A Some Notions from Functional Analysis

In this appendix we put together some notions from Functional Analysis. For proofs we refer to standard textbooks on Functional Analysis, e.g.,

- J. B. Conway: A Course in Functional Analysis. Springer 1990.
- M. Reed, B. Simon: Functional Analysis. Academic Press 1980.
- Rudin: Functional Analysis, 1973.
- D. Werner: Funktionalanalysis. 4., überarb. Aufl., Springer 2002.

A.1 A short reminder on topological and metric spaces

A **topological space** is a set S in which a collection τ of subsets (called **open sets**) has been specified, with the following properties:

(O1) S and $\emptyset$ are open,

(O2) the intersection of any two open sets is open,

(O3) the union of every collection of open sets is open.

Such a collection τ is called a **topology** on S.

For a subset A of a topological space S, we recall the following notions. A is **closed** if and only if its complement in S is open. The **closure** $\overline{A}$ of A is the intersection of all closed sets that contain A. The **interior** A° of A is the union of all open sets that are subsets of A. A **neighborhood** of A is any open set that contains A. The set A is **compact** if every open covering of A has a finite subcovering. If σ is the collection of all intersections $A \cap U$, with $U \in \tau$, then σ is a topology on A; we call this the topology that A **inherits** from S.

A sequence (x_n) in a topological space S **converges** to a point $x \in S$ (or $\lim_{n\to\infty} x_n = x$) if every neighborhood of x contains all but finitely many of the points x_n. Observe that x in general is not unique. A subset D a topological space S is called **dense** in S if every $x \in S$ is limit of elements in D.

A function $f : S \to \widetilde{S}$ between topological spaces (S, τ) and $(\widetilde{S}, \widetilde{\tau})$ is called **continuous** if for each open subset $B \subseteq \widetilde{S}$ the set $f^{-1}(B) = \{x \in S : f(x) \in B\}$ is open in S. In particular, if $f : S \to \widetilde{S}$ is continuous and $x \in S$, convergence of (x_n) to x in S implies convergence of $(f(x_n))$ to $f(x)$ in $\widetilde{S}$. The converse statement is not true in general.

(S, τ) is a **Hausdorff space**, and τ is a **Hausdorff topology**, if distinct points of S have disjoint neighborhoods.

Now we come to the definition of a metric space.

A **metric space** is a set M, endowed with a real-valued function d on $M \times M$ which satisfies

(D1) $d(x, y) \in [0, \infty)$ for all $x, y \in M$,

(D2) $d(x, y) = 0$ if and only if $x = y$,

(D3) $d(x, y) = d(y, x)$ for all $x, y \in M$,

(D4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

In any metric space X, the **open ball** with center at x and radius r is the set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

By declaring a subset of a metric space to be open if and only if it is a (possibly empty) union of open balls, a topology is obtained. If not stated otherwise the topology on a metric space will always be the one just described.

It is not hard to show that a sequence (x_n) in a metric space (M, d) converges to $x \in M$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. A sequence in (x_n) in (M, d) is called a **Cauchy sequence** if for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \varepsilon$. It is easy to see that any convergent sequence in a metric space is a Cauchy sequence. The converse is not always true. Therefore we define: A metric space in which all Cauchy sequences converge is called **complete**.

Let (M, d), $(\widetilde{M}, \widetilde{d})$ be metric spaces. Then one can prove that a function f from M to $\widetilde{M}$ is continuous if and only if for all $x \in M$, convergence of (x_n) to x with respect to d implies convergence of $(f(x_n))$ to $f(x)$ with respect to $\widetilde{d}$.

A.2 Normed spaces and continuous linear operators

A complex vector space X is said to be a **normed space** if to every $x \in X$ there is associated a real number $\|x\|$, called the **norm** of x, in such a way that

(N1) $\|x\| \in [0, \infty)$ for all $x \in X$,

(N2) $\|x\| = 0 \;\Leftrightarrow\; x = 0$,

(N3) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{C}$, and

(N4) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The word “norm” is also used to denote the function that maps x to $\|x\|$.

Every normed space may be regarded as a metric space, in which the distance $d(x, y)$ between x and y is $\|x - y\|$. A **Banach space** is a normed space which is complete with respect to the metric defined by its norm.

A mapping $T : X \to Y$ between two normed spaces X and Y is called a **linear operator** if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$$

for all $x, y \in X$ and all $\lambda, \mu \in \mathbb{C}$. It is called **bounded** if there exists $C \geq 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \qquad \text{for all } x \in X.$$

Continuous operators on normed spaces are especially nice operators as the next theorem shows:

A.2.1 Theorem *Let T be a linear operator between two normed spaces. The following are equivalent:*

(i) T is continuous. *(ii) T is continuous at 0.* *(iii) T is bounded.*

For normed spaces X, Y we consider

$$\mathcal{L}(X, Y) := \{T : X \to Y : T \text{ is linear and bounded}\}. \tag{A.1}$$

With respect to the algebraic operations

$$(S + T)x = Sx + Tx, \qquad (\lambda T)x = \lambda(Tx),$$

$\mathcal{L}(X, Y)$ is a vector space. Endowed with the **operator norm**

$$\|T\|_{X\to Y} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y, \tag{A.2}$$

$\mathcal{L}(X, Y)$ becomes a normed space. If Y is complete, then $\mathcal{L}(X, Y)$ is complete also.

The following theorem is about extending continuous linear operators from a dense subspace to the whole space.

A.2.2 Theorem *If D is a dense linear subspace of the normed space X, if Y is a Banach space and $T \in \mathcal{L}(D, Y)$, then there exists a unique $\widetilde{T} \in \mathcal{L}(X, Y)$ with $\widetilde{T}_{|D} = T$. In addition, $\|\widetilde{T}\| = \|T\|$.*

Finally we quote the three main classical theorems on bounded linear operators:

A.2.3 Uniform Boundedness Principle *Let X be a Banach space, Y be normed space, I some index set, and $T_i \in \mathcal{L}(X, Y)$ for $i \in I$. If*

$$\sup_{i\in I} \|T_i x\| < \infty \qquad \textit{for all } x \in X,$$

then

$$\sup_{i \in I} \|T_i\| < \infty.$$

A.2.4 Open Mapping Theorem *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ be surjective. Then T is open, i.e., T maps open sets to open sets.*

A.2.5 Closed Graph Theorem *Suppose X, Y are Banach spaces and $T : X \to Y$ is a linear operator satisfying the following: whenever $(x_k, Tx_k) \to (x, y)$ in $X \times Y$, then $y = Tx$. Then T is bounded.*

A.3 Continuous linear functionals, the dual space, and the adjoint operator

The space $X' = \mathcal{L}(X, \mathbb{C})$ is called **dual space** of X. Its elements are called continuous linear functionals. The dual space of a normed space, endowed with the norm

$$\|x'\|_{X'} = \sup_{\|x\|_X \le 1} |x'(x)| \tag{A.3}$$

is always a Banach space.

The following important theorem guarantees that the dual space of X is rich enough.

A.3.1 Hahn-Banach Theorem *Let X be a normed space and U a linear subspace of X. For each continuous linear functional $u' \in U'$ there exists a continuous linear functional $x' \in X'$ such that*

$$x'_{|U} = u', \qquad \|x'\| = \|u'\|.$$

In other words: Each continuous linear functional can be extended with same norm.

Let X, Y be normed spaces and $T \in \mathcal{L}(X, Y)$. The **adjoint** operator $T' : Y' \to X'$ is defined by

$$(T'y')x = y'(Tx). \tag{A.4}$$

The mapping $T \mapsto T'$ is linear and isometric, i.e., $\|T\| = \|T'\|$. In general it is not surjective. Moreover, $(ST)' = T'S'$ for $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$.

A.4 Hilbert Spaces

Let X be a complex vector space. A mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called **scalar product**, if for all $x, y, z \in X$ and all $\alpha \in \mathbb{C}$

(S1) $\langle x, x \rangle \ge 0$,

(S2) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(S3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(S4) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,

(S5) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

It is straightforward to show that (S3) and (S4) imply

(S4') $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$

and (S3) and (S5) imply

(S5') $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

A.4.1 Exercise Let X be a complex vector space with scalar product and let $x, y \in X$. If $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, then $x = y$.

A.4.2 Cauchy-Schwarz inequality *If X is a complex vector space with scalar product, then*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

for all $x, y \in X$. Equality holds if and only if x, y are linearly independent.

For $x \in X$, we set

$$\|x\| := \langle x, x\rangle^{1/2}. \tag{A.5}$$

Then $x \mapsto \|x\|$ defines a norm on X and the Cauchy-Schwarz inquality reads

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

The norm (A.5) induces a metric on X. If X is complete with respect to this metric, then X is called a **Hilbert space**.

A.4.3 Exercise Let X be a complex vector space with scalar product and $x, y \in X$. Then for all $x, y \in X$

(a) Polarization: $\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$,

(b) Parallelogram identity: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|x\|^2$.

If X is a normed space with norm $\|\cdot\|$ satisfying the parallelogram identity, then one can show that $\|\cdot\|$ is induced by a scalar product.

A.4.4 Exercise Let X be a complex vector space with scalar product. Then for all $x \in X$

$$\|x\| = \sup_{\|y\|\leq 1} |\langle x, y\rangle|.$$

We come to the notion of orthogonality. Let X be a complex vector space with scalar product. To vectors $x, y \in X$ are called **orthogonal**, in symbols $x \perp y$, if $\langle x, y\rangle = 0$. The set

$$A^{\perp} = \{x \in X : x \perp a \ \forall a \in A\}$$

is called **orthogonal complement** of $A \subseteq X$.

With these definitions, one can prove the following simple facts.

(a) Pythagoras' theorem: $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2$.

(b) $A^{\perp}$ is a closed subspace of X, $\{0\}^{\perp} = H$, and $H^{\perp} = \{0\}$.

(c) $A \subseteq B$ implies $B^{\perp} \subseteq A^{\perp}$.

(d) $A \subseteq (A^{\perp})^{\perp}$, $A^{\perp} = \overline{A}^{\perp}$, $A^{\perp} = (spanA)^{\perp}$.

Given a point x in a Hilbert space X and a closed subspace Y of X, we can always find a unique point in Y such that the distance of x to that point is minimal. In fact, the following two theorems holds.

A.4.5 Best Approximation Theorem *Let X be a Hilbert space and K some non-void, closed, and convex subset of X. Then for each $x \in X$ there exists a unique best approximation in K, i.e. there exists a unique $y \in K$ with*

$$\|x - y\| = \inf\{\|x - k\| : k \in K\} =: d(x, K).$$

A.4.6 Projection Theorem *Let X be a Hilbert space and Y a closed subspace of X. Then each $x \in X$ can be written in a unique way as $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$. y is called the **orthogonal projection** of x on Y and is denoted by $P_Y x$.*

A.4.7 Corollary *Let X be a Hilbert space and Y a closed subspace of X. Then $(Y^{\perp})^{\perp} = Y$.*

If X is a vector space with scalar product, then for each $x \in X$ the expression

$$\phi_x(y) = \langle y, x \rangle$$

defined a continuous linear functional with norm $\|\phi_x\| = \|x\|$, because $|\phi_x(y)| = |\langle y, x\rangle| \leq \|y\|\|x\|$ for all $y \in H$ and $\phi_x(x) = \|x\|^2$. The next theorem shows that this procedure gives us all continuous linear functionals on a Hilbert space.

A.4.8 Riesz representation theorem *Let X be a Hilbert space. Then for each continuous linear functional ϕ on X there exists a unique $x = x_\phi \in X$ with $\phi(y) = \langle y, x_\phi \rangle$ for all $y \in X$. The mapping $\phi \mapsto x_\phi$ is conjugate linear, isometric (i.e. $\|\phi\| = \|x_\phi\|$) and bijectiv.*

Let X, Y be Hilbert spaces and $T \in B(X, Y)$. The **Hilbert space adjoint** $T^* : Y \to X$ of T is defined by

$$\langle x, T^* y\rangle_X = \langle Tx, y\rangle_Y \qquad \forall x \in X \ \forall y \in Y.$$

One can show that T^* is well defined, linear, and bounded. Moreover $\|T^*\| = \|T\|$.

Let X, Y be Hilbert spaces and $U : X \to Y$ a linear operator. Then U is called **isometrie**, if $\|Ux\|_Y = \|x\|_X$ for all $x \in X$. It is clear that this implies continuity of U. Moreover, it is not hard to see that U is an isometry if and only if $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in X$. Finally, a surjective isometry is also called a **unitary operator**. In this case, $U^* = U^{-1}$.

B Measure Theory and Lebesgue Integration

In this appendix we collect some results from measure theory and integration, which we will need in this lecture. For proofs we refer to textbooks and lecture notes, e.g.,

- N. Henze: Skriptum zur Vorlesung Stochastik II.
- J. Elstrodt: Maß- und Integrationstheorie. 3., erweiterte Aufl., Springer 2002.
- F. Jones: Lebesgue integration on Euclidean space, Jones and Bartlett Publishers 1993.

As motivation we recall the idea of the Riemann integral. It is named after B. Riemann (1826-1866). We consider a bounded function $f : [a, b] \to [0, \infty)$. For each partition Z of $[a, b]$ into finitely many subintervals I_k we write down the lower and upper sums

$$U_R(f, Z) = \sum_k \lambda(I_k) \inf_{x \in I_k} f(x), \qquad O_R(f, Z) = \sum_k \lambda(I_k) \sup_{x \in I_k} f(x).$$

Here $\lambda(I_k)$ denotes the length of the interval I_k. If $\sup_Z U_R(f, Z) = \inf_Z O_R(f, Z)$, then f is called Riemann integrable and the Riemann integral $\int_a^b f(x)dx$ is defined by $\sup_Z U_R(f, Z)$.

The construction of the Riemann integral is simple and concrete. It has (at least) one crucial drawback: the criteria for interchanging limits and integration are not satisfactory at all. Therefore we use the more general Lebesgue integral, named after its inventor H. Lebesgue (1875-1941).

The main idea of the Lebesgue integral is to divide the range of the function f in subintervals. Let $n \in \mathbb{N}$ and $J_{n,k} = [\frac{k}{n}, \frac{k+1}{n})$ for $k = 0, 1, 2, \ldots$. Now we study the preimage of the intervals $J_{n,k}$ under f, i.e., the sets $E_{n,k} = f^{-1}(J_{n,k}) = \{x \in [a, b] : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\}$. If we can measure the „length“ of $E_{k,n}$, then we can write down the Lebesgue lower and upper sums

$$U_L(f, n) = \sum_{k=0}^{\infty} \lambda(E_{n,k}) \frac{k}{n}, \qquad O_L(f, n) = \sum_{k=0}^{\infty} \lambda(E_{n,k}) \frac{k+1}{n}.$$

Since in general the sets $E_{k,n}$ can be complicated, it is not clear a priori how one can measure the length of such sets. Therefore we study the notion of measurability of sets first.

B.1 Measurability

As motivation for the notion of measurability we consider the set $\mathbb{R}$ of real numbers. Our goal is to assign a „length“ to as many subsets of $\mathbb{R}$ as possible. If A is such a „measuable“ subset of $\mathbb{R}$, then we denote the length of A by $\lambda(A)$. We will ask the following properties from our notion of length:

1. For an interval $[a, b]$ with $b \geq a$ it is easy to define its length: we simply set $\lambda([a, b]) = b - a$. (In particular, a single point $\{a\}$ has length 0.)
2. If A, B are two disjoint subsets of $\mathbb{R}$ with known length then $\lambda(A \cup B)$ should be equal to $\lambda(A) + \lambda(B)$. Or more general: If $A_1, A_2, \ldots$ are countably many pairwise disjoint subsets of $\mathbb{R}$ with known length, then $\lambda(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \lambda(A_j)$. If this is the case we say the length is σ-additive. (In particular, each countable subset of $\mathbb{R}$ has length 0.)
3. If $A \subseteq B$, then $\lambda(B \setminus A) = \lambda(B) - \lambda(A)$.
4. The length of a set does not change if we translate the set (translation invariance).

One can show that it is impossible to assign a length to each subset of $\mathbb{R}$ such that 1. to 4. are satisfied. Hence we will restrict ourselves to a subset of the power set of $\mathbb{R}$, named the Borel σ-algebra after E. Borel (1871-1956).

Measurable Sets

First we introduce the notion of a σ-algebra over some non-void set Ω.

B.1.1 Definition Let Ω be a non-void set and $\mathcal{A}$ a subset of the power set $\mathcal{P}(\Omega)$ of Ω with

(1) $\emptyset \in \mathcal{A}$,

(2) $A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A}$,

(3) $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{j \in \mathbb{N}} A_j \in \mathcal{A}$.

Then $\mathcal{A}$ is calledσ-**algebra** over Ω. Each set in $\mathcal{A}$ is called $\mathcal{A}$-**measurable**.

The simplest examples of σ-algebras over a set Ω are $\{\emptyset, \Omega\}$ and $\mathcal{P}(\Omega)$. Hence for each set Ω there is at least one σ-algebra. Moreover the intersection of σ-algebras over Ω is also a σ-algebra over Ω (proof?). As important example we consider the **Borel σ-algebra** $\mathcal{B}$ over $\mathbb{R}$. This σ-algebra is defined as the intersection of all σ-algebras, that contain all finite real intervals. Sometimes $\mathcal{B}$ is also called the σ-algebra of **Borel sets**.

B.1.2 Exercise Let Ω be a non-void set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a σ-Algebra. Then:
(a) $\Omega \in \mathcal{A}$, (b) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$, (c) $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcap_{j \in \mathbb{N}} A_j \in \mathcal{A}$.

B.1.3 Exercise Let Ω_1, Ω_2 be non-void sets and $f : \Omega_1 \to \Omega_2$ a function.
(a) If $\mathcal{A}_2$ is a σ-algebra over Ω_2, then $f^{-1}(\mathcal{A}_2) = \{f^{-1}(A) : A \in \mathcal{A}_2\}$ is a σ-algebra over Ω_1.
(b) If $\mathcal{A}_1$ is σ-algebra over Ω_1, then $\{B \in \mathcal{P}(\Omega_2) : f^{-1}(B) \in \mathcal{A}_1\}$ is a σ-algebra over Ω_2.

Now we procede toward the definition of measure.

B.1.4 Definition Let $\mathcal{A}$ be σ-algebra over a set Ω and $\mu : \mathcal{A} \to [0, \infty]$ with

(1) $\mu(\emptyset) = 0$,

(2) μ is σ-**additive**, d.h. $A_1, A_2, \dots \in \mathcal{A}$ are disjoint $\Rightarrow \mu(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} \mu(A_j)$.

Then μ is called **measure** on $\mathcal{A}$ and $(\Omega, \mathcal{A}, \mu)$ is called **measure space**.

B.1.5 Example (a) Let Ω be a set and $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ be defined by

$$\mu(A) = \begin{cases} |A|, & \text{falls } A \text{ finite,} \\ \infty, & \text{else.} \end{cases}$$

Then $(\Omega, \mathcal{P}(\Omega), \mu)$ is a measure space. μ is called **counting measure** on Ω.
(b) Let $\mathcal{A}$ be a σ-algebra over a set Ω and $x \in \Omega$. Let $\delta_x : \mathcal{A} \to \{0, 1\}$ be given by

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Then $(\Omega, \mathcal{A}, \delta_x)$ is a measure space. δ_x is called **Dirac measure**.
(c) Let $\mathcal{B}$ be the Borel σ-algebra over $\mathbb{R}$. Then there is a unique measure $\beta : \mathcal{B} \to [0, \infty]$ with the additional properties

(3) $\beta((a, b)) = \beta([a, b]) = b - a$, provided $a < b$,

(4) β is translation invariant, i.e., for all $x \in \mathbb{R}$ and all $A \in \mathcal{B}$ we have $\beta(x + A) = \beta(A)$.

β is called **Lebesgue-Borel measure** on $\mathbb{R}$. Hence the Lebesgue-Borel measure satisfies all four properties of a notion of length mentioned in the introduction. For the (rather long) proof of existence and uniqueness of β we refer to the literature.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A set $N \in \mathcal{A}$ is called μ-**nullset**, if $\mu(N) = 0$. If there is a μ-nullset N, such that some statement holds for all $\omega \in \Omega \setminus N$ then we say that this statement hold for μ-**almost all** $\omega \in \Omega$, or simply **almost everywhere (a.e.)**.

B.1.6 Exercise (a) The union of countably many μ-nullsets is a μ-nullset.
(b) If $A \subseteq \mathbb{R}$ is countable, then A is a β-nullset.

One might guess that each subset of a μ-nullset also has measure zero. But in general such a subset even is not an element σ-algebra associated to the measure. An example for this phenomenon is the Borel σ-algebra $\mathcal{B}$. Therefore we introduce the notion of a complete measure space: A measure space $(\Omega, \mathcal{A}, \mu)$ is called **complete**, if each subset of μ-nullset is an element of $\mathcal{A}$. (This definition has nothing to do with the notion of a complete metric space!)

B.1.7 Theorem and Definition *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathcal{N} = \{N \subseteq A : A \in \mathcal{A}, \mu(A) = 0\}$ the set of all subsets of μ-nullsets. We define $\widetilde{\mathcal{A}} := \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$ and $\widetilde{\mu} : \mathcal{A} \to [0, \infty]$ where $\widetilde{\mu}(A \cup N) = \mu(A)$ für $A \in \mathcal{A}$, $N \in \mathcal{N}$. Then $\widetilde{\mathcal{A}}$ is a σ-algebra over Ω, $\widetilde{\mu}$ is well defined and $(\Omega, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is a complete measure space, called **completion** of $(\Omega, \mathcal{A}, \mu)$.*

The completion of the Borel' σ-algebra over $\mathbb{R}$ is called the **σ-algebra of Lebesgue sets** over $\mathbb{R}$ and is denoted by mit $\mathcal{L}$. One can show that $\mathcal{L}$ is a proper subset of $\mathcal{P}(\mathbb{R})$. The associated measure $\widetilde{\beta}$ is called **Lebesgue measure** and is denoted by λ.

Measureable functions

Now we come to the important notion of measurable functions.

B.1.8 Definition Let Ω be a set and let $\mathcal{A}$ be a σ-algebra over Ω.

(a) A function $f : \Omega \to [-\infty, \infty]$ is called **$\mathcal{A}$-measurable**, if $f^{-1}([a, b)) \in \mathcal{A}$ for $-\infty \leq a < b \leq \infty$.

(b) A function $f : \Omega \to \mathbb{C}$ is called **$\mathcal{A}$-measurable**, if $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{A}$-measurable.

Instead of $\mathcal{B}$-measurable ($\mathcal{L}$-measurable) we say Borel measurable (Lebesgue measurable). If $f : \mathbb{R} \to [-\infty, \infty]$ or. $f : \mathbb{R} \to \mathbb{C}$ is Borel measurable, then f is Lebesgue measurable, since $\mathcal{B}$ is a subset of $\mathcal{L}$. If $f : \mathbb{R} \to \mathbb{C}$ is continuous, then f is Borel measurable and hence Lebesgue measurable.

B.1.9 Proposition *Let Ω be a set and $\mathcal{A}$ a σ-algebra over Ω.*

(a) If $f, g : \Omega \to [-\infty, \infty]$ are $\mathcal{A}$-measurable and if $\alpha \in \mathbb{C}$, then αf, $f + g$, $f \cdot g$, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$ are $\mathcal{A}$-measurable.

(b) If $f_n : \Omega \to [-\infty, \infty]$ is $\mathcal{A}$-measurable, then $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ are $\mathcal{A}$-measurable. In particular, $\lim_{n\to\infty} f_n$ is $\mathcal{A}$-measurable, if the limit exists pointwise in $[-\infty, \infty]$.

B.1.10 Definition Let $\mathcal{A}$ be a σ-algebra over the setΩ.

(a) The function

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in \Omega \setminus A \end{cases}$$

is called **indicator function** of the set $A \subseteq \Omega$.

(b) A function of the form $\phi = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$ with $\alpha_k \in \mathbb{C}$ and $A_k \in \mathcal{A}$ is called **$\mathcal{A}$-step**

function.

B.1.11 Proposition *Let $\mathcal{A}$ be a σ-algebra over a set Ω. Then:*

(a) $\mathcal{A}$-step funcions are $\mathcal{A}$-measurable.

(b) If $f : \Omega \to [-\infty, \infty]$ or $f : \Omega \to \mathbb{C}$ is $\mathcal{A}$-measurable, dann there is a sequence (ϕ_n) of $\mathcal{A}$-step functions with $f(x) = \lim_{n\to\infty} \phi_n(x)$ for all $x \in \Omega$.

(c) If $f : \Omega \to [0, \infty]$ is $\mathcal{A}$-measurable, then (ϕ_n) from (b) can be chosen such that $0 \leq \phi_1 \leq \phi_2 \leq \dots$.

(d) If $f : \Omega \to \mathbb{C}$ is $\mathcal{A}$-measurable and bounded, then (ϕ_n) from (b) can be chosen such that $\phi_n \to f$ uniformly in Ω.

Idea of the proof: (c) Construction of ϕ_n: Decompose $[0, n)$ in intervals $I_{k,n}$ of length $\frac{1}{n}$, set $E_{k,n} := \{x \in \Omega : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\}$ and $\phi_n = \sum_{k=0}^{n^2-1} \frac{k}{n} \chi_{E_{k,n}}$.
(b) If $f : \Omega \to [-\infty, \infty]$, apply (c) to $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$.
If $f : \Omega \to \mathbb{C}$, consider $\operatorname{Re} f$ and $\operatorname{Im} f$.

B.2 The μ-integral

This section contains the main definitions and theorems concerning the μ-integral. Proofs can be found in the literature.

Integrability

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. First we define the μ-integral for non-negative step functions: If $f = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ is $\mathcal{A}$-step function with $\alpha_k \geq 0$ for all k, then we define

$$\int_\Omega f d\mu := \sum_{k=1}^{n} \alpha_k \mu(E_k) \in [0, \infty].$$

This definition is independent of the choice of the particular form of f.

Next we consider a $\mathcal{A}$-measurable function $f : \Omega \to [0, \infty]$. By Proposition B.1.11 there is a sequence (ϕ_n) of step functions with $0 \leq \phi_1 \leq \phi_2 \leq \dots$ and $f = \lim_{n\to\infty} \phi_n$. The sequence $(\int_\Omega \phi_n d\mu)_{n\in\mathbb{N}}$ of non-negative real numbers is increasing. Hence it has a limit in $[0, \infty]$. We define

$$\int_\Omega f d\mu := \lim_{n\to\infty} \int_\Omega \phi_n d\mu.$$

This definition is independent of the choice of the sequence (ϕ_n). For the proof of this non-trivial statement we refer to the literature.

Now we define the notion of μ-integrability.

B.2.1 Definition Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.
(a) A function $f : \Omega \to [0, \infty]$ is called μ-**integrable**, if f is $\mathcal{A}$-measurable and $\int_\Omega f d\mu < \infty$.
(b) A function $f : \Omega \to \mathbb{R}$ is called μ-**integrable**, if $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ are μ-integrable. In this case

$$\int_\Omega f d\mu := \int_\Omega f_+ d\mu - \int_\Omega f_- d\mu.$$

(c) A function $f : \Omega \to \mathbb{C}$ is called μ-**integrable**, if $\operatorname{Re} f$ and $\operatorname{Im} f$ are μ-integrable. In this case

$$\int_\Omega f d\mu := \int_\Omega \operatorname{Re} f d\mu + i \int_\Omega \operatorname{Im} f d\mu.$$

We say that a function f defined of $\mathbb{R}$ is **Lebesgue integrable**, if f is integrable with respect to the Lebesgue measure λ.

B.2.2 Proposition *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{C}$ a function. Then the following are equivalent:*
(1) f is μ-integrable.
(2) f is $\mathcal{A}$-measurable and $|f|$ is μ-integrable.
(3) f is $\mathcal{A}$-measurable and there is a μ-integrable $g : \Omega \to [0, \infty)$ with $|f| \leq g$.

B.2.3 Proposition *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.*
(a) If $f, g : \Omega \to \mathbb{C}$ are μ-integrable and $\alpha \in \mathbb{C}$, then αf, $f + g$ are μ-integrable and

$$\int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu, \qquad \int_\Omega f + g d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu.$$

(b) If $f, g; \Omega \to \mathbb{R}$ are μ-integrable and $f \leq g$, then

$$\int_\Omega f d\mu \leq \int_\Omega g d\mu.$$

(c) If $f : \Omega \to \mathbb{C}$ is μ-integrable, then

$$\left| \int_\Omega f d\mu \right| \leq \int_\Omega |f| d\mu.$$

(d) Let $f : \Omega \to [0, \infty]$ be $\mathcal{A}$-measurable. Then $\int_\Omega f d\mu = 0$ if and only if $f(x) = 0$ for μ-almost every $x \in \Omega$.

B.2.4 The space $L_1(\Omega, \mu)$: Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and

$$\mathcal{L}_1(\Omega, \mu) := \{f \mid f : \Omega \to \mathbb{C} \ \mu\text{-integrable}\}.$$

Then $\mathcal{L}_1(\Omega, \mu)$ is a complex vector space (by Proposition B.2.3 (a)). The space

$$\mathcal{N}(\Omega, \mu) := \{f \mid f : \Omega \to \mathbb{C} \ \text{ is } \mathcal{A}\text{-measurable and } f(x) = 0 \text{ for } \mu\text{-almost all } x \in \Omega\}$$

is a subspace of $\mathcal{L}_1(\Omega,\mu)$. Hence

$$f \sim g \;:\Leftrightarrow\; f-g \in \mathcal{N}(\Omega,\mu) \;\Leftrightarrow\; f(x)=g(x) \text{ for } \mu\text{-almost all } x \in \Omega$$

defines a equivalence relation on $\mathcal{L}_1(\Omega,\mu)$. For $f,g \in \mathcal{L}_1(\Omega,\mu)$ with $f \sim g$ we have by B.2.3 (d) that

$$\int_\Omega |f| d\mu = \int_\Omega |g| d\mu.$$

Hence the following definition makes sense:

$$L_1(\Omega,\mu) := \mathcal{L}_1(\Omega,\mu)/\mathcal{N}(\Omega,\mu), \qquad \|[f]\|_1 := \|[f]\|_{L_1(\Omega,\mu)} := \int_\Omega |f| d\mu.$$

Instead of $[f]$ we write f, i.e., we identify functions if they are equal μ-almost everywhere.

The mapping $f \mapsto \|f\|_1$ is a **Norm** on $L_1(\Omega,\mu)$, i.e. for all $f,g \in L_1(\Omega,\mu)$ and $\alpha \in \mathbb{C}$

- (N1) $\|f\|_1 \geq 0$.
- (N2) $\|f\|_1 = 0$ if and only if $f = 0$ (μ-almost everywhere).
- (N3) $\|\alpha f\|_1 = |\alpha| \|f\|_1$.
- (N4) $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ (triangle inequality).

B.2.5 Substitution Let $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$. If $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue integrable, then $f(\cdot - b)$ and $f(a\cdot)$ are Lebesgue integrable and

$$\int_{\mathbb{R}} f(\cdot - b) d\lambda = \int_{\mathbb{R}} f d\lambda \qquad \text{and} \qquad \int_{\mathbb{R}} f(a\cdot) d\lambda = \frac{1}{|a|} \int_{\mathbb{R}} f d\lambda.$$

B.2.6 Riemann integral and Lebesgue integral (a) If $f : [a,b] \to \mathbb{C}$ is Riemann integrable, then $\widetilde{f} : \mathbb{R} \to \mathbb{C}$, defined by $\widetilde{f} = f$ on $[a,b]$ and $\widetilde{f} = 0$ otherwise, is Lebesgue integrable and

$$\int_a^b f(x)dx = \int_{\mathbb{R}} \widetilde{f} d\lambda.$$

(This statement does **not** hold for improper Riemann integrals!) If $f \in L_1(\mathbb{R})$, we also write $\int_{\mathbb{R}} f(x)dx$ for $\int_{\mathbb{R}} f d\lambda$.
(b) Let f the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and $\int_{\mathbb{R}} f d\lambda = 0$. Indeed, f is a $\mathcal{L}$-step function and $\lambda(\mathbb{Q}) = 0$, since $\mathbb{Q}$ is countable. But $f|_{[0,1]}$ is not Riemann integrable.

Theorems on convergence

This section contains important theorems on interchanging limits and integration.

B.2.7 Fatou's Lemma *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $f_n : \Omega \to [0, \infty]$ are $\mathcal{A}$-measurable, then*

$$\int_\Omega \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

B.2.8 Theorem on monotone convergence (Beppo Levi) *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $f_n : \Omega \to [0, \infty]$ are $\mathcal{A}$-measurable with $0 \leq f_1 \leq f_2 \leq \ldots$ and $f(x) := \lim_{n\to\infty} f(x) \in [0, \infty]$, then f is $\mathcal{A}$-measurable and*

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

B.2.9 Theorem on dominated convergence (Lebesgue) *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If $f_n, f : \Omega \to \mathbb{C}$ are $\mathcal{A}$-measurable and $f(x) = \lim_{n\to\infty} f_n(x)$ for μ-almost all $x \in M$. If the functions f_n are μ-integrable and if there is a μ-integrable $g : \Omega \to [0, \infty)$ with $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and μ-almost all $x \in \Omega$, then f is μ-integrable and*

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

As an application we consider measures with densities.

B.2.10 Measures with densities Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\rho : \Omega \to [0, \infty]$ a $\mathcal{A}$-measurable function. Then

$$\nu(A) = \int_A \rho d\mu = \int_\Omega \rho \cdot \chi_A \, d\mu, \qquad A \in \mathcal{A},$$

defines a measure on $\mathcal{A}$. Here the σ-additivity follows from the Theorem on monotone convergence B.2.8. Using the same theorem B.2.8 one can show that for a $\mathcal{A}$-measurable function $f : \Omega \to \mathbb{C}$ the following holds: If $f \cdot \rho$ is μ-integrable, then f is ν-integrable and

$$\int_\Omega f d\nu = \int_\Omega f \cdot \rho \, d\mu.$$

Product measure and Fubini's Theorem

We want to define a measure on the product of two measure spaces with the property that the measure of the product of two sets is the product of the measures of the sets. First we define the notion of a σ-finite measure space.

B.2.11 Definition A measure space $(\Omega, \mathcal{A}, \mu)$ is called **σ-finite**, it there is a sequence $(E_k) \subseteq \mathcal{A}$ with $\mu(E_k) < \infty$ and $\bigcup_{k=1}^\infty E_k = \Omega$.

The measure space $(\mathbb{R}, \mathcal{L}, \lambda)$ is σ-finite, since $\mathbb{R} = \bigcup_{k\in\mathbb{N}}[-k, k]$.

For the rest of this section, $(\Omega_1, \mathcal{A}_1, \mu_1)$, $(\Omega_2, \mathcal{A}_2, \mu_2)$ are always σ-finite measure spaces.

B.2.12 Theorem and Definition *Let $\mathcal{A}_1 \otimes \mathcal{A}_2$ be the σ-algebra auf $\Omega_1 \times \Omega_2$ generated by the sets $A_1 \times A_2$, $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$. Then there is a unique measure $\mu : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, \infty]$ with $\mu(A_1 \times A_2) = \mu(A_1)\mu(A_2)$. $\mu_1 \otimes \mu_2 := \mu$ is called* ***product measure*** *of μ_1 and μ_2.*

The completion of $(\mathbb{R}^2, \mathcal{L}\otimes\mathcal{L}, \lambda\otimes\lambda)$ we denote by $(\mathbb{R}^2, \mathcal{L}^2, \lambda^2)$. Repeating this procedure, we define $(\mathbb{R}^N, \mathcal{L}^N, \lambda^N)$ in the obvious way. A function $f : \mathbb{R}^N \to \mathbb{C}$ is called Lebesgue-measurable or Lebesgue-integrable, if f is $\mathcal{L}^N$-measurable or λ^N-integrable, respectively.

Tonelli's Theorem and Fubini's Theorem deal with interchanging the order of integration.

B.2.13 Tonelli's Theorem *Let $f : \Omega_1 \times \Omega_2 \to [0, \infty]$ be $\mu_1 \otimes \mu_2$-measurable. Then the function $x \mapsto f(x, y)$ is μ_1-measurable for almost all $y \in \Omega_2$. Moreover, $y \mapsto \int_{\Omega_1} f(x, y)d\mu_1(x)$ is μ_2-measurable and*

$$\int_{\Omega_1\times\Omega_2} f\, d(\mu_1 \otimes \mu_2) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y).$$

B.2.14 Fubini's Theorem *Let $f : \Omega_1 \times \Omega_2 \to \mathbb{C}$ be $\mu_1 \otimes \mu_2$-integrable. Then the function $x \mapsto f(x, y)$ is μ_1-integrable for almost all $y \in \Omega_2$. Moreover, $y \mapsto \int_{\Omega_1} f(x, y)d\mu_1(x)$ is μ_2-integrable and*

$$\int_{\Omega_1\times\Omega_2} f\, d(\mu_1 \otimes \mu_2) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y).$$

C The Lebesgue spaces $L^p(\Omega, \mu)$

This appendix contains the basic facts about the Lebesgue spaces $L^p(\Omega, \mu)$ where $1 \leq p \leq \infty$. For more details and the proofs see the references given at the beginning of Appendix A and B, in particular

- F. Jones: Lebesgue integration on Euclidean space, Jones and Bartlett Publishers 1993.
- D. Werner: Funktionalanalysis. 4., überarb. Aufl., Springer 2002.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. For $p \in [1, \infty)$, $L^p(\Omega, \mu)$ will denote the set of all complex-valued μ-measurable functions on Ω such that $|f|^p$ is μ-integrable. $L^\infty(\Omega, \mu)$ will be the set of all complex-valued μ-measurable functions f on Ω such that for some $B > 0$, the set $\{x : |f(x)| > B\}$ has μ-measure zero. Two functions in $L^p(\Omega, \mu)$ will be considered equal if they are equal μ-almost everywhere. The notation $L^p(\mathbb{R}^N)$ will be reserved for the space $L^p(\mathbb{R}^N, \lambda^N)$. The space $L^p(\mathbb{Z})$ equipped with counting measure will be denoted by $\ell^p(\mathbb{Z})$ or simply ℓ^p.

For $p \in [1, \infty)$, we define

$$\|f\|_{L^p(\Omega,\mu)} = \left(\int_\Omega |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \qquad f \in L^p(\Omega, \mu), \tag{C.1}$$

and for $p = \infty$ by

$$\|f\|_{L^\infty(\Omega,\mu)} = \inf\big\{B > 0 : \mu\big(\{x : |f(x)| > B\}\big) = 0\big\}, \qquad f \in L^\infty(\Omega, \mu). \tag{C.2}$$

It is well-known that Minkowski's (or the triangle) inequality

$$\|f + g\|_{L^p(\Omega,\mu)} \leq \|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)} \tag{C.3}$$

holds for all f, g in $L^p(\Omega, \mu)$, whenever $1 \leq p \leq \infty$. Since in addition $\|f\|_{L^p(\Omega,\mu)} = 0$ implies that $f = 0$ (μ-a.e.), the L^p spaces are normed linear spaces for $1 \leq p \leq \infty$. (Recall that we agreed to identify functions that are equal μ-almost everywhere.) For all $1 \leq p \leq \infty$, it can be shown that every Cauchy sequence in $L^p(\Omega, \mu)$ is convergent, and hence the spaces $L^p(\Omega, \mu)$ are complete. Therefore, the L^p spaces are Banach spaces. For any $p \in (1, \infty)$ we will use the notation $p' = \frac{p}{p-1}$. Moreover $1' = \infty$ and $\infty' = 1$

so that $p'' = p$ for all $p \in [1, \infty]$. Hölder's inequality says that for all $p \in [1, \infty]$ and all measurable functions f, g on (Ω, μ) we have

$$\|fg\|_{L^1(\Omega,\mu)} \leq \|f\|_{L^p(\Omega,\mu)} \|g\|_{L^{p'}(\Omega,\mu)}. \tag{C.4}$$

It is a well-known fact that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for all $1 \leq p < \infty$. Furthermore, the L^p norm of a function can be obtained via duality when $1 \leq p \leq \infty$ as follows:

$$\|f\|_{L^p(\Omega\mu)} = \sup_{\|g\|_{p'}=1} \left| \int_\Omega fg d\mu \right|. \tag{C.5}$$

Continuous functions with compact support in $\mathbb{R}^N$ are dense in $L^p(\mathbb{R}^N)$, if $1 \leq p < \infty$.