &= \left(x - \nu - u^{-1}(x) \right) \left\{ \varrho(x) \int_\nu^\infty \frac{v\left(u^{-1}(x), z\right)}{z - \nu} dz \right\}^{-1} . \end{aligned} \tag{6.24}$$

By *(a5)*, this quantity satisfies $\kappa_{\nu,x}^2 > 0$ and $d_\nu^x \neq 0$ for $\kappa^2 \neq \kappa_{\nu,x}^2$.

Lemma 6.6 *Assume (a1)–(a3), (a7) and (a10). Then*

$$\lim_{\xi\to\nu^+,\,\eta\to 0^+} D_\pm(x,\kappa,\xi-i\eta+u^{-1}(x)) = d^x_\nu\,. \tag{6.25}$$

Proof: By (3.17), (6.8) and Lemma 3.4,

$$\begin{aligned}
&D_+(x,\kappa,\xi-i\eta+u^{-1}(x)) = x-\xi-u^{-1}(x)+i\eta\\
&\quad -\kappa^2\varrho(x)[\mathcal{G}(u^{-1}(x),\xi-i\eta+u^{-1}(x))+2i\pi v(u^{-1}(x),\xi-i\eta)]\,,\\
&D_-(x,\kappa,\xi-i\eta+u^{-1}(x)) = x-\xi-u^{-1}(x)+i\eta\\
&\quad -\kappa^2\varrho(x)\mathcal{G}(u^{-1}(x),\xi-i\eta+u^{-1}(x))
\end{aligned}$$

for $\eta>0$, $\xi-i\eta\in\Omega$ which is the sufficient range of variables as $\nu+u^{-1}(x)\in\Omega$ by *(a10)*. Under assumption *(a10)*, there exists $A>0$ such that $v(u^{-1}(x),\cdot)$ is holomorphic in the set $\{\zeta\in\mathbb{C}||\zeta-\nu|<2A\}$. Taking into account *(a7)* then

$$\lim_{\xi\to\nu^+,\,\eta\to 0^+} v(u^{-1}(x),\xi-i\eta) = v(u^{-1}(x),\nu)=0\,.$$

Let us write

$$\begin{aligned}
\mathcal{G}(u^{-1}(x),\xi-i\eta+u^{-1}(x)) &= \int_\nu^{\nu+A}\frac{v(u^{-1}(x),z)-v(u^{-1}(x),\xi-i\eta)}{z-\xi+i\eta}\,dz\\
&+v(u^{-1}(x),\xi-i\eta)\int_\nu^{\nu+A}\frac{dz}{z-\xi+i\eta}+\int_{A+\nu}^\infty\frac{v(u^{-1}(x),z)}{z-\xi+i\eta}\,dz\,.
\end{aligned}\tag{6.26}$$

For the first and third integral, dominated convergence theorem can be used giving (recall *(a7)*)

$$\int_\nu^\infty\frac{v(u^{-1}(x),z)}{z-\nu}\,dz$$

as the limit of their sum as $\xi\to\nu^+$, $\eta\to 0^+$. The second integral

$$\begin{aligned}
\int_\nu^{\nu+A}\frac{dz}{z-\xi+i\eta} &= \frac{1}{2}\ln\frac{(\nu+A-\xi)^2+\eta^2}{(\xi-\nu)^2+\eta^2}\\
&+ i\left(\arctan\frac{\eta}{\nu+A-\xi}+\arctan\frac{\eta}{\xi-\nu}-\pi\right)
\end{aligned}$$

for $\nu<\xi<\nu+A$, $\eta>0$. As $|v(u^{-1}(x),\xi-i\eta)|\le c\sqrt{(\xi-\nu)^2+\eta^2}$ for $\xi-i\eta$ in a neighborhood of ν with a suitable constant c, the limit of the second term in (6.26) is zero. Now (6.25) is seen. ∎

Lemma 6.7 *Let $0 < |\kappa| < \delta_2$, $x \in I_1$ and $d^x_\nu \neq 0$. Then the function g_x has no poles in $\{\zeta \in \mathbf{D}_{\nu,\theta_0} \,|\, |\zeta - \nu| < \varepsilon_0\}$ with a constant $\varepsilon_0 > 0$ and the limit*

$$\begin{aligned} w_{\nu,x} &:= \lim_{\zeta\to 0,\, \zeta\in\mathbf{D}_{0,\theta_0}} \frac{g_x(\nu+\zeta)}{\zeta^{p_{\nu,x}}} \qquad (6.27) \\ &= \frac{\kappa^2\varrho(x)A_{\nu,x}}{D_+\left(x,\kappa,\nu+u^{-1}(x)\right)D_-\left(x,\kappa,\nu+u^{-1}(x)\right)} = \frac{\kappa^2\varrho(x)A_{\nu,x}}{{d^x_\nu}^2}\,. \end{aligned}$$

Proof: The poles of $g_x(\zeta)$ come only from the zeroes of $D_\pm\left(x,\kappa,\zeta+u^{-1}(x)\right)$. If $d^x_\nu \neq 0$ then

$$\lim_{\zeta\to\nu,\, \zeta\in\mathbf{D}_{\nu,\theta_0}} g_x(\zeta) = 0 \qquad (6.28)$$

by (6.10) and Lemma 6.6. By Lemma 6.1, g_x is meromorphic in $\mathbf{D}_{\nu,\theta_0}$ so its only possible singularities there are isolated poles; they also do not accumulate at ν due to (6.28). Thus $g_x(\zeta)$ has no poles in a small neighborhood of $\zeta = \nu$ in $\mathbf{D}_{\nu,\theta_0}$. By *(a13)*, we therefore have

$$g_x(\nu+\zeta) \underset{\zeta\to 0}{\sim} \frac{\kappa^2\varrho(x)A_{\nu,x}}{{d^x_\nu}^2}\zeta^{p_{\nu,x}}\,. \quad \blacksquare$$

Now we can formulate the main theorem of this section:

Theorem 6.8 *Assume (a1)–(a7), (a10)–(a13). Then for every $x \in I_1$ and $\kappa \in \mathbb{R}$ satisfying $\varrho(x) > 0$, $d^x_\nu \neq 0$, $0 < |\kappa| < \delta_2$ we have the following asymptotic behaviour:*

$$\mathcal{U}(t,x) \underset{t\to\infty}{\sim} w_{\nu,x}e^{-i[\nu+u^{-1}(x)]t}e^{-i\pi(p_{\nu,x}+1)/2}\Gamma\left(p_{\nu,x}+1\right)t^{-(p_{\nu,x}+1)},$$

where Γ is the gamma function.

Proof: It is sufficient to apply [13, Theorem 2.1(b)] to (6.12) with the help of Lemmas 6.1-6.7 and we obtain the desired result. $\blacksquare$

Acknowledgment

The research has been partially supported by GA ASCR and Czech Ministry of Education under the contracts A1048801 and ME170. M.H. was supported by Grant-in-Aid 11740109 for Encouragement of Young Scientists from Japan Society for the Promotion of Science.

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