Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop

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Abstract

In this paper we investigate the operator $H_\beta = -\Delta - \beta\delta(\cdot - \Gamma)$ in $L^2(\mathbb{R}^2)$, where $\beta > 0$ and Γ is a closed C^4 Jordan curve in $\mathbb{R}^2$. We obtain the asymptotic form of each eigenvalue of H_β as β tends to infinity. We also get the asymptotic form of the number of negative eigenvalues of H_β in the strong coupling asymptotic regime.

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1. Introduction

In this paper we study the Schrödinger operator with a δ-interaction on a loop. Let $\Gamma : [0, L] \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2$ be a closed C^4 Jordan curve which is parametrized by the arc length. Let $\gamma : [0, L] \to \mathbb{R}^2$ be the signed curvature of Γ. For $\beta > 0$, we define

$$q_\beta(f, f) = \|\nabla f\|^2_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma |f(x)|^2 \, dS \quad \text{for} \quad f \in H^1(\mathbb{R}^2). \tag{1.1}$$

By H_β we denote the self-adjoint operator associated with the form q_β. The operator H_β is formally written as $-\Delta - \beta\delta(\cdot - \Gamma)$. Since Γ is compact in $\mathbb{R}^2$, we have $\sigma_{\text{ess}}(H_\beta) = [0, \infty)$ by [3, Theorem 3.1]. Our main purpose is to study the asymptotic behaviour of the negative eigenvalues of H_β as β tends to infinity. We define

$$S = -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2((0, L)) \tag{1.2}$$

with the domain

$$P = \{\varphi \in H^2((0,L)); \quad \varphi(L) = \varphi(0), \quad \varphi'(L) = \varphi'(0)\}. \tag{1.3}$$

For $j \in \mathbb{N}$, we denote by μ_j the jth eigenvalue of S counted with multiplicity. Our main results are the following.

THEOREM 1. *Let n be an arbitrary integer. There exists $\beta(n) > 0$ such that*

$$\sharp\sigma_{\mathrm{d}}(H_\beta) \geqslant n \quad \textit{for} \quad \beta \geqslant \beta(n).$$

For $\beta \geqslant \beta(n)$ we denote by $\lambda_n(\beta)$ the nth eigenvalue of H_β counted with multiplicity. Then $\lambda_n(\beta)$ admits an asymptotic expansion of the form

$$\lambda_n(\beta) = -\frac{1}{4}\beta^2 + \mu_n + \mathcal{O}(\beta^{-1}\log\beta) \quad \textit{as} \quad \beta \to \infty. \tag{1.4}$$

THEOREM 2. *The function $\beta \mapsto \sharp\sigma_{\mathrm{d}}(H_\beta)$ admits an asymptotic expansion of the form*

$$\sharp\sigma_{\mathrm{d}}(H_\beta) = \frac{L}{2\pi}\beta + \mathcal{O}(\log\beta) \quad \textit{as} \quad \beta \to \infty. \tag{1.5}$$

The Schrödinger operator with a singular interaction has been studied by numerous authors (see [1-3] and the references therein). The basic concepts of the theory are summarized in the monograph [1]. A particular case of a δ-interaction supported by a curve attracted much less attention (see [3, 4] and a recent paper [5]). In [3] some upper bounds to the number of eigenvalues for a more general class of operators (with β dependent on the arc length parameter) were obtained by the Birman-Schwinger argument (see [3, Theorems 3.4, 3.5, and 4.2]). As it is usually the case with the Birman-Schwinger technique, these bounds are sharp for small positive β (see [3, Example 4.1]) while they give a poor estimate in the semiclassical regime. On the contrary, our estimate (1.5) is close to optimal for large positive β. Our main tools to prove Theorem 1 and Theorem 2 are the Dirichlet-Neumann bracketing and approximate operators with separated variables.

2. Proof of Theorem 1

Let us prepare some quadratic forms and operators which we need in the sequel. For this purpose, we first need the following result.

LEMMA 2.1. *Let Φ_a be the map*

$$[0,L) \times (-a,a) \ni (s,u) \mapsto (\Gamma_1(s) - u\Gamma_2'(s), \Gamma_2(s) + u\Gamma_1'(s)) \in \mathbb{R}^2.$$

Then there exists $a_1 > 0$ such that the map Φ_a is injective for any $a \in (0, a_1]$.

Proof. We extend Γ to a periodic function with period L, which we denote by $\tilde\Gamma(s) = (\tilde\Gamma_1(s), \tilde\Gamma_2(s))$. Since Γ is a closed C^4 Jordan curve, we have $\tilde\Gamma \in C^4(\mathbb{R})$. We extend γ

to a function $\tilde\gamma$ on $\mathbb{R}$ by using the formula $\tilde\gamma(s) = \tilde\Gamma_1''(s)\tilde\Gamma_2'(s) - \tilde\Gamma_2''(s)\tilde\Gamma_1'(s)$. Then $\tilde\gamma(\cdot)$ is periodic with period L and $\tilde\gamma \in C^2(\mathbb{R})$. By Φ we denote the map

$$\mathbb{R}^2 \ni (s,u) \mapsto (\tilde\Gamma_1(s) - u\tilde\Gamma_2'(s), \tilde\Gamma_2(s) + u\tilde\Gamma_1'(s)) \in \mathbb{R}^2.$$

Let $J\Phi$ be the Jacobian matrix of Φ. We put

$$\gamma_+ = \max_{[0,L]} |\gamma(\cdot)|.$$

We have

$$\det J\Phi(s,u) = 1 + u\tilde\gamma(s) \geqslant \frac{1}{2} \quad \text{for} \quad (s,u) \in \mathbb{R} \times [-\tfrac{1}{2\gamma_+}, \tfrac{1}{2\gamma_+}]. \tag{2.1}$$

In addition, there exists a constant $M > 0$ such that

$$|\partial_y^\alpha \Phi_j(y)| \leqslant M \quad \text{on} \quad \mathbb{R} \times [-\tfrac{1}{2\gamma_+}, \tfrac{1}{2\gamma_+}] \tag{2.2}$$

for any $1 \leqslant |\alpha| \leqslant 2$ and $j = 1,2$, where $y = (s,u)$ and $\Phi(y) = (\Phi_1(y), \Phi_2(y))$. Combining [8, Lemma 3.6] with (2.1) and (2.2), we claim that there exists $a_0 \in (0, \frac{1}{2\gamma_+})$ such that Φ is injective on $[k - a_0, k + a_0] \times [-a_0, a_0]$ for all $k \in \mathbb{R}$. We put

$$\tau = \min_{p \in [a_0, L/2]} \min_{t \in [0,L]} |\tilde\Gamma(t) - \tilde\Gamma(t+p)|. \tag{2.3}$$

Since $\tilde\Gamma$ is injective on $[0,L)$ and $\tilde\Gamma(\cdot)$ has period L, we have $\tau > 0$. Put $a_1 = \min\{a_0, \tau/4\}$. Let us show that Φ is injective on $[0,L) \times (-a_1, a_1)$. We first prove the following claim.
(i) *Assume that* $\Phi(s_1,u_1) = \Phi(s_2,u_2)$, $|s_1 - s_2| \leqslant \frac{L}{2}$, *and* $(s_1,u_1), (s_2,u_2) \in \mathbb{R} \times (-a_1, a_1)$. *Then we have* $(s_1,u_1) = (s_2,u_2)$.

Since $\Phi(s_1,u_1) = \Phi(s_2,u_2)$ and $|\tilde\Gamma_j'(\cdot)| \leqslant 1$ on $\mathbb{R}$ for $j = 1,2$, we obtain

$$|\tilde\Gamma_1(s_1) - \tilde\Gamma_1(s_2)| = |u_1\tilde\Gamma_2'(s_1) - u_2\tilde\Gamma_2'(s_2)| \leqslant 2a_1,$$

$$|\tilde\Gamma_2(s_1) - \tilde\Gamma_2(s_2)| = |u_1\tilde\Gamma_1'(s_1) - u_2\tilde\Gamma_1'(s_2)| \leqslant 2a_1.$$

So we have $|\tilde\Gamma(s_1) - \tilde\Gamma(s_2)| \leqslant 2\sqrt{2}a_1$, and therefore

$$|\tilde\Gamma(s_1) - \tilde\Gamma(s_2)| < \tau.$$

This together with (2.3) implies that $|s_1 - s_2| < a_0$. Since Φ is injective on $[s_1 - a_0, s_1 + a_0] \times [-a_0, a_0]$ and $\Phi(s_1,u_1) = \Phi(s_2,u_2)$, we get $(s_1,u_1) = (s_2,u_2)$. In this way we proved (i).

Next we shall prove the following implication.
(ii) *Assume that* $\Phi(s_1,u_1) = \Phi(s_2,u_2)$, $s_1 \leqslant s_2$, *and* $(s_1,u_1), (s_2,u_2) \in [0,L) \times (-a_1, a_1)$. *Then we have* $s_2 - s_1 \leqslant \frac{L}{2}$.

We prove this by contradiction. Assume that $s_2 - s_1 > \frac{L}{2}$. We put $s_3 = s_2 - L$. Then we get $0 < s_1 - s_3 < \frac{L}{2}$ and $\Phi(s_3, u_2) = \Phi(s_1, u_1)$. As in the proof of (i) we obtain $(s_1, u_1) = (s_3, u_2)$ which violates the fact that $0 < s_1 - s_3 < \frac{L}{2}$, so we proved (ii).

Combining (i) with (ii), we conclude that Φ is injective on $[0, L) \times (-a_1, a_1)$. $\square$

Let $0 < a < a_1$. Let Σ_a be the strip of width $2a$ enclosing Γ:

$$\Sigma_a = \Phi([0, L) \times (-a, a)).$$

Then $\mathbb{R}^2 \backslash \Sigma_a$ consists of two connected components which we denote by Λ_a^{in} and Λ_a^{out}, where Λ_a^{in} is compact. We define

$$q^+_{a,\beta}(f, f) = \|\nabla f\|^2_{L^2(\Sigma_a)} - \beta \int_\Gamma |f(x)|^2 \, dS \quad \text{for} \quad f \in H^1_0(\Sigma_a),$$

$$q^-_{a,\beta}(f, f) = \|\nabla f\|^2_{L^2(\Sigma_a)} - \beta \int_\Gamma |f(x)|^2 \, dS \quad \text{for} \quad f \in H^1(\Sigma_a).$$

Let $L^+_{a,\beta}$ and $L^-_{a,\beta}$ be the self-adjoint operators associated with the forms $q^+_{a,\beta}$ and $q^-_{a,\beta}$, respectively. By using the Dirichlet-Neumann bracketing (see [7, XIII.15, Proposition 4]), we obtain

$$(-\Delta^{\text{N}}_{\Lambda_a^{\text{in}}}) \oplus L^-_{a,\beta} \oplus (-\Delta^{\text{N}}_{\Lambda_a^{\text{out}}}) \leqslant H_\beta \leqslant (-\Delta^{\text{D}}_{\Lambda_a^{\text{in}}}) \oplus L^+_{a,\beta} \oplus (-\Delta^{\text{D}}_{\Lambda_a^{\text{out}}}) \tag{2.4}$$

in $L^2(\Lambda_a^{\text{in}}) \oplus L^2(\Sigma_a) \oplus L^2(\Lambda_a^{\text{out}})$. In order to estimate the negative eigenvalues of H_β, it is sufficient to estimate those of $L^+_{a,\beta}$ and $L^-_{a,\beta}$ because the other operators involved in (2.4) are positive.

To this aim we introduce two operators in $L^2((0, L) \times (-a, a))$ which are unitarily equivalent to $L^+_{a,\beta}$ and $L^-_{a,\beta}$, respectively. We define

$$\begin{aligned} Q^+_a = \{\varphi \in H^1((0, L) \times (-a, a)); \quad & \varphi(L, \cdot) = \varphi(0, \cdot) \quad \text{on} \quad (-a, a), \\ & \varphi(\cdot, a) = \varphi(\cdot, -a) = 0 \quad \text{on} \quad (0, L)\}, \end{aligned}$$

$$Q^-_a = \{\varphi \in H^1((0, L) \times (-a, a)); \quad \varphi(L, \cdot) = \varphi(0, \cdot) \quad \text{on} \quad (-a, a)\},$$

$$\begin{aligned} b^+_{a,\beta}(f, f) = & \int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^a \left| \frac{\partial f}{\partial u} \right|^2 duds \\ & + \int_0^L \int_{-a}^a V(s, u) |f|^2 \, dsdu - \beta \int_0^L |f(s, 0)|^2 \, ds \quad \text{for} \quad f \in Q^+_a, \end{aligned}$$

$$\begin{aligned} b^-_{a,\beta}(f, f) = & \int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^a \left| \frac{\partial f}{\partial u} \right|^2 duds \\ & + \int_0^L \int_{-a}^a V(s, u) |f|^2 \, dsdu - \beta \int_0^L |f(s, 0)|^2 \, ds \\ & - \frac{1}{2} \int_0^L \frac{\gamma(s)}{1 + a\gamma(s)} |f(s, a)|^2 \, ds + \frac{1}{2} \int_0^L \frac{\gamma(s)}{1 - a\gamma(s)} |f(s, -a)|^2 \, ds \end{aligned}$$

for $f \in Q_a^-$, where

$$V(s,u) = \frac{1}{2}(1+u\gamma(s))^{-3}u\gamma''(s) - \frac{5}{4}(1+u\gamma(s))^{-4}u^2\gamma'(s)^2 - \frac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2.$$

Let $B_{a,\beta}^+$ and $B_{a,\beta}^-$ be the self-adjoint operators associated with the forms $b_{a,\beta}^+$ and $b_{a,\beta}^-$, respectively. Then we have the following result.

LEMMA 2.2. *The operators $B_{a,\beta}^+$ and $B_{a,\beta}^-$ are unitarily equivalent to $L_{a,\beta}^+$ and $L_{a,\beta}^-$, respectively.*

Proof. We prove the assertion only for $B_{a,\beta}^-$ because that for $B_{a,\beta}^+$ is similar. Given $f \in L^2(\Sigma_a)$, we define

$$(U_a f)(s,u) = (1+u\gamma(s))^{1/2} f(\Phi_a(s,u)), \quad (s,u) \in (0,L)\times(-a,a). \tag{2.5}$$

From Lemma 2.1, we infer that U_a is a unitary operator from $L^2(\Sigma_a)$ to $L^2((0,L)\times(-a,a))$. Since Γ is a closed C^4 Jordan curve, U_a is a bijection from $H^1(\Sigma_a)$ to Q_a^-. Using an integration by parts, we obtain

$$\begin{aligned} &q_{a,\beta}^-(f,g) - b_{a,\beta}^-(U_a f, U_a g) \\ &= -\frac{1}{2}\int_{-a}^{a} \left[(1+u\gamma(s))^{-3}\gamma'(s)(U_a f)(s,u)\overline{(U_a g)(s,u)}\right]_{s=-a}^{s=a} du. \end{aligned}$$

Since $U_a f$ and $U_a g$ as elements of Q_a^- satisfy the periodicity condition, we get

$$q_{a,\beta}^-(f,g) = b_{a,\beta}^-(U_a f, U_a g) \quad \text{for} \quad f,\, g \in H^1(\Sigma_a).$$

This together with the first representation theorem (see [6, Theorem VI.2.1]) implies that

$$U_a^* B_{a,\beta}^- U_a = L_{a,\beta}^-.$$

This completes the proof of the lemma. □

Next we estimate $B_{a,\beta}^+$ and $B_{a,\beta}^-$ by operators with separated variables. We put

$$\gamma'_+ = \max_{[0,L]} |\gamma'(\cdot)|, \quad \gamma''_+ = \max_{[0,L]} |\gamma''(\cdot)|,$$

$$V_+(s) = \frac{1}{2}(1-a\gamma_+)^{-3}a\gamma''_+ - \frac{5}{4}(1+a\gamma_+)^{-4}a^2(\gamma'_+)^2 - \frac{1}{4}(1+a\gamma_+)^{-2}\gamma(s)^2,$$

$$V_-(s) = -\frac{1}{2}(1-a\gamma_+)^{-3}a\gamma''_+ - \frac{5}{4}(1-a\gamma_+)^{-4}a^2(\gamma'_+)^2 - \frac{1}{4}(1-a\gamma_+)^{-2}\gamma(s)^2.$$

If $0 < a < \frac{1}{2}\gamma_+$, we can define

$$\begin{aligned} \tilde{b}_{a,\beta}^+(f,f) =& (1-a\gamma_+)^{-2}\int_0^L\int_{-a}^{a}\left|\frac{\partial f}{\partial s}\right|^2 duds + \int_0^L\int_{-a}^{a}\left|\frac{\partial f}{\partial u}\right|^2 duds \\ &+ \int_0^L\int_{-a}^{a} V_+(s)|f|^2\, duds - \beta\int_0^L |f(s,0)|^2\, ds \quad \text{for} \quad f \in Q_a^+, \end{aligned}$$

$$\begin{aligned}\tilde{b}^-_{a,\beta}(f,f) =&(1+a\gamma_+)^{-2}\int_0^L\int_{-a}^a\left|\frac{\partial f}{\partial s}\right|^2 duds+\int_0^L\int_{-a}^a\left|\frac{\partial f}{\partial u}\right|^2 duds\\&+\int_0^L\int_{-a}^a V_-(s)|f|^2\,duds-\beta\int_0^L|f(s,0)|^2\,ds\\&-\gamma_+\int_0^L(|f(s,a)|^2+|f(s,-a)|^2)\,ds\quad\text{for}\quad f\in Q^-_a.\end{aligned}$$

Then we have

$$b^+_{a,\beta}(f,f)\leqslant\tilde{b}^+_{a,\beta}(f,f)\quad\text{for}\quad f\in Q^+_a,\tag{2.6}$$

$$\tilde{b}^-_{a,\beta}(f,f)\leqslant b^-_{a,\beta}(f,f)\quad\text{for}\quad f\in Q^-_a.\tag{2.7}$$

Let $\tilde{H}^+_{a,\beta}$ and $\tilde{H}^-_{a,\beta}$ be the self-adjoint operators associated with the forms $\tilde{b}^+_{a,\beta}$ and $\tilde{b}^-_{a,\beta}$ respectively. Let $T^+_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^+_{a,\beta}(f,f)=\int_{-a}^a|f'(u)|^2\,du-\beta|f(0)|^2,\quad f\in H^1_0((-a,a)).$$

Let finally $T^-_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^-_{a,\beta}(f,f)=\int_{-a}^a|f'(u)|^2\,du-\beta|f(0)|^2-\gamma_+(|f(a)|^2+|f(-a)|^2),\quad f\in H^1((-a,a)).$$

We define

$$U^+_a=-(1-a\gamma_+)^{-2}\frac{d^2}{ds^2}+V_+(s)\quad\text{in}\quad L^2((0,L))\quad\text{with the domain}\quad P,$$

$$U^-_a=-(1+a\gamma_+)^{-2}\frac{d^2}{ds^2}+V_-(s)\quad\text{in}\quad L^2((0,L))\quad\text{with the domain}\quad P.$$

Then we have

$$\begin{aligned}\tilde{H}^+_{a,\beta}&=U^+_a\otimes 1+1\otimes T^+_{a,\beta},\\\tilde{H}^-_{a,\beta}&=U^-_a\otimes 1+1\otimes T^-_{a,\beta}.\end{aligned}\tag{2.8}$$

Next we consider the asymptotic behaviour of each eigenvalue of $U^\pm_a$ as a tends to zero. Let $\mu^\pm_j(a)$ be the jth eigenvalue of $U^\pm_a$ counted with multiplicity. The following proposition is needed to prove Theorem 2 as well as Theorem 1.

Proposition 2.3. *There exists $C_1>0$ such that*

$$|\mu^+_j(a)-\mu_j|\leqslant C_1aj^2\tag{2.9}$$

and

$$|\mu^-_j(a)-\mu_j|\leqslant C_1aj^2\tag{2.10}$$

for $j \in \mathbb{N}$ *and* $0 < a < \frac{1}{2\gamma_+}$, *where* C_1 *is independent of* j, a.

Proof. We define

$$S_0 = -\frac{d^2}{ds^2} \quad \text{in} \quad L^2((0,L)) \quad \text{with the domain} \quad P.$$

Notice that the jth eigenvalue of S_0 counted with multiplicity is $4[\frac{j}{2}]^2(\frac{\pi}{L})^2$. Since

$$\|S - S_0\|_{\mathcal{B}(L^2((0,L)))} \leqslant \frac{1}{4}\gamma_+^2,$$

the min-max principle (see [7, Theorem XIII.2]) implies that

$$\left|\mu_j - 4[j/2]^2(\pi/2)^2\right| \leqslant \frac{1}{4}\gamma_+^2 \quad \text{for} \quad j \in \mathbb{N}. \tag{2.11}$$

Since

$$\begin{aligned} U_a^+ - (1-a\gamma_+)^{-2}S =& \frac{1}{2}(1-a\gamma_+)^{-3}a\gamma_+'' - \frac{5}{4}(1+a\gamma_+)^{-4}a^2(\gamma_+')^2 \\ &+ a\gamma_+(1+a\gamma_+)^{-2}(1-a\gamma_+)^{-2}\gamma(s)^2, \end{aligned}$$

we infer that there exists $C_0 > 0$ such that

$$\|U_a^+ - (1-a\gamma_+)^{-2}S\|_{\mathcal{B}(L^2((0,L)))} \leqslant C_0 a \quad \text{for} \quad 0 < a < \frac{1}{2\gamma_+}.$$

This together with the min-max principle implies that

$$|\mu_j^+(a) - (1-a\gamma_+)^{-2}\mu_j| \leqslant C_0 a \quad \text{for} \quad 0 < a < \frac{1}{2\gamma_+}.$$

Hence we get

$$|\mu_j^+(a) - \mu_j| \leqslant C_0 a + \frac{a\gamma_+(2-a\gamma_+)}{(1-a\gamma_+)^2}|\mu_j|.$$

Combining this with (2.11) we arrive at (2.9).

The proof of (2.10) is similar. □

Next we estimate the first eigenvalue of $T_{a,\beta}^+$.

Proposition 2.4. *Assume that* $\beta a > \frac{8}{3}$. *Then* $T_{a,\beta}^+$ *has only one negative eigenvalue, which we denote by* $\zeta_{a,\beta}^+$. *It satisfies the inequalities*

$$-\frac{1}{4}\beta^2 < \zeta_{a,\beta}^+ < -\frac{1}{4}\beta^2 + 2\beta^2 \exp\left(-\frac{1}{2}\beta a\right).$$

Proof. Let $k > 0$. We will show that $-k^2$ is an eigenvalue of $T_{a,\beta}^+$ if and only if

$$g_{a,\beta}(k) := \log(\beta - 2k) - \log(\beta + 2k) + 2ka = 0.$$

Assume that $-k^2$ is an eigenvalue of $T^+_{a,\beta}$. Notice that

$$\begin{aligned}\mathcal{D}(T^+_{a,\beta}) = \{\varphi \in H^1_0((-a,a));\quad & \varphi|_{(0,a)} \in H^2((0,a)),\\ & \varphi|_{(-a,0)} \in H^2((-a,0)),\\ & \varphi'(+0) - \varphi'(-0) = -\beta\varphi(0)\}.\end{aligned}$$

Let a nonzero ψ be the eigenfunction of $T^+_{a,\beta}$ associated with the eigenvalue $-k^2$, then we have

(**i**) $-\psi''(u) = -k^2\psi(u)$ on $(-a,0)\cup(0,a)$.
(**ii**) $\psi(\pm a) = 0$.
(**iii**) $\psi'(+0) - \psi'(-0) = -\beta\psi(0)$.

Since $T^+_{a,\beta}$ commutes with the parity operator $f(x) \mapsto f(-x)$, the ground state ψ satisfies $\psi(u) = \psi(-u)$ on $[0,a]$. Combining this with (**i**), we infer that ψ is of the form

$$\psi(u) = \begin{cases} C_1 e^{ku} + C_2 e^{-ku}, & u \in (0,a),\\ C_2 e^{ku} + C_1 e^{-ku}, & u \in (-a,0).\end{cases} \tag{2.12}$$

Note that (**ii**) is equivalent to

$$C_2 = -C_1 e^{2ka}.$$

In addition, (**iii**) is equivalent to

$$(2k+\beta)C_1 - (2k-\beta)C_2 = 0.$$

Thus the equation for C_1 and C_2 becomes

$$\begin{pmatrix} 2k+\beta & -(2k-\beta)\\ e^{2ka} & 1\end{pmatrix}\begin{pmatrix} C_1\\ C_2\end{pmatrix} = 0. \tag{2.13}$$

Since $(C_1, C_2) \neq (0,0)$, we get

$$\det\begin{pmatrix} 2k+\beta & -(2k-\beta)\\ e^{2ka} & 1\end{pmatrix} = 0$$

which is equivalent to $g_{a,\beta}(k) = 0$.

To check the converse, assume that $g_{a,\beta}(k) = 0$. Then (2.13) has a solution $(C_1, C_2) \neq (0,0)$. It is easy to see that the function ψ from (2.12) satisfies (**i**)-(**iii**) and $\psi \in \mathcal{D}(T^+_{a,\beta})$.

Let us show that $g_{a,\beta}(\cdot)$ has a unique zero in $(0, \beta/4)$. We have $g_{a,\beta}(0) = 0$. Since

$$\frac{d}{dk} g_{a,\beta}(k) = \frac{-4\beta}{\beta^2 - 4k^2} + 2a,$$

we claim that $g_{a,\beta}(\cdot)$ is monotone increasing on $(0, \frac{1}{2}\sqrt{\beta^2 - \frac{2\beta}{a}})$ and is monotone decreasing on $(\frac{1}{2}\sqrt{\beta^2 - \frac{2\beta}{a}}, \frac{1}{2}\beta)$. Moreover, we have

$$\lim_{k\to\frac{\beta}{2}-0} g_{a,\beta}(k) = -\infty.$$

Hence the function $g_{a,\beta}(\cdot)$ has a unique zero in $(0, \beta/2)$. Since $a\beta > \frac{8}{3}$, we have $\frac{1}{2}\sqrt{\beta^2 - \frac{2\beta}{a}} \geqslant \frac{\beta}{4}$. Consequently, the solution k has the form $k = \frac{\beta}{2} - s$, $0 < s \leqslant \frac{\beta}{4}$. Taking into account the relation $g_{a,\beta}(k) = 0$, we get

$$\log 2s = \log(2\beta - s) - \beta a + 2as < \log 2\beta - \frac{1}{2}a\beta.$$

So we obtain $s < \beta \exp(-\frac{1}{2}a\beta)$. This completes the proof of Proposition 2.4. □

Next we estimate the first eigenvalue of $T^-_{a,\beta}$.

Proposition 2.5. *Let $a\beta > 8$ and $\beta > \frac{8}{3}\gamma_+$. Then $T^-_{a,\beta}$ has a unique negative eigenvalue $\zeta^-_{a,\beta}$, and moreover, we have*

$$-\frac{1}{4}\beta^2 - \frac{2205}{16}\beta^2 \exp\left(-\frac{1}{2}\beta a\right) < \zeta^-_{a,\beta} < -\frac{1}{4}\beta^2.$$

Proof. Let us first show that $T^-_{a,\beta}$ has a unique negative eigenvalue. Let $k > 0$. As in the proof of Proposition 2.4, we infer that $-k^2$ is an eigenvalue of $T^-_{a,\beta}$ if and only if

$$\frac{ke^{ka} - \gamma_+}{ke^{-ka} + \gamma_+} = \frac{2k + \beta}{2k - \beta}. \tag{2.14}$$

Since the left side of (2.14) is positive for $k \geqslant \gamma_+$ and the right side of (2.14) is negative for $0 < k < \frac{\beta}{2}$, (2.14) has no solution in $[\gamma_+, \frac{\beta}{2})$. We put

$$g(k) = \frac{ke^{ka} - \gamma_+}{ke^{-ka} + \gamma_+} \quad \text{and} \quad h(k) = \frac{2k + \beta}{2k - \beta}.$$

Then we get $\lim_{k\to\infty} g(k) = \infty$ and

$$g'(k) = \frac{\gamma_+(e^{ka} - e^{-ka}) + 2k^2 a + ka\gamma_+(e^{ka} + e^{-ka})}{(ke^{-ka} + \gamma_+)^2} > 0 \quad \text{for} \quad k > 0.$$

Thus $g(k)$ is monotone increasing on $(0, \infty)$. On the other hand, $h(k)$ is monotone decreasing on $(\beta/2, \infty)$,

$$\lim_{k\to\frac{\beta}{2}+0} h(k) = \infty, \quad \lim_{k\to\infty} h(k) = 1.$$

Hence (2.14) has a unique solution in $(\beta/2, \infty)$. Since $h(k)$ is monotone decreasing on $(0, \beta/2)$ and $g(0) = h(0)$, we claim that (2.14) has no solution in $(0, \beta/2)$.

Next we show that $g(k) > \frac{2k+\beta}{2k-\beta}$ for $k \geqslant \frac{3}{4}\beta$. We have $\frac{2k+\beta}{2k-\beta} \leqslant 5$ for $k \geqslant \frac{3}{4}\beta$. For $k \geqslant \frac{3}{4}\beta$, we get

$$\begin{aligned} g(k) &\geqslant g\left(\frac{3}{4}\beta\right) \\ &= \frac{\frac{3}{4}\beta\exp(\frac{3}{4}a\beta) - \gamma_+}{\frac{3}{4}\beta\exp(-\frac{3}{4}a\beta) + \gamma_+} \end{aligned}$$

since $\gamma_+ < \frac{3}{8}\beta < \frac{3}{8}\beta\exp(\frac{3}{4}a\beta)$

$$\begin{aligned}
&\geqslant \frac{\frac{3}{8}\beta\exp(\frac{3}{4}a\beta)}{\frac{3}{4}\beta\exp(-\frac{3}{4}a\beta)+\frac{3}{8}\beta} \\
&= \frac{\exp(\frac{3}{4}a\beta)}{2\exp(-\frac{3}{4}a\beta)+1}
\end{aligned}$$

since $a\beta > 8$

$$\begin{aligned}
&\geqslant \frac{e^6}{2e^{-6}+1} \\
&> 5.
\end{aligned}$$

So (2.14) has no solution in $[\frac{3}{4}\beta,\infty)$. Hence, the solution k of (2.14) is of the form $k=\frac{\beta}{2}+s$, $0<s<\frac{1}{4}\beta$. From (2.14), we get

$$\begin{aligned}
\frac{5\beta}{4s} &\geqslant \frac{2k+\beta}{2k-\beta} \\
&= \frac{ke^{ka}-\gamma_+}{ke^{-ka}+\gamma_+}
\end{aligned}$$

since $\gamma_+ < \frac{3}{8}\beta < \frac{3}{8}\beta\exp(\frac{1}{2}\beta a)$ and $ke^{ka} \geqslant \frac{1}{2}\beta\exp(\frac{1}{2}\beta a)$

$$\geqslant \frac{\frac{1}{8}\beta\exp(\frac{1}{2}\beta a)}{ke^{-ka}+\gamma_+}$$

since $ke^{-ka} < k < \frac{3}{4}\beta$ and $\gamma_+ < \frac{3}{8}\beta$

$$\begin{aligned}
&\geqslant \frac{\frac{1}{8}\beta\exp(\frac{1}{2}\beta a)}{\frac{9}{8}\beta} \\
&= \frac{1}{9}\exp(\tfrac{1}{2}\beta a).
\end{aligned}$$

Thus we get $s \leqslant \frac{45}{4}\beta\exp(-\frac{1}{2}\beta a)$, which gives $k^2 \geqslant \frac{\beta^2}{4}$ and

$$\begin{aligned}
k^2 =&\frac{\beta^2}{4}+\beta s+s^2 \\
\leqslant&\frac{\beta^2}{4}+\frac{45}{4}\beta^2\exp\left(-\frac{1}{2}\beta a\right)+\left(\frac{45}{4}\right)^2\beta^2\exp\left(-\beta a\right) \\
\leqslant&\frac{\beta^2}{4}+\frac{45}{4}\beta^2\exp\left(-\frac{1}{2}\beta a\right)+\left(\frac{45}{4}\right)^2\beta^2\exp\left(-\frac{1}{2}\beta a\right) \\
=&\frac{\beta^2}{4}+\frac{2205}{16}\exp\left(-\frac{1}{2}\beta a\right).
\end{aligned}$$

This completes the proof of Proposition 2.5. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We put $a(\beta) = 6\beta^{-1}\log\beta$. Let $\xi^{\pm}_{\beta,j}$ be the jth eigenvalue of $T^{\pm}_{a(\beta),\beta}$. From Propositions 2.4 and 2.5, we have

$$\xi^{\pm}_{\beta,1} = \zeta^{\pm}_{a(\beta),\beta} \quad \text{and} \quad \xi^{\pm}_{\beta,2} \geqslant 0.$$

From (2.8), we infer that $\{\xi^{\pm}_{\beta,j} + \mu^{\pm}_k(a(\beta))\}_{j,k\in\mathbb{N}}$ is a sequence of all eigenvalues of $\tilde{H}^{\pm}_{a(\beta),\beta}$ counted with multiplicity. From Proposition 2.3, we have

$$\xi^{\pm}_{\beta,j} + \mu^{\pm}_k(a(\beta)) \geqslant \mu^{\pm}_1(a(\beta)) = \mu_1 + \mathcal{O}(\beta^{-1}\log\beta) \tag{2.15}$$

for $j \geqslant 2$ and $k \geqslant 1$. For $j \in \mathbb{N}$, we define

$$\tau^{\pm}_{\beta,j} = \zeta^{\pm}_{a(\beta),\beta} + \mu^{\pm}_j(a(\beta)). \tag{2.16}$$

From Propositions 2.3–2.5, we get

$$\tau^{\pm}_{\beta,j} = -\frac{1}{4}\beta^2 + \mu_j + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty. \tag{2.17}$$

Let $n \in \mathbb{N}$. Combining (2.15) with (2.17), we claim that there exists $\beta(n) > 0$ such that

$$\tau^{+}_{\beta,n} < 0, \quad \tau^{+}_{\beta,n} < \xi^{+}_{\beta,j} + \mu^{+}_k(a(\beta)), \quad \text{and} \quad \tau^{-}_{\beta,n} < \xi^{-}_{\beta,j} + \mu^{-}_k(a(\beta))$$

for $\beta \geqslant \beta(n)$, $j \geqslant 2$, and $k \geqslant 1$. Hence the jth eigenvalue of $\tilde{H}^{\pm}_{a(\beta),\beta}$ counted with multiplicity is $\tau^{\pm}_{\beta,j}$ for $j \leqslant n$ and $\beta \geqslant \beta(n)$. Let $\beta \geqslant \beta(n)$ and denote by $\kappa^{\pm}_j(\beta)$ the jth eigenvalue of $L^{\pm}_{a(\beta),\beta}$. From (2.4), (2.6), and the min–max principle we obtain

$$\tau^{-}_{\beta,j} \leqslant \kappa^{-}_j(\beta) \quad \text{and} \quad \kappa^{+}_j(\beta) \leqslant \tau^{+}_{\beta,j} \quad \text{for} \quad 1 \leqslant j \leqslant n, \tag{2.18}$$

so we have $\kappa^{+}_n(\beta) < 0$. Hence the min–max principle and (2.4) imply that H_β has at least n eigenvalues in $(-\infty, \kappa^{+}_n(\beta))$. For $1 \leqslant j \leqslant n$, we denote by $\lambda_j(\beta)$ the jth eigenvalue of H_β. We have

$$\kappa^{-}_j(\beta) \leqslant \lambda_j(\beta) \leqslant \kappa^{+}_j(\beta) \quad \text{for} \quad 1 \leqslant j \leqslant n.$$

This together with (2.17) and (2.18) implies that

$$\lambda_j(\beta) = -\frac{1}{4}\beta^2 + \mu_j + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty \quad \text{for} \quad 1 \leqslant j \leqslant n.$$

This completes the proof of Theorem 1. $\square$

3. Proof of Theorem 2

For a self-adjoint operator A, we define

$$N^-(A) = \sharp\{\sigma_{\rm d}(A) \cap (-\infty, 0)\}.$$

From (2.4), we have $N^-(L^-_{a,\beta}) \geqslant \sharp\sigma_{\rm d}(H_\beta) \geqslant N^-(L^+_{a,\beta})$. On the other hand, Lemma 2.2, (2.6), and (2.7) imply that $N^-(\tilde{H}^-_{a,\beta}) \geqslant N^-(L^-_{a,\beta})$ and $N^-(L^+_{a,\beta}) \geqslant N^-(\tilde{H}^+_{a,\beta})$. In this way we get

$$N^-(\tilde{H}^+_{a,\beta}) \leqslant \sharp\sigma_{\rm d}(H_\beta) \leqslant N^-(\tilde{H}^-_{a,\beta}). \tag{3.1}$$

Recall the relation (2.16). We define

$$K^\pm_\beta = \{j \in \mathbb{N};\quad \tau^\pm_{\beta,j} < 0\}$$

and use the following proposition to estimate $N^-(\tilde{H}^\pm_{a,\beta})$.

Proposition 3.1. *We have*

$$\sharp K^\pm_\beta = \frac{L}{2\pi}\beta + \mathcal{O}(\log\beta) \quad as \quad \beta \to \infty.$$

Proof. We choose $C_2 > 0$ such that $-\frac{1}{4}C_2^2 \leqslant -1 - \frac{1}{4}\gamma_+^2$. Let $\beta \geqslant \max\{2, C_2\}$. Then we have $\frac{1}{4}(\beta - C_2)^2 < \frac{1}{4}\beta^2 - 1 - \frac{1}{4}\gamma_+^2$. We get

$$K^+_\beta = \{j \in \mathbb{N};\quad \mu^+_j(a(\beta)) < -\zeta^+_{a(\beta),\beta}\}$$

by using Propositions 2.3 and 2.4

$$\supset \{j \in \mathbb{N};\quad \mu_j + C_1 a(\beta) j^2 < \tfrac{1}{4}\beta^2 - 2\beta^2\exp(-\tfrac{1}{2}\beta a(\beta))\}$$

since $\mu_j \leqslant [j/2]^2(\pi/L)^2 + \frac{1}{4}\gamma_+^2$

$$\supset \{j \in \mathbb{N};\quad 4[j/2]^2(\pi/L)^2 + C_1(\beta^{-1}\log\beta)j^2 < \tfrac{1}{4}\beta^2 - \tfrac{2}{\beta} - \tfrac{1}{4}\gamma_+^2\}$$

since $\beta \geqslant 2$

$$\begin{aligned} &\supset \{j \in \mathbb{N};\quad j^2(\pi/L)^2 + C_1(\beta^{-1}\log\beta)j^2 < \tfrac{1}{4}\beta^2 - 1 - \tfrac{1}{4}\gamma_+^2\} \\ &\supset \{j \in \mathbb{N};\quad j^2(\pi/L)^2 + C_1(\beta^{-1}\log\beta)j^2 \leqslant \tfrac{1}{4}(\beta - C_2)^2\} \\ &= \{j \in \mathbb{N};\quad j \leqslant \tfrac{1}{2}(\beta - C_2)((\pi/L)^2 + C_1\beta^{-1}\log\beta)^{-1/2}\}. \end{aligned}$$

Furthermore, from

$$\tfrac{1}{2}(\beta - C_2)((\pi/L)^2 + C_1\beta^{-1}\log\beta)^{-1/2} = \tfrac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta \to \infty,$$

we infer that

$$\sharp K^+_\beta \geqslant \tfrac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta\to\infty. \tag{3.2}$$

Similarly we get

$$\begin{aligned} K^-_\beta =& \{j\in\mathbb{N};\quad \mu^-_j(a(\beta)) < -\zeta^-_{a(\beta),\beta}\} \\ \supset& \{j\in\mathbb{N};\quad \mu_j - C_1 a(\beta) j^2 < \tfrac14\beta^2 + \tfrac{2205}{4\beta} + \tfrac14\gamma_+^2\} \end{aligned}$$

since $2(j-1)\geqslant j$ for $j\geqslant 2$

$$\begin{aligned} =& \{1\}\cup\{j\geqslant 2;\quad (j-1)^2(\pi/L)^2 - 4C_1(\beta^{-1}\log\beta)(j-1)^2 < \tfrac14\beta^2 + \tfrac{2205}{4\beta} + \tfrac14\gamma_+^2\} \\ =& \{1\}\cup\{j\geqslant 2;\quad j < 1 + (\tfrac14\beta^2 + \tfrac{2205}{4\beta} + \tfrac14\gamma_+^2)((\pi/L)^2 - 4C_1\beta^{-1}\log\beta)^{-1/2}\}. \end{aligned}$$

However,

$$1 + (\tfrac14\beta^2 + \tfrac{2205}{4\beta} + \tfrac14\gamma_+^2)((\pi/L)^2 - 4C_1\beta^{-1}\log\beta)^{-1/2} = \tfrac{L\beta}{2\pi} + \mathcal{O}(\log\beta)$$

as $\beta\to\infty$, which leads to

$$\sharp K^-_\beta \leqslant \tfrac{L\beta}{2\pi} + \mathcal{O}(\log\beta) \quad \text{as} \quad \beta\to\infty. \tag{3.3}$$

Since $\tau^-_{\beta,j} < \tau^+_{\beta,j}$, we get $K^-_\beta \supset K^+_\beta$. Combining this with (3.2) and (3.3), we get the assertion of Proposition 3.1. $\square$

We also need the following result to estimate the second eigenvalue of $T^-_{a,\beta}$.

Proposition 3.2. *Let $0 < a < \frac{1}{\sqrt{2}\gamma_+}$ and $\beta > 0$. Then $T^-_{a,\beta}$ has no eigenvalue in $[0, \min\{\frac{\pi^2}{16a^2}, \frac{\beta\gamma_+}{2}, \beta^2\})$.*

Proof. Let $k>0$. As in the proof of Proposition 2.4, we infer that k^2 is an eigenvalue of $T^-_{a,\beta}$ if and only if k solves either

$$\tan ka = \tfrac{k}{\gamma_+} \tag{3.4}$$

or

$$\tan ka = \tfrac{\beta + 2k\gamma_+}{\beta\gamma_+ - 2k^2}\beta. \tag{3.5}$$

For $k\in(0,\frac{\pi}{4a})$, we have

$$\tan ka < \sqrt{2}\sin ka < \sqrt{2}ka < \tfrac{k}{\gamma_+}. \tag{3.6}$$

Thus (3.4) has no solution in $(0,\frac{\pi}{4a})$. For $k\in(0,\min\{\frac{\pi}{4a}, \frac{\sqrt{\beta\gamma_+}}{\sqrt{2}}, \beta\})$, we have

$$\frac{\beta + 2k\gamma_+}{\beta\gamma_+ - 2k^2}\beta - \frac{k}{\gamma_+} = \frac{\beta\gamma_+(\beta-\gamma_+) + 2k(\gamma_+)^2\beta + 2k^3}{(\beta\gamma_+ - 2k^2)\gamma_+} > 0.$$

This together with (3.6) implies that (3.5) has no solution in $(0, \min\{\frac{\pi}{4a}, \frac{\sqrt{\beta\gamma_+}}{\sqrt{2}}, \beta\})$. Consequently, $T^-_{a,\beta}$ has no eigenvalue in $(0, \min\{\frac{\pi^2}{16a^2}, \frac{\beta\gamma_+}{2}, \beta^2\})$.

Next we show that 0 is not an eigenvalue of $T^-_{a,\beta}$. As in the proof of Proposition 2.4, we infer that 0 is an eigenvalue of $T^-_{a,\beta}$ if and only if either $\gamma_+ a = 1$ or $\beta(\gamma_+ a - 1) = 2\gamma_+$ holds. Since $0 < a < \frac{1}{\sqrt{2}\gamma_+}$ and $\beta > 0$, we have $\gamma_+ a < 1$ and $\beta(\gamma_+ a - 1) < 2\gamma_+$. Hence 0 is not an eigenvalue of $T^-_{a,\beta}$, and the proof is complete. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Let us first show that

$$N^-(\tilde{H}^-_{a(\beta),\beta}) = \sharp K^-_\beta \quad \text{for sufficiently large} \quad \beta > 0. \tag{3.7}$$

Recall that $\{\xi^-_{\beta,j} + \mu^-_k(a(\beta))\}_{j,k\in\mathbb{N}}$ is a sequence of all eigenvalues of $\tilde{H}^-_{a(\beta),\beta}$ counted with multiplicity. From Proposition 3.2, we have

$$\xi^-_{\beta,2} \geqslant \min\left\{\frac{\pi}{4a(\beta)}, \frac{\sqrt{\beta\gamma_+}}{\sqrt{2}}, \beta\right\}.$$

This together with (2.10) implies that there exists $\beta_0 > 0$ such that $\xi^-_{\beta,2} + \mu^-_1(a(\beta)) > 0$ for $\beta \geqslant \beta_0$. We obtain

$$\xi^-_{\beta,j} + \mu^-_k(a(\beta)) > 0 \quad \text{for} \quad j \geqslant 2, \quad k \geqslant 1, \quad \text{and} \quad \beta \geqslant \beta_0.$$

Thus we get

$$\begin{aligned} N^-(\tilde{H}^-_{a(\beta),\beta}) =& \sharp\{(j,k) \in \mathbb{N}^2; \quad \xi^-_{\beta,j} + \mu^-_k(a(\beta)) < 0\} \\ =& \sharp\{j \in \mathbb{N}; \quad \tau^-_{\beta,j} < 0\} \\ =& \sharp K^-_\beta \quad \text{for} \quad \beta \geqslant \beta_0. \end{aligned}$$

In this way we obtain (3.7). From (3.1), we get

$$\sharp K^+_\beta \leqslant \sharp\sigma_{\mathrm{d}}(H_\beta) \leqslant N^-(\tilde{H}^-_{a(\beta),\beta}).$$

This together with (3.7) and Proposition 3.1 implies the assertion of Theorem 2. □

Remark 3.3. We can also prove (1.5) in the case that γ is a C^4 curve which is not self-intersecting. Indeed, it suffices to use the following operators $\hat{H}^\pm_{a,\beta}$ instead of $\tilde{H}_{a,\beta} = U^\pm_a \otimes 1 + 1 \otimes T^\pm_{a,\beta}$:

$$\hat{H}_{a,\beta} := \hat{U}^\pm_a \otimes 1 + 1 \otimes T^\pm_{a,\beta} \quad \text{in} \quad L^2((0,L)) \otimes L^2((-a,a)) = L^2((0,L) \times (-a,a)),$$

$$\hat{U}^+_a := -(1 - a\gamma_+)^{-2}\frac{d^2}{ds^2} + V_+(s) \quad \text{in} \quad L^2((0,L))$$

with the Dirichlet boundary condition,

$$\hat{U}_a^- := -(1 + a\gamma_+)^{-2} \frac{d^2}{ds^2} + V_-(s) \quad \text{in} \quad L^2((0, L))$$

with the Neumann boundary condition.

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