# FEYNMAN MAPS WITHOUT IMPROPER INTEGRALS*)

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The Feynman maps introduced first by Truman are examined. The domain considered here consists of the Fresnel-integrable functions in the sense of Albeverio and Hoegh-Krohn; extensions to wider classes of functions will be studied in the following paper. The original definition of the F-maps is slightly modified: we start from the underlying measures on the Hilbert space of paths in order to avoid use of improper integrals. Some new properties of the F-maps are derived. In particular, the dominated convergence theorem is shown to be not valid for the $F_1$-map (or Feynman integral); this fact is of a certain importance for the classical limit of quantum mechanics.

## 1. INTRODUCTION

Mathematical concepts born in physics often prove their fertility in two steps: first on a more or less formal level and after that by finding a suitable mathematical framework and examining the original idea rigorously. So the delta function represents itself an extremely useful computational tool; on the other hand, full power of this concept (which certainly goes beyond Dirac's intention) was not revealed before formulation of the distribution theory. One can therefore understand easily why there is so much temptation in mathematical theory of Feynman integrals, or more exactly, in various attempts to construct such a theory. A substantial progress achieved in this field during recent years is reported e.g. in the monographs [1, 2] and review papers [3–5][1]).

The efforts are mainly concentrated around the problem of expressing dynamics by means of path integrals. As to the simplest case of a single spinless particle corresponding to the free Hamiltonian $H_0 = -(\hbar^2/2m)\,\Delta$ which interacts with an external field described by a potential $V$, the celebrated Feynman result (cf. [6, 7]. chap. 3) states that the wave function at a given time $t$ is given by [2])

$$(1)\qquad \left(\exp\left(-\frac{\mathrm{i}}{\hbar}(H_0+V)\,t\right)\psi\right)(x)=\int_{\Gamma_x}\exp\left(\frac{\mathrm{i}}{\hbar}S(\gamma)\right)\psi(\gamma(0))\,\mathscr{D}\gamma\,,$$

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[1]) We pretend neither to an exhaustive exposition of the path-integration problems nor to completeness of the list of references — at the present time this seems to be a task rather for a monograph writer.

[2]) For the sake of simplicity we shall further always set $\hbar = m = 1$.

where $\psi$ is a wave function at the initial time $t = 0$,

$$S(\gamma) = \int_0^t [\tfrac{1}{2}m|\dot{\gamma}(\tau)|^2 - V(\gamma(\tau))] \, d\tau \tag{2}$$

is the classical action along the path $\gamma$ and $\Gamma_x$ is the space of all paths ending in the point $x$[3]. The central problem is to give meaning to the formal expression on the rhs of eq. (1), or more generally, to

$$\int_{\Gamma_x} \exp\left(\frac{i}{2s} \int_0^t |\dot{\gamma}(\tau)|^2 \, d\tau\right) f(\gamma) \, \mathscr{D}\gamma \,, \tag{3}$$

where $f$ is a complex-valued function on the path space $\Gamma_x$ and $s$ is a real parameter. The first possible way was proposed by Feynman in his original paper. He replaces the set of all paths by a subset of polygonal paths (velocity of the particle is assumed to be constant in the time intervals $(ti/n, t(i+1)/n)$, $i = 0, 1, \ldots, n-1$), in which case (3) can be defined naturally as an integral over the corresponding finite-dimensional vector space of paths. The construction is completed by taking the limit $n \to \infty$. It is clear that we need not assume equidistant partitions of $[0, t]$ only; every sequence of partitions such that the subinterval lengths tend to zero would serve as well. The most important property of this definition is the following: for *cylindrical functions* (which are, roughly speaking, those depending on a "finite" number of variables only) the relation

$$\int_{\Gamma_x} \exp\left(\frac{i}{2s} \int_0^t |\dot{\gamma}(\tau)|^2 \, d\tau\right) f(\gamma(\tau_0), \ldots, \gamma(\tau_{n-1})) \, \mathscr{D}\gamma = \tag{4}$$

$$= \prod_{i=0}^{n-1} (2\pi \, is(\tau_{i+1} - \tau_i))^{-3/2} \int_{\mathbb{R}^{3n}} \exp\left(\frac{i}{2s} \sum_{i=0}^{n-1} |\gamma^{i+1} - \gamma^i|^2 (\tau_{i+1} - \tau_i)^{-1}\right) .$$

$$. f(\gamma^0, \ldots, \gamma^{n-1}) \, d\gamma^0, \ldots, d\gamma^{n-1}$$

is valid, where $0 = \tau_0 < \tau_1 < \ldots < \tau_n = t$. This relation interprets in such a natural way that it seems to be reasonable to require validity of an analogous formula for every definition of the functional integral (3).

Let us notice that one of the constructions of the *Wiener measure* starts just from formula (4) with $s = -i$ (cf. [9] or [8], sec. X. 11). In this case, however, the non-existence of the Lebesgue-type measure in an infinite-dimensional path space ([1], Appendix A; [10], chap. 1) does not hinder one from treating this functional integral in terms of the measure theory: loosely speaking, singularities of the exponential term and of $\mathscr{D}\gamma$ cancel one another, and the formal expression $\exp(-\frac{1}{2} \int_0^t |\dot{\gamma}(\tau)|^2 . d\tau) \mathscr{D}\gamma$ can be replaced by $dw(\gamma)$, where $w$ is the Wiener measure. On the other hand,

[3]) This is the standard quantum-mechanical convention. On the other hand, people more inclined to the probability theory often write *the same* formula using the space of paths with fixed origins — cf. [8], sec. X. 11, [1], p. 291 and Note I added in proof.

these considerations do not apply to the Feynman integral, where the exponential term behaves in a different way. One might overcome this difficulty by defining the F-integral as a limit (with $s$ arriving at the real axis from below), if appropriate path space measures existed in the open lower complex halfplane of $s$ (this is essentially the proposal of Gel'fand and Yaglom [11]). Unfortunately, there are no such measures as shown by Cameron [12]: a finite measure $\mu^{(s)}$ such that integrals of all cylindrical functions w.r.t. $\mu^{(s)}$ are expressed by the rhs of eq. (4) exists iff $s = -i\sigma$, $\sigma > 0$.

The Wiener integral itself can also be used for treating the F-integral: either the latter is determined directly by some sort of analytic continuation of the former (an extensive list of references concerning this matter is given in [2]) or the problem under consideration is reformulated (essentially again by analytic continuation) so that the F-integrals are replaced by W-integrals. The last mentioned method represents a backbone of Euclidean approach to constructive quantum field theory which has developed so successfully in recent years [13, 14].

The second group of definitions follows the original idea of Feynman and determines the integrals (3) "sequentially", i.e. as a limit of some sequence of "finite-dimensional" integrals (see again [2] for further references). In this way Nelson [9] was able to derive a rigorous version of eq. (1) (the analogous relation for the heat equation, the so-called *Feynman-Kac formula*, was deduced by Kac in 1951 – cf. [8], sec. X. 11). In order to make use of the Lie-Trotter formula Nelson was forced to define the rhs of eq. (1) in a way which differs slightly from the Feynman heuristic proposal: the function $\exp(-i \int_0^t V(\gamma(\tau))\, d\tau)$ was replaced in the $n$-th approximative integral by the Riemannian sum $\exp\left(-i \sum_{j=0}^{n-1} V(\gamma(\tau_{j+1}))\,(\tau_{j+1} - \tau_j)\right)$, where $\tau_j = jt/n$. In this way the validity of eq. (1) was established for potentials $V$ belonging to $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. This result was further discussed and extended – see e.g. [15] for the case of the harmonic oscillator. Generally speaking, the sequential definitions (handled more or less rigorously) are the most popular in physical literature.

Important progress was achieved by Ito [19], who assumed functions defined on a Hilbert space of paths. His method is again based on a limiting technique, but more clever than that of [11]. The F-integral is in [19] defined as a limit of mean values of the "integrated function" w.r.t. certain sequences of Gaussian measures. In this way Ito derived a rigorous version of eq. (1) (which is now often called the *Feynman-Ito formula*) for potentials which are Fourier transforms of bounded measures on $\mathbb{R}$ (this class is denoted as $\mathscr{F}(\mathbb{R})$), for linear potential and potential of the harmonic oscillator.

Recently a new group of definitions has appeared which makes use of Fourier transformation. The first of them to be mentioned is that of DeWitt-Morette [16, 17] which replaces the non-existing Feynman measure by a "prodistribution" determined by its Fourier transform (equal to $\exp(-\frac{1}{2}i\, W(x, x'))$; here $W$ is a bilinear form on the dual $X'$ of the path space $X$, which is assumed to be a locally convex Haussdorf space). This method combined with the product operator formalism can be applied to calculations of various path-integral expressions of physical interest [1].

More important for us is the definition of Albeverio and Hoegh-Krohn [2, 3]. It assumes again the path space to be a Hilbert space $\mathscr{H}$, at the same time the class of "integrable" functions is restricted to Fourier transforms of finite complex measures on $\mathscr{H}$. The F-integral is at that price expressed by a simple and elegant formula which gives values coincident with those of the Ito method. In this framework one more way was shown how the Feynman-Ito formula for potentials $V \in \mathscr{F}(\mathbb{R}^d)$ can be proved [2, 18].

The original definition of Albeverio and Hoegh-Krohn (AH) does not make it possible to "integrate" some physically important functions, as for example the exponential function corresponding to the harmonic-oscillator potentials. It has led these authors to the more general definition of the F-integral w.r.t. a (not necessarily positive) quadratic form, in which the exponential term entering the "measure" can contain a part corresponding to the potential energy as well. An alternative way to handle some non-bounded potentials was proposed by Truman [18], who extended the original definition of [2] by means of *polygonal-path approximations*. He also showed that these approximations are expressed via integrals of exponents of the exact classical action along the polygonal paths; in this sense his approach is closer to the heuristic considerations of Feynman than the Nelson-type approximations.

The same author has formulated another appealing idea [21] by generalizing the definition of the F-integral from [20] to the concept of *Feynman maps*. By this notion a certain family of maps from a set of functions of the path space $\mathscr{H}$ into $\mathbb{R}$ is understood, which is indexed by numbers $s$ from the lower complex halfplane; the cases $s = 1, -\mathrm{i}$ refer to the F-integral and W-integral, respectively. This approach makes it possible to treat both the important path integrals on the same footing (for a certain class of functions); moreover, it unifies in some sense the sequential methods with those based on analytic continuation.

On the other hand, some objections can be stated. Firstly, the finite-dimensional approximations used in the definition of the F-maps in [21] are not given by means of some AH-type expressions, but via integrals analogous to the rhs of eq. (4). Consequently, if the "integrated" function is such that its cylindrical approximations are not L-integrable (such a situation occurs frequently and represents no pathology), then the approximations to the F-map value contain improper integrals. It certainly means no harm as far as we know how to calculate them. However, principal values of multidimensional integrals represent an extremely touchy business (cf. a simple example in sec. 1·2 of [1], which shows that two quite reasonable choices of the limiting procedure in a two-dimensional integral can give completely different results), and we prefer to stay on the solid ground of the measure theory. Secondly, the definition under consideration uses for approximative purposes only those polygonal paths which refer to equidistant partitions of the given time interval.

This seems to be a discriminative assumption: it may happen that a function "integrable" w.r.t. the given prescription would not occur to be "integrable" in the approximation carried out using arbitrary polygonal paths. This circumstance is stressed by the absence of the dominated convergence theorem for F-integrals (see below), which could assure independence of the choice of polygonal-path approximation.

The above considerations determine the main line of this paper. We shall examine here the AH-type definition of the F-maps; its "polygonal" extensions properties and applications are left to the next paper. First we review for further use results about the algebra of "Fresnel-integrable" functions [2, 3, 18]. In the third section we define the F-maps and discuss their properties. Some of them are connected closely to those obtained in [21], only presentation (and consequently, some of the assumptions) differs. The others are new, as e.g. the "Fubini theorem" for F-maps.

We also present a simple example illustrating that the AH-integral does not fulfil the dominated convergence theorem. This invalidates the theorem concerning the classical limit of quantum mechanics deduced in [18], the proof of which is based on this very assumption[4]). Fortunately, there are other methods of treating this problem – see [23–25] and references quoted therein.

## 2. THE ALGEBRA $\mathscr{F}(\mathscr{H})$

Let $\mathscr{H}$ be a *real separable Hilbert space of paths* (to be specified later) and $\mathscr{M}(\mathscr{H})$ the *set of all complex Borel measures on* $\mathscr{H}$ *with* $|\mu|(\mathscr{H}) < \infty$. Here $|\mu|$ is the total variation of $\mu$: $|\mu|(\mathsf{A}) = \sup\{\sum_k |\mu(\mathsf{A}_k)| : \{\mathsf{A}_k\}$ finite system of disjoint Borel sets, $\bigcup_k \mathsf{A}_k = \mathsf{A}\}$; it is a non-negative measure on $\mathscr{H}$. Any linear combination of $\mu$, $\nu \in \mathscr{M}(\mathscr{H})$ belongs again to $\mathscr{M}(\mathscr{H})$, since the definition of $|\mu|$ implies easily $|\alpha\mu|(\mathsf{A}) = |\alpha|\,|\mu|(\mathsf{A})$, $|\mu + \nu|(\mathsf{A}) \leqq |\mu|(\mathsf{A}) + |\nu|(\mathsf{A})$ for all $\alpha \in \mathbb{C}$ and any $\mathsf{A} \in \mathscr{B}$, the system of Borel sets in $\mathscr{H}$. Let $\{\mu_n\}$ be a sequence $\subset \mathscr{M}(\mathscr{H})$ such that $|\mu_n - \mu_m| \times (\mathscr{H}) \to 0$ with $n, m \to \infty$. Using standard arguments one can prove that $\mu(\mathsf{A}) = \lim_{n\to\infty} \mu_n(\mathsf{A})$ exists for each Borel $\mathsf{A}$ and that $\mu$ defined in this way belongs to $\mathscr{M}(\mathscr{H})$. Thus the space $\mathscr{M}(\mathscr{H})$ equipped with the norm $|.|(\mathscr{H})$ is Banach.

We shall show further that $\mathscr{M}(\mathscr{H})$ can be equipped naturally with an algebraic structure. For this purpose assume first $\mu$, $\nu \in \mathscr{M}(\mathscr{H})$. The product measure $\mu \otimes \nu$ on $(\mathscr{H} \times \mathscr{H}, \mathscr{B} \otimes \mathscr{B})$ is defined in the standard way; it is finite because $|\mu \otimes \nu| \,.\, (\mathscr{H} \times \mathscr{H}) = |\mu|(\mathscr{H})\,|\nu|(\mathscr{H})$ (cf. [26], sec. III. 11, lemma 11). Further $f : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ belongs to $L(\mathscr{H} \times \mathscr{H}, \mu \otimes \nu)$ iff $|f| \in L(\mathscr{H} \times \mathscr{H}, |\mu \otimes \nu|)$ and the "complex Fubini theorem" holds in this case ([36], sec. III. 11, th. 13):

[4]) There is a weak form of the dominated convergence theorem [21]; however, its assumptions are such that they hardly could be verified in the cases of physical interest.

(5) $$\int_{\mathscr{H}\times\mathscr{H}} f(\gamma,\gamma')\,\mathrm{d}(\mu\otimes\nu)(\gamma,\gamma') = \int_{\mathscr{H}}\mathrm{d}\mu(\gamma)\int_{\mathscr{H}} f(\gamma,\gamma')\,\mathrm{d}\nu(\gamma') = \\ = \int_{\mathscr{H}}\mathrm{d}\nu(\gamma')\int_{\mathscr{H}} f(\gamma,\gamma')\,\mathrm{d}\mu(\gamma)\,.$$

Notice that in the mentioned theorem finiteness of $\mu$, $\nu$ is substantial in contrast to the usual Fubini theorem where both the measures are non-negative but may be $\sigma$-finite ([26], sec. III. 11, [22], sec. IX. 2).

Let us define convolution of $\mu, \nu \in \mathscr{M}(\mathscr{H})$ : we denote $\mathsf{A} - \gamma = \{\gamma' - \gamma : \gamma' \in \mathsf{A}\}$ and set

(6) $$(\mu * \nu)(\mathsf{A}) = \int_{\mathscr{H}} \mu(\mathsf{A} - \gamma)\,\mathrm{d}\nu(\gamma)\,, \quad \mathsf{A}\in\mathfrak{B}\,.$$

The following properties are easily derived:

(i) $\mu * \nu$ is a complex Borel measure: it holds $\mu(\mathsf{A} - \gamma) = \int_{\mathscr{H}}\chi_{\mathsf{A}-\gamma}(\gamma')\,\mathrm{d}\mu(\gamma') = \int_{\mathscr{H}}\chi_{\mathsf{A}}(\gamma + \gamma')\,\mathrm{d}\mu(\gamma')$, $\chi_{\mathsf{A}}$ being the characteristic function of $\mathsf{A}$. The Fubini theorem (5) then implies

(7) $$(\mu * \nu)(\mathsf{A}) = \int_{\mathscr{H}\times\mathscr{H}} \chi_{\mathsf{A}}(\gamma + \gamma')\,\mathrm{d}(\nu\otimes\mu)(\gamma,\gamma')\,.$$

The set $\{(\gamma,\gamma') : \gamma+\gamma' \in \mathsf{A}\}$ is Borel in $\mathscr{H}\times\mathscr{H}$ for any $\mathsf{A}\in\mathfrak{B}$ (cf. [31], theorem 1.10), thus the last integral makes sense, further the set function $\mu * \nu : \mathfrak{B} \to \mathbb{C}$ is obviously $\sigma$-additive and $(\mu * \nu)(\emptyset) = 0$.

(ii) The mapping $(\mu, \nu) \mapsto \mu * \nu$ is clearly bilinear and commutative: $(\mu * \nu)(\mathsf{A}) = (\nu * \mu)(\mathsf{A})$ follows from (7). Further this relation together with the image measure theorem (or change-of-variable theorem, cf. [26], sec. III. 10, prop. 8) imply

$$(\mu * \nu)(\mathsf{A}) = \int_{\mathsf{A}} \mathrm{d}(\mu\otimes\nu)(\varphi^{(-1)}(\gamma,\gamma'))\,,$$

where $\varphi : \mathscr{H}\times\mathscr{H} \to \mathscr{H}$ is defined by $\varphi(\gamma,\gamma') = \gamma + \gamma'$. Then we have for any $f \in \mathsf{L}(\mathscr{H}, \mu * \nu)$

(8) $$\int_{\mathscr{H}} f(\gamma)\,\mathrm{d}(\mu * \nu)(\gamma) = \int_{\mathscr{H}\times\mathscr{H}} f(\gamma + \gamma')\,\mathrm{d}(\mu\otimes\nu)(\gamma,\gamma') = \\ = \int_{\mathscr{H}}\mathrm{d}\mu(\gamma)\int_{\mathscr{H}}\mathrm{d}\nu(\gamma')\,f(\gamma + \gamma')\,.$$

In particular, this relation implies $(\mu * (\nu * \varrho))(\mathsf{A}) = ((\mu * \nu) * \varrho)(\mathsf{A})$ for any $\mathsf{A}\in\mathfrak{B}$, i.e. *associativity* of the convolution.

(iii) Finally, the inequality

(9) $$|\mu * \nu|(\mathscr{H}) \leqq |\mu|(\mathscr{H})\,|\nu|(\mathscr{H})$$

holds, which in particular shows that $*$ maps $\mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H})$ into $\mathcal{M}(\mathcal{H})$. It can be obtained with the help of the following expression for total variation: $|\mu|(\mathsf{A}) = \sup\{|\int_{\mathsf{A}} g(\gamma)\,\mathrm{d}\mu(\gamma)| : g \text{ Borel}, |g(\gamma)| \leqq 1\}$ ([26], sec. III.2; [27], §29). Relation (8) implies

$$\left|\int_{\mathcal{H}} g(\gamma)\,\mathrm{d}(\mu * \nu)(\gamma)\right| \leqq \int_{\mathcal{H}} \left|\int_{\mathcal{H}} g(\gamma + \gamma')\,\mathrm{d}\nu(\gamma')\right| \mathrm{d}|\mu|(\gamma) \leqq$$
$$\leqq \int_{\mathcal{H}} \mathrm{d}|\mu|(\gamma) \int_{\mathcal{H}} |g(\gamma + \gamma')|\,\mathrm{d}|\nu|(\gamma') \leqq |\mu|(\mathcal{H})\,|\nu|(\mathcal{H})$$

if $|g(\gamma)| \leqq 1$, so (9) is valid. Thus we arrive at the following assertion:

**Proposition 1:** The space $\mathcal{M}(\mathcal{H})$ equipped with the norm $|.|(\mathcal{H})$ and the product $*$ is a commutative Banach algebra.

Up to now we have not made use of the Hilbert structure of $\mathcal{H}$. Assume now the set $\mathcal{F}(\mathcal{H}) = \{f : f(\gamma) = \int_{\mathcal{H}} e^{i(\gamma,\gamma')}\,\mathrm{d}\mu(\gamma'), \mu \in \mathcal{M}(\mathcal{H})\}$, where $(.,.)$ is the inner product in $\mathcal{H}$. Continuity of $(\gamma,.)$ implies continuity of $e^{i(\gamma,.)}$ so the latter is Borel measurable and $f$ is well-defined for each $\mu \in \mathcal{M}(\mathcal{H})$. Further $\mathcal{F}(\mathcal{H})$ is a vector space w.r.t. pointwise addition and scalar multiplication.

We shall show that the B-algebra structure of $\mathcal{M}(\mathcal{H})$ can be isomorphically transferred to $\mathcal{F}(\mathcal{H})$. The crucial point here is to prove bijectivity of the mapping $\mu \mapsto f$; in view of linearity it is sufficient to check that $f = 0$ implies $\mu_f = 0$. Let us remark that the hint to this proof given in [2] seems not to work; especially the implication: if $\mu\{y : (x, y) \leqq \alpha\} = 0$, then $\mu(\mathsf{A}) = 0$ for all closed convex $\mathsf{A}$, is in no case obvious for complex $\mu$.

One can, however, use the known result about injectivity of Fourier transformation for positive measures on $\mathcal{H}$ ([30], theorem 2.1, [31], sec. I. 3)[5]). If $f = 0$, then $\frac{1}{2}(f(\gamma) + f(-\gamma)) = (1/2i)(f(\gamma) - f(-\gamma)) = 0$ for all $\gamma \in \mathcal{H}$, so

$$\int_{\mathcal{H}} \cos(\gamma, \gamma')\,\mathrm{d}\mu_f(\gamma') = \int_{\mathcal{H}} \sin(\gamma, \gamma')\,\mathrm{d}\mu_f(\gamma') = 0\,.$$

If $g$ is real-valued and $\nu$ complex, then $\int g\,\mathrm{d}\nu = 0$ implies $\int g\,\mathrm{d}\,\mathrm{Re}\,\nu = \int g\,\mathrm{d}\,\mathrm{Im}\,\nu = 0$, thus the above equalities give

$$(*) \qquad \int_{\mathcal{H}} e^{i(\gamma,\gamma')}\,\mathrm{d}\,\mathrm{Re}\,\mu_f(\gamma') = \int_{\mathcal{H}} e^{i(\gamma,\gamma')}\,\mathrm{d}\,\mathrm{Im}\,\mu_f(\gamma') = 0$$

for all $\gamma \in \mathcal{H}$. We shall assume e.g. the signed measure $\varrho = \mathrm{Re}\,\mu_f$; the argument concerning $\mathrm{Im}\,\mu_f$ would be the same. Let $\varrho = \varrho_1 - \varrho_2$ be the Jordan decomposition of $\varrho$. The positive measures $\varrho_1, \varrho_2$ have disjoint supports so that $\varrho_1 \neq \varrho_2$ unless

[5]) This assertion can be proved also using injectivity of F-transformation of promeasures [22] as was done in the preprint version of this paper [32]. In this case, however, one meets an additional complication of checking that a bounded measure in the sense of [22] corresponds injectively to every finite Borel measure on separable $\mathcal{H}$.

both these measures are zero; in the first case the mentioned assertion implies $\mathscr{F}\varrho = \mathscr{F}\varrho_1 - \mathscr{F}\varrho_2 \neq 0$. This contradicts (*) and thus $\varrho = \operatorname{Re}\mu_f = 0$. Analogously $\operatorname{Im}\mu_f = 0$ holds, so $\mu_f = 0$.

The abbreviation $\mu_f$ for the measure corresponding to $f \in \mathscr{F}(\mathscr{H})$ makes therefore sense and we shall use it whenever it proves to be convenient. The above-mentioned statement together with other properties of $\mathscr{F}(\mathscr{H})$ is given by the following

**Proposition 2.** The space $\mathscr{F}(\mathscr{H})$ is a functional Banach algebra with unity w.r.t. the norm $\|.\|_0 : \|f\|_0 = |\mu_f|(\mathscr{H})$. Each $f \in \mathscr{F}(\mathscr{H})$ is norm continuous and bounded, $\|f\|_\infty \leqq \|f\|_0$. If $h : \mathbb{C} \to \mathbb{C}$ is an entire function and $f \in \mathscr{F}(\mathscr{H})$, then the composed mapping $h \circ f$ belongs to $\mathscr{F}(\mathscr{H})$ as well.

*Proof:* The Fourier transformation $\mathscr{F}$ is linear and maps $\mathscr{M}(\mathscr{H})$ bijectively onto $\mathscr{F}(\mathscr{H})$, further this isomorphism is isometric, $|\mu|(\mathscr{H}) = \|\mathscr{F}\mu\|_0$, so the space $\mathscr{F}(\mathscr{H})$ is Banach. The convolution is transformed by $\mathscr{F}$ into pointwise multiplication: relations (5), (8) give

$$(\mathscr{F}(\mu * \nu))(\gamma) = \int_{\mathscr{H}} e^{i(\gamma,\gamma')} \, d(\mu * \nu)(\gamma') =$$

$$= \int_{\mathscr{H}\times\mathscr{H}} e^{i(\gamma,\gamma'+\gamma'')} \, d(\mu \otimes \nu)(\gamma', \gamma'') = (\mathscr{F}\mu)(\gamma)(\mathscr{F}\nu)(\gamma)$$

for all $\mu$, $\nu \in \mathscr{M}(\mathscr{H})$, $\gamma \in \mathscr{H}$. Further the B-algebra $\mathscr{M}(\mathscr{H})$ contains unity, because the Dirac measure $\mu_e : \mu_e(\{0\}) = 1$, $\mu_e(\mathscr{H} - \{0\}) = 0$ belongs to $\mathscr{M}(\mathscr{H})$ and for all $\gamma \in \mathscr{H}$ $(\mathscr{F}\mu_e)(\gamma) = 1$. Since $\mathscr{H}$ is first countable w.r.t. the norm topology (even second countable), a function $f : \mathscr{H} \to \mathbb{C}$ is continuous if it is sequentially continuous. Let $\gamma$ be the norm limit of a sequence $\{\gamma_n\} \subset \mathscr{H}$, then $(\gamma, \gamma') = \lim_{n\to\infty} (\gamma_n, \gamma')$ so that $\exp(i(\gamma, \gamma')) = \lim_{n\to\infty} \exp(i(\gamma_n, \gamma'))$ for all $\gamma' \in \mathscr{H}$. If $f \in \mathscr{F}(\mathscr{H})$, the dominated convergence theorem implies

$$f(\gamma) = \int_{\mathscr{H}} \exp(i(\gamma, \gamma')) \, d\mu_f(\gamma') = \lim_{n\to\infty} \int_{\mathscr{H}} \exp(i(\gamma_n, \gamma')) \, d\mu_f(\gamma') = \lim_{n\to\infty} f(\gamma_n) .$$

Consequently, $f$ is norm continuous. Further the inequality

$$|f(\gamma)| \leqq \int_{\mathscr{H}} d|\mu_f|(\gamma') \quad \text{gives} \quad \|f\|_\infty = \sup_{\gamma\in\mathscr{H}} |f(\gamma)| \leqq |\mu_f|(\mathscr{H}) = \|f\|_0 .$$

Finally, let $h$ be expressed by the series $h(z) = \sum_{n=0}^{\infty} a_n z^n$ with the infinite radius of convergence, then $h \circ f = \sum_{n=0}^{\infty} a_n f^n$. The sequence $\{a_n f^n\}_{n=0}$ is absolutely summable, $\sum_{n=0}^{\infty} \|a_n f^n\|_0 \leqq \sum_{n=0}^{\infty} |a_n| \, \|f\|_0^n < \infty$, and because $\mathscr{F}(\mathscr{H})$ is Banach, it is also summable, i.e. $\sum_{n=0}^{N} a_n f^n$ converges in $\|\cdot\|_0$-norm to some element of $\mathscr{F}(\mathscr{H})$ (cf. [8], th. III. 3). ■

## 3. THE FEYNMAN MAPS

Now we are in position to formulate the main definition. Let us denote $\mathbb{C}_F = \{z \in \mathbb{C} : z \neq 0, \operatorname{Im} z \leqq 0\}$ and $\mathbb{C}_F^0 = \{z : \operatorname{Im} z < 0\}$. To any $s \in \mathbb{C}_F$ we define the mapping $I_s : \mathscr{F}(\mathscr{H}) \to \mathbb{C}$ by

$$I_s(f) = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma\|^2\right) \mathrm{d}\mu_f(\gamma) \tag{10}$$

and call it **$\mathbf{F}_s$-map**. In particular, $I_1(\cdot) \equiv I(\cdot)$ is called **F-integral**; this definition coincides precisely with the first definition of the F-integral in [2][6]. Basic properties of the F-maps are the following:

(i) $I_s(f)$ is *well-defined*: the Hilbert space norm is continuous, so $g_s(\cdot) = \exp . (-\frac{1}{2}\mathrm{i}s\|\cdot\|^2)$ is continuous too, and therefore Borel measurable; further $|g_s(\gamma)| \leqq \exp(\frac{1}{2}\|\gamma\|^2 \operatorname{Im} s) \leqq 1$ implies $g_s \in \mathrm{L}(\mathscr{H}, \mu_f)$ for each $f \in \mathscr{F}(\mathscr{H})$. Moreover, $I_s(\cdot)$ is a *linear functional* which is obviously *bounded*, $\|I_s\| = 1$, and *normalized*, because to the unit function $e$ the normalized $\{0\}$-supported Dirac measure corresponds so that $I_s(e) = 1$. On the other hand, $I_s(\cdot)$ is not positive w.r.t. the natural involution in $\mathscr{F}(\mathscr{H})$ unless $s$ is purely imaginary; this is clear from relations (11, 12) below.

(ii) Let us take a *finite-dimensional* $\mathscr{H} = \mathbb{R}^n$ with the standard norm $|\cdot|$ and express

$$I'_s(f) = (2\pi\,\mathrm{i}s)^{-n/2} \int_{\mathbb{R}^n} \exp\left(\frac{\mathrm{i}}{2s}|x|^2\right) f(x)\, \mathrm{d}m(x) \tag{11}$$

for $f : f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,y)}\, \mathrm{d}\mu_f(y)$, $\mu_f \in \mathscr{M}(\mathbb{R}^n)$, where $m$ is the Lebesgue measure on $\mathbb{R}^n$. Integral (11) is assumed to exist for all $s \in \mathbb{C}_F$, i.e. $f \in \mathrm{L}(\mathbb{R}^n)$. If $s \in \mathbb{C}_F^0$, then by the Fubini theorem one obtains

$$I'_s(f) = (2\pi\,\mathrm{i}s)^{-n/2} \int_{\mathbb{R}^n} \mathrm{d}\mu_f(y) \prod_{j=1}^{n} \int_{\mathbb{R}} \exp\left(\frac{\mathrm{i}}{2s}x_j^2 + \mathrm{i}x_j y_j\right) \mathrm{d}x_j\,; \tag{12}$$

evaluating the last integral ([28], 3.896.4) we get

$$I'_s(f) = \int_{\mathbb{R}^n} \exp\left(-\frac{\mathrm{i}s}{2}|y|^2\right) \mathrm{d}\mu_f(y) = I_s(f)\,.$$

For real $s$ one cannot apply the Fubini theorem directly, because $\{x, y\} \mapsto \exp(\mathrm{i}/2s\,|x|^2 + \mathrm{i}(x, y))$ does not belong to $\mathrm{L}(\mathbb{R}^n \times \mathbb{R}^n, m \otimes \mu_f)$. Thus we express $I'_s(f)$ as follows

$$I'_s(f) = \lim_{\alpha \to \infty} (2\pi\,\mathrm{i}s)^{-n/2} \int_{C_\alpha} \exp\left(\frac{\mathrm{i}}{2s}|x|^2\right) f(x)\, \mathrm{d}m(x)\,, \tag{13}$$

[6]) Other $I_s(\cdot)$, $s > 0$, may be called F-integrals as well. Their properties are analogous to those of $I_1(\cdot)$, because they can be obtained one from the other through changing the Hilbert space norm by a non-zero multiplicative constant.

where $C_\alpha = \{x : |x_j| \leqq \alpha\}$, $j = 1, 2, \ldots, n$. Then

$$I'_s(f) = \lim_{\alpha \to \infty} \int_{\mathbb{R}^n} \exp\left(-\frac{is}{2}|y|^2\right) K^n_s(y, \alpha)\, d\mu_f(y),$$

where

$$K^n_s(y, \alpha) = (2\pi\, is)^{-n/2} \int_{C_\alpha} \exp\left(\frac{i}{2s}|x + sy|^2\right) dm(x).$$

Concerning the last integral the following assertion is valid (see Appendix C): *there exists* $K^n_s > 0$ *to any non-zero s such that* $|K^n_s(y, \alpha)| \leqq K^n_s$ *for all* $\alpha \geqq 0$, $y \in \mathbb{R}^n$, *and* $\lim_{\alpha \to \infty} K^n_s(y, \alpha) = 1$. Using it together with the dominated convergence theorem, we arrive again at relation (12).

*Remark*: Considerations of Appendix C do not employ the assumed integrability of $f$. Thus if the relation (13) is regarded as the *definition* of $I'_s(f)$, then (12) is valid for all $f \in \mathscr{F}(\mathbb{R}^n)$. This is essentially the way in which improper integrals appear in the original F-map definition [21]. Let us call to mind here the example quoted in the introduction, which shows how much these considerations are sensitive to the limiting prescription: if $n = 2$ and $f$ is the unit function on $\mathbb{R}^2$, then $I_1(f) = 1$ as well as $I'_1(f)$ in the sense of (13). However, if the blowing-up square $C_\alpha$ is replaced by the circle $\{x : |x| \leqq \alpha\}$, then the corresponding expression equals $\lim_{\alpha \to \infty} (1 - \exp . (\frac{1}{2} i\alpha^2))$, i.e. the principal value does not exist at all.

(iii) An assertion analogous to the *Fubini theorem* was deduced in [2] for the F-integrals. It can be generalized for the F-maps: let $\mathscr{H}$ decompose into an orthogonal sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ so that for all $\gamma = \gamma_1 + \gamma_2 \in \mathscr{H}$ we have $\|\gamma\|^2 = \|\gamma_1\|^2 + \|\gamma_2\|^2$. If $f \in \mathscr{F}(\mathscr{H})$, $f(\gamma) = \int_{\mathscr{H}} \exp(i(\gamma, \gamma'))\, d\mu(\gamma')$, we write $d\mu(\gamma) = d\mu(\gamma_1, \gamma_2)$ and define

$$\mu_{\gamma_2}(A_1) \equiv \tilde{\mu}_{\gamma_2}(A_1 \times \mathscr{H}_2), \quad \tilde{\mu}_{\gamma_2}(A) \equiv \int_A \exp(i(\gamma_2, \gamma'_2)\, d\mu(\gamma'_1, \gamma'_2)$$

for each $\gamma_2 \in \mathscr{H}_2$ and Borel $A_1 \subset \mathscr{H}_1$. The mapping $\mu_{\gamma_2}(\cdot)$ is clearly $\sigma$-additive, $\mu_{\gamma_2}(\emptyset) = 0$, $|\mu_{\gamma_2}(A)| \leqq |\mu(A \times \mathscr{H}_2)| \leqq |\tilde{\mu}|(\mathscr{H})$, so $\mu_{\gamma_2} \in \mathscr{M}(\mathscr{H}_1)$; further

$$\int_{\mathscr{H}_1} g(\gamma_1)\, d\mu_{\gamma_2}(\gamma_1) = \int_{\mathscr{H}} \tilde{g}(\gamma)\, d\tilde{\mu}_{\gamma_2}(\gamma), \quad \tilde{g}(\gamma_1, \gamma_2) \equiv g(\gamma_1)$$

for any $g \in L(\mathscr{H}_1, \mu_{\gamma_2})$. The Borel measure $\tilde{\mu}_{\gamma_2}$ on $\mathscr{H}$ is absolutely continuous w.r.t. $\mu$, and therefore the last integral can be expressed by means of $\mu$ and the Radon-Nikodym derivative $\gamma' \mapsto \exp(i(\gamma_2, \gamma'_2))$ (cf. [26], sec. III. 10, cor. 6). Thus we obtain

$$\int_{\mathscr{H}_1} g(\gamma_1)\, d\mu_{\gamma_2}(\gamma_1) = \int_{\mathscr{H}} \tilde{g}(\gamma') \exp(i(\gamma_2, \gamma'_2))\, d\mu(\gamma'_1, \gamma'_2). \tag{14}$$

In particular, this equality with $g(\gamma_1') = \exp(\mathrm{i}(\gamma_1, \gamma_1'))$ shows that $f_{\gamma_2}(\cdot) \equiv f(\cdot, \gamma_2)$ belongs to $\mathscr{F}(\mathscr{H}_1)$ for any fixed $\gamma_2$ and

$$I_s(f_{\gamma_2}) = \int_{\mathscr{H}_1} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_1\|^2\right) \mathrm{d}\mu_{\gamma_2}(\gamma_1)\,.$$

Applying further (14) to $g(\gamma_1) = \exp(-\mathrm{i}s/2\|\gamma_1\|^2)$ one obtains

$$I_s(f_{\gamma_2}) = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_1\|^2 + \mathrm{i}(\gamma_2, \gamma_2')\right) \mathrm{d}\mu(\gamma_1', \gamma_2')\,.$$

This integral can be in the same way as above expressed as

$$I_s(f_{\gamma_2}) = \int_{\mathscr{H}_2} \exp(\mathrm{i}(\gamma_2, \gamma_2'))\, \mathrm{d}\nu_s(\gamma_2')\,,$$

where $\nu_s$ is the complex Borel measure on $\mathscr{H}_2$ determined by the relation

$$\nu_s(\mathsf{A}) = \int_{\mathscr{H}_1 \times \mathsf{A}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_1\|^2\right) \mathrm{d}\mu(\gamma_1, \gamma_2)\,.$$

Hence the function $h_s : h_s(\gamma_2) = I_s(f_{\gamma_2})$ belongs to $\mathscr{F}(\mathscr{H}_2)$ and

$$I_s(h_s) = \int_{\mathscr{H}_2} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_2\|^2\right) \mathrm{d}\nu_s(\gamma_2) = \tag{15}$$

$$= \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_2\|^2 - \frac{\mathrm{i}s}{2}\|\gamma_1\|^2\right) \mathrm{d}\mu(\gamma_1, \gamma_2) = I_s(f)\,.$$

It is also clear that the order in which the "integrations" are performed is irrelevant.

*Remark:* The central argument of the presented proof (deduction of relation (14)) is *not* based on the Fubini theorem as stated in [2], because the measure $\mu$ is not in general a product measure on $\mathscr{H}_1 \times \mathscr{H}_2$ (cf. [8], sec. I. 4.). In fact, $\mu_f$ is a product measure iff $f$ factorizes, $f(\gamma) = f_1(\gamma_1) f_2(\gamma_2)$ for all $\gamma_i \in \mathscr{H}_i$ as can be easily seen.

(iv) A function $f : \mathscr{H} \to \mathbb{C}$ is called **cylindrical** or **tame** if there exists a finite-dimensional projection $P$ on $\mathscr{H}$ such that $f \circ P = f$, i.e. $f(P\gamma) = f(\gamma)$ for each $\gamma \in \mathscr{H}$. The function $f$ is in such a case said to have basis (to be based) in $P\mathscr{H}$. The subset of all cylindrical functions in $\mathscr{F}(\mathscr{H})$ is denoted $\mathscr{F}^t(\mathscr{H})$. For $f \in \mathscr{F}^t(\mathscr{H})$ the above results can be used: we decompose $\mathscr{H}$ into the orthogonal sum of $\mathscr{H}_1 = (I - P)\,\mathscr{H}$, $\mathscr{H}_2 = P\mathscr{H}$ and define $\tilde{f} : \tilde{f}(\gamma_2) = f(0, \gamma_2) = f(P\gamma)$, $\gamma \in \mathscr{H}$, then $f(\gamma) = \tilde{f}(\gamma_2) = e_1(\gamma_1)\tilde{f}(\gamma_2)$, where $e_1$ is the unit function on $\mathscr{H}_1$. Now (i) and (ii) imply $I_s(f_{\gamma_2}) = I_s(\tilde{f}(\gamma_2)\, e_1) = \tilde{f}(\gamma_2)\, I_s(e_1) = \tilde{f}(\gamma_2)$ so that $I_s(f) = I_s(\tilde{f})$. Further if $\tilde{f} \in \mathrm{L}(P\mathscr{H}, m)$, then according to (ii) $I_s(\tilde{f})$ can be expressed in the form (11), and we obtain therefore

$$(16) \qquad I_s(f) = (2\pi\,\mathrm{i}s)^{-\frac{1}{2}\dim P\mathscr{H}} \int_{P\mathscr{H}} \exp\left(\frac{\mathrm{i}}{2s}\,\|P\gamma\|^2\right) f(P\gamma)\,\mathrm{d}m(P\gamma)\,;$$

here $m$ is again the Lebesgue measure on $P\mathscr{H}$.

(v) If $a \in \mathbb{R}^n$ and $R$ is a linear orthogonal transformation on $\mathscr{H} = \mathbb{R}^n$, then

$$I_s(f) = I'_s(f) = (2\pi\,\mathrm{i}s)^{-n/2} \int_{\mathbb{R}^n} \exp\left(\frac{\mathrm{i}}{2s}\,|Rx + a|^2\right) f(Rx + a)\,\mathrm{d}m(x)$$

for each $f \in \mathscr{F}(\mathbb{R}^n) \cap \mathrm{L}(\mathbb{R}^n)$, because the Lebesgue measure is Euclidean-invariant: $m(R\mathsf{A} + a) = m(\mathsf{A})$. This equality can be rewritten as

$$(17) \quad \exp\left(\frac{\mathrm{i}}{2s}\,|a|^2\right) I_s(f_{R,a}) = I_s(f)\,, \quad f_{R,a}(x) = \exp\left(\frac{\mathrm{i}}{s}\,(x, R^{-1}a)\right) f(Rx + a)$$

if $f_{R,a} \in \mathscr{F}(\mathbb{R}^n) \cap \mathrm{L}(\mathbb{R}^n)$ too (this is an additional assumption if $s$ is non-real and $a \neq 0$ because of the real exponent present in this case). In the same way we obtain

$$(18) \qquad |\det B|\, I_s(f_B) = I_s(f)\,, \quad f_B(x) = \exp\left(\frac{\mathrm{i}}{2s}\,|Bx|^2 - \frac{\mathrm{i}}{2s}\,|x|^2\right) f(Bx)$$

for any regular linear operator $B$ on $\mathbb{R}^n$ assuming that both $f$, $f_B$ belong to $\mathscr{F}(\mathbb{R}^n) \cap \cap\, \mathrm{L}(\mathbb{R}^n)$.

*Remark:* Relation (18) shows that the second formula from (P 4), sec. 2 in [3] is not valid even in the finite-dimensional case; it holds for isometric $T$ only – cf. (20) below. There is the obvious confusion here with Proposition 4.3 of [2]: for the F-integrals w.r.t. a quadratic form the determinant is included into normalization.

(vi) We shall verify further that property (17) is preserved *in the infinite-dimensional case* if $s$ is real non-zero. The deduction is based again on the image measure theorem. Consider first the translations: let $f \in \mathscr{F}(\mathscr{H})$ and define

$$f_\alpha : f_\alpha(\gamma) = \exp\left(\frac{\mathrm{i}}{s}\,(\gamma, \alpha)\right) f(\gamma + \alpha)$$

for given $\alpha \in \mathscr{H}$. It holds

$$f_\alpha(\gamma) = \int_{\mathscr{H}} \exp\left(\frac{\mathrm{i}}{s}\,(\gamma, \alpha) + \mathrm{i}(\gamma + \alpha, \gamma')\right) \mathrm{d}\tilde{\mu}_f(\gamma') = \int_{\mathscr{H}} \exp\left(\mathrm{i}(\gamma, \gamma'')\right) \mathrm{d}\mu_\alpha(\gamma'')\,,$$

where $\mu_\alpha(\mathsf{A}) = \int_{\mathsf{A}+\alpha s^{-1}} \exp\left(\mathrm{i}(\alpha, \gamma')\right) \mathrm{d}\mu_f(\gamma')$; further $\mathscr{H} + \alpha s^{-1} = \mathscr{H}$ so

$$I_s(f_\alpha) = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\,\|\gamma' + \alpha s^{-1}\|^2 + \mathrm{i}(\alpha, \gamma')\right) \mathrm{d}\mu_f(\gamma') = \exp\left(-\frac{\mathrm{i}s}{2}\,\|\alpha\|^2\right) I_s(f)\,,$$

i.e.

$$\exp\left(\frac{\mathrm{i}}{2s}\|\alpha\|^2\right) I_s(f_\alpha) = I_s(f)\,, \quad f_\alpha(\gamma) \equiv \exp\left(\frac{\mathrm{i}}{s}(\gamma, \alpha)\right) f(\gamma + \alpha)\,. \tag{19}$$

Analogously, if $U$ is a regular isometric operator on $\mathscr{H}$ and $f \in \mathscr{F}(\mathscr{H})$, then we have

$$f(U\gamma) = \int_{\mathscr{H}} \exp\left(\mathrm{i}(\gamma, U^{-1}\gamma')\right) \mathrm{d}\mu_f(\gamma') = \int_{\mathscr{H}} \exp\left(\mathrm{i}(\gamma, \gamma'') \,\mathrm{d}\mu_f(U\gamma'')\right.$$

so

$$I_s(f) = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|U^{-1}\gamma'\|^2\right) \mathrm{d}\mu_f(\gamma') = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma''\|^2\right) \mathrm{d}\mu_f(U\gamma'')\,,$$

i.e.

$$I_s(f_U) = I_s(f)\,, \quad f_U(\gamma) \equiv f(U\gamma)\,. \tag{20}$$

The last formula, however, holds for non-real $s$ too as the proof shows. Under some additional assumptions the validity of (19) can also be extended to non-real $s$. Finally, the Cameron-Martin-type formula (18) generalizes to the infinite-dimensional case if a special class of operators $B$ is considered. We postpone these matters to the next paper.

(vii) Let $\{E_n\}$ be a sequence of orthogonal projections on $\mathscr{H}$ such that $\operatorname{s-lim}_{n\to\infty} E_n = \mathrm{I}$. The restriction $f_n = f \upharpoonright E_n\mathscr{H}$ of a given $f \in \mathscr{F}(\mathscr{H})$ can be expressed as follows:

$$f(E_n\gamma) = \int_{\mathscr{H}} \exp\left(\mathrm{i}(E_n\gamma, \gamma'\right) \mathrm{d}\mu_f(\gamma') = \int_{\mathscr{H}} \exp\left(\mathrm{i}(E_n\gamma, E_n\gamma')\right) \mathrm{d}\mu_f(F_n\gamma', E_n\gamma')\,,$$

where $F_n = I - E_n$. In analogy with the proof of (iii) we introduce the Borel measure $\mu_n$ on $E_n\mathscr{H}$ by $\mu_n(\mathsf{A}) = \mu_f(F_n\mathscr{H} \times \mathsf{A})$. Clearly $\mu_n \in \mathscr{M}(E_n\mathscr{H})$, further the image measure theorem gives

$$\int_{E_n\mathscr{H}} g(\gamma_2')\,\mathrm{d}\mu_n(\gamma_2') = \int_{\mathscr{H}} g(E_n\gamma')\,\mathrm{d}\mu_f(F_n\gamma', E_n\gamma') \tag{21}$$

for any Borel $g : E_n\mathscr{H} \to \mathbb{C}$. In particular, for $g(\gamma_2') = \exp\left(\mathrm{i}(E_n\gamma, \gamma_2'\right)$ we get $f_n \in \mathscr{F}(E_n\mathscr{H})$. Applying further relation (21) to $g(\gamma_2') = \exp\left(-(\mathrm{i}s/2)\|\gamma_2'\|^2\right)$ we obtain

$$I_s(f_n) = \int_{E_n\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|\gamma_2'\|^2\right) \mathrm{d}\mu_n(\gamma_2') = \int_{\mathscr{H}} \exp\left(-\frac{\mathrm{i}s}{2}\|E_n\gamma'\|^2\right) \mathrm{d}\mu_f(\gamma')\,.$$

Finally, $\exp\left(-(\mathrm{i}s/2)\|\gamma\|^2\right) = \lim_{n\to\infty} \exp\left(-(\mathrm{i}s/2)\|E_n\gamma\|^2\right)$ due to the assumption so the dominated convergence theorem yields

$$\lim_{n\to\infty} I_s(f_n) = I_s(f)\,. \tag{22}$$

(vii) So far we have discussed the $f$-dependence of $I_s(f)$. Let now in turn $f \in \mathscr{F}(\mathscr{H})$ be fixed. The standard condition under which integral (10) can be differentiated w.r.t. the parameter $s$ verifies easily: it holds

$$\left| -\frac{\mathrm{i}}{2} \|\gamma\|^2 \exp\left( -\frac{\mathrm{i}s}{2} \|\gamma\|^2 \right) \right| \leqq \tfrac{1}{2} \|\gamma\|^2 \exp\left( \tfrac{1}{2} \|\gamma\|^2 \operatorname{Im} s \right)$$

and the rhs belongs to $\mathrm{L}(\mathscr{H}, \mu_f)$ if $\operatorname{Im} s < 0$ so the function $s \mapsto I_s(f)$ is differentiable in each $\{s : \operatorname{Im} s < s_1 < 0\}$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} I_s(f) = -\frac{\mathrm{i}}{2} \int_{\mathscr{H}} \|\gamma\|^2 \exp\left( -\frac{\mathrm{i}s}{2} \|\gamma\|^2 \right) \mathrm{d}\mu_f(\gamma) \,. \tag{23}$$

Consequently, the function $s \mapsto I_s(f)$ is single-valued analytic in the open lower halfplane $\mathbb{C}_F^0$. Moreover, this function is continuous in $\mathbb{C}_F$ due to the dominated convergence theorem. Finally, the relation

$$\lim_{\substack{s \to 0 \\ s \in \mathbb{C}_F}} I_s(f) = f(0) \tag{24}$$

holds; one can use it to define $I_0(\cdot)$ if necessary.

Concluding this section we bring together the obtained results. The $F_s$-maps defined by (10) have the following properties:

**Theorem 1.** (a) $I_s(\cdot)$ is a normalized linear functional and $\|I_s\| = 1$ for each $s \in \mathbb{C}_F$. (b) Let $\{E_n\}$ be a sequence of orthoprojections on $\mathscr{H}$ which converges strongly to the unit operator. If $f \in \mathscr{F}(\mathscr{H})$, then $\lim_{n\to\infty} I_s(f \circ E_n) = I_s(f)$, where $(f \circ E_n)(\gamma) \equiv \equiv f(E_n \gamma)$. (c) For each $f \in \mathscr{F}(\mathscr{H})$ the function $s \mapsto I_s(f)$ is single-valued analytic in $\mathbb{C}_F^0$ and continuous in $\mathbb{C}_F$; moreover, relation (24) holds.

**Theorem 2.** Let $f$ be a tame function, $f \in \mathscr{F}^t(\mathscr{H})$, based in a subspace $P\mathscr{H}$, and let further $f \restriction P\mathscr{H}$ belong to $\mathrm{L}(P\mathscr{H}, m)$, $m$ being the Lebesgue measure, then $I_s(f)$ is expressed by (16). In particular, for a finite-dimensional $\mathscr{H} = \mathbb{R}^n$ and $f \in \mathscr{F} \, . \, (\mathbb{R}_\gamma) \cap \mathrm{L}(\mathbb{R}^n)$ relations (11), (12) are valid.

**Theorem 3.** Let $\mathscr{H}$ decompose into an orthogonal sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ and $f \in \mathscr{F}(\mathscr{H})$. We denote $f(\gamma) = f(\gamma_1, \gamma_2)$, $\gamma_i \in \mathscr{H}_i$, then the functions $f_{\gamma_2}(\cdot) = f(\cdot, \gamma_2)$ and $f'_{\gamma_1}(\cdot) = = f(\gamma_1, .)$ belong to $\mathscr{F}(\mathscr{H}_1)$ and $\mathscr{F}(\mathscr{H}_2)$ for all $\gamma_2 \in \mathscr{H}_2$, $\gamma_1 \in \mathscr{H}_1$, respectively, further the functions $h_s : h_s(\gamma_2) = I_s(f_{\gamma_2})$ and $h'_s : h'_s(\gamma_1) = I_s(f'_{\gamma_1})$ belong to $\mathscr{F}(\mathscr{H}_2)$ and $\mathscr{F}(\mathscr{H}_1)$, respectively, and finally $I_s(f) = I_s(h_s) = I_s(h'_s)$.

**Theorem 4.** (a) Let $s$ be real non-zero, $f \in \mathscr{F}(\mathscr{H})$, then $I_s(f)$ transforms under translations of $\mathscr{H}$ according to (19). Furthermore, for $s \in \mathbb{C}_F$ and $U$ regular isometric relation (20) is valid. In particular, these relations express transformation properties of the F-integral w.r.t. Euclidean motions of $\mathscr{H}$.

(b) If $\mathscr{H} = \mathbb{R}^n$ and $f \in \mathscr{F}(\mathbb{R}^n) \cap \mathrm{L}(\mathbb{R}^n)$, then the formula expressing transformation properties under translations holds for $s \in \mathbb{C}_F^0$ as well – cf. (17). Moreover, if $f, f_B \in \mathscr{F}(\mathbb{R}^n) \cap \mathrm{L}(\mathbb{R}^n)$, then $I_s(f)$ transforms under "change of variables" mediated by a regular operator $B$ on $\mathbb{R}^n$ according to (18).

## 4. CONCLUSIONS

The study of the F-maps started here will be continued in our forthcoming paper. We shall specify there the path space and extend the F-maps defined above by means of a general polygonal-path approximation; further we shall discuss properties and applications of the obtained extensions.

As the last item here we shall make a comment on one more property of the F-integrals. Evaluation of an integral is often simplified if the integrated function represents a limit of some sequence of functions, the integrals of which are known. In fact, this method is one of the most used for the "usual" integrals, where powerful sufficient conditions are available for convergence of the corresponding sequence of integrals, among them especially the dominated convergence theorem. We have no such assertions for the F-integrals, though there exists e.g. a treatment of the classical limit of quantum mechanics based on the assumption of its validity (see Introduction), to say nothing of the non-rigorous path-integral calculations.

We shall show that a dominated-convergence-type theorem *is not valid* for the F-integrals, even if the simplest finite-dimensional case and substantially stronger assumption about the function sequence are considered. It is clear that we must avoid situations when $\lim_{n\to\infty} \|f_n - f\|_0 = 0$ formulating the counterexample, otherwise $I_s(f) = \lim_{n\to\infty} I_s(f_n)$ would follow from Theorem 1.

**Example:** Let $\mathscr{H} = \mathbb{R}$ and $\mu_a : \mu_a(\mathrm{J}) = (2a)^{-1} \int_{\mathrm{J}_a} \exp(\frac{1}{2}\mathrm{i}x^2)\,\mathrm{d}x$, where $\mathrm{J}_a = \mathrm{J} \cap (-a, a)$. Obviously $\mu_a \in \mathscr{M}(\mathbb{R})$ for each $a > 0$ and the corresponding functions $f_a$ are bounded by the unit function, $|f_a(x)| \leqq |\mu_a|(\mathbb{R}) = 1 = e(x)$, which is "integrable" (cf. Proposition 2). It holds

$$f_a(x) = \int_{\mathbb{R}} \exp(\mathrm{i}xy)\,\mathrm{d}\mu_a(y) = (2a)^{-1} \exp(-\tfrac{1}{2}\mathrm{i}x^2) \int_{-a}^{a} \exp(\tfrac{1}{2}\mathrm{i}\,(x + y)^2)\,\mathrm{d}y$$

and according to the assertion proved in the Appendix there exists a constant $M = (2\pi)^{1/2} K_1^1$ such that $|f_a(x)| \leqq M/2a$ for all $x \in \mathbb{R}$, $a > 0$, i.e.

$$\lim_{a\to\infty} \|f_a\|_\infty = 0\,. \tag{25}$$

On the other hand, $I(f_a) = \int_{\mathbb{R}} \exp(-\frac{1}{2}ix^2)\,d\mu_a(x) = 1$ for each $a > 0$, and therefore

$$\lim_{a\to\infty} I(f_a) = 1 \neq 0\,. \tag{26}$$

Notice that the net $\{f_a\}$ converges to the zero function according to (25) not only pointwise (everywhere in $\mathbb{R}$), but even uniformly, and yet $\{I(f_a)\}$ does not converge to $I(0)$. At the same time relations (25), (26) show that the functional $I(\cdot)$ is not bounded w.r.t. the uniform norm $\|\cdot\|_\infty$ on $\mathscr{F}(\mathscr{H})$.

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## APPENDIX

Assume first the integral $C(a, b) = \int_a^b \exp(ibt^2)\,dt$ for $a, b > 0$. Since the function $z \mapsto \exp(ibz^2)$ is entire, $C(a, b)$ can be evaluated by contour integration. A suitable closed contour follows the real axis from 0 to $a$, then it makes a circular arc anticlockwise and returns to the origin along the halfline $\{z : \arg z = \frac{1}{4}\pi\}$. Consequently, we have

$$C(a, b) = -ia\int_0^{\pi/4} \exp(iba^2\,e^{2i\varphi})\,e^{i\varphi}\,d\varphi + \exp\left(\frac{\pi i}{4}\right)\int_0^a \exp(-bt^2)\,dt\,. \tag{*}$$

The first integral on the rhs (call it $J_1$) can be estimated as follows

$$|J_1| \leqq a\int_0^{\pi/4} |\exp(iba^2\,e^{2i\varphi})|\,d\varphi = \tfrac{1}{2}a\left(\int_0^{a^{-3/2}} + \int_{a^{-3/2}}^{\pi/2}\right)\exp(-ba^2\sin\xi)\,d\xi$$

assuming $a \geqq (\pi/2)^{-2/3}$. Using further $\sin\xi \geqq \frac{1}{2}\xi \geqq \frac{1}{2}a^{-3/2}$ for $\xi \in (a^{-3/2}, \pi/2)$ we get

$$|J_1| \leqq \tfrac{1}{2}a^{-1/2} + \frac{\pi a}{4}\exp(-\tfrac{1}{2}ba^{1/2})\,. \tag{**}$$

The second rhs integral in (*) can be estimated easily as well as $C(a, b)$ for small $a$: we obtain

$$|C(a, b)| \leqq \begin{cases} a & \ldots\ldots\ldots\ldots\ldots\ldots\ a \leqq (\pi/2)^{-2/3} \\ \frac{1}{2}\left(\frac{\pi}{b}\right)^{1/2} + \frac{1}{2}a^{-1/2} + \frac{\pi a}{4}\exp(-\frac{1}{2}ba^{1/2}) \ldots\ a \geqq (\pi/2)^{-2/3} \end{cases} \tag{+}$$

so $|C(a, b)|$ is for every $b$ majorized by a constant independent of $a$. The relation (**) further implies

$$\lim_{a\to\infty} C(a, b) = \exp\left(\frac{\pi i}{4}\right)\int_0^\infty \exp(-bt^2)\,dt = \left(\frac{\pi i}{4b}\right)^{1/2}. \tag{++}$$

Since $C(a, -b) = \overline{C(a, b)}$, relations $(+)$, $(++)$ are valid for $b < 0$ as well. Assume now

$$K_s^1(y, \alpha) = (2\pi\, \mathrm{i}s)^{-1/2} \int_{-\alpha}^{\alpha} \exp\left(\frac{\mathrm{i}}{2s}(x + sy)^2\right) \mathrm{d}x =$$

$$= (2\pi\, \mathrm{i}s)^{-1/2} \left(C\left(\alpha + sy, \frac{1}{2s}\right) + C\left(\alpha - sy, \frac{1}{2s}\right)\right).$$

This expression is majorized according to $(+)$ by a constant $K_s^1$ which depends on $s$ only, further $(++)$ implies $\lim_{\alpha\to\infty} K_s^1(y, \alpha) = 1$. Finally the Fubini theorem implies $K_s^n(y, \alpha) = \prod_{j=1}^{n} K_s^1(y_j, \alpha)$ so $K_s^n = (K_s^1)^n$ and $\lim_{\alpha\to\infty} K_s^n(y, \alpha) = 1$.

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