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Positive Representations of Split Real Quantum Groups: The Universal R Operator

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The universal R operator for the positive representations of split real quantum groups is computed, generalizing the formula of compact quantum groups $\mathcal{U}_q(\mathfrak{g})$ by Kirillov–Reshetikhin and Levendorskiĭ–Soibelman, and the formula in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by Faddeev, Kashaev, and Bytsko-Teschner. Several new functional relations of the quantum dilogarithm are obtained, generalizing the quantum exponential relations and the pentagon relations. The quantum Weyl element and Lusztig's isomorphism in the positive setting are also studied in detail. Finally, we introduce a C^*-algebraic version of the split real quantum group in the language of multiplier Hopf algebras, and consequently the definition of R is made rigorous as the canonical element of the Drinfeld's double $\mathbf{U}$ of certain multiplier Hopf algebra $\mathbf{Ub}$. Moreover, a ribbon structure is introduced for an extension of $\mathbf{U}$.

1 Introduction

In this paper, we construct the universal R operator for the positive representations of split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$, generalizing the formula of the R operator in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by Faddeev [7], Kashaev [14], and Bytsko–Teschner [1], as well as the universal R matrix computed independently by Kirillov–Reshetikhin [16] and

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Levendorskiĭ–Soibelman [19] for compact quantum group $\mathcal{U}_q(\mathfrak{g})$ associated to simple Lie algebra $\mathfrak{g}$ of all type.

The notion of the *positive principal series representations*, or simply *positive representations*, was introduced in [9] as a new research program devoted to the representation theory of split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$. It uses the concept of modular double for quantum groups [6, 7], and has been studied for $U_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by Teschner et al. [1, 22, 23]. Explicit construction of the positive representations $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ associated to a simple Lie algebra $\mathfrak{g}$ has been obtained for the simply laced case in [11] and non-simply laced case in [12], where the generators of the quantum groups are realized by positive essentially self-adjoint operators. Furthermore, the so-called *transcendental relations* of the (rescaled) generators:

$$\tilde{\mathbf{e}}_i = \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \tilde{\mathbf{f}}_i = \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \tilde{K}_i = K_i^{\frac{1}{b_i^2}} \tag{1.1}$$

give the self-duality between different parts of the modular double, while in the non-simply laced case, new explicit analytic relations between the quantum group and its Langland's dual have been observed [12].

Motivated by the detailed study in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by Teschner *et al.*, a natural problem is to find the universal R matrix so that it gives a braiding of the positive representations $\mathcal{P}_\lambda$ of the split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$. Since positive representations are infinite-dimensional, instead of acting by a "matrix", a natural setting will be realizing R as a *unitary operator* acting on $\mathcal{P}_{\lambda_1}\otimes\mathcal{P}_{\lambda_2}$ such that the usual properties are satisfied:

(1) *Braiding relation*:

$$\Delta'(X)R := (\sigma\circ\Delta)(X)R = R\Delta(X), \quad \sigma(x\otimes y) = y\otimes x. \tag{1.2}$$

(2) *Quasi-triangularity*:

$$(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}, \tag{1.3}$$

$$(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}. \tag{1.4}$$

Here the coproduct Δ acts on R in a natural way on the generators, and we have also used the standard leg notation. These together imply the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.5}$$

The expression of R in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is particularly simple, and is given by

$$R = q^{\frac{H\otimes H}{4}} g_b(\mathbf{e}\otimes\mathbf{f}) q^{\frac{H\otimes H}{4}}, \tag{1.6}$$

where

$$E = \frac{\mathbf{i}}{q-q^{-1}}\mathbf{e}, \quad F = \frac{\mathbf{i}}{q-q^{-1}}\mathbf{f}, \quad K = q^H \tag{1.7}$$

are the usual generators, and $g_b(x)$ is the remarkable *quantum dilogarithm function*, central to the study of split real quantum groups. See also [1] for a discussion of the "universal" aspect of this operator.

On the other hand, the universal R matrix in the compact case is given explicitly by products of the form

$$\mathbf{\Omega}^{\frac{1}{2}} \prod_\alpha \mathrm{Exp}_{q^{-2}}((1-q^{-2})E_\alpha \otimes F_\alpha)\mathbf{\Omega}^{\frac{1}{2}}, \tag{1.8}$$

where $\mathbf{\Omega} = q^{\sum (d\cdot A^{-1})_{ij} H_i\otimes H_j}$ with d such that dA is the symmetrized Cartan matrix, and q corresponds to the short root. Here, $\mathrm{Exp}_q(x)$ is the quantum exponential function, and E_α are the root vectors of $\mathfrak{g}$, given by the Lusztig's isomorphism T_k on the simple root vectors, which can be written as certain composition of q-commutators, and play a crucial role in the theory of Lusztig's canonical basis [21].

Therefore, a natural proposal will be replacing the expression (1.8) by

$$\mathbf{\Omega}^{\frac{1}{2}} \prod_\alpha g_b(\mathbf{e}_\alpha \otimes \mathbf{f}_\alpha)\mathbf{\Omega}^{\frac{1}{2}}, \tag{1.9}$$

thus generalizing both equations. More precisely, by absorbing d into the definition of q_i (cf. Definition 2.1), we have the following Main Theorem:

Main Theorem. Let $\mathfrak{g}_\mathbb{R}$ be the split real form of a simple Lie algebra $\mathfrak{g}$. Let $w_0 = s_{i_1}s_{i_2}\ldots s_{i_N}$ be a reduced expression of the longest element of the Weyl group. Then the universal R operator for the positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ acting on $\mathcal{P}_{\lambda_1}\otimes\mathcal{P}_{\lambda_2} \simeq L^2(\mathbb{R}^N)\otimes L^2(\mathbb{R}^N)$ is a unitary operator given by

$$R = \prod_{ij} q_i^{\frac{1}{2}(A^{-1})_{ij}H_i\otimes H_j} \prod_{k=1}^N g_b(\mathbf{e}_{\alpha_k}\otimes \mathbf{f}_{\alpha_k}) \prod_{ij} q_i^{\frac{1}{2}(A^{-1})_{ij}H_i\otimes H_j}, \tag{1.10}$$

where $\mathbf{e}_{\alpha_k} := T_{i_1}T_{i_2}\ldots T_{i_{k-1}}\mathbf{e}_{i_k}$ are given by the Lusztig's isomorphism in Theorem 4.9, similarly for $\mathbf{f}_{\alpha_k}$. The product is such that the term $k=1$ appears on the rightmost position. □

In particular, by the properties of the transcendental relations [11, 12] as well as the self-duality of $g_b(x)$, the universal R operator simultaneously serves as an R operator for the modular double counterpart.

The main difficulty lies in the fact that, in order for the expression (1.10) to be well-defined, we need both $\mathbf{e}_\alpha$ and $\mathbf{f}_\alpha$ to be positive essentially self-adjoint, so that we can apply functional calculus. Following the approach by Kirillov and Reshetikhin [16] and Levendorskiĭ and Soibelman [19], the main technical result is that these nonsimple basis can actually be obtained by conjugations on the generators by means of the quantum Weyl elements w_i, which is unitary in the setting of positive representations (cf. Corollary 4.10).

Theorem 1.1. The operators $\mathbf{e}_{\alpha_k}$ and $\mathbf{f}_{\alpha_k}$ corresponding to nonsimple roots are positive essentially self-adjoint under the positive representations, and satisfy the transcendental relations. □

Because of the nice properties enjoyed by the *rescaled* generators $\mathbf{e}_i$ and $\mathbf{f}_i$, we find it instructive throughout the paper to stick with these variables rather than the original E_i and F_i as defined in (1.7).

Another difficulty lies in the fact that since the representations are infinite-dimensional, we can no longer work with formal power series, and the usual Drinfeld's double construction trick does not really work anymore. Instead, using hard technical analysis, we discover explicitly certain (considerably new) functional relations (cf. Proposition 3.1–3.3) of the quantum dilogarithm function $g_b(x)$, and prove directly the braiding relations and the quasi-triangular relations of the R operator.

In order to compute the quantum Weyl elements, we have to compute the branching rules for $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))\subset\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$. It turns out that the branching rules are particularly simple, and remarkably they resemble both the decomposition of the tensor product representation $P_\alpha\otimes P_\beta$ (cf. [23]) and the Peter–Weyl-type decomposition of $L^2(\mathrm{SL}_q^+(2,\mathbb{R}))$ (cf. [10]) with exactly the same Plancherel measure. (cf. Theorem 4.7).

Theorem 1.2. Fix any positive representation $\mathcal{P}_\lambda\simeq L^2(\mathbb{R}^N)$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$. Restricting to a representation of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))\subset\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ corresponding to the simple root α_i, we have the following unitary equivalence:

$$\mathcal{P}_\lambda\simeq L^2(\mathbb{R}^{N-2})\otimes\int_{\mathbb{R}_+}P_\gamma\,\mathrm{d}\mu(\gamma), \tag{1.11}$$

where P_γ is the positive representation of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ with parameter $\gamma \in \mathbb{R}_+$, and $d\mu(\gamma) = |S_{b_i}(Q_i + 2\gamma)|^2\, d\gamma$. □

Furthermore, we also encounter the calculation of the ribbon element v, and the element u which exists for any (regular) quasi-triangular Hopf algebra. Therefore, it is strongly suggested that there is an underlying algebraic structure enveloping all the calculations so far. In particular, the expression for the universal R operator suggests that it is a canonical element of certain algebra with a "continuous basis", very similar to the analysis that has been done for the quantum plane in our previous work [10]. Therefore, we proceed to construct the split real quantum group in the C^*-algebraic setting, and show that in fact the satisfactory answer lies in the language of a *multiplier Hopf algebra*, introduced by van Daele [26]. Consequently, all the calculations made so far are rigorously defined and simplified by the following (cf. Corollary 6.15):

Theorem 1.3. The universal R operator from the Main Theorem can be considered as (the projection of) the canonical element of the Drinfeld's double (cf. [3, 4]) $\mathcal{D}(\mathbf{Ub})$ of the multiplier Hopf algebraic version of the Borel subalgebra $\mathbf{Ub}$. □

Finally, we remark that the ribbon element v calculated are also of certain interest, since the expression involves the number $Q = b + b^{-1}$, which implies that there is no classical limit as $b \to 0$. Hence, this ribbon element differs from the one usually considered in compact quantum group, and it is well known that the ribbon structure of Hopf algebra is needed to construct quantum topological invariant by the Reshetikhin–Turaev construction [24, 25]. Therefore, this may serve as evidence for the possibility of constructing new classes of topological invariants.

The paper is organized as follows. Section 2 serves as the technical backbone of the paper. We fix the notation by recalling the definition of $\mathcal{U}_q(\mathfrak{g})$ associated to simple Lie algebra $\mathfrak{g}$ of general type. Next, we recall the main properties and construction of the positive representations considered in [9, 11, 12], and write down explicitly a particular expression for the rank=2 case. Since the paper involves a lot of technical computations, we review in detail the definition and properties of the quantum dilogarithm function G_b and its variant g_b, which summarizes old and new results from [1, 10, 12] that is needed in this paper. Finally, we recall the construction of the universal R matrices by [16, 19] in the compact quantum group case, as well as the universal R operator by [1] in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.

In Section 3, we extend the quantum exponential relations and pentagon relations of $g_b(x)$ to more generalized setting involving certain q-commutators. These new functional relations are what we needed to prove the properties of the R operator. In Section 4, we proceed to construct the quantum Weyl elements so that conjugations by them realize Lusztig's isomorphism. It involves calculating the ribbon element, and the branching rules of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))\subset\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$. In Section 5, we state the main theorem about the universal R operator, and prove the braiding relations and quasi-triangularity in the simply laced case, while we only give several remarks on the nonsimply laced case to avoid getting too technical. Finally, in Section 6, we introduce the notion of a multiplier Hopf algebra, and by finding certain Hopf pairing, we show that the universal R operator can actually be regarded as the canonical element of a Drinfeld's double construction of the Borel subalgebra as a multiplier Hopf algebra, and we introduce a ribbon structure in the extension of the split real quantum group.

2 Preliminaries

Throughout the paper, we will fix once and for all $q=e^{\pi \mathbf{i} b^2}$ with $\mathbf{i}=\sqrt{-1}$, $0<b^2<1$ and $b^2\in\mathbb{R}\setminus\mathbb{Q}$. We also denote by $Q=b+b^{-1}$.

2.1 Definition of $\mathcal{U}_q(\mathfrak{g})$

In order to fix the convention we use throughout the paper, we recall the definition of the quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$, where $\mathfrak{g}$ is of general type [2]. Let $I=\{1,2,\ldots,n\}$ denote the set of nodes of the Dynkin diagram of $\mathfrak{g}$ where $n=\mathrm{rank}(\mathfrak{g})$.

Definition 2.1. Let $(-,-)$ be the inner product of the root lattice. Let α_i, $i\in I$ be the positive simple roots, and we define

$$a_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}, \tag{2.1}$$

$$q_i:=q^{\frac{1}{2}(\alpha_i,\alpha_i)}:=e^{\pi \mathbf{i} b_i^2}, \tag{2.2}$$

where $A=(a_{ij})$ is the Cartan matrix. We will let α_1 be the short root in type B_n and the long root in type C_n, F_4 and G_2.

We choose

$$\frac{1}{2}(\alpha_i,\alpha_i)=\begin{cases}1 & i \text{ is long root or in the simply laced case,}\\ \frac{1}{2} & i \text{ is short root in type } B,C,F,\\ \frac{1}{3} & i \text{ is short root in type } G_2,\end{cases} \tag{2.3}$$

and $(\alpha_i,\alpha_j)=-1$ when i, j are adjacent in the Dynkin diagram.

Therefore in the case when $\mathfrak{g}$ is of type B_n, C_n and F_4, if we define $b_l = b$, and $b_s = \frac{b}{\sqrt{2}}$ we have the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root,} \\ e^{\pi i b_s^2} = q^{\frac{1}{2}} & i \text{ is short root.} \end{cases} \tag{2.4}$$

In the case when $\mathfrak{g}$ is of type G_2, we define $b_l = b$, and $b_s = \frac{b}{\sqrt{3}}$, and we have the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root,} \\ e^{\pi i b_s^2} = q^{\frac{1}{3}} & i \text{ is short root.} \end{cases} \tag{2.5}$$

□

Definition 2.2. Let $A = (a_{ij})$ denote the Cartan matrix. Then $\mathcal{U}_q(\mathfrak{g})$ with $q = e^{\pi i b_l^2}$ is the algebra generated by E_i, F_i and $K_i^{\pm 1}$, $i \in I$ subject to the following relations:

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \tag{2.6}$$

$$K_i F_j = q_i^{-a_{ij}} F_j K_i, \tag{2.7}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.8}$$

together with the Serre relations for $i \neq j$:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}![k]_{q_i}!} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \tag{2.9}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}![k]_{q_i}!} F_i^k F_j F_i^{1-a_{ij}-k} = 0, \tag{2.10}$$

where $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$. □

To deal with operator representations, we also define H_i so that $K_i = q_i^{H_i}$, and it will be convenient to adjoin $K_i^{\frac{1}{2}}$, such that the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ is given by

$$\Delta(E_i) = K_i^{-\frac{1}{2}} \otimes E_i + E_i \otimes K_i^{\frac{1}{2}}, \tag{2.11}$$

$$\Delta(F_i) = K_i^{-\frac{1}{2}} \otimes F_i + F_i \otimes K_i^{\frac{1}{2}}, \tag{2.12}$$

$$\Delta(K_i) = K_i \otimes K_i, \tag{2.13}$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1, \tag{2.14}$$

$$S(E_i) = -q_i E_i, \quad S(F_i) = -q_i^{-1} F_i, \quad S(K_i) = K_i^{-1}. \tag{2.15}$$

We define $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ to be the real form of $\mathcal{U}_q(\mathfrak{g})$ induced by the star structure

$$E_i^* = E_i, \quad F_i^* = F_i, \quad K_i^* = K_i. \tag{2.16}$$

Finally, according to the results of [11, 12], we define the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ to be

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R}) := \mathcal{U}_q(\mathfrak{g}_\mathbb{R}) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{g}_\mathbb{R}) \quad \mathfrak{g} \text{ is simply laced}, \tag{2.17}$$

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R}) := \mathcal{U}_q(\mathfrak{g}_\mathbb{R}) \otimes \mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_\mathbb{R}) \quad \text{otherwise}, \tag{2.18}$$

where $\tilde{q} = e^{\pi i b_s^{-2}}$, and ${}^L\mathfrak{g}_\mathbb{R}$ is the Langland's dual of $\mathfrak{g}_\mathbb{R}$ obtained by interchanging the long roots and short roots of $\mathfrak{g}_\mathbb{R}$.

2.2 Positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$

In [9, 11, 12], a special class of representations for $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$, called the positive representation, is defined. The generators of the quantum groups are realized by positive essentially self-adjoint operators, and also satisfy the so-called *transcendental relations*, relating the quantum group with its modular double counterpart. More precisely, we have

Theorem 2.3. Let

$$\mathbf{e}_i := 2\sin(\pi b_i^2) E_i, \quad \mathbf{f}_i := 2\sin(\pi b_i^2) F_i. \tag{2.19}$$

Note that $2\sin(\pi b_i^2) = (\frac{i}{q_i - q_i^{-1}})^{-1} > 0$. Then there exists a representation $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ parametrized by the $\mathbb{R}_+$-span of the cone of positive weights $\lambda \in P_\mathbb{R}^+$, or equivalently by $\lambda \in \mathbb{R}_+^n$ where $n = \text{rank}(\mathfrak{g})$, such that

(1) The generators $\mathbf{e}_i, \mathbf{f}_i$, and K_i are represented by positive essentially self-adjoint operators acting on $L^2(\mathbb{R}^{l(w_0)})$, where $l(w_0)$ is the length of the longest element $w_0 \in W$ of the Weyl group.

(2) Define the transcendental generators:

$$\tilde{\mathbf{e}}_i := \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \tilde{\mathbf{f}}_i := \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \tilde{K}_i := K_i^{\frac{1}{b_i^2}}. \tag{2.20}$$

Then

(a) if $\mathfrak{g}$ is simply laced, the generators $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$, and $\tilde{K}_i$ are obtained by replacing b with b^{-1} in the representations of the generators $\mathbf{e}_i, \mathbf{f}_i$, and K_i.

(b) If $\mathfrak{g}$ is of type B, C, F, and G, then the generators $\tilde{E}_i, \tilde{F}_i$, and $\tilde{K}_i$ with

$$\tilde{\mathbf{e}}_i := 2\sin(\pi b_i^{-2})\tilde{E}_i, \quad \tilde{\mathbf{f}}_i := 2\sin(\pi b_i^{-2})\tilde{F}_i \tag{2.21}$$

generate $\mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$ defined in the previous section.

(3) The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ and $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i, \tilde{K}_i$ commute weakly up to a sign. □

The positive representations are constructed for each reduced expression $w_0 \in W$ of the longest element of the Weyl group, and representations corresponding to different reduced expressions are unitary equivalent.

Definition 2.4. Fix a reduced expression of $w_0 = s_{i_1} \dots s_{i_N}$. Let the coordinates of $L^2(\mathbb{R}^N)$ be denoted by $\{u_i^k\}$ so that i is the corresponding root index, and k denotes the sequence this root is appearing in w_0 from the right. Also denote by $\{v_j\}_{j=1}^N$ the same set of coordinates counting from the left, $v(i,k)$ the index such that $u_i^k = v_{v(i,k)}$, and $r(k)$ the root index corresponding to v_k. □

Example 2.5. The coordinates of $L^2(\mathbb{R}^6)$ for A_3 corresponding to $w_0 = s_3s_2s_1s_3s_2s_3$ is given by

$$(u_3^3, u_2^2, u_1^1, u_3^2, u_2^1, u_3^1) = (v_1, v_2, v_3, v_4, v_5, v_6).$$ □

Definition 2.6. Denote by

$$[u_s + u_l]e(-p_s - p_l) := e^{\pi b_s(-u_s - 2p_s) + \pi b_l(-u_l - 2p_l)} + e^{\pi b_s(u_s - 2p_s) + \pi b_l(u_l - 2p_l)}, \tag{2.22}$$

where u_s (resp. u_l) is a linear combination of the variables corresponding to short roots (resp. long roots). The parameters λ_i are also considered in both cases. Similarly p_s (resp. p_l) are linear combinations of the p shifting of the short roots (resp. long roots) variables. This applies to all simple $\mathfrak{g}$, with the convention given in Definition 2.1. □

Theorem 2.7 ([11, 12]). For a fixed reduced expression of w_0, the positive representation $\mathcal{P}_\lambda$ is given by

$$\mathbf{f}_i = \sum_{k=1}^{n}\left[-\sum_{j=1}^{v(i,k)-1} a_{i,r(j)}v_j - u_i^k - 2\lambda_i\right] e(p_i^k), \tag{2.23}$$

$$K_i = \mathrm{e}^{-\pi(\sum_{k=1}^{l(w_0)} a_{i,r(k)}b_{r(k)}v_k + 2b_i\lambda_i)}, \tag{2.24}$$

and by taking $w_0 = w's_i$ so that the simple reflection for root i appears on the right, the action of $\mathbf{e}_i$ is given by

$$\mathbf{e}_i = [u_i^1]e(-p_i^1). \tag{2.25}$$

□

In this paper, it is instructive to recall the explicit expression in the case of ranks 1 and 2. For details of the construction and the other cases please refer to [11, 12].

Proposition 2.8 ([1, 23]). The positive representation P_λ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is given by

$$\mathbf{e} = [u-\lambda]e(-p) = \mathrm{e}^{\pi b(-u+\lambda-2p)} + \mathrm{e}^{\pi b(u-\lambda-2p)},$$

$$\mathbf{f} = [-u-\lambda]e(p) = \mathrm{e}^{\pi b(u+\lambda+2p)} + \mathrm{e}^{\pi b(-u-\lambda+2p)},$$

$$K = \mathrm{e}^{-2\pi bu}.$$

(Note that it is unitary equivalent to the canonical form (2.23)–(2.25) by $u \mapsto u+\lambda$.) □

Proposition 2.9 ([11]). The positive representation $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(3,\mathbb{R}))$ with parameters $\lambda = (\lambda_1, \lambda_2)$, corresponding to the reduced expression $w_0 = s_2s_1s_2$, acting on $f(u,v,w) \in L^2(\mathbb{R}^3)$, is given by

$$\mathbf{e}_1 = [v-w]e(-p_v) + [u]e(-p_v+p_w-p_u),$$

$$\mathbf{e}_2 = [w]e(-p_w),$$

$$\mathbf{f}_1 = [-v+u-2\lambda_1]e(p_v),$$

$$\mathbf{f}_2 = [-2u+v-w-2\lambda_2]e(p_w) + [-u-2\lambda_2]e(p_u),$$

$$K_1 = \mathrm{e}^{-\pi b(-u+2v-w+2\lambda_1)},$$

$$K_2 = \mathrm{e}^{-\pi b(2u-v+2w+2\lambda_2)}.$$

□

Proposition 2.10 ([12]). The positive representation $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ with parameters $\lambda=(\lambda_1,\lambda_2)$, where $\mathfrak{g}_\mathbb{R}$ is of type B_2, corresponding to the reduced expression $w_0=s_1s_2s_1s_2$, acting on $f(t,u,v,w)\in L^2(\mathbb{R}^4)$, is given by

$$\mathbf{e}_1=[t]e(-p_t-p_u+p_w)+[u-v]e(-p_u-p_v+p_w)+[v-w]e(-p_v),$$

$$\mathbf{e}_2=[w]e(-p_w),$$

$$\mathbf{f}_1=[2\lambda_1-t]e(p_t)+[2\lambda_1-2t+u-v]e(p_v),$$

$$\mathbf{f}_2=[2\lambda_2+2t-u]e(p_u)+[2\lambda_2+2t-2u+2v-w]e(p_w),$$

$$K_1=\mathrm{e}^{\pi b_s(2\lambda_1-2t-2v)}\,\mathrm{e}^{\pi b(u+w)},$$

$$K_2=\mathrm{e}^{\pi b(2\lambda_2-2u-2w)}\,\mathrm{e}^{\pi b_s(2t+2v)}.$$

In this case (cf. Definition 2.6), u_s are linear combinations of $\{t,v\}$, while u_l are linear combinations of $\{u,w\}$. Similarly for p_s and p_l. □

We will omit the case of type G_2 for simplicity.

2.3 Quantum dilogarithm $G_b(x)$ and $g_b(x)$

First introduced by Faddeev [6, 7], (See also [8]), the quantum dilogarithm $G_b(x)$ and its variants $g_b(x)$ play a crucial role in the study of positive representations of split real quantum groups, and also appear in many other areas of mathematics and physics. In this subsection, let us recall the definition and some properties of the quantum dilogarithm functions [1, 10, 23] that is needed in the calculations in this paper.

Definition 2.11. The quantum dilogarithm function $G_b(x)$ is defined on $0\le \mathrm{Re}(z)\le Q$ by

$$G_b(x)=\bar{\zeta}_b\exp\left(-\int_\Omega \frac{\mathrm{e}^{\pi tz}}{(\mathrm{e}^{\pi bt}-1)(\mathrm{e}^{\pi b^{-1}t}-1)}\frac{\mathrm{d}t}{t}\right), \tag{2.26}$$

where

$$\zeta_b=\mathrm{e}^{\frac{\pi\mathrm{i}}{2}\left(\frac{b^2+b^{-2}}{6}+\frac{1}{2}\right)}, \tag{2.27}$$

and the contour goes along $\mathbb{R}$ with a small semicircle going above the pole at $t=0$. This can be extended meromorphically to the whole complex plane with poles at $x=-nb-mb^{-1}$ and zeros at $x=Q+nb+mb^{-1}$, for $n,m\in\mathbb{Z}_{\ge 0}$. □

The quantum dilogarithm $G_b(x)$ satisfies the following properties:

Proposition 2.12. *Self-duality*:

$$G_b(x) = G_{b^{-1}}(x); \tag{2.28}$$

Functional equations:

$$G_b(x + b^{\pm 1}) = (1 - e^{2\pi \mathbf{i} b^{\pm 1} x}) G_b(x); \tag{2.29}$$

Reflection property:

$$G_b(x) G_b(Q - x) = e^{\pi \mathbf{i} x(x-Q)}; \tag{2.30}$$

Complex conjugation:

$$\overline{G_b(x)} = \frac{1}{G_b(Q - \bar{x})}, \tag{2.31}$$

in particular

$$\left| G_b\left(\frac{Q}{2} + \mathbf{i}x\right) \right| = 1 \quad \text{for } x \in \mathbb{R}. \tag{2.32}$$

Asymptotic properties:

$$G_b(x) \sim \begin{cases} \bar{\zeta}_b & \mathrm{Im}(x) \to +\infty, \\ \zeta_b e^{\pi \mathbf{i} x(x-Q)} & \mathrm{Im}(x) \to -\infty. \end{cases} \tag{2.33}$$

□

Lemma 2.13 (q-binomial theorem)**.** For positive self-adjoint variables U, V with $UV = q^2 VU$, we have:

$$(U + V)^{\mathbf{i}b^{-1}t} = \int_C \binom{\mathbf{i}t}{\mathbf{i}\tau}_b U^{\mathbf{i}b^{-1}(t-\tau)} V^{\mathbf{i}b^{-1}\tau}\, d\tau, \tag{2.34}$$

where the q-beta function (or q-binomial coefficient) is given by

$$\binom{t}{\tau}_b = \frac{G_b(-\tau) G_b(\tau - t)}{G_b(-t)}, \tag{2.35}$$

and C is the contour along $\mathbb{R}$ that goes above the pole at $\tau=0$ and below the pole at $\tau=t$. □

Lemma 2.14 (tau-beta theorem). We have

$$\int_C e^{-2\pi\tau\beta}\frac{G_b(\alpha+\mathbf{i}\tau)}{G_b(Q+\mathbf{i}\tau)}\,d\tau=\frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha+\beta)}, \tag{2.36}$$

where the contour C goes along $\mathbb{R}$ and goes above the poles of $G_b(Q+\mathbf{i}\tau)$ and below those of $G_b(\alpha+\mathbf{i}\tau)$. By the asymptotic properties of G_b, the integral converges for $\mathrm{Re}(\beta)>0, \mathrm{Re}(\alpha+\beta)<Q$. □

Generalizing the delta distribution results from [10, Corollary 3.13], we have the following proposition:

Proposition 2.15. For $f(x)$ entirely analytic and rapidly decreasing (faster than any exponential) along the real direction, we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}}\frac{G_b(\epsilon+\mathbf{i}x-mb-nb^{-1})G_b(Q+mb+nb^{-1}-2\epsilon)}{G_b(Q+\mathbf{i}x-\epsilon)}f(x)\,dx \tag{2.37}$$

$$=\sum_{\substack{kb+lb^{-1}<mb+nb^{-1}\\ k,l>0}} r_{kl}\,f(-\mathbf{i}(kb+lb^{-1})), \tag{2.38}$$

where the constants r_{kl} is the residue of the integrand at $-\mathbf{i}(kb+lb^{-1})$. □

Finally, we will need the new integral transformation obtained in [10]:

Proposition 2.16. The 3–2 relation is given by

$$\int_C G_b(\alpha+\mathbf{i}\tau)G_b(\beta-\mathbf{i}\tau)G_b(\gamma-\mathbf{i}\tau)\,e^{-2\pi\mathbf{i}(\beta-\mathbf{i}\tau)(\gamma-\mathbf{i}\tau)}\,d\tau=G_b(\alpha+\gamma)G_b(\alpha+\beta), \tag{2.39}$$

where the contour goes along $\mathbb{R}$ and separates the poles for $\mathbf{i}\tau$ and $-\mathbf{i}\tau$. By the asymptotic properties for G_b, the integral converges for $\mathrm{Re}(\alpha-\beta-\gamma)<\frac{Q}{2}$. □

We will also need another important variant of the quantum dilogarithm.

Definition 2.17. The function $g_b(x)$ is defined by

$$g_b(x)=\frac{\bar{\zeta}_b}{G_b(\frac{Q}{2}+\frac{\log x}{2\pi\mathbf{i}b})}, \tag{2.40}$$

where log takes the principal branch of x. □

Lemma 2.18 ([1, (3.31), (3.32)]). We have the following Fourier transformation formula:

$$\int_{\mathbb{R}+\mathbf{i}0} \frac{\mathrm{e}^{-\pi \mathbf{i} t^2}}{G_b(Q+\mathbf{i}t)} X^{\mathbf{i}b^{-1}t}\,\mathrm{d}t = g_b(X), \tag{2.41}$$

$$\int_{\mathbb{R}+\mathbf{i}0} \frac{\mathrm{e}^{-\pi Q t}}{G_b(Q+\mathbf{i}t)} X^{\mathbf{i}b^{-1}t}\,\mathrm{d}t = g_b^*(X), \tag{2.42}$$

where X is a positive operator and the contour goes above the pole at $t=0$. □

We will need the following properties of $g_b(x)$.

Lemma 2.19. By (2.32), $|g_b(x)|=1$ when $x\in\mathbb{R}_+$, hence $g_b(X)$ is a unitary operator for any positive operator X. Furthermore, by (2.28) and Lemma 2.18, we have the self-duality of $g_b(x)$ given by

$$g_b(X) = g_{b^{-1}}(X^{\frac{1}{b^2}}). \tag{2.43}$$

□

Lemma 2.20. If $UV=q^2VU$ where U and V are positive self-adjoint operators, then

$$g_b(U)g_b(V) = g_b(U+V), \tag{2.44}$$

$$g_b(U)^* V g_b(U) = q^{-1}UV + V, \tag{2.45}$$

$$g_b(V)Ug_b(V)^* = U + q^{-1}UV. \tag{2.46}$$

Note that (2.44) and (2.45) together imply the pentagon relation

$$g_b(V)g_b(U) = g_b(U)g_b(q^{-1}UV)g_b(V). \tag{2.47}$$

□

If $UV=q^4VU$, then we apply the lemma twice and obtain

$$g_b(U)^* V g_b(U) = V + [2]_q q^2 VU + q^4 VU^2, \tag{2.48}$$

$$g_b(V)Ug_b(V)^* = U + [2]_q q^{-2}UV + q^{-4}UV^2. \tag{2.49}$$

where $[2]_q = q+q^{-1}$.

As a consequence of the above lemma, we also have the following:

Lemma 2.21 ([12, 27]). If $UV = q^2 VU$ where U and V are positive essentially self-adjoint operators, then $U + V$ is positive essentially self-adjoint, and

$$(U+V)^{\frac{1}{b^2}} = U^{\frac{1}{b^2}} + V^{\frac{1}{b^2}}. \tag{2.50}$$

□

2.4 Universal R matrices for $\mathcal{U}_q(\mathfrak{g})$

For $q := e^{h/2}$, it is known [5, 13] that for the quantum group $\mathcal{U}_h(\mathfrak{g})$ as a $\mathbb{C}[[h]]$-algebra completed in the h-adic topology, one can associate certain canonical, invertible element R in an appropriate completion of $(\mathcal{U}_h(\mathfrak{g}))^{\otimes 2}$ such that the braiding relation and quasi-triangularity (1.2)–(1.4) are satisfied.

For the quantum groups $\mathcal{U}_h(\mathfrak{g})$ associated to the simple Lie algebra $\mathfrak{g}$, an explicit multiplicative formula has been computed independently in [16, 19], where the central ingredient involves the quantum Weyl group which induces Lusztig's isomorphism T_i. Explicitly, let

$$[U, V]_q := qUV - q^{-1}VU \tag{2.51}$$

be the q-commutator.

Definition 2.22 ([16, 20]). Define

$$T_i(K_j) = K_j K_i^{-a_{ij}}, \quad T_i(E_i) = -F_i K_i^{-1}, \quad T_i(F_i) = -K_i E_i, \tag{2.52}$$

$$T_i(E_j) = (-1)^{a_{ij}} \frac{1}{[-a_{ij}]_{q_i}!} \left[\left[E_i, \ldots [E_i, E_j]_{q_i^{\frac{a_{ij}}{2}}} \right]_{q_i^{\frac{a_{ij}+2}{2}}} \cdots \right]_{q_i^{\frac{-a_{ij}-2}{2}}}, \tag{2.53}$$

$$T_i(F_j) = \frac{1}{[-a_{ij}]_{q_i}!} \left[\left[F_i, \ldots [F_i, F_j]_{q_i^{\frac{a_{ij}}{2}}} \right]_{q_i^{\frac{a_{ij}+2}{2}}} \cdots \right]_{q_i^{\frac{-a_{ij}-2}{2}}}. \tag{2.54}$$

□

Note that we have slightly modified the notation and scaling used in [16].

Proposition 2.23 ([20, 21]). The operators T_i satisfy the Weyl group relations:

$$\underbrace{T_i T_j T_i \cdots}_{-a'_{ij}+2} = \underbrace{T_j T_i T_j \cdots}_{-a'_{ij}+2}, \tag{2.55}$$

where $-a'_{ij} = \max\{-a_{ij}, -a_{ji}\}$. Furthermore, for α_i, α_j simple roots, and an element $w = s_{i_1} \cdots s_{i_k} \in W$ such that $w(\alpha_i) = \alpha_j$, we have

$$T_{i_1} \cdots T_{i_k}(X_i) = X_j \tag{2.56}$$

for $X = E, F, K$. □

Definition 2.24 ([18]). Define the (upper) quantum exponential function as

$$\mathrm{Exp}_q(x) = \sum_{k=0}^{\infty} \frac{z^k}{\lceil k \rceil_q!}, \tag{2.57}$$

where $\lceil k \rceil_q = \frac{1-q^k}{1-q}$, so that

$$\lceil k \rceil_{q^2}! = [k]_q! q^{\frac{k(k-1)}{2}}. \tag{2.58}$$

□

Theorem 2.25 ([16, 19]). Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression of the longest element of the Weyl group. Then the universal R matrix is given by

$$R = \mathbf{Q}^{\frac{1}{2}} \hat{R}(i_N | s_{i_1} \cdots s_{i_N-1}) \cdots \hat{R}(i_2 | s_{i_1}) \hat{R}(i_1) \mathbf{Q}^{\frac{1}{2}}, \tag{2.59}$$

where

$$\mathbf{Q} := q^{\sum_{i,j=1}^n (d \cdot A^{-1})_{ij} H_i \otimes H_j}, \tag{2.60}$$

d is such that $d_i A_{ij}$ is the symmetrized Cartan matrix, $q = q_s$, and

$$\hat{R}(i) := \mathrm{Exp}_{q_i^{-2}}((1 - q_i^{-2}) E_i \otimes F_i), \tag{2.61}$$

$$\hat{R}(i_l | s_{i_1} \cdots s_{i_{l-1}}) := (T_{i_1}^{-1} \otimes T_{i_1}^{-1}) \cdots (T_{i_{l-1}}^{-1} \otimes T_{i_{l-1}}^{-1}) \hat{R}(i_1). \tag{2.62}$$

□

In both studies [16, 19], the expression for the R matrix is obtained from the canonical element of the Drinfeld double of $\mathcal{U}_h(\mathfrak{b}_+)$ generated by E_i's and H_i's. The Lusztig's isomorphism gives the ordered basis of $\mathcal{U}_h(\mathfrak{b}_+)$, and there exists a dual pairing between $\mathcal{U}_h(\mathfrak{b}_+)$ and $\mathcal{U}_h(\mathfrak{b}_-)$ of this basis involving the quantum factorials $[k]_q!$, hence the expression (2.61).

2.5 Universal R operator for $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$

In the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, an expression of the R operator is computed in [1]. It is given formally by

$$R = q^{\frac{H\otimes H}{4}} g_b(\mathbf{e}\otimes\mathbf{f}) q^{\frac{H\otimes H}{4}}, \tag{2.63}$$

where we recall

$$\mathbf{e} := 2\sin(\pi b^2)E, \quad \mathbf{f} := 2\sin(\pi b^2)F, \quad K := q^H. \tag{2.64}$$

The operator R acts naturally on $P_{\lambda_1}\otimes P_{\lambda_2}$ by means of the positive representation. Note that the remarkable fact about this expression is the positivity of the argument $\mathbf{e}\otimes\mathbf{f}$ inside the quantum dilogarithm g_b which makes the expression a well-defined operator. In fact it is clear that R acts as a unitary operator by Lemma 2.19 of the properties of $g_b(x)$. Furthermore, by the transcendental relations (2.20) and self-duality (2.43) of g_b, the expression (2.63) is invariant under the change of $b \longleftrightarrow b^{-1}$:

$$R = \tilde{R} := \tilde{q}^{\frac{\tilde{H}\otimes\tilde{H}}{4}} g_{b^{-1}}(\tilde{\mathbf{e}}\otimes\tilde{\mathbf{f}}) \tilde{q}^{\frac{\tilde{H}\otimes\tilde{H}}{4}}. \tag{2.65}$$

Hence in fact it simultaneously serves as the R operator of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.

The properties as an R operator imply certain functional equations for the quantum dilogarithm g_b. While the quasi-triangular relations (1.3)–(1.4) are equivalent to (2.44), the braiding relation

$$\Delta'(X)R = R\Delta(X)$$

implies the following:

Lemma 2.26. We have

$$(\mathbf{e}\otimes K^{-1} + 1\otimes\mathbf{e})g_b(\mathbf{e}\otimes\mathbf{f}) = g_b(\mathbf{e}\otimes\mathbf{f})(\mathbf{e}\otimes K + 1\otimes\mathbf{e}), \tag{2.66}$$

and similarly

$$(\mathbf{f}\otimes 1 + K\otimes\mathbf{f})g_b(\mathbf{e}\otimes\mathbf{f}) = g_b(\mathbf{e}\otimes\mathbf{f})(\mathbf{f}\otimes 1 + K^{-1}\otimes\mathbf{f}). \tag{2.67}$$

□

Proof. By definition

$$\begin{aligned}&\Delta'(\mathbf{e})R = R\Delta(\mathbf{e})\\ &\iff \Delta'(\mathbf{e})q^{\frac{1}{4}H\otimes H}g_b(\mathbf{e}\otimes\mathbf{f})q^{\frac{1}{4}H\otimes H} = q^{\frac{1}{4}H\otimes H}g_b(\mathbf{e}\otimes\mathbf{f})q^{\frac{1}{4}H\otimes H}\Delta(\mathbf{e})\\ &\iff (\mathbf{e}\otimes K^{-1}+1\otimes\mathbf{e})g_b(\mathbf{e}\otimes\mathbf{f}) = g_b(\mathbf{e}\otimes\mathbf{f})(\mathbf{e}\otimes K+1\otimes\mathbf{e})\end{aligned} \tag{2.68}$$

and similarly for the other statement using $\Delta(\mathbf{f})$. ■

3 Generalized Pentagon Relations for $g_b(x)$

It turns out that the exponential and pentagon relations (2.44)–(2.47) are not enough to show the properties of the universal R matrix. In this section, following techniques from [18], we derive more general functional equations for $g_b(x)$ which generalizes the pentagon relation as well as the quantum exponential relation.

3.1 Simply laced case

Proposition 3.1. Let U and V be positive self-adjoint operators such that $c := \frac{UV-VU}{q-q^{-1}}$ is also positive self-adjoint, and $Uc = q^2cU$, $Vc = q^{-2}cV$. Then

$$g_b(V)Ug_b^*(V) = U + c, \tag{3.1}$$

$$g_b^*(U)Vg_b(U) = c + V, \tag{3.2}$$

which also implies

$$g_b(V)g_b(U) = g_b(U)g_b(c)g_b(V). \tag{3.3}$$

□

Note that if $UV = q^2VU$, these reduce to the usual pentagon relations (2.45)–(2.47).

Proof. By induction, we calculate formally

$$\begin{aligned}VU &= UV - (q-q^{-1})c,\\ V^nU &= V^{n-1}UV - (q-q^{-1})V^{n-1}c\end{aligned}$$

$$
\begin{aligned}
&= V^{n-2}UV^2 - (q-q^{-1})V^{n-2}cV + (q-q^{-1})V^{n-1}c \\
&= \cdots \\
&= UV^n - (q-q^{-1})(q^{2-2n} + q^{4-2n} + \cdots + 1)cV^{n-1} \\
&= UV^n - q(1-q^{-2n})cV^{n-1} \\
&= UV^n + q(1-q^{2n})cq^{-2n}V^{n-1}.
\end{aligned}
$$

Hence by virtue of functional calculus, we can replace the power by complex powers $\mathbf{i}b^{-1}t$, and apply the integration formula for $g_b(x)$. We obtain

$$
\begin{aligned}
g_b(V)U &= Ug_b(V) + qc\int_{\mathbb{R}+\mathbf{i}0} (1-q^{2\mathbf{i}b^{-1}t})q^{-2\mathbf{i}b^{-1}t}\,\mathrm{e}^{-\pi\mathbf{i}t^2}\frac{V^{\mathbf{i}b^{-1}t-1}}{G_b(Q+\mathbf{i}t)}\,\mathrm{d}t \\
&= Ug_b(V) + qc\int_{\mathbb{R}+\mathbf{i}0} (1-\mathrm{e}^{-2\pi b(t-\mathbf{i}b)})\,\mathrm{e}^{2\pi b(t-\mathbf{i}b)}\,\mathrm{e}^{-\pi\mathbf{i}(t-\mathbf{i}b)^2}\frac{V^{\mathbf{i}b^{-1}t}}{G_b(Q+\mathbf{i}t+b)}\,\mathrm{d}t \\
&= Ug_b(V) + qc\int_{\mathbb{R}+\mathbf{i}0} \frac{(1-\mathrm{e}^{-2\pi b(t-\mathbf{i}b)})\,\mathrm{e}^{2\pi bt}q^{-2}\,\mathrm{e}^{-2\pi bt}q}{(1-\mathrm{e}^{2\pi\mathbf{i}b(Q+\mathbf{i}t)})}\,\mathrm{e}^{-\pi\mathbf{i}t^2}\frac{V^{\mathbf{i}b^{-1}t}}{G_b(Q+\mathbf{i}t)}\,\mathrm{d}t \\
&= (U+c)g_b(V).
\end{aligned}
$$

Hence

$$g_b(V)Ug_b^*(V) = U + c$$

and

$$g_b(V)g_b(U)g_b^*(V) = g_b(U+c) = g_b(U)g_b(c).$$

Similarly, we also have

$$g_b^*(U)Vg_b(U) = c + V. \qquad \blacksquare$$

3.2 Nonsimply laced case

In the nonsimply laced case, more q-commutators are involved. By applying the same techniques in the previous subsections repeatedly, we have the following relations.

Proposition 3.2. Let U and V be positive operators and define c and d to be

$$c := \frac{[U,V]}{q-q^{-1}}, \quad d := \frac{q^{-1}cV - qVc}{q^2-q^{-2}},$$

such that c and d are positive self-adjoint, and the following relations hold:

$$Uc = q^4 cU, \quad cd = q^4 dc, \quad dV = q^4 Vd.$$

Then we have

$$g_b(V)Ug_b^*(V) = U + c + d. \tag{3.4}$$

Similarly, we have

$$g_b^*(U)Vg_b(U) = d' + c + V, \tag{3.5}$$

where

$$d' := \frac{q^{-1}Uc - qcU}{q^2 - q^{-2}},$$

with

$$Vc = q^{-4}cV, \quad cd' = q^{-4}d'c, \quad d'U = q^{-4}Ud'.$$

Note that when $UV = q^4 VU$, these reduce to the relations (2.48)–(2.49). □

Even more generally for the type G_2 case, by defining $e := \frac{q^{-2}dV - q^2 Vd}{q^3 - q^{-3}}$ such that e is positive self-adjoint and

$$Uc = q^6 cU, \quad cd = q^6 dc, \quad de = q^6 ed, \quad eV = q^6 Ve,$$

we have

$$g_b(V)Ug_b^*(V) = U + c + d + e.$$

Similar relations also hold for the other q-commutators d' and e'.

Finally, we have the following useful functional relations generalizing the q-exponential relation.

Proposition 3.3. Let U, c, d, d' be as in Proposition 3.2. Let $q = e^{\pi i b_s^2}$ and $q^2 = e^{\pi i b_l^2}$. Then we have

$$g_{b_s}(U + c) = g_{b_s}(U)g_{b_l}(d')g_{b_s}(c), \tag{3.6}$$

$$g_{b_l}(U + c + d) = g_{b_l}(U)g_{b_s}(c)g_{b_l}(d). \tag{3.7}$$

□

Using Propositions 3.1 and 3.2, these two relations are related by the transcendental relations in virtue with the approach in [12], where the long roots and short roots are interchanged, and using the self-duality (2.43) of $g_b(x)$.

The functional relations in the case of compact quantum exponential function using power series can be found in [17], where some of the generalized functional relations for type G_2 case have been computed. We will leave the analogue of these functional relations of $g_b(x)$ in the case of type G_2 to the interested reader.

4 Quantum Weyl Element and Lusztig's Isomorphism

The starting point of the present work is the observation of the positivity appearing in the root vectors $\mathbf{e}_{ij}$ corresponding to the nonsimple roots $\alpha_i+\alpha_j$. They are given by composition of certain q-commutators of simple root generators $\mathbf{e}_i$ and $\mathbf{e}_j$, and in turn is given by the Lusztig's isomorphism. Therefore to prove positivity, we show that Lusztig's isomorphism can actually be implemented by conjugations of certain elements w_i, which is known as the quantum Weyl elements. In the compact case this is done in [16, 19] by means of semi-simplicity of $\mathcal{U}_q(\mathfrak{sl}_2)$-submodules in $\mathcal{U}_q(\mathfrak{g})$-modules. In the current paper, we show that the w_i can actually be implemented as a unitary operator, hence preserving positivity. The construction requires explicit calculation of the ribbon element u and v in Section 4.2, as well as the branching rules of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))\subset\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ as positive representations in Section 4.3 since we no longer have obvious semi-simplicity.

4.1 Positivity of $\mathbf{e}_{ij}$

It is well-known that the Lusztig's isomorphism T_i defined in Definition 2.22 essentially gives the generators of the canonical basis of $\mathcal{U}_q(\mathfrak{g})$. In the present case of positive representations, we also require the generators to be positive essentially self-adjoint.

In the simply laced case, we observe the following:

Proposition 4.1. Fix a positive representation $\mathcal{P}_\lambda$. Then

$$\mathbf{e}_{ij}:=\frac{[\mathbf{e}_j,\mathbf{e}_i]_{q^{\frac12}}}{q-q^{-1}}=\frac{q^{\frac12}\mathbf{e}_j\mathbf{e}_i-q^{-\frac12}\mathbf{e}_i\mathbf{e}_j}{q-q^{-1}} \tag{4.1}$$

is positive essentially self-adjoint, and also satisfies the transcendental relations

$$\tilde{\mathbf{e}}_{ij}:=\mathbf{e}_{ij}^{\frac{1}{b^2}}=\frac{\tilde{q}^{\frac12}\tilde{\mathbf{e}}_j\tilde{\mathbf{e}}_i-\tilde{q}^{-\frac12}\tilde{\mathbf{e}}_i\tilde{\mathbf{e}}_j}{\tilde{q}-\tilde{q}^{-1}}. \tag{4.2}$$

□

Proof. Without loss of generality, we can choose $w_0 = w' s_j s_i s_j$. Then it suffices to look at the representation in the case of type A_2 given by Proposition 2.9. We obtain

$$\mathbf{e}_{ij} = e^{-\pi b(v-2w+2p_v+2p_w)} + e^{-\pi b(v+2p_v+2p_w)} \tag{4.3}$$

$$+ e^{-\pi b(u-w+2p_u+2p_v)} + e^{-\pi b(-u-w+2p_u+2p_v)}, \tag{4.4}$$

which is evidently positive. Since each term q^2 commute with the terms on its right, by Lemma 2.21, the operator is essentially self-adjoint, and satisfy the transcendental relation. ■

We have similar observations in the nonsimply laced as well. Again it suffices to consider rank 2 case.

Proposition 4.2. In general, define the operators

$$\mathbf{e}_{ij} = (-1)^{a_{ij}} [[\mathbf{e}_i, \ldots [\mathbf{e}_i, \mathbf{e}_j]_{q_i^{\frac{a_{ij}}{2}}}]_{q_i^{\frac{a_{ij}+2}{2}}} \ldots]_{q_i^{\frac{-a_{ij}-2}{2}}} \prod_{k=1}^{-a_{ij}} (q_i^k - q_i^{-k})^{-1}. \tag{4.5}$$

Then it is positive essentially self-adjoint, and satisfy the generalized transcendental relations, where $\mathbf{e}_{ij}^{\frac{1}{b_i^2}}$ is given by the same expression as $\mathbf{e}_{ij}$ with all $\mathbf{e}_i$ replaced by $\tilde{\mathbf{e}}_i$, q_i replaced by $\tilde{q}_i$, and a_{ij} replaced by a_{ji}. □

Proof. These are calculated directly from the explicit expression of the positive representations of type B_2, G_2 and also the transcendental relations using expressions of type C_2 given in [12]. ■

We note that $\mathbf{e}_{ij} = T_i(\mathbf{e}_j)$ up to some constant. Therefore, if we can show that T_i are given by *inner automorphism* of some unitary element, then both positivity and transcendental relations of the remaining generators in higher rank will be immediate. This is achieved by the use of the quantum Weyl elements described in Section 4.4.

Finally, we define $\mathbf{f}_{ij}$ with the exact same formula (4.5) with $\mathbf{e}$ replaced by $\mathbf{f}$. Then using the Weyl element w_0 derived in Section 4.4 we see that it also satisfies all the properties enjoyed by $\mathbf{e}_{ij}$.

4.2 Calculation of the ribbon element u and v for $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$

In this section, we restrict the attention to a fixed positive representation P_λ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Let the R operator be given by (2.63). Explicitly, it can be written as

$$R = \mathbf{Q}^{\frac{1}{2}}\left(\int_{\mathbb{R}+\mathbf{i}0} \frac{e^{-\pi \mathbf{i} t^2}\mathbf{e}^{\mathbf{i}b^{-1}t}\otimes \mathbf{f}^{\mathbf{i}b^{-1}t}}{G_b(Q+\mathbf{i}t)}\, dt\right)\mathbf{Q}^{\frac{1}{2}}, \tag{4.6}$$

where

$$\mathbf{Q} = q^{\frac{H\otimes H}{2}} = \sum_{n=0}^{\infty}\left(\frac{\pi \mathbf{i} b^2}{2}\right)^n \frac{H^n\otimes H^n}{n!}. \tag{4.7}$$

We will write R informally as $R=\sum_k \alpha_k\otimes\beta_k$.

We wish to calculate the element

$$u = m^{op}\circ(1\otimes S)R = \sum_k S(\beta_k)\alpha_k, \tag{4.8}$$

which is crucial in the analysis of quasi-triangular Hopf algebras. Here, we will first calculate the expression formally using an extension of the antipode S. In Section 6, we will then define u rigorously as an element in certain multiplier Hopf-* algebra.

From the expression of u, it means we need to calculate the action of $\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}$, in other words we need to calculate the action of $\mathbf{e}^{\mathbf{i}b^{-1}t}$ and $\mathbf{f}^{\mathbf{i}b^{-1}t}$ under the positive representation P_λ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Furthermore we also need the actual effect of the antipode. We introduce the expression

$$S(\mathbf{e}) = e^{\pi \mathbf{i} b Q}\mathbf{e} = -q\mathbf{e}, \quad S(\mathbf{f}) = e^{-\pi \mathbf{i} b Q}\mathbf{f} = -q^{-1}\mathbf{f}, \quad S(H) = -H \tag{4.9}$$

consistent with the usual definition, and define S on the complex powers by

$$S(\mathbf{e}^{\mathbf{i}b^{-1}t}) := e^{-\pi Q t}\mathbf{e}^{\mathbf{i}b^{-1}t}, \quad S(\mathbf{f}^{\mathbf{i}b^{-1}t}) := e^{\pi Q t}\mathbf{f}^{\mathbf{i}b^{-1}t}. \tag{4.10}$$

Again the definition is rigorous once we impose the setting of multiplier Hopf-* algebra in Section 6.

Lemma 4.3. The action of $\mathbf{e}^{\mathbf{i}b^{-1}t}$ and $\mathbf{f}^{\mathbf{i}b^{-1}t}$ on $f(x)$ is given by

$$\mathbf{e}^{\mathbf{i}b^{-1}t}\cdot f(x)=e^{\pi\mathbf{i}(x-\lambda)t}\,e^{\frac{-\pi\mathbf{i}t^2}{2}}\frac{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)}{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda-\mathbf{i}t)}f(x-t), \tag{4.11}$$

$$\mathbf{f}^{\mathbf{i}b^{-1}t}\cdot f(x)=e^{\pi\mathbf{i}(x+\lambda)t}\,e^{\frac{\pi\mathbf{i}t^2}{2}}\frac{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)}f(x+t). \tag{4.12}$$

Note that these actions are unitary transformations. □

Proof.

$$\begin{aligned}
\mathbf{e}&=e^{-\pi bx+\pi b\lambda-2\pi bp}+e^{\pi bx-\pi b\lambda-2\pi bp}\\
&=g_b^*(e^{-2\pi b(x-\lambda)})\,e^{\pi bx-\pi b\lambda-2\pi bp}g_b(e^{-2\pi b(x-\lambda)})\\
&=g_b^*(e^{-2\pi b(x-\lambda)})\,e^{\pi\mathbf{i}(\frac{x^2}{2}-\lambda x)}\,e^{-2\pi bp}\,e^{-\pi\mathbf{i}(\frac{x^2}{2}-\lambda x)}g_b(e^{-2\pi b(x-\lambda)}),\\
\mathbf{e}^{\mathbf{i}b^{-1}t}\cdot f(x)&=g_b^*(e^{-2\pi b(x-\lambda)})\,e^{\pi\mathbf{i}(\frac{x^2}{2}-\lambda x)}\,e^{-2\pi\mathbf{i}tp}\,e^{-\pi\mathbf{i}(\frac{x^2}{2}-\lambda x)}g_b(e^{-2\pi b(x-\lambda)})\cdot f(x)\\
&=g_b^*(e^{-2\pi b(x-\lambda)})\,e^{\pi\mathbf{i}(\frac{x^2}{2}-\lambda x)}\,e^{-\pi\mathbf{i}(\frac{(x-t)^2}{2}-\lambda(x-t))}g_b(e^{-2\pi b(x-\lambda-t)})\,f(x-t)\\
&=e^{\pi\mathbf{i}(x-\lambda)t}\,e^{\frac{-\pi\mathbf{i}t^2}{2}}\frac{g_b(e^{-2\pi b(x-\lambda-t)})}{g_b(e^{-2\pi b(x-\lambda)})}f(x-t)\\
&=e^{\pi\mathbf{i}(x-\lambda)t}\,e^{\frac{-\pi\mathbf{i}t^2}{2}}\frac{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)}{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda-\mathbf{i}t)}f(x-t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{f}&=e^{-\pi bx-\pi b\lambda+2\pi bp}+e^{\pi bx+\pi b\lambda+2\pi bp}\\
&=g_b(e^{-2\pi b(x+\lambda)})\,e^{\pi bx+\pi b\lambda+2\pi bp}g_b^*(e^{-2\pi b(x+\lambda)})\\
&=g_b(e^{-2\pi b(x+\lambda)})\,e^{-\pi\mathbf{i}(\frac{x^2}{2}+\lambda x)}\,e^{2\pi bp}\,e^{\pi\mathbf{i}(\frac{x^2}{2}+\lambda x)}g_b^*(e^{-2\pi b(x+\lambda)}),\\
\mathbf{f}^{\mathbf{i}b^{-1}t}\cdot f(x)&=g_b(e^{-2\pi b(x+\lambda)})\,e^{-\pi\mathbf{i}(\frac{x^2}{2}+\lambda x)}\,e^{2\pi\mathbf{i}tp}\,e^{\pi\mathbf{i}(\frac{x^2}{2}+\lambda x)}g_b^*(e^{-2\pi b(x+\lambda)})\cdot f(x)\\
&=g_b(e^{-2\pi b(x+\lambda)})\,e^{-\pi\mathbf{i}(\frac{x^2}{2}+\lambda x)}\,e^{\pi\mathbf{i}(\frac{(x+t)^2}{2}+\lambda(x+t))}g_b^*(e^{-2\pi b(x+\lambda+t)})\,f(x+t)
\end{aligned}$$

$$= e^{\pi \mathbf{i}(x+\lambda)t} e^{\frac{\pi \mathbf{i}t^2}{2}} \frac{g_b(e^{-2\pi b(x+\lambda)})}{g_b(e^{-2\pi b(x+\lambda+t)})} f(x+t)$$

$$= e^{\pi \mathbf{i}(x+\lambda)t} e^{\frac{\pi \mathbf{i}t^2}{2}} \frac{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)} f(x+t). \qquad \blacksquare$$

Hence combining, we have

$$\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t} = e^{\pi \mathbf{i}(x+\lambda)t} e^{\frac{\pi \mathbf{i}t^2}{2}} \frac{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)} e^{\pi \mathbf{i}(x+t-\lambda)t} e^{\frac{-\pi \mathbf{i}t^2}{2}} \frac{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)} f(x)$$

$$= e^{\pi \mathbf{i}t(2x+t)} \frac{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)} \frac{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)} f(x).$$

Theorem 4.4. The element $u = \sum S(\beta_k)\alpha_k = m^{op}(1\otimes S)R$ is given by

$$u = e^{2\pi \mathbf{i}(\lambda^2+\frac{Q^2}{4})} K^{\frac{Q}{b}}. \tag{4.13}$$

$\square$

Proof. First note that $H\mathbf{e} = \mathbf{e}H + 2\mathbf{e}$ implies

$$H^n\mathbf{e} = \mathbf{e}(H+2)^n$$

$$H^n\mathbf{e}^{\mathbf{i}b^{-1}t} = \mathbf{e}^{\mathbf{i}b^{-1}t}(H+2\mathbf{i}b^{-1}t)^n.$$

Similarly

$$H^n\mathbf{f}^{\mathbf{i}b^{-1}t} = \mathbf{f}^{\mathbf{i}b^{-1}t}(H-2\mathbf{i}b^{-1}t)^n.$$

Note that H commutes with $\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}$.

Hence using the "continuous basis" (4.6)–(4.7)

$$(H^n\otimes H^n)(e^{\mathbf{i}b^{-1}t}\otimes f^{\mathbf{i}b^{-1}t})(H^m\otimes H^m) = H^n e^{\mathbf{i}b^{-1}t}H^m \otimes H^n f^{\mathbf{i}b^{-1}t}H^m,$$

$$m^{op}(1\otimes S) = S(H^n\mathbf{f}^{\mathbf{i}b^{-1}t}H^m)H^n\mathbf{e}^{\mathbf{i}b^{-1}t}H^m$$

$$= (-1)^m(-1)^n H^m e^{\pi Qt}\mathbf{f}^{\mathbf{i}b^{-1}t}H^nH^n\mathbf{e}^{\mathbf{i}b^{-1}t}H^m$$

$$= (-1)^m(-1)^n H^m e^{\pi Qt}\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}(H+2\mathbf{i}b^{-1}t)^{2n}H^m$$

$$= e^{\pi Qt}\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}(-H^2)^m(-H^2-4\mathbf{i}b^{-1}tH+4b^{-2}t^2)^n.$$

Hence

$$m^{op}(1\otimes S)R=\left(\int_{\mathbb{R}+\mathbf{i}0}\frac{e^{-\pi\mathbf{i}t^2+\pi Qt}}{G_b(Q+\mathbf{i}t)}\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}K^{-\mathbf{i}b^{-1}t}e^{\pi\mathbf{i}t^2}\,dt\right)q^{-\frac{H^2}{2}}$$
$$=\left(\int_{\mathbb{R}+\mathbf{i}0}\frac{e^{\pi Qt}}{G_b(Q+\mathbf{i}t)}\mathbf{f}^{\mathbf{i}b^{-1}t}\mathbf{e}^{\mathbf{i}b^{-1}t}K^{-\mathbf{i}b^{-1}t}\,dt\right)q^{-\frac{H^2}{2}},$$

and the action on $f(x)$ is given by ($K=e^{-2\pi bx}=q^{2\mathbf{i}b^{-1}x}$, so $H=2\mathbf{i}b^{-1}x$):

$$u=\int_{\mathbb{R}+\mathbf{i}0}e^{-(\pi\mathbf{i}b^2)(2\mathbf{i}b^{-1}x)^2/2}\cdot e^{-2\pi bx(-\mathbf{i}b^{-1}t)}\,e^{\pi\mathbf{i}t(2x+t)}$$
$$\cdot\frac{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)}\frac{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)}{G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)}\frac{e^{\pi Qt}}{G_b(Q+\mathbf{i}t)}\,dt$$
$$=\frac{\int_{\mathbb{R}+\mathbf{i}0}e^{2\pi\mathbf{i}(x+t)^2+2\pi Qt}G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)G_b(-\mathbf{i}t)\,dt}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)}$$
$$=e^{2\pi\mathbf{i}(\lambda^2+\frac{Q^2}{4})}\,e^{-2\pi Qx}$$
$$\cdot\frac{\int_{\mathbb{R}+\mathbf{i}0}e^{-2\pi\mathbf{i}(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+it)(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)}G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda+\mathbf{i}t)G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda+\mathbf{i}t)G_b(-\mathbf{i}t)\,dt}{G_b(\frac{Q}{2}+\mathbf{i}x+\mathbf{i}\lambda)G_b(\frac{Q}{2}+\mathbf{i}x-\mathbf{i}\lambda)}$$
$$=e^{2\pi\mathbf{i}(\lambda^2+\frac{Q^2}{4})}K^{\frac{Q}{b}},$$

where in the last line we used the 3–2 relations from Proposition 2.16. ■

Remark 4.5. Letting $l=-\frac{Q}{2}+\mathbf{i}\lambda$, one can rewrite this expression as

$$u=q^{-2\frac{l}{b}(\frac{l}{b}+\frac{Q}{b})}K^{\frac{Q}{b}},\tag{4.14}$$

and compare with the expression from the compact case [16] on the $(2j+1)$-dimensional module V_j:

$$u=q^{-2j(j+1)}K.\tag{4.15}$$

□

Now one can check that the following is satisfied: $S^2(a)=uau^{-1}$:

$$S^2(\mathbf{e}^{\mathbf{i}b^{-1}t})=e^{-2\pi Qt}\mathbf{e}^{\mathbf{i}b^{-1}t}=K^{\frac{Q}{b}}\mathbf{e}^{\mathbf{i}b^{-1}t}K^{-\frac{Q}{b}}=u\mathbf{e}^{\mathbf{i}b^{-1}t}u^{-1},$$
$$S^2(\mathbf{f}^{\mathbf{i}b^{-1}t})=e^{2\pi Qt}\mathbf{f}^{\mathbf{i}b^{-1}t}=K^{\frac{Q}{b}}\mathbf{f}^{\mathbf{i}b^{-1}t}K^{-\frac{Q}{b}}=u\mathbf{f}^{\mathbf{i}b^{-1}t}u^{-1}.$$

Definition 4.6. The ribbon element v is defined to be the constant operator acting on P_λ as multiplication by

$$v = e^{2\pi \mathbf{i}(\lambda^2 + \frac{Q^2}{4})}, \tag{4.16}$$

such that $u = vK^{\frac{Q}{b}}$. □

4.3 Branching rules for $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R})) \subset \mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$

In [16, 19], the quantum Weyl element is defined by decomposing $\mathcal{U}_q(\mathfrak{g})$ into irreducible $\mathcal{U}_{q_i}(\mathfrak{sl}_2)$ submodules corresponding to simple roots α_i, which exists because the algebra involved is semisimple. In the current setting of positive representations, which is infinite-dimensional, it is not at all clear whether the same decomposition is possible. It turns out that the branching rules are particularly simple, and they remarkably resemble both the decomposition of the tensor product representation $P_\alpha \otimes P_\beta$ (cf. [23]) and the Peter–Weyl-type decomposition of $L^2(\mathrm{SL}_q^+(2,\mathbb{R}))$ (cf. [10]) with exactly the same Plancherel measure $\mathrm{d}\mu(\gamma) = |S_b(Q+2\gamma)|^2 d\gamma$.

Let $q_i = e^{\pi \mathbf{i} b_i^2}$ and $Q_i = b_i + b_i^{-1}$.

Theorem 4.7. Fix any positive representation $\mathcal{P}_\lambda \simeq L^2(\mathbb{R}^N)$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$, where $N = l(w_0)$. As a representation of $\mathcal{U}_{q_i\tilde{q}_i}(\mathfrak{sl}(2,\mathbb{R})) \subset \mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ corresponding to the simple root α_i,

$$\mathcal{P}_\lambda \simeq L^2(\mathbb{R}^{N-2}) \otimes \int_{\mathbb{R}_+} P_\gamma \, \mathrm{d}\mu(\gamma) \tag{4.17}$$

is a unitary equivalence, where P_γ is the positive representation of $\mathcal{U}_{q_i\tilde{q}_i}(\mathfrak{sl}(2,\mathbb{R}))$ with parameter $\gamma \in \mathbb{R}_+$, and the Plancherel measure is given by

$$\mathrm{d}\mu(\gamma) = |S_{b_i}(Q_i + 2\gamma)|^2 \, \mathrm{d}\gamma, \tag{4.18}$$

where $S_b(x) = G_b(x)\, e^{\frac{\pi \mathbf{i}}{2} x(Q-x)}$. □

Proof. Using the same techniques as in [10], it suffices to diagonalize the Casimir element.

By taking $w_0 = w's_i$ so that the simple reflection for root i appears on the right, the action of $\mathbf{e}_i$ is the standard action (using the notation from Section 2.2)

$$\mathbf{e}_i = [u_i^1]e(-p_i^1),$$

while the action of $\mathbf{f}_i$ and K_i are given by (2.23) and (2.24), respectively.

Note that $\mathbf{e}_i$ commutes with the terms of $\mathbf{f}_i$ for $k>1$. Then the rescaled Casimir element $\mathbf{c}_i$ for this $\mathcal{U}_{q_i\tilde{q}_i}(\mathfrak{sl}(2,\mathbb{R}))$ representation

$$\mathbf{c}_i := \left(\frac{\mathbf{i}}{q_i - q_i^{-1}}\right)^{-2} C_i = \mathbf{f}_i\mathbf{e}_i - (q_iK_i + q_i^{-1}K_i^{-1}) \tag{4.19}$$

is given by

$$\begin{aligned}\mathbf{c}_i &= \sum_{k=1}^{n}\left[-\sum_{j=1}^{v(i,k)-1} a_{i,r(j)}v_j - u_i^k - 2\lambda_i\right] e(p_i^k)[u_i^1]e(-p_i^1) - (q_iK + q_i^{-1}K^{-1})\\ &= \sum_{k=2}^{n}\left[-\sum_{j=1}^{v(i,k)-1} a_{i,r(j)}v_j - u_i^k - 2\lambda_i\right][u_i^1]e(p_i^k - p_i^1) + 2\cosh\left(\pi\mathbf{b}\cdot\left(\sum_{j=1}^{v(i,1)-1} a_{i,r(j)}v_j + 2\lambda_i\right)\right).\end{aligned}$$

Here we used the notation $(\mathbf{b}\cdot -)$ so that variables corresponding to short (resp. long) root get multiplied by b_s (resp. b_l) (cf. Definition 2.6). Applying the transformation by multiplication by $g_{b_i}^*(2u_i^1)$, using Lemma 2.20, will eliminate the $[u_i^1]$ factor:

$$\simeq \sum_{k=2}^{n}\left[-\sum_{j=1}^{v(i,k)-1} a_{i,r(j)}v_j - u_i^k - 2\lambda_i\right] e(p_i^k - p_i^1) + 2\cosh\left(\pi\mathbf{b}\cdot\left(\sum_{j=1}^{v(i,1)-1} a_{i,r(j)}v_j + 2\lambda_i\right)\right).$$

Now we know from the explicit expression that the terms from $k=2$ to $k=n$ q_i^2-commute successively [11, 12]. Hence there exists transformations by certain g_{b_i}, where the arguments are given by the differences of the factors, that the above operator is unitary equivalent to just the first term:

$$\simeq 2\cosh\left(\pi\mathbf{b}\cdot\left(\sum_{j=1}^{v(i,1)-1} a_{i,r(j)}v_j + 2\lambda_i\right)\right) + e^{\pi\mathbf{b}\cdot\left(-\sum_{j=1}^{v(i,n)-1} a_{i,r(j)}v_j - u_i^n - 2\lambda_i + 2p_i^n - 2p_i^1\right)}.$$

Now we can apply simple unitary transformations to simplify the expression. (For a review, see [10, Section 6.1].) First, apply the transformation $u_i^1 \mapsto u_i^1 - u_i^n$ to get rid of p_i^1. Then apply

$$p_i^n \mapsto p_i^n + \lambda_i - \frac{1}{2}\left(\sum_{j=1}^{v(i,n)-1} a_{i,r(j)}v_j - u_i^n\right),$$

so that the last term becomes simply $e^{2\pi b_i p_i^n}$. Finally apply

$$u_i^n \mapsto u_i^n - \lambda_i - \frac{1}{2}\sum_{j=1, v_j\neq u_i^n}^{v(i,1)-1} a_{i,r(j)} v_j,$$

and we arrive at

$$\mathbf{c}_i \simeq e^{2\pi b_i u_i^n} + e^{-2\pi b_i u_i^n} + e^{2\pi b_i p_i^n}.$$

We know from [10, 14] that this is unitary equivalent to

$$\mathbf{c}_i \simeq \int_{\mathbb{R}^+} (e^{2\pi b_i \gamma} + e^{-2\pi b_i \gamma})\, d\mu(\gamma),$$

with the measure given by $d\mu(\gamma) = |S_{b_i}(Q_i + 2\gamma)|^2\, d\gamma$.

Finally, by reversing the transformations above, skipping the variables u_i^n, we obtain an explicit expression of the action $\mathbf{e}_i, \mathbf{f}_i$ involving only the last variable in

$$L^2(\mathbb{R}^{N-2}) \otimes \int_{\mathbb{R}_+} P_\gamma\, d\mu(\gamma). \qquad \blacksquare$$

4.4 Unitary action of the Weyl element w_i

Following the compact case in [16], we adjoin an element w to $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ such that it satisfies the following:

$$w\mathbf{e}w^{-1} = \mathbf{f}, \tag{4.20}$$

$$w\mathbf{f}w^{-1} = \mathbf{e}, \tag{4.21}$$

$$wKw^{-1} = K^{-1}, \tag{4.22}$$

with the Hopf algebra structure

$$\Delta w = R^{-1}(w\otimes w), \tag{4.23}$$

$$S(w) = wK^{-\frac{Q}{b}}, \tag{4.24}$$

$$\epsilon(w) = 1, \tag{4.25}$$

so that in addition it satisfies

$$w^2 = v = uK^{-\frac{Q}{b}}, \tag{4.26}$$

which implies

$$S(w)w = u. \tag{4.27}$$

On the positive representations considered in Proposition 2.8, we define the action of w on $P_\lambda = L^2(\mathbb{R})$ as a unitary operator

$$w \cdot f(x) = e^{\pi \mathbf{i}(\lambda^2 + \frac{Q^2}{4})} f(-x), \tag{4.28}$$

so that all the above properties are satisfied.

Now in the general case, consider the positive representation $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$. For each simple roots α_i, using the branching rules of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ from Theorem 4.7, we define the action of w_i on $\mathcal{P}_\lambda$ as

$$w_i := \mathrm{Id}_{N-2} \otimes \int_{\mathbb{R}_+} w_i^\gamma \, d\mu(\gamma), \tag{4.29}$$

where w_i^γ acts as (4.28) on P_γ. It is clear that w_i is a unitary operator since the branching rules of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is a unitary equivalence.

Now, we can follow the approach in [KR] and calculate the action of $w_i \mathbf{e}_j w_i^{-1}$ and $w_i \mathbf{f}_j w_i^{-1}$, while we also have

$$w_i K_j w_i^{-1} = K_j K_i^{-a_{ij}}. \tag{4.30}$$

We will do the calculations mainly for $\mathbf{e}_i$, while those for $\mathbf{f}_i$ is similar.

Let

$$\bar{\mathbf{e}}_i := q_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} \mathbf{e}_i,$$

$$\bar{\mathbf{f}}_i := q_i^{\frac{1}{2}} K_i^{\frac{1}{2}} \mathbf{f}_i,$$

so that the R operator can be expressed as

$$R_i = g_b(\bar{\mathbf{e}}_i \otimes \bar{\mathbf{f}}_i) q^{\frac{H_i \otimes H_i}{2}}. \tag{4.31}$$

Note that $\bar{\mathbf{e}}_i$ and $\bar{\mathbf{f}}_i$ are still positive essentially self-adjoint and satisfy the transcendental relations.

For any Hopf algebra A, one can define the adjoint action of A on itself by

$$a \circ b = \sum_i a^i b S(a_i), \tag{4.32}$$

where $\Delta(a) = \sum_i a^i \otimes a_i$. Then the action $w_i \circ \bar{\mathbf{e}}_j$ can be calculated exactly as in [KR], taking into account the new antipode, and we still obtain

$$w_i \circ \bar{\mathbf{e}}_j = w_i \bar{\mathbf{e}}_j K_i^{\frac{1}{2}a_{ij}} w_i^{-1}. \tag{4.33}$$

On the other hand, $V_{ij} = \{(\bar{\mathbf{e}}_i)^n \circ \bar{\mathbf{e}}_j\}_{n=0}^{-a_{ij}}$ is an irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module with highest weight $-a_{ij}$. Since w_i flips the action of E_i and F_i by definition, the adjoint action maps the lowest weight vector to highest weight vector. In particular, we have

$$w_i \circ \bar{\mathbf{e}}_j = c_{ij} \bar{\mathbf{e}}_i^{-a_{ij}} \circ \bar{\mathbf{e}}_j \tag{4.34}$$

for some constant c_{ij}. Note that this equation also holds for the modular double counterpart $\tilde{\mathbf{e}}_j$. Hence the constant c_{ij} is uniquely determined by the fact that $w_i \mathbf{e}_j w_i^{-1}$ is positive and satisfy the transcendental relation. Now, it is easy to calculate that

$$\bar{\mathbf{e}}_i^{-a_{ij}} \circ \bar{\mathbf{e}}_j = (-1)^{a_{ij}} \mathbf{e}_{ij} K_i^{\frac{a_{ij}}{2}} K_j^{-\frac{1}{2}}, \tag{4.35}$$

where $\mathbf{e}_{ij}$ is defined in Propositions 4.1 and 4.2, and

$$\begin{aligned} w_i \bar{\mathbf{e}}_j K_i^{\frac{a_{ij}}{2}} w_i^{-1} &= w_i q_j^{\frac{1}{2}} K_j^{-\frac{1}{2}} \mathbf{e}_j K_i^{\frac{a_{ij}}{2}} w_i^{-1} \\ &= w_i \mathbf{e}_j w_i^{-1} q_j^{-\frac{1}{2}} K_j^{-\frac{1}{2}}. \end{aligned}$$

Hence, combining we have

$$w_i \mathbf{e}_j w_i^{-1} = c'_{ij} \mathbf{e}_{ij} K_i^{\frac{a_{ij}}{2}}.$$

The constant can now be easily determined by positivity to be $c'_{ij} = q_i^{-\frac{a_{ij}^2}{4}}$.

Definition 4.8. We define $w'_i := w_i q_i^{\frac{H_i^2}{4}}$ and

$$T_i(a) = w'_i a (w'_i)^{-1}. \tag{4.36}$$

□

The operators T_i resemble the Lusztig's isomorphisms [20], while taking positivity into account. We have

Theorem 4.9. The operators T_i are given on the generators by

$$T_i(\mathbf{e}_i) = q_i \mathbf{f}_i K_i^{-1} = q_i^{-1} K_i^{-1} \mathbf{f}_i, \tag{4.37}$$

$$T_i(\mathbf{f}_i) = q_i^{-1} K_i \mathbf{e}_i = q_i \mathbf{e}_i K_i, \tag{4.38}$$

$$T_i(\mathbf{e}_j) = \mathbf{e}_{ij} \quad \text{for } i, j \text{ adjacent}, \tag{4.39}$$

$$T_i(\mathbf{f}_j) = \mathbf{f}_{ij} \quad \text{for } i, j \text{ adjacent}, \tag{4.40}$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}. \tag{4.41}$$

In particular, Proposition 2.23 is still satisfied. Furthermore, the same relations also hold for the modular double counterpart $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$, and $\tilde{K}_i$. □

Proof. By definition,

$$\begin{aligned}
T_i(\mathbf{e}_i) &= w_i q_i^{\frac{H_i^2}{4}} \mathbf{e}_i q_i^{-\frac{H_i^2}{4}} w_i^{-1} \\
&= w_i \mathbf{e}_i q_i^{\frac{(H_i+2)^2}{4}} q_i^{-\frac{H_i^2}{4}} w_i^{-1} \\
&= w_i \mathbf{e}_i K_i q_i w_i^{-1} \\
&= q_i \mathbf{f}_i K_i^{-1} = q_i^{-1} K_i^{-1} \mathbf{f}_i, \\
T_i(\mathbf{e}_j) &= w_i q_i^{\frac{H_i^2}{4}} \mathbf{e}_j q_i^{-\frac{H_i^2}{4}} w_i^{-1} \\
&= w_i \mathbf{e}_i q_i^{\frac{(H_i+a_{ij})^2}{4}} q_i^{-\frac{H_i^2}{4}} w_i^{-1} \\
&= q_i^{-\frac{a_{ij}^2}{4}} \mathbf{e}_{ij} K_i^{\frac{a_{ij}}{2}} K_i^{-\frac{a_{ij}}{2}} q_i^{\frac{a_{ij}^2}{4}} \\
&= \mathbf{e}_{ij},
\end{aligned}$$

and similarly for the calculations of $\mathbf{f}$. The action T_i only differs from Lusztig's isomorphism by certain scaling, hence Proposition 2.23 is still satisfied due to positivity that restricts the scaling.

Finally, since w_i depends only on the root system, and independent of the interchange $b_i \longleftrightarrow b_i^{-1}$ (cf. (4.28)), all the previous arguments work for the tilde variables. ■

Corollary 4.10. Under the positive representations $\mathcal{P}_\lambda$, the operators $T_{i_1}\dots T_{i_k}(X_j)$, where $X=\mathbf{e},\mathbf{f}$ or K, are positive essentially self-adjoint, and satisfy the transcendental relations. □

5 Universal *R* Operator

We are now in the position to define the universal R operator in the flavor of Sections 2.4 and 2.5, generalizing the respective formula.

Theorem 5.1. Let $\mathfrak{g}_\mathbb{R}$ be the split real form of a simple Lie algebra $\mathfrak{g}$. Let $w_0=s_{i_1}s_{i_2}\dots s_{i_N}$ be a reduced expression of the longest element of the Weyl group. Then the universal R operator for the positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ acting on $\mathcal{P}_{\lambda_1}\otimes\mathcal{P}_{\lambda_2}\simeq L^2(\mathbb{R}^N)\otimes L^2(\mathbb{R}^N)$ is given by

$$R=\prod_{ij} q_i^{\frac{1}{2}(A^{-1})_{ij}H_i\otimes H_j}\prod_{k=1}^{N} g_b(\mathbf{e}_{\alpha_k}\otimes\mathbf{f}_{\alpha_k})\prod_{ij} q_i^{\frac{1}{2}(A^{-1})_{ij}H_i\otimes H_j}, \tag{5.1}$$

where $\mathbf{e}_{\alpha_k}=T_{i_1}T_{i_2}\dots T_{i_{k-1}}\mathbf{e}_{i_k}$, similarly for $\mathbf{f}_{\alpha_k}$. The product is such that the term $k=1$ appears on the rightmost position. Furthermore, R is a unitary operator. □

Remark 5.2. By Corollary 4.10, the generators $\mathbf{e}_{\alpha_k}\otimes\mathbf{f}_{\alpha_k}$ are positive, hence the expression is well defined. By Lemma 2.19, it is clear that R is a unitary operator. By commuting the last factor, R can also be written as

$$R=\prod_{ij} q_i^{\frac{1}{2}(A^{-1})_{ij}H_i\otimes H_j}\prod_{k=1}^{N} g_b(\bar{\mathbf{e}}_{\alpha_k}\otimes\bar{\mathbf{f}}_{\alpha_k}), \tag{5.2}$$

where $\bar{\mathbf{e}}_{\alpha_k}=q_{i_k}^{-\frac{1}{2}}K_{\alpha_k}^{\frac{1}{2}}\mathbf{e}_{\alpha_k}$ and $\bar{\mathbf{f}}_{\alpha_k}=q_{i_k}^{-\frac{1}{2}}K_{\alpha_k}^{-\frac{1}{2}}\mathbf{f}_{\alpha_k}$. Note that the symmetrizing factor d of the Cartan matrix is absorbed in the definition of the q_i's. □

By general theory developed in [16, 19], the R operator can be written in terms of the root components as follows. By abuse of notation, let $w_0=w_{i_1}\dots w_{i_N}$. Then

$$\begin{aligned} R^{-1}(w_0\otimes w_0) &= \Delta(w_0)\\ &= \Delta(w_{i_1})\dots\Delta(w_{i_N})\\ &= R_{i_1}^{-1}(w_{i_1}\otimes w_{i_1})\dots R_{i_N}^{-1}(w_{i_N}\otimes w_{i_N}), \end{aligned}$$

or

$$R=(w_0\otimes w_0)(w_{i_N}\otimes w_{i_N})^{-1}R_{i_N}\dots(w_{i_1}\otimes w_{i_1})^{-1}R_{i_1}. \tag{5.3}$$

It turns out, not surprisingly, that it suffices to prove the braiding relations and quasi-triangularity relations in the case of rank = 2. It is known that the braiding relations and quasi-triangularity relations imply

$$(S\otimes S)(R)=R, \tag{5.4}$$

In rank = 2 case, this means that the expression of R corresponding to the Coxeter relation (2.55) for the change of words of w_0 is the same. Therefore, the definition given in Theorem 5.1 does not depend on the choice of reduced expression, hence the expression of R is uniquely defined.

In the next section, we will show that this R operator arises as the canonical element of certain Drinfeld's double construction. Hence the braiding relation and the quasi-triangularity will be automatic from the formal algebraic manipulation. However, it is still instructive to see explicitly how the functional equations of the quantum dilogarithm $g_b(x)$ play a role in the calculation of these properties.

5.1 Braiding relations in simply laced case

Consider the case of type A_2, and choose $w_0=s_1s_2s_1$. The universal R operator is given explicitly by

$$R=\mathbf{Q}^{\frac12}g_b(\mathbf{e}_2\otimes\mathbf{f}_2)g_b(\mathbf{e}_{12}\otimes\mathbf{f}_{12})g_b(\mathbf{e}_1\otimes\mathbf{f}_1)\mathbf{Q}^{\frac12}, \tag{5.5}$$

where

$$\mathbf{Q}=q^{\frac23H_1\otimes H_1+\frac13H_1\otimes H_2+\frac13H_2\otimes H_1+\frac23H_2\otimes H_2}. \tag{5.6}$$

We will show that

$$\Delta'(\mathbf{e}_1)R=R\Delta(\mathbf{e}_1). \tag{5.7}$$

The other cases are similar.

$$\begin{aligned}\Delta'(\mathbf{e}_1)\mathbf{Q}^{\frac12}&=(\mathbf{e}_1\otimes K_1^{-\frac12}+K_1^{\frac12}\otimes\mathbf{e}_1)\mathbf{Q}^{\frac12}\\&=\mathbf{Q}^{\frac12}(\mathbf{e}_1\otimes K_1^{-1}+1\otimes\mathbf{e}_1).\end{aligned}$$

Next we have

$$(\mathbf{e}_1 \otimes K_1^{-1} + 1 \otimes \mathbf{e}_1) g_b(\mathbf{e}_2 \otimes \mathbf{f}_2) = g_b(\mathbf{e}_2 \otimes \mathbf{f}_2)(\mathbf{e}_1 \otimes K_1^{-1} + \mathbf{e}_{12} \otimes q^{\frac{1}{2}} K_1^{-1}\mathbf{f}_2 + 1 \otimes \mathbf{e}_1),$$

where we used the generalized pentagon relation (3.3) with

$$\frac{[\mathbf{e}_2 \otimes \mathbf{f}_2, \mathbf{e}_1 \otimes K_1^{-1}]}{q - q^{-1}} = \mathbf{e}_{12} \otimes q^{\frac{1}{2}} K_1^{-1}\mathbf{f}_2,$$

and the fact that $1 \otimes \mathbf{e}_1$ commute with $\mathbf{e}_2 \otimes \mathbf{f}_2$. Then we have by (3.3) again

$$(\mathbf{e}_1 \otimes K_1^{-1} + \mathbf{e}_{12} \otimes q^{\frac{1}{2}} K_1^{-1}\mathbf{f}_2 + 1 \otimes \mathbf{e}_1) g_b(\mathbf{e}_{12} \otimes \mathbf{f}_{12}) = g_b(\mathbf{e}_{12} \otimes \mathbf{f}_{12})(\mathbf{e}_1 \otimes K_1^{-1} + 1 \otimes \mathbf{e}_1),$$

where we used $\mathbf{e}_1\mathbf{f}_{!2} = \mathbf{f}_{12}\mathbf{e}_1 + q^{\frac{1}{2}}(q - q^{-1})K_1^{-1}\mathbf{f}_2$ such that

$$\frac{[1 \otimes \mathbf{e}_1, \mathbf{e}_{12} \otimes \mathbf{f}_{12}]}{q - q^{-1}} = \mathbf{e}_{12} \otimes q^{\frac{1}{2}} K_1^{-1}\mathbf{f}_2.$$

Finally, by Lemma 2.26,

$$(\mathbf{e}_1 \otimes K_1^{-1} + 1 \otimes \mathbf{e}_1) g_b(\mathbf{e}_1 \otimes \mathbf{f}_1) = g_b(\mathbf{e}_1 \otimes \mathbf{f}_1)(\mathbf{e}_1 \otimes K_1 + 1 \otimes \mathbf{e}_1)$$

and

$$\begin{aligned}(\mathbf{e}_1 \otimes K_1 + 1 \otimes \mathbf{e}_1)\mathbf{\Omega}^{\frac{1}{2}} &= \mathbf{\Omega}^{\frac{1}{2}}(\mathbf{e}_1 \otimes K_1^{\frac{1}{2}} + K_1^{-\frac{1}{2}} \otimes \mathbf{e}_1) \\ &= \mathbf{\Omega}^{\frac{1}{2}} \Delta(\mathbf{e}_1).\end{aligned}$$

Recall the expression of R given by (5.3). What we have shown is that (by abuse of notation, write $w_i := w_i \otimes w_i$):

$$\begin{aligned}w_0 w_1^{-1} R_1 w_2^{-1} R_2 w_1^{-1} R_1 \Delta(E_1) &= \Delta'(E_1) w_0 w_1^{-1} R_1 w_2^{-1} R_2 w_1^{-1} R_1, \\ w_0 w_2^{-1} R_2 w_1^{-1} R_1 w_2^{-1} R_2 \Delta(E_1) &= \Delta'(E_1) w_0 w_2^{-1} R_2 w_1^{-1} R_1 w_2^{-1} R_2,\end{aligned}$$

or simplifying:

$$w_1^{-1} R_1 w_2^{-1} R_2 \Delta(E_1) = \Delta(q^{\frac{1}{2}} E_2 K_2^{\frac{1}{2}}) w_1^{-1} R_1 w_2^{-1} R_2, \tag{5.8}$$

$$w_1^{-1} R_1 w_2^{-1} R_2 \Delta(F_1) = \Delta(q^{\frac{1}{2}} F_2 K_2^{-\frac{1}{2}}) w_1^{-1} R_1 w_2^{-1} R_2, \tag{5.9}$$

and also

$$w_1^{-1}R_1\Delta(E_1)=\Delta'(F_1)w_1^{-1}R_1. \tag{5.10}$$

Applying this repeatedly, we can show the braiding relation for all other simply laced type.

5.2 Quasi-triangularity relations in simply laced case

Again let us work with $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(3,\mathbb{R}))$. We will prove the first relation

$$(\Delta\otimes 1)R=R_{13}R_{23},$$

the second one is similar. We have

$$\begin{aligned}(\Delta\otimes 1)R &= \Delta(\mathbf{Q}^{\frac12})(g_b(\Delta\mathbf{e}_2\otimes\mathbf{f}_2)g_b(\Delta\mathbf{e}_{12}\otimes\mathbf{f}_{12})g_b(\Delta\mathbf{e}_1\otimes\mathbf{f}_1)\Delta(\mathbf{Q}^{\frac12})\\
&= \mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}g_b(\mathbf{e}_2\otimes K_2^{\frac12}\otimes\mathbf{f}_2+K_2^{-\frac12}\otimes\mathbf{e}_2\otimes\mathbf{f}_2)\\
&\quad\cdot g_b(\mathbf{e}_{12}\otimes K_1^{\frac12}K_2^{\frac12}\otimes\mathbf{f}_{12}+K_2^{-\frac12}\mathbf{e}_1\otimes K_1^{\frac12}\mathbf{e}_2\otimes\mathbf{f}_{12}+K_1^{-\frac12}K_2^{-\frac12}\otimes\mathbf{e}_{12}\otimes\mathbf{f}_{12})\\
&\quad\cdot g_b(\mathbf{e}_1\otimes K_1^{\frac12}\otimes\mathbf{f}_1+K_1^{-\frac12}\otimes\mathbf{e}_1\otimes\mathbf{f}_1)\mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}\\
&= \mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}g_b(\mathbf{e}_2\otimes K_2^{\frac12}\otimes\mathbf{f}_2)g_b(K_2^{-\frac12}\otimes\mathbf{e}_2\otimes\mathbf{f}_2)\\
&\quad\cdot g_b(\mathbf{e}_{12}\otimes K_1^{\frac12}K_2^{\frac12}\otimes\mathbf{f}_{12})g_b(K_2^{-\frac12}\mathbf{e}_1\otimes K_1^{\frac12}\mathbf{e}_2\otimes\mathbf{f}_{12})g_b(K_1^{-\frac12}K_2^{-\frac12}\otimes\mathbf{e}_{12}\otimes\mathbf{f}_{12})\\
&\quad\cdot g_b(\mathbf{e}_1\otimes K_1^{\frac12}\otimes\mathbf{f}_1)g_b(K_1^{-\frac12}\otimes\mathbf{e}_1\otimes\mathbf{f}_1)\mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}\\
&= \mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}g_b(\mathbf{e}_2\otimes K_2^{\frac12}\otimes\mathbf{f}_2)g_b(\mathbf{e}_{12}\otimes K_1^{\frac12}K_2^{\frac12}\otimes\mathbf{f}_{12})\\
&\quad\cdot g_b(K_2^{-\frac12}\otimes\mathbf{e}_2\otimes\mathbf{f}_2)g_b(K_2^{-\frac12}\mathbf{e}_1\otimes K_1^{\frac12}\mathbf{e}_2\otimes\mathbf{f}_{12})g_b(\mathbf{e}_1\otimes K_1^{\frac12}\otimes\mathbf{f}_1)\\
&\quad\cdot g_b(K_1^{-\frac12}K_2^{-\frac12}\otimes\mathbf{e}_{12}\otimes\mathbf{f}_{12})g_b(K_1^{-\frac12}\otimes\mathbf{e}_1\otimes\mathbf{f}_1)\mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}\\
&= \mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}g_b(\mathbf{e}_2\otimes K_2^{\frac12}\otimes\mathbf{f}_2)g_b(\mathbf{e}_{12}\otimes K_1^{\frac12}K_2^{\frac12}\otimes\mathbf{f}_{12})\\
&\quad\cdot g_b(\mathbf{e}_1\otimes K_1^{\frac12}\otimes\mathbf{f}_1)g_b(K_2^{-\frac12}\otimes\mathbf{e}_2\otimes\mathbf{f}_2)\\
&\quad\cdot g_b(K_1^{-\frac12}K_2^{-\frac12}\otimes\mathbf{e}_{12}\otimes\mathbf{f}_{12})g_b(K_1^{-\frac12}\otimes\mathbf{e}_1\otimes\mathbf{f}_1)\mathbf{Q}_{13}^{\frac12}\mathbf{Q}_{23}^{\frac12}\end{aligned}$$

$$= \mathbf{Q}_{13}^{\frac{1}{2}} g_b(\mathbf{e}_2 \otimes 1 \otimes \mathbf{f}_2) g_b(\mathbf{e}_{12} \otimes 1 \otimes \mathbf{f}_{12}) g_b(\mathbf{e}_1 \otimes 1 \otimes \mathbf{f}_1) \mathbf{Q}_{13}^{\frac{1}{2}}$$
$$\cdot\, \mathbf{Q}_{23}^{\frac{1}{2}} g_b(1 \otimes \mathbf{e}_2 \otimes \mathbf{f}_2) g_b(1 \otimes \mathbf{e}_{12} \otimes \mathbf{f}_{12}) g_b(1 \otimes \mathbf{e}_1 \otimes \mathbf{f}_1) \mathbf{Q}_{23}^{\frac{1}{2}}$$
$$= R_{13} R_{23}.$$

Where in the fourth line we used

$$\frac{[K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 \otimes \mathbf{f}_2, \mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \otimes \mathbf{f}_1]}{q - q^{-1}} = K_2^{-\frac{1}{2}} \mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \mathbf{e}_2 \otimes \mathbf{f}_{12}.$$

By the relation $R_{12} = w_1 R_2 w_1^{-1}$, the above calculation is also equivalent to the following relation of the quantum dilogarithms:

$$g_b(K_2^{-\frac{1}{2}} \mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \mathbf{e}_2 \otimes \mathbf{f}_{12})$$
$$= g_b^*(K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 \otimes \mathbf{f}_2) g_b(\mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \otimes \mathbf{f}_1) g_b(K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 \otimes \mathbf{f}_2) g_b^*(\mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \otimes \mathbf{f}_1)$$
$$= g_b^*(\mathbf{e}_{12} \otimes K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \mathbf{f}_{12}) g_b(K_2^{\frac{1}{2}} \mathbf{f}_2 \otimes K_2^{-\frac{1}{2}} \mathbf{e}_2 \otimes 1) g_b(\mathbf{e}_{12} \otimes K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \mathbf{f}_{12})$$
$$\times\, g_b^*(K_2^{\frac{1}{2}} \mathbf{f}_2 \otimes K_2^{-\frac{1}{2}} \mathbf{e}_2 \otimes 1),$$

or after rewriting, that $g_b(K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 \otimes \mathbf{f}_2) g_b(K_2^{\frac{1}{2}} \mathbf{f}_2 \otimes K_2^{-\frac{1}{2}} \mathbf{e}_2 \otimes 1)$ commute with $g_b(\mathbf{e}_{12} \otimes K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \mathbf{f}_{12}) g_b(\mathbf{e}_1 \otimes K_1^{\frac{1}{2}} \otimes \mathbf{f}_1)$. Symbolically, using superscript for the corresponding root, and the leg notation for the operators, we present this relation informally as

$$G_{23}^2 G_{21}^{'2} G_{13}^{12} G_{13}^1 = G_{13}^{12} G_{13}^1 G_{23}^2 G_{21}^{'2}, \tag{5.11}$$

which resembles the so-called "Tetrahedron Equation" [15]. It suffices to apply this relation, together with (5.3) repeatedly to obtain the quasi-triangular relation in higher rank.

5.3 Remarks on the nonsimply laced case

The relations for the nonsimply laced case can also be done along the same line. What we have found is that the braiding relations amount to the generalized pentagon relations of g_b given by Proposition 3.2, and the same relations apply to all higher rank case.

On the other hand, the quasi-triangularity is more difficult. For type B_2, it is equivalent to the generalized exponential relation given in Proposition 3.3, which is needed to break down the coproduct of $\mathbf{e}_{21}$ and $\mathbf{e}_{12}$. For simplicity, let

$$\mathbf{e}'_3 := \mathbf{e}_{121} = \mathbf{e}_{2^{-1}1} = \frac{q^{\frac{1}{2}}\mathbf{e}_2\mathbf{e}_1 - q^{-\frac{1}{2}}\mathbf{e}_1\mathbf{e}_2}{q-q^{-1}}, \tag{5.12}$$

$$\mathbf{e}_X := \mathbf{e}_{12} = \frac{\mathbf{e}'_3\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}'_3}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{5.13}$$

(Recall $q = e^{\pi \mathbf{i} b_l^2} = q_2$ and $q^{\frac{1}{2}} = e^{\pi \mathbf{i} b_s^2} = q_1$.) Then R is given by

$$R = \mathbf{\Omega}^{\frac{1}{2}} g_b(\mathbf{e}_2 \otimes \mathbf{f}_2) g_b(\mathbf{e}'_3 \otimes \mathbf{f}'_3) g_b(\mathbf{e}_X \otimes \mathbf{f}_X) g_b(\mathbf{e}_1 \otimes \mathbf{f}_1)\mathbf{\Omega}^{\frac{1}{2}}. \tag{5.14}$$

Proposition 3.3 then implies

$$\begin{aligned} g_{b_s}(\Delta(\mathbf{e}'_3) \otimes \mathbf{f}'_3) &= g_{b_s}(\mathbf{e}'_3 \otimes K_3^{\frac{1}{2}} \otimes \mathbf{f}'_3) g_{b_l}(\mathbf{e}_X K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 K_X^{\frac{1}{2}} \otimes \mathbf{f}'^{\,2}_3) \\ &\quad \cdot g_{b_s}(\mathbf{e}_1 K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 K_1^{\frac{1}{2}} \otimes \mathbf{f}'_3) g_{b_s}(K_3^{-\frac{1}{2}} \otimes \mathbf{e}'_3 \otimes \mathbf{f}'_3), \\ g_{b_l}(\Delta(\mathbf{e}_X) \otimes \mathbf{f}_X) &= g_{b_l}(\mathbf{e}_X \otimes K_X^{\frac{1}{2}} \otimes \mathbf{f}_X) g_{b_l}(\mathbf{e}_1^2 K_2^{-\frac{1}{2}} \otimes \mathbf{e}_2 K_1 \otimes \mathbf{f}_X) \\ &\quad \cdot g_{b_s}(\mathbf{e}_1 K_3^{-\frac{1}{2}} \otimes \mathbf{e}'_3 K_1^{\frac{1}{2}} \otimes \mathbf{f}_X) g_{b_l}(K_X^{-\frac{1}{2}} \otimes \mathbf{e}_X \otimes \mathbf{f}_X), \end{aligned}$$

and together with the generalized pentagon relations the quasi-triangularity can be proved. Again these can be rephrased as a generalized tetrahedron equation using the quantum Weyl element. We conjecture that these are all we need to prove the higher rank case, as well as the case in type G_2.

6 $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ as a Quasi-Triangular Multiplier Hopf Algebra

So far, we have worked on the algebraic calculation quite formally. From the explicit expression of the R operator in Theorem 5.1, it motivates us to define R as the canonical element of certain Drinfeld's double construction. The accurate language to use here turns out to be the so-called multiplier Hopf algebra [26] and its Drinfeld's double construction [3], which gives the notion of a quasi-triangular multiplier Hopf algebra introduced by Zhang [28].

Let us recall the basic definitions. For further details please refer to [26].

Definition 6.1. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Then the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is the C^*-algebra of operators

$$M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : b\mathcal{A} \subset \mathcal{A}, \mathcal{A}b \subset \mathcal{A}\}. \tag{6.1}$$

In particular, $\mathcal{A}$ is an ideal of $M(\mathcal{A})$. □

Definition 6.2. A multiplier Hopf *-algebra is a C^*-algebra $\mathcal{A}$ together with the antipode S, the counit ϵ, and the coproduct map

$$\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A}), \tag{6.2}$$

all of which can be extended to a map from $M(\mathcal{A})$, such that the usual properties of a Hopf algebra holds on the level of $M(\mathcal{A})$. □

Definition 6.3. A quasi-triangular multiplier Hopf algebra is a multiplier Hopf algebra $\mathcal{A}$ together with an invertible element $R \in M(\mathcal{A} \otimes \mathcal{A})$ such that

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \in M(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}), \tag{6.3}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \in M(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}), \tag{6.4}$$

$$\Delta'(a)R = R\Delta(a) \in M(\mathcal{A} \otimes \mathcal{A}) \quad \forall a \in M(\mathcal{A}), \tag{6.5}$$

$$(\epsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \epsilon)(R) = 1 \in M(\mathcal{A}). \tag{6.6}$$

□

Furthermore, the element $u := m^{op}(1 \otimes S)(R)$ will be an invertible element in $M(\mathcal{A})$ such that

$$S^2(a) = uau^{-1} \quad \forall a \in M(\mathcal{A}). \tag{6.7}$$

Definition 6.4. A ribbon multiplier Hopf algebra is a quasi-triangular multiplier Hopf algebra $\mathcal{A}$ that possesses a central ribbon element $v \in M(\mathcal{A})$, such that

$$v^2 = uS(u), \quad S(v) = v, \quad \epsilon(v) = 1, \tag{6.8}$$

$$\Delta(v) = (R_{21}R_{12})^{-1}(v \otimes v) \tag{6.9}$$

hold in $M(\mathcal{A})$. □

6.1 The Borel subalgebra $\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$

Let us fix a positive representation $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$, and fix a reduced expression $w_0 = s_{i_1} \dots s_{i_N}$ of the longest element of the Weyl group. Motivated from the compact case, as well as the expression of R, it is intuitive to choose a "basis" given by

$$\prod_{i=1}^{n} H_i^{m_i} \prod_{k=1}^{N} \mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1} t_k} = H_1^{m_1} \dots H_n^{m_n} \mathbf{e}_{\alpha_N}^{\mathbf{i}b_{i_N}^{-1} t_N} \dots \mathbf{e}_{\alpha_1}^{\mathbf{i}b_{i_1}^{-1} t_1}. \tag{6.10}$$

Here, $N = l(w_0)$, $n = \text{rank}(\mathfrak{g})$, while $t_i \in \mathbb{R}$, and as before

$$\mathbf{e}_{\alpha_k} := T_{i_1} T_{i_2} \dots T_{i_{k-1}} \mathbf{e}_{i_k}. \tag{6.11}$$

Following the approach in [10] for the harmonic analysis of the quantum plane, we give the following definition.

Definition 6.5. We define the C^*-algebraic version of the Borel subalgebra

$$\mathbf{Ub} := \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}}^+)$$

as the operator norm closure of the linear span of all bounded operators on $L^2(\mathbb{R}^N)$ of the form

$$\overrightarrow{F} := F_0(\mathbf{H}) \prod_{k=1}^{N} \int_C \frac{F_k(t_k)}{G_{b_{i_k}}(Q_{i_k} + \mathbf{i}t_k)} \mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1} t_k} \, dt_k, \tag{6.12}$$

where $\mathbf{e}_{\alpha_k}$ is given by (6.11) and

$$F_0(\mathbf{H}) := F_0(\mathbf{i}b_1 H_1, \dots, \mathbf{i}b_n H_b) \tag{6.13}$$

is a smooth compactly supported functions on the positive operators $\mathbf{i}b_k H_k$, $F_k(t_k)$ are entire analytic functions that have rapid decay along the real direction (i.e., for fixed y_0, $F_k(x + \mathbf{i}y_0)$ decays faster than any exponential function in x). Finally, the contour C is along the real axis which goes above the pole of G_b at $t_k = 0$. □

Since $\mathbf{e}_{\alpha_k}$ are positive essentially self-adjoint, $\mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1} t}$ is unitary, and by the decay properties of F_k, the operator $\overrightarrow{F}$ is indeed a well-defined bounded operator acting on $L^2(\mathbb{R}^N)$. Furthermore, since the positive representations are injective, the definition of this algebra does not depend on the choice of the parameter λ. Finally, by Proposition 6.8 below, the usual complex conjugation gives the star structure of $\mathbf{Ub}$.

Remark 6.6. Definition 6.5 is compatible with the modular double counterpart. In other words, we obtain the same algebra when we replace all variables $\mathbf{e}_{\alpha_k}, \mathbf{i}b_i H_i$ with $\tilde{\mathbf{e}}_{\alpha_k}, \mathbf{i}b_i^{-1}\tilde{H}_i$ due to the transcendental relations. Hence **Ub** can indeed be called the modular double of the Borel subalgebra. □

Proposition 6.7. The map defined by

$$\Delta(\overrightarrow{F}) = F_0(\Delta(\mathbf{H}))\prod_{k=1}^{N}\int_C \frac{F_k(t_k)}{G_{b_{i_k}}(Q_{i_k}+\mathbf{i}t_k)}\Delta(\mathbf{e}_{\alpha_k})^{\mathbf{i}b_{i_k}^{-1}t_k}\,\mathrm{d}t_k \tag{6.14}$$

is a coproduct $\Delta : \mathcal{A} \to M(\mathcal{A}\otimes\mathcal{A})$, where $\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$. □

Proof. Coassociativity is immediate since the expression is the same as in the usual case. The factors $G_{b_{i_k}}(Q_{i_k}+\mathbf{i}t_k)$ are needed in order to define the coproduct in the sense of a multiplier Hopf algebra. This follows from the use of the q-binomial formula (Lemma 2.13), or in the nonsimply laced case, the generalized exponential relation (Proposition 3.3), which basically says that $\Delta(\mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}t_k}) = \Delta(\mathbf{e}_{\alpha_k})^{\mathbf{i}b_{i_k}^{-1}t_k}$ cancels the factors $G_{b_i}(Q_{i_k}+\mathbf{i}t_k)$ and introduce two new factors $G_{b_i}(Q_{i_k}+i\tau_1)G_{b_i}(Q_{i_k}+i\tau_2)$ in the respective factors for $\mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}\tau_1}\otimes\mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}\tau_2}$. For nonsimple roots, the extra integration can be shown to be holomorphic due to meromorphicity of G_b as well as application of the delta distribution rules (Proposition 2.15). ■

For the term $\prod_{k=1}^{N}\mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}t}$ to deserve to be called a "basis", it suffices to show that we can interchange the order of the generators. Only the rank $=2$ cases need to be considered, and we show this for the simply laced type as follows.

Proposition 6.8. In type A_2, we have

$$\frac{\mathbf{e}_2^{\mathbf{i}b^{-1}t}\mathbf{e}_1^{\mathbf{i}b^{-1}s}}{G_b(Q+\mathbf{i}t)G_b(Q+\mathbf{i}s)} = \int_C \frac{e^{2\pi\mathbf{i}(s+t)\tau-\pi\mathbf{i}st-\frac{5}{2}\pi\mathbf{i}\tau^2}\mathbf{e}_1^{\mathbf{i}b^{-1}(s-\tau)}\mathbf{e}_{21}^{\mathbf{i}b^{-1}\tau}\mathbf{e}_2^{\mathbf{i}b^{-1}(t-\tau)}}{G_b(Q+\mathbf{i}s-\mathbf{i}\tau)G_b(Q+\mathbf{i}\tau)G_b(Q+\mathbf{i}t-\mathbf{i}\tau)}\,\mathrm{d}\tau, \tag{6.15}$$

where the contour separate the poles of $\tau=0$ and $\tau=s,t$.

$$\frac{\mathbf{e}_{12}^{\mathbf{i}b^{-1}t}}{G_b(Q+\mathbf{i}t)} = \int_C \frac{e^{\pi Q(\tau-t)+\pi\mathbf{i}\tau t-\frac{3}{2}\pi\mathbf{i}\tau^2}\mathbf{e}_1^{\mathbf{i}b^{-1}\tau}\mathbf{e}_{21}^{\mathbf{i}b^{-1}(t-\tau)}\mathbf{e}_2^{\mathbf{i}b^{-1}\tau}}{G_b(Q+\mathbf{i}t-\mathbf{i}\tau)G_b(Q+\mathbf{i}\tau)}\,\mathrm{d}\tau, \tag{6.16}$$

where the contour separate the poles of $\tau=0$ and $\tau=t$. □

Note that by taking $s, t \to -\mathbf{i}b$, one recovers the standard relation

$$\mathbf{e}_2\mathbf{e}_1 = q\mathbf{e}_1\mathbf{e}_2 - (q-q^{-1})q^{\frac{1}{2}}\mathbf{e}_{21}, \tag{6.17}$$

$$\mathbf{e}_{12} = q^{\frac{1}{2}}\mathbf{e}_1\mathbf{e}_2 - q\mathbf{e}_{21}, \tag{6.18}$$

by means of Proposition 2.15. Also, the factors $G_b(Q+\mathbf{i}t)$ implies that the holomorphicity condition for **Ub** is still satisfied.

Proof. By the generalized pentagon relation (3.3), we have

$$g_b(q^{-\frac{1}{2}}K_2^{\frac{1}{2}}\mathbf{e}_2)g_b(q^{\frac{1}{2}}K_1^{-\frac{1}{2}}\mathbf{e}_1) = g_b(q^{\frac{1}{2}}K_1^{-\frac{1}{2}}\mathbf{e}_1)g_b(K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}\mathbf{e}_{21})g_b(q^{-\frac{1}{2}}K_2^{\frac{1}{2}}\mathbf{e}_2).$$

Now expand the relation using Lemma 2.18, and equate the powers of K_1 and K_2 we will obtain (6.15).

Next, using again the generalized pentagon relation again, written as

$$g_b^*(q^{\frac{1}{2}}K_2^{-\frac{1}{2}}\mathbf{e}_2)g_b(q^{-\frac{1}{2}}K_1^{\frac{1}{2}}\mathbf{e}_1)g_b(q^{\frac{1}{2}}K_2^{-\frac{1}{2}}\mathbf{e}_2)g_b^*(q^{-\frac{1}{2}}K_1^{\frac{1}{2}}\mathbf{e}_1) = g_b(K_2^{-\frac{1}{2}}K_1^{\frac{1}{2}}\mathbf{e}_{12}),$$

expanding by Lemma 2.18 and equating again the powers of K_1 and K_2, and using the first equation to interchange $\mathbf{e}_1$ and $\mathbf{e}_2$, the integral can be evaluated explicitly and we obtain (6.16). ■

The interchange relation for type B_n, C_n, and F_4 can be obtained along the same line by combining Propositions 3.2 and 3.3. For the G_2 case, one can also obtain the interchange relation for the generators $\mathbf{e}_1$ and $\mathbf{e}_2$ using the general form of the pentagon equation of g_b generalizing the one given in [17], however, explicit interchange relations for the nonsimple root basis have not been computed.

As a corollary, we can now define the antipode.

Definition 6.9. The antipode is defined on the generators by (cf. (4.9))

$$S(H_i) = -H_i \tag{6.19}$$

$$S(\mathbf{e}_i^{\mathbf{i}b_i^{-1}t}) = e^{-\pi Q_i t}\mathbf{e}_i \tag{6.20}$$

and extended anti-homomorphically. □

So, for example, we have $S(\mathbf{e}_{12}^{\mathbf{i}b^{-1}t}) = e^{-2\pi Qt}\mathbf{e}_{21}^{\mathbf{i}b^{-1}t}$.

Corollary 6.10. The antipode is a map from **Ub** to **Ub**. Furthermore, **Ub** also possesses a natural star structure given by complex conjugation, such that the analytic properties are satisfied. □

Proof. The antipode is well defined by the interchange relation from Proposition 6.8, while the star structure follows from the complex conjugation properties (2.31) of $G_b(x)$. ■

Finally, we define the counit

$$\epsilon(\overrightarrow{F}) = F_0(\mathbf{0}) \in \mathbb{C}, \tag{6.21}$$

by setting all H_i to be zero.

Corollary 6.11. The C^*-algebra **Ub** is a multiplier Hopf *-algebra in the sense of Definition 6.2. □

6.2 Hopf pairing and Drinfeld's double

For two Hopf algebra $\mathcal{A}, \mathcal{A}'$, a pairing is called a Hopf pairing if for $a \in \mathcal{A}$, $b, c \in \mathcal{A}'$,

$$\langle a, bc\rangle = \langle \Delta(a), b \otimes c\rangle = \sum \langle a^i, b\rangle \langle a_i, c\rangle, \tag{6.22}$$

$$\langle S(a), b\rangle = \langle a, S(b)\rangle, \tag{6.23}$$

$$\langle a, 1\rangle = \epsilon(a), \quad \langle 1, b\rangle = \epsilon(b), \tag{6.24}$$

where $\Delta(a) = \sum a^i \otimes a_i$. Moreover, it can be extended naturally to the multiplier algebra $M(\mathcal{A})$. Let $\mathbf{Ub}^-$ be the multiplier Hopf algebra generated in the above sense by $\mathbf{i}b_i H_i'$ and $\mathbf{f}_{\alpha_k}$ with the opposite coproduct. Then we define the pairing on the generators (we used the modified generator, cf. Remark 5.2) as

Proposition 6.12. There exists a Hopf pairing given by

$$\langle (\mathbf{i}b_i H_i)^n, (\mathbf{i}b_i H_i')^m\rangle = \delta_{mn} n! \frac{\mathbf{i}}{\pi}, \tag{6.25}$$

$$\langle \bar{\mathbf{e}}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}s}, \bar{\mathbf{f}}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}t}\rangle = \delta(s-t) G_{b_{i_k}}(Q_{i_k} + \mathbf{i}t)\, e^{\pi \mathbf{i} t^2}, \tag{6.26}$$

or more generally, denoting $\overrightarrow{F} \in \mathbf{Ub}$, $\overrightarrow{F'} \in \mathbf{Ub}^-$,

$$\langle \overrightarrow{F}, \overrightarrow{F'} \rangle = \langle F_0(\mathbf{H}), F_0'(\mathbf{H}') \rangle \prod_{k=1}^{N} \int \frac{F_{i_k}(t_k) F'_{i_k}(t_k)\, e^{\pi \mathbf{i} t_k^2}}{G_{b_{i_k}}(Q_{i_k} + \mathbf{i} t_k)}\, dt_k. \tag{6.27}$$

□

Proof. We will show that the definition is consistent with the Hopf pairing between simple root generators:

$$\begin{aligned}
\langle \bar{\mathbf{e}}^{\mathbf{i}b^{-1}s}, \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t} \rangle &= \langle \bar{\mathbf{e}}^{\mathbf{i}b^{-1}s}, \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_1} \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_2} \rangle \\
&= \langle \Delta(\bar{\mathbf{e}}^{\mathbf{i}b^{-1}s}), \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_1} \otimes \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_2} \rangle \\
&= \left\langle \int_C \frac{G_b(-\mathbf{i}s + \mathbf{i}\tau) G_b(-\mathbf{i}\tau)}{G_b(-\mathbf{i}s)} \bar{\mathbf{e}}^{\mathbf{i}b^{-1}\tau} \otimes K^{\mathbf{i}b^{-1}\tau} \bar{\mathbf{e}}^{\mathbf{i}b^{-1}(s-\tau)}, \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_1} \otimes \bar{\mathbf{f}}^{\mathbf{i}b^{-1}t_2} \right\rangle \\
&= \int_C \frac{G_b(-\mathbf{i}s + \mathbf{i}\tau) G_b(-\mathbf{i}\tau)}{G_b(-\mathbf{i}s)} \delta(\tau - t_1)\, e^{\pi \mathbf{i} t_1^2} G_b(Q + \mathbf{i}t_1) \delta(s - \tau - t_2) \\
&\quad \times e^{\pi \mathbf{i} t_2^2} G_b(Q + \mathbf{i}t_2)\, d\tau \\
&= \frac{G_b(-\mathbf{i}t_2) G_b(-\mathbf{i}t_1) G_b(Q + \mathbf{i}t_1) G_b(Q + \mathbf{i}t_2)}{G_b(-\mathbf{i}s)}\, e^{\pi \mathbf{i} t_1^2 + \pi \mathbf{i} t_2^2} \delta(s - t_1 - t_2) \\
&= e^{\pi \mathbf{i} s^2} G_b(Q + \mathbf{i}s) \delta(s - (t_1 + t_2)).
\end{aligned}$$

The other cases are similar. The properties involving antipode are easy to check if we choose the reverse ordering of the basis of $\mathbf{Ub}^-$. The properties of ϵ are trivial. ■

Now we recall the Drinfeld's double construction in the setting of multiplier Hopf algebra.

Definition 6.13 ([3]). The Drinfeld's double $\mathcal{D}$ of multiplier Hopf algebra $\mathcal{A}$ and its dual $\mathcal{A}'$ is a Hopf algebra with underlying vector space $\mathcal{A} \otimes \mathcal{A}'$, comultiplication $\Delta_{\mathcal{A}} \otimes \Delta_{\mathcal{A}'}^{op}$, and product given by

$$(a \otimes x)(b \otimes y) = \sum ab_{(2)} \otimes x_{(2)} y \langle b_{(1)}, S_{\mathcal{A}'}^{-1}(x_{(3)}) \rangle \langle b_{(3)}, x_{(1)} \rangle. \tag{6.28}$$

□

Then it is known [4] that the Drinfeld's double $\mathcal{D}$ is a quasi-triangular multiplier Hopf algebra, where R is given by the canonical element, which is the unique element in

$M(\mathcal{D}\otimes\mathcal{D})$ such that

$$\langle R, b\otimes a\rangle = \langle a, b\rangle, \quad a\in\mathcal{A}, b\in\mathcal{A}'. \tag{6.29}$$

Definition 6.14. We define

$$\mathbf{U} := \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{g}_\mathbb{R}) := \mathcal{D}(\mathbf{Ub})/(H_i' = (A^{-1})_{ij}H_j) \tag{6.30}$$

to be the Drinfeld's double of the Borel subalgebra **Ub** modulo the Cartan subalgebra $\mathfrak{h}\subset\mathbf{Ub}^-$. □

Corollary 6.15. **U** is a quasi-triangular multiplier Hopf algebra. The canonical element is given precisely by R as in Theorem 5.1. □

Proof. This follows directly from the explicit expression of R, the integral expression of $g_b(x)$ from (2.41), and the Hopf pairing we are using. ■

Finally, we note that R acts as a unitary operator on the positive representations $\mathcal{P}_{\lambda_1}\otimes\mathcal{P}_{\lambda_2}$ giving the braiding structure.

6.3 The ribbon structure of $\hat{\mathcal{U}}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$

In Section 4.2, we have computed in the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ the element $u = m^{op}(1\otimes S)(R)$ to be

$$u = vK^{\frac{Q}{b}},$$

which is now clear that it lies in the multiplier algebra $M(\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{sl}(2,\mathbb{R})))$ in the sense defined in the previous subsection. Let us adjoin the unitary operators $w_1,\ldots,w_n$ defined in (4.29) to the algebra **U**, and call this $\hat{\mathcal{U}}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$.

Proposition 6.16. Define $v = w_0^2$ where $w_0 = w_{i_1}\ldots w_{i_N}$ with $s_{i_1}\ldots s_{i_N}$ a reduced expression of the longest element. Then v is a ribbon element, making $\hat{\mathcal{U}}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ a ribbon multiplier Hopf algebra. □

The properties of v follows directly from the coproduct properties of w_i, and the fact that w_0^2 commute with all the generators $\mathbf{e}_i, \mathbf{f}_i, K$, so that v is central. Furthermore, the operator u can now be expressed as

$$u = v \prod_{i=1}^{n} K_i^{\frac{Q_i}{b_i}}, \tag{6.31}$$

which is again clear that it lies in the multiplier Hopf algebra $M(\mathbf{U})$.

With the involvement of Q^2 in the expression of v (cf. (4.16)), this means that there are no classical limit as $b \to 0$, and we believe that this observation opens up a possibility of finding a new class of quantum topological invariants, where the ribbon structure plays a crucial role [24, 25].

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