ON S-TRANSFORMATION IN THE STRONG COUPLING THEORY*)

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S-transformation is a unitary transformation accomplishing the main part of the diagonalization of the interaction Hamiltonian in strong coupling theories. For a compact symmetry group the action of the S-transformation on the integrals of motion is derived.

1. INTRODUCTION

It was a current procedure of the old strong coupling theory [1, 2] to diagonalize first the interaction part of the Hamiltonian and consider the kinetic part as a perturbation to the first order. A consequent group-theoretical formulation of this procedure was given in [3].

In the static strong coupling approximation the interaction Hamiltonian H_I is a matrix in the vector space of the spin and internal symmetry bare states of the heavy particle which is assumed fixed. For the case of the Yukawa-type coupling of mesons to the heavy particle, the elements of the matrix H_I depend linearly on the space-averaged meson field operators – so-called bound-meson variables $\boldsymbol{q}$. For the well-known model of one static bare nucleon interacting with pseudoscalar π-mesons of mass μ, the reduced Hamiltonian is [1, 2]

$$H = H_0 + H_I$$

where

$$H_0 = \tfrac{1}{2}\sum_{a=1}^{3}\sum_{i=1}^{3}(p_{ai}^2 + \mu^2 q_{ai}^2)\, I$$

and

$$H_I = g\sum_{a=1}^{3}\sum_{i=1}^{3}\sigma_a \otimes \tau_i q_{ai}\,;$$

here g is the coupling constant, σ_a and τ_i are the spin and isotopic spin Pauli matrices, $I = \sigma_0 \otimes \tau_0$ the 4×4 unit matrix, and ordinary quantum commutation relations

$$[q_{ai}, p_{bj}] = \mathrm{i}\delta_{ab}\,\delta_{ij}$$

hold. The strong coupling limit means $g \to \infty$. The symmetry group of the Hamiltonian H is $SU(2)_J \otimes SU(2)_T$.

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S-transformation is the $\boldsymbol{q}$-dependent part of the unitary transformation U accomplishing the diagonalization of the matrix H_I. The diagonalization is completed by an additional $\boldsymbol{q}$-independent unitary matrix Z so that

$$H'_I = UH_IU^{-1}$$

with

$$U = ZS\,.$$

Through the unitary transformation S (or U) a new quantum mechanical representation is obtained, with a wave function $\Psi' = S\Psi$ (or $\Psi' = U\Psi$) instead of the former state vector Ψ.

In order to calculate the excitation energies of isobars in the limit of strong coupling it is necessary to know how the S-transformation acts on the integrals of motion: the total angular momentum

$$\hat{J}_a = \tfrac{1}{2}\sigma_a \otimes \tau_0 + \mathscr{L}_a I\,, \quad a = 1, 2, 3\,,$$

and the total isotopic spin

$$\hat{T}_i = \sigma_0 \otimes \tfrac{1}{2}\tau_i + \mathscr{T}_i I\,, \quad i = 1, 2, 3\,.$$

Here

(1a) $$\mathscr{L}_a = \sum_{bcj} \varepsilon_{abc} q_{bj} p_{cj}$$

and

(1b) $$\mathscr{T}_i = \sum_{bjk} \varepsilon_{ijk} q_{bj} p_{bk}$$

are contributions to the quantities mentioned above from the bound mesons. Using an explicit parametrization of S and $\mathscr{L}_a$ by means of Euler angles, the S-transformation of $\mathscr{L}_a$ was derived in the form (see Appendix 2 of [2])

(2a) $$\mathscr{L}'_a = S\mathscr{L}_a I S^{-1} = \mathscr{L}_a I - \tfrac{1}{2}\sigma'_a$$

where

(2b) $$\sigma'_a = S\sigma_a \otimes \tau_0 S^{-1}\,.$$

Hence the total angular momentum transforms as follows [1]):

$$\hat{J}'_a = S\hat{J}_a S^{-1} = \tfrac{1}{2}\sigma'_a + \mathscr{L}'_a = \mathscr{L}_a I\,.$$

The eigenvalues of $\hat{J}'^2$ and $\hat{J}'_3$ in the representation Ψ' are of course the same as those of $\hat{J}^2$ and $\hat{J}_3$ in the representation Ψ.

[1]) A special case of this transformation was studied also in [5].

Relations of the form (2) may be anticipated also in cases of larger symmetry where the derivation with explicit parametrization is difficult. In view of our interest in the S-transformation for the case of the rather complicated $SU(3)$-symmetric strong coupling theory [3, 4] we give here a generalization of relations (2) under reasonable general assumptions.

2. DERIVATION

Suppose that the meson variables $\boldsymbol{q} = (q_A)$ form a vector space V and transform under some unitary representation of the compact symmetry group K of the Hamiltonian. Further, let for each $\boldsymbol{q} \in V$ exist a transformation $R_{\boldsymbol{q}}$ of this representation such that

$$\boldsymbol{q} = R_{\boldsymbol{q}} \mathring{\boldsymbol{q}}$$

where $\mathring{\boldsymbol{q}}$ is some fixed vector from V.

Let us write down the equation expressing the invariance of the interaction Hamiltonian $H_I = H_I(\boldsymbol{q})$ under the transformations from K with Hermitian generators (integrals of motion)

$$\hat{G}_k = G_k + \mathscr{G}_k I\,, \quad k = 1, \ldots, N\,;$$

matrices G_k refer to states of the heavy particle and $\mathscr{G}_k$ are differential operators in $\boldsymbol{q}$-space, analogous to (1); the equation

$$\exp\left(\mathrm{i} \sum_k \gamma_k \hat{G}_k\right) H_I(\boldsymbol{q}) \exp\left(-\mathrm{i} \sum_k \gamma_k \hat{G}_k\right) = H_I(\boldsymbol{q})$$

may be written also in the form

$$\exp\left(i \sum_k \gamma_k G_k\right) H_I(\boldsymbol{q}) \exp\left(-i \sum_k \gamma_k G_k\right) = \tag{3}$$

$$= \exp\left(-\mathrm{i} \sum_k \gamma_k \mathscr{G}_k\right) H_I(\boldsymbol{q}) \exp\left(\mathrm{i} \sum_k \gamma_k \mathscr{G}_k\right).$$

On the left-hand side of Eq. (3) the unitary matrix

$$S = \exp\left(\mathrm{i} \sum_k \gamma_k G_k\right) \tag{4}$$

in the space of heavy particle states has the same effect on $H_I(\boldsymbol{q})$ as the corresponding transformation in the space of meson variables $\boldsymbol{q}$ on the right-hand side of Eq. (3). If the parameters γ_k are chosen in such a way that

$$\exp\left(-\mathrm{i} \sum_k \gamma_k \mathscr{G}_k\right) q_A \exp\left(\mathrm{i} \sum_k \gamma_k \mathscr{G}_k\right) = (R_{\boldsymbol{q}}^{-1} \boldsymbol{q})_A = \mathring{q}_A$$

holds, then S transforms $H_I(\boldsymbol{q})$ into a constant matrix $H_I(\mathring{\boldsymbol{q}})$,

$$S\, H_I(\boldsymbol{q})\, S^{-1} = H_I(\mathring{\boldsymbol{q}})\,.$$

This can be taken as the first step in the diagonalization of $H_I(\boldsymbol{q})$. Usually $\mathring{\boldsymbol{q}}$ is chosen in the simplest way; the diagonalization is then completed by a $\boldsymbol{q}$-independent unitary matrix Z.

According to Eq. (4) S appears to be a finite-dimensional unitary representation of the compact group K whose generators are the Hermitian matrices G_k. We assume S to be a faithful representation of K in the m-dimensional representation space. In Eq. (4) a finite K-transformation was given; with infinitesimal parameters $\delta\gamma_k$ we get

$$S = I + \mathrm{i} \sum_k \delta\gamma_k G_k \tag{5}$$

where I is the $m \times m$ unit matrix.

Since the matrices G_k generally do not commute with one another, the transformation (4) cannot be written in the form of a product of single transformations $\exp(\mathrm{i}\gamma_k G_k)$. However, for the compact groups $U(n)$, $SU(n)$, or $SO(n)$ parametrizations analogous to the Euler angles for $SU(2)$ [6] are known [7]. We suppose the existence of such a parametrization for the representation

$$S = S(\alpha_1, \ldots, \alpha_N) \equiv S(\alpha_j)\,.$$

Let us look first for relations among the two sets of parameters describing the infinitesimal transformation (5), i.e. among $\delta\gamma_k$ and $\mathrm{d}\alpha_j$. In the case of $SU(2)$, they are familiar as the Euler kinematical equations of classical mechanics [8]. They can be derived by putting together two S-transformations, a finite one and an infinitesimal one:

$$S(\alpha_j)\,(I + \mathrm{i} \sum_k \delta\gamma_k G_k) = S(\alpha_j + \mathrm{d}\alpha_j)\,.$$

Expanding $S(\alpha_j + \mathrm{d}\alpha_j)$ to the first order and using the unitarity of the matrix S,

$$SS^\dagger = S^\dagger S = I\,,$$

a matrix equation follows,

$$\sum_k \delta\gamma_k G_k = -\mathrm{i} \sum_j S^\dagger \frac{\partial S}{\partial \alpha_j}\,\mathrm{d}\alpha_j \equiv \mathrm{i} \sum_j \frac{\partial S^\dagger}{\partial \alpha_j}\, S\,\mathrm{d}\alpha_j\,, \tag{6}$$

the derivatives $\partial S/\partial \alpha_j$ being taken in the point $(\alpha_1, \ldots, \alpha_N)$.

Solving these linear equations for $\delta\gamma_k$, linear relations

$$\delta\gamma_k = \sum_j X_{kj}\,\mathrm{d}\alpha_j \tag{7}$$

can be obtained, where $X = (X_{kj})$ is a regular $N \times N$ matrix. It is a transformation of contravariant N-vectors (the differentials $\mathrm{d}\alpha_j$) from which the transformation of covariant N-vectors $\partial/\partial\alpha_j$ follows:

$$\frac{\partial}{\partial\gamma_k} = \sum_j (X^{-1})_{jk} \frac{\partial}{\partial\alpha_j}. \tag{8}$$

Our aim is to find the S-transformation of the differential operators (8) which are proportional to the bound-meson generators

$$\mathscr{G}_k \equiv -\mathrm{i}\frac{\partial}{\partial\gamma_k}.$$

Substituting (7) into Eq. (6) gives

$$\sum_k G_k \sum_j X_{kj}\,\mathrm{d}\alpha_j = \mathrm{i}\sum_j \frac{\partial S^\dagger}{\partial\alpha_j} S\,\mathrm{d}\alpha_j .$$

Since the differentials $\mathrm{d}\alpha_j$ are linearly independent and arbitrary, it holds that[2])

$$\sum_k X_{kj} G_k = \mathrm{i}\frac{\partial S^\dagger}{\partial\alpha_j} S . \tag{9}$$

Multiplying Eq. (9) by S from the left, by $S^\dagger$ from the right, and adding the term

$$\mathrm{i}I\frac{\partial}{\partial\alpha_j} = \mathrm{i}SS^\dagger\frac{\partial}{\partial\alpha_j}$$

to both sides, we get

$$\mathrm{i}I\frac{\partial}{\partial\alpha_j} + \sum_k X_{kj} S G_k S^\dagger = \mathrm{i}S\frac{\partial}{\partial\alpha_j} S^\dagger .$$

Finally, multiplying this equation by $-(X^{-1})_{jl}$, summing over j and using (8) the final set of equations is obtained

$$S\left(-\mathrm{i}\frac{\partial}{\partial\gamma_l} I\right) S^\dagger = -\mathrm{i}\frac{\partial}{\partial\gamma_l} I - SG_l S^\dagger ,$$

or

$$S\mathscr{G}_l I S^{-1} = \mathscr{G}_l I - SG_l S^{-1} , \quad l = 1, \ldots, N . \tag{10}$$

It is obvious that the same relations hold for the complete transformation $U = ZS$ as well.

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[2]) These equations could be used for calculation of the elements X_{kj} for a given parametrization of S.

3. CONCLUSION

In contrast to ref. [2] this derivation is rather general. It exploits only the regularities holding for the compact group K and the existence of its parametrization $(\alpha_1, \ldots, \alpha_N)$. In a paper by one of us, ref. [4], equations (10) for the group $K = SU(2)_J \otimes SU(3)$ are needed to calculate isobar energies. They have a form similar to (2)

$$S\mathscr{L}_a I S^{-1} = \mathscr{L}_a I - S\,\tfrac{1}{2}\sigma_a \otimes I_8 S^{-1}\,. \quad a = 1, 2, 3\,, \tag{11}$$

and

$$S\mathscr{F}_i I S^{-1} = \mathscr{F}_i I - S\sigma_0 \otimes F_i S^{-1}\,, \quad i = 1, \ldots, 8\,, \tag{12}$$

where F_i are the generators of the octet representation of $SU(3)$; I_8 and I are the 8×8 and 16×16 unit matrices, respectively.

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References

[1] Wentzel G.: Helv. Phys. Acta *13* (1940), 269; *14* (1941), 633.
[2] Pauli W., Dancoff S. M.: Phys. Rev. *62* (1942), 85.
[3] Bednář M., Tolar J.: Nucl. Phys. *B 5* (1968), 255.
[4] Tolar J.: to be published.
[5] Pauli W.: Helv. Phys. Acta *12* (1939), 147.
[6] Edmonds A. R.: Angular Momentum in Quantum Mechanics. Princeton University Press, Princeton 1957.
[7] Murnaghan F. D.: The Unitary and Rotation Groups. Spartan Books, Washington D.C. 1962.
[8] Goldstein H.: Classical Mechanics. Addison-Wesley Publishing Co., Inc., Reading (Mass.) and London 1950.