

Representation of the quantum plane, its quantum double, and harmonic analysis on $GL_q^+(2, \mathbb{R})$

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Abstract We give complete detail of the description of the GNS representation of the quantum plane \mathcal{A} and its dual $\widehat{\mathcal{A}}$ as a von Neumann algebra. In particular, we obtain a rather surprising result that the multiplicative unitary W is manageable in this quantum semigroup context. We study the quantum double group construction introduced by Woronowicz, and using Baaj and Vaes' construction of the multiplicative unitary W_m , we give the GNS description of the quantum double $\mathcal{D}(\mathcal{A})$ which is equivalent to $GL_q^+(2, \mathbb{R})$. Furthermore, we study the fundamental corepresentation $T^{\lambda, t}$ and its matrix coefficients, and show that it can be expressed by the b -hypergeometric function. We also study the regular corepresentation and representation induced by W_m and prove that the space of L^2 functions on the quantum double decomposes into the continuous series representation of $U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ with the quantum dilogarithm $|S_b(Q + 2i\alpha)|^2$ as the Plancherel measure. Finally, we describe certain representation theoretic meaning of integral transforms involving the quantum dilogarithm function.

Keywords GNS representations · Locally compact quantum groups · Quantum double · Split real quantum groups · Quantum dilogarithm

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1 Introduction

The quantum plane \mathcal{A}_q as a Hopf algebra is generated by self-adjoint elements A, B satisfying $AB = q^2BA$ and the coproduct

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes A + 1 \otimes B \quad (1.1)$$

for $q = e^{\pi i b^2}$ with $|q| = 1$. The coproduct reflects the fact that it is the quantum version of the classical $ax + b$ group, which is the group of affine transformations on the real line \mathbb{R} . Sometimes, known as the quantum $ax + b$ group, it has been studied for example in [35, 46]. The main problem arose from the fact that A and B are realized as unbounded operators, and that $\Delta(B)$ is in general not self-adjoint. This poses quite some problems in the well definedness of the algebra on the C^* -algebraic level.

A class of well-behaved “integrable” representations is studied in [10, 38] where instead we impose the condition that both A, B are *positive* self-adjoint. In this case, the operators can be realized as the “canonical representation”:

$$A = e^{2\pi bs}, \quad B = e^{2\pi bp}, \quad (1.2)$$

where they act on $\mathcal{H}_{\text{irr}} := L^2(\mathbb{R})$ as unbounded operators. This simplifies a lot of functional analytic problems, because now all the operators considered are positive essentially self-adjoint, with the help of certain transformations that can be carried out by the quantum dilogarithm function $g_b(x)$. On the other hand, with positivity, we can define using functional calculus a wide class of functions on \mathcal{A}_q , and in this way, we can define the C^* -algebra $\mathcal{A} := C^\infty(\mathcal{A}_q)$ of “functions vanishing at infinity” for \mathcal{A}_q , which is expressed using an integral transform

$$f := \iint f(s, t) A^{ib^{-1}s} B^{tb^{-1}t} ds dt, \quad (1.3)$$

where $f(s, t)$ satisfies certain analytic properties. It turns out that this definition encodes the modular double counterpart as well. By definition, the other half $\mathcal{A}_{\tilde{q}}$, first introduced by Faddeev [5], is generated by \tilde{A}, \tilde{B} satisfying $\tilde{A}\tilde{B} = \tilde{q}^2\tilde{B}\tilde{A}$ with the same coproduct, where $\tilde{q} = e^{\pi ib^{-2}}$, and they are related to the quantum plane by $\tilde{A} = A^{1/b^2}, \tilde{B} = B^{1/b^2}$. This $b \longleftrightarrow b^{-1}$ duality is manifest in the definition of \mathcal{A} and subsequently present in all later calculations involved.

The first main result of this paper is the derivation of the Haar functional on \mathcal{A} (cf. Theorem 4.7). It can be expressed simply by

$$h \left(\iint f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) = f(0, iQ) \quad (1.4)$$

where $Q := b + b^{-1}$. This comes as a surprise as there is no classical Haar measure on the $ax + b$ semigroup due to the lack of inverse. With \mathcal{A} a C^* -algebra equipped with a left invariant Haar weight, we can carry out the so-called Gelfand–Naimark–Segal (GNS) construction, which essentially represents \mathcal{A} naturally on a Hilbert space \mathcal{H} induced by its own multiplication. In this paper, we put \mathcal{A} in the context of the theory of locally compact quantum group in the von Neumann setting [24, 25], in which the modular theory (Tomita–Takesaki’s Theory) for von Neumann algebra is studied, and the main ingredient, the multiplicative unitary W , can be defined as a unitary operator on $\mathcal{H} \otimes \mathcal{H}$.

The multiplicative unitary W is a unitary operator on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ that satisfies the pentagon equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}. \quad (1.5)$$

For every quantum group, a multiplicative unitary can be constructed using the coproduct; however, not every multiplicative unitary is related to a quantum group. In [45], Woronowicz introduced the notion of *manageability* that describes a class of

well-behaved multiplicative unitary. The second main result of this paper is that this multiplicative unitary W obtained for the quantum plane above is in fact manageable (cf. Theorem 2.21). This is rather striking because in [46], it is mentioned that manageability is the property that distinguishes quantum groups from quantum semigroups; however, as noted above, we have been restricting ourselves to positive operators. The main reason is the fact that the GNS construction provides us with a “bigger” Hilbert space, so that there is more freedom of choice for the operators to satisfy the manageability condition. As a by-product of this discrepancy, it turns out that we obtain a new transformation rule for the quantum dilogarithm function (cf. Proposition 9.3) that is not available in the literature.

The motivation for the study of the quantum plane comes from the quantum double group construction introduced in [30] for compact quantum groups, which is the dual version of the Drinfeld double construction. In [34], it is shown that the quantum double construction of the quantum $az + b$ group gives rise to $GL_q(2, \mathbb{C})$ for certain root of unity q . It turns out that this construction can be carried over to locally compact quantum groups. In order to carry out a similar recipe for the quantum plane for general q with $|q| = 1$, it is necessary to define the dual space $\widehat{\mathcal{A}}$ in the same setting as \mathcal{A} on the C^* -algebraic level. The third main result of the paper is the derivation of the non-degenerate pairing between \mathcal{A} and $\widehat{\mathcal{A}}$ on the C^* -algebraic level, which remarkably involves the quantum dilogarithm function in place of the q -factorial. Following [24], we can then describe the GNS representation for $\widehat{\mathcal{A}}$ acting on the same GNS space \mathcal{H} for \mathcal{A} , its modular theory, and its multiplicative unitary as before.

After defining the dual space, we can apply the quantum double group construction and obtain a new algebra $\mathcal{D}(\mathcal{A})$ generated by $\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ which is precisely the $GL_q^+(2, \mathbb{R})$ quantum (semi)group as Hopf algebra, where the generators are again restricted to positive self-adjoint elements. We note also that the relations for $GL_q^+(2, \mathbb{R})$ is a special case for the two parameter deformations $GL_{p,q}(2, \mathbb{C})$ observed in [4, 37] where $p := 1, q := q^2$. On the other hand, the relations involved unmistakably resemble the quantum Minkowski spacetime relations defined in [9], but this time in the non-compact setting where $|q| = 1$ and the variables z_{ij} are positive self-adjoint. Therefore, the quantum plane can be seen as a building block toward this “split quantum Minkowski spacetime” \mathcal{M}_q . Hence, from the properties of $\mathcal{D}(\mathcal{A})$, the quantum Minkowski spacetime can be easily extended to the C^* -algebraic and von Neumann algebraic level, and hence $L^2(GL_q^+(2, \mathbb{R}))$ is well defined. Motivated from the representation theory of classical $SL^+(2, \mathbb{R})$, we define the matrix coefficients $T_{s,\alpha}^{\lambda,t}(z)$ for the fundamental corepresentation of $\mathcal{D}(\mathcal{A})$ (Definition 7.10). For $t = \lambda$, it is explicitly given by

$$T_{s,\alpha}^{\lambda}(z) = \frac{1}{2\pi b} \left(\int_0^\infty (xz_{11} + z_{21})^{b^{-1}(l-i\alpha)} (xz_{12} + z_{22})^{b^{-1}(l+i\alpha)} x^{b^{-1}(-l+is)} \frac{dx}{x} \right) N_{\frac{\rho}{2b}}, \tag{1.6}$$

which can be seen to be a generalization of the matrix coefficient in the compact case [11]. On the other hand, a closed form expression (cf. Corollary 7.14) using the

b -hypergeometric function F_b is found, which clearly gives a quantum analogue of the classical formula to the matrix coefficients for $SL(2, \mathbb{R})$.

To complete the description of $\mathcal{D}(\mathcal{A})$ in the von Neumann setting, it is necessary to compute its multiplicative unitary \mathbf{W}_m . A derivation is obtained from the construction given in [1, 16] for general quantum groups where \mathbf{W}_m is obtained as a product of 6 W 's of the base quantum group and its dual. Restricting to the present simpler setting and using the properties of W , we simplify the expression to just 4 W 's. Explicitly, they are given by

$$\mathbf{W} = W_{13}V''_{32}\widehat{W}_{24}V^*_{32} \tag{1.7}$$

(cf. Proposition 7.49). This is encouraging since it has been shown in certain context that the R matrix (satisfying the hexagon or Yang Baxter equation) can be expressed in 4 R 's [26, Ex 7.3.3], or 4 W 's [17] from the smaller group using instead the Heisenberg Double construction.

With the new multiplicative unitary constructed, it is straightforward, despite tedious, to construct both the left and right regular corepresentations for $\mathcal{D}(\mathcal{A})$. Furthermore, Frenkel [8] has constructed a non-degenerate pairing between \mathcal{M}_q with $U_q(\mathfrak{gl}(2, \mathbb{R}))$; hence, from the corepresentation, we can derive the fundamental representation $\mathcal{P}_{\lambda,t}$ from $T^{\lambda,t}$, the left and right regular representation from \mathbf{W}_m of $U_q(\mathfrak{gl}(2, \mathbb{R}))$ as well as its modular double on $L^2(GL_q^+(2, \mathbb{R}))$ by the pairing. The final main result (cf. Theorem 8.15) of this paper is the quantum ‘‘Peter Weyl theorem’’

$$L^2(GL_q^+(2, \mathbb{R})) \simeq \int_{\mathbb{R}} \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_{\lambda,s/2} \otimes \mathcal{P}_{\lambda,-s/2} |S_b(Q + 2i\lambda)|^2 d\lambda ds \tag{1.8}$$

as a representation of $U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))_L \otimes U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))_R$, which is a generalization of the statement for $U_q(\mathfrak{sl}(2, \mathbb{R}))$ announced in [31]; however, the details of the proof are never published. The remarkable fact is that the quantum dilogarithm appears as the Plancherel measure for $L^2(GL_q^+(2, \mathbb{R}))$, which comes from the spectral analysis of the Casimir operators. In short, it states that the operator

$$\mathbf{C} = e^{2\pi bx} + e^{-2\pi bx} + e^{-2\pi bp} \tag{1.9}$$

can be diagonalized as multiplication operator

$$\mathbf{C} = e^{2\pi b\lambda} + e^{-2\pi b\lambda}, \tag{1.10}$$

acting on the Hilbert space $L^2(\mathbb{R}_+, |S_b(Q + 2i\lambda)|^2 d\lambda)$, using certain eigenfunction transform (cf. Theorem 8.10). Furthermore, we note that only the fundamental series \mathcal{P}_λ appears in the decomposition of the regular representation. Called the ‘‘self-dual’’ principal series representation, they are known [31] to be closed under tensor product, which is rather interesting since the same does not hold in the classical group setting [33].

As a corollary, we have expressed the multiplicative unitary canonically as a direct integral of the fundamental representations (cf. Corollary 8.13):

$$W_m = \int_{\mathbb{R}} \int_{\mathbb{R}_+}^{\oplus} T^{\lambda, t/2} d\mu(\lambda) dt, \tag{1.11}$$

which generalizes the canonical definition for the compact quantum group given in [30]. An interesting problem will be to investigate its classical limit. As we know, the functions on the full group $SL(2, \mathbb{R})$ contain both the continuous and discrete series representation; hence, the above results may give an insight into the decomposition of the functions on the *positive* semigroup and its decomposition, and distinguish the principal continuous series as the fundamental component. Furthermore, we also know that classically $SL(2, \mathbb{R}) \simeq SU(1, 1)$, but the behaviors change drastically in the quantum level. Therefore, another interesting problem will be a comparison with several known harmonic analysis on the quantum $SU(1, 1)$ group in the operator algebraic setting, where the discrete series and the so-called *strange series* also play a role [13, 23].

Another aspect of this paper is the study of several properties of the quantum dilogarithm function G_b . The quantum dilogarithm function played a prominent role in this quantum theory. This function and its many variants are being studied [12, 21, 36, 44] and applied to vast amount of different areas; for example, the construction of the “ $ax + b$ ” quantum group by Woronowicz et al. [30, 46], the harmonic analysis of the non-compact quantum group $U_q(\mathfrak{sl}(2, \mathbb{R}))$ and its modular double [2, 31, 32], the q -deformed Toda chains [22], current algebra and Virasoro algebra on the lattice [6], and hyperbolic knot invariants [18]. Recently, attempts have also been made to cluster algebra [7, 21] and quantization of the Teichmüller space [3, 10, 19]. One of the important properties of this function is its invariance under the duality $b \leftrightarrow b^{-1}$ that helps encoding the details of the modular double in \mathcal{A} and also relates, for example, to the self-duality of Liouville theory [31]. The classical limit and several relations of G_b are studied in the previous paper [14]. In this paper, we give a proof of the important relation (cf. Corollary 3.11)

$$\lim_{\epsilon \rightarrow 0} \int_C \frac{G_b(Q - \epsilon)G_b(iz)}{G_b(Q + iz - \epsilon)} f(z) dz = f(0) \tag{1.12}$$

with suitable contour C , which is important to make sense rigorously of certain calculations involving the Haar functional, as well as the non-degenerate pairing needed to obtain the regular representations. This statement and its variants is often assumed, for example in [44], but is never proved.

As noted above, there are various version of the quantum dilogarithm. Their relations and visualization of their graphs are presented in [15]. We choose this particular definition G_b, g_b by Teschner [2] for various reasons. First of all, it gives a closed form expression for the q -Binomial Theorem (Lemma 3.8), so that it gives precisely the quantum analogue of the classical Gamma function $\Gamma(x)$, see [14]. Furthermore, various transformations such as the Tau–Beta Theorem (Lemma 3.9), the “45-relation”

(Lemma 3.10), and the “69-relation” (Proposition 9.2) can be written in closed form without any extra constants and exponentials. Finally, the multiplicative unitary constructed above can be expressed in closed form by g_b in a very simple way. Another popular version $G(a_+, a_-; z)$ is given by Ruijsenaars [36] and their relation is given by (cf. Lemma 3.2)

$$G_b(z) = G\left(b, b^{-1}; iz - \frac{iQ}{2}\right) e^{\frac{\pi i}{2}z(z-Q)}. \tag{1.13}$$

We have chosen the pair (a_+, a_-) to be (b, b^{-1}) which demonstrates the self-duality of the Liouville theory in the physical literature. We believe similar treatments in our paper can be made for a general pair with $q = e^{\pi i \frac{a_+}{a_-}}$.

It is well known that by studying the matrix coefficients of various classical matrix groups, we recover certain functional relationships between different special functions; for example, the hypergeometric functions ${}_mF_n$. Various examples can be found, e.g., in [43]. In this quantum setting, it is not surprising that all these various calculations involving the quantum dilogarithm will provide us certain representation theoretic meaning of their relations by integral transforms. By applying the classical limit for G_b obtained in [14] (cf. Theorem 3.5), we recover Barnes’ first and second lemma for the Gamma function. Furthermore, a relation involving G_b that is not commonly used is observed (cf. Proposition 9.3):

$$\int_C G_b(\alpha + i\tau)G_b(\beta - i\tau)G_b(\gamma - i\tau)e^{-2\pi i(\beta-i\tau)(\gamma-i\tau)}d\tau = G_b(\alpha + \gamma)G_b(\alpha + \beta). \tag{1.14}$$

These will be briefly discussed in the last section.

As a side note, the frequent use of the term “modular” in this paper requires some clarification. The modular double introduced by Faddeev refers in the general case to the quantum groups related by the transformation of the modular group for the complex parameter

$$q = e^{\pi i\tau} \mapsto \tilde{q} = e^{\pi i \frac{a\tau+b}{c\tau+d}}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{1.15}$$

while the modular theory of Tomita–Takesaki refers in the classical case to the modularity of the Haar measure; for example, the modular function Δ that relates the left translates of the right Haar measure

$$\mu(g^{-1}A) = \Delta(g)\mu(A), \tag{1.16}$$

where $g \in G$ and A are a Borel set in G .

The present paper is organized as follows. In Sect. 2, we recall several technical results and motivations that is needed in later sections, including the Mellin transform, the GNS construction, and the significance of the multiplicative unitary. We also

collect several results concerning the calculations involving q -commuting variables. In Sect. 3, we recall the definition and properties of the quantum dilogarithm function G_b , and describe its integral transformation formula. In Sect. 4, we define the quantum plane \mathcal{A} in the Hopf algebra and C^* -algebraic level, the modular double, and describe completely its GNS construction and the multiplicative unitary. Section 5 deals with the dual space $\widehat{\mathcal{A}}$; we derive a non-degenerate pairing and obtain its GNS construction on the same space as \mathcal{A} . In Sect. 6, we develop two useful transformation that shed light to the action of the multiplicative unitary, as well as the action of the quantum plane on a Hilbert space. In Sect. 7, we carry out the quantum double group construction, study its fundamental corepresentation, express explicitly the new multiplicative unitary, and the corepresentation induced by it. In Sect. 8, we look at the dual picture and obtain a pairing with $U_q(\mathfrak{gl}(2, \mathbb{R}))$, derive the regular representations, and prove the main theorem on the decomposition of $L^2(GL_q^+(2, \mathbb{R}))$ into principal continuous series of $U_q(\mathfrak{gl}(2, \mathbb{R}))$. Finally, the last section discuss certain integral transformations of G_b arising from representation theoretic calculations.

2 Preliminaries

In this section, we recall several technical results that will be needed in the description of the quantum plane. We will first remind the definitions and properties of the Mellin transform which serves as a motivation to define the quantum plane algebra in Sect. 4. Then, we describe the details of weights and multiplier algebra of a C^* -algebra, and its GNS construction and modular theory in the von Neumann setting. Next, we explain the significance of the multiplicative unitary that is important in the study of locally compact quantum groups. Finally, we collect several technical results concerning the calculations involving q -commuting variables.

2.1 Mellin transform

In this subsection, let us recall the Mellin transform of a function and its properties.

Theorem 2.1 *Let $f(x)$ be a continuous function on the half line $0 < x < \infty$. Then, its Mellin transform is defined by*

$$\phi(s) := (\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x) dx, \tag{2.1}$$

whenever the integration is absolutely convergent for $a < \operatorname{Re}(s) < b$. By the Mellin inversion theorem, $f(x)$ is recovered from $\phi(s)$ by

$$f(x) := (\mathcal{M}^{-1}\phi)(x) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \phi(s) ds, \tag{2.2}$$

where $c \in \mathbb{R}$ is any values in between a and b .

We have the following analyticity theorem [29]:

Theorem 2.2 (Strip of analyticity) *If $f(x)$ is a locally integrable function on $(0, \infty)$ such that it has decay property:*

$$f(x) = \begin{cases} O(x^{-a-\epsilon}) & x \rightarrow 0^+ \\ O(x^{-b+\epsilon}) & x \rightarrow +\infty \end{cases} \quad (2.3)$$

for every $\epsilon > 0$ and some $a < b$, then the Mellin transform defines an analytic function $(\mathcal{M}f)(s)$ in the strip

$$a < \operatorname{Re}(s) < b.$$

(Analytic continuation) Assume $f(x)$ behaves algebraically for $x \rightarrow 0^+$, i.e.,

$$f(x) \sim \sum_{k=0}^{\infty} A_k x^{a_k}, \quad (2.4)$$

where $\operatorname{Re}(a_k)$ increases monotonically to ∞ as $k \rightarrow \infty$. Then, the Mellin transform $(\mathcal{M}f)(s)$ can be analytically continued into $\operatorname{Re}(s) \leq a = -\operatorname{Re}(a_0)$ as a meromorphic function with simple poles at the points $s = -a_k$ with residue A_k .

A similar analytic property holds for the continuation to the right half plane.

(Growth) If $f(x)$ is a holomorphic function of the complex variable x in the sector $-\alpha < \arg x < \beta$ where $0 < \alpha, \beta \leq \pi$, and satisfy the growth property (2.3) uniformly in any sector interior to the above sector, then $(\mathcal{M}f)(s)$ has exponential decay in $a < \operatorname{Re}(s) < b$ with

$$(\mathcal{M}f)(s) = \begin{cases} O(e^{-(\beta-\epsilon)t}) & t \rightarrow +\infty \\ O(e^{(\alpha-\epsilon)t}) & t \rightarrow -\infty \end{cases} \quad (2.5)$$

for any $\epsilon > 0$ uniformly in any strip interior to $a < \operatorname{Re}(s) < b$, where $s = \sigma + it$.

(Parseval's Formula)

$$\int_0^{\infty} f(x)g(x)x^{z-1}dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\mathcal{M}f)(s)(\mathcal{M}g)(z-s)ds, \quad (2.6)$$

where $\operatorname{Re}(s) = c$ lies in the common strip for $\mathcal{M}f$ and $\mathcal{M}g$. In particular, we have

$$\int_0^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |(\mathcal{M}f)\left(\frac{1}{2} + it\right)|^2 dt. \quad (2.7)$$

2.2 Weight and multiplier algebra of a C^* -algebra

In this subsection, we recall the definition of weights on a C^* -algebra and the language of multiplier algebra. Most of the notions are adopted from [24,40]. Let \mathcal{A} be a C^* -algebra and \mathcal{A}^+ its positive self-adjoint elements.

Definition 2.3 A weight on a C^* -algebra \mathcal{A} is a function $\phi : \mathcal{A}^+ \rightarrow [0, \infty]$ such that

$$\phi(x + y) = \phi(x) + \phi(y), \tag{2.8}$$

$$\phi(rx) = r\phi(x), \tag{2.9}$$

for $x, y \in \mathcal{A}^+, r \in \mathbb{R}_{>0}$.

Definition 2.4 Given a weight ϕ on \mathcal{A} , we define

$$\mathcal{M}_\phi^+ = \{a \in \mathcal{A}^+ : \phi(a) < \infty\}, \tag{2.10}$$

$$\mathcal{N}_\phi = \{a \in \mathcal{A} : \phi(a^*a) < \infty\}, \tag{2.11}$$

$$\mathcal{M}_\phi = \text{span}\{y^*x : x, y \in \mathcal{N}_\phi\}. \tag{2.12}$$

Then, it is known that $\mathcal{M}_\phi^+ = \mathcal{M}_\phi \cap \mathcal{A}^+$ and that ϕ extends uniquely to a map $\mathcal{M}_\phi \rightarrow \mathbb{C}$. A weight is called faithful iff $\phi(a) = 0 \implies a = 0$ for every $a \in \mathcal{A}^+$.

Next, we recall a useful notion of a multiplier algebra. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space \mathcal{H} .

Definition 2.5 If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ as operators, then the multiplier algebra $M(\mathcal{A})$ of \mathcal{A} is the C^* -algebra of operators

$$M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : b\mathcal{A} \subset \mathcal{A}, \mathcal{A}b \subset \mathcal{A}\}. \tag{2.13}$$

In particular, \mathcal{A} is an ideal of $M(\mathcal{A})$.

Example 2.6 Important examples include

$$M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H}), \tag{2.14}$$

where $\mathcal{K}(\mathcal{H})$ are compact operators on \mathcal{H} , and

$$M(C_0(X)) = C_b(X), \tag{2.15}$$

where X is a locally compact Hausdorff space, $C_0(X)$ is the algebra of \mathbb{C} -valued functions on X vanishing at infinity equipped with the sup-norm, and $C_b(X)$ is the C^* -algebra of all bounded continuous functions on X .

Proposition 2.7 *Let \mathcal{A} and \mathcal{B} be C^* -algebras. A homomorphism $\phi : \mathcal{A} \longrightarrow M(\mathcal{B})$ is called non-degenerate if the linear span of $\phi(\mathcal{A})\mathcal{B}$ and of $\mathcal{B}\phi(\mathcal{A})$ are both equal to \mathcal{B} . Then, ϕ extends uniquely to a homomorphism $M(\mathcal{A}) \longrightarrow M(\mathcal{B})$. In particular, by taking $\mathcal{B} = \mathbb{C}$, every weight ω on \mathcal{A} has a unique extension to $M(\mathcal{A})$.*

Using the notion of a multiplier algebra, the concept of a multiplier Hopf algebra is introduced in [41] (see also [40]). In particular, the coproduct Δ of \mathcal{A} will be a non-degenerate homomorphism $\Delta : \mathcal{A} \longrightarrow M(\mathcal{A} \otimes \mathcal{A})$. The coassociativity is well defined from the proposition above.

Given a multiplier Hopf $*$ algebra \mathcal{A} with coproduct Δ , we can define left and right invariance of a functional.

Definition 2.8 A linear functional h on \mathcal{A} is called a left invariant Haar functional if it satisfies

$$(1 \otimes h)(\Delta x) = h(x) \cdot 1_{M(\mathcal{A})}, \quad (2.16)$$

where $1_{M(\mathcal{A})}$ is the unital element in $M(\mathcal{A})$.

Similarly, a right invariant Haar functional satisfies

$$(h \otimes 1)(\Delta x) = h(x) \cdot 1_{M(\mathcal{A})}. \quad (2.17)$$

2.3 GNS representation and Tomita–Takesaki’s theory

Let us recall the main objects in the study of GNS representation of a C^* -algebra (see, e.g., [24]).

Definition 2.9 A GNS representation of a C^* -algebra \mathcal{A} with a weight ϕ is a triple

$$(\mathcal{H}, \pi, \Lambda),$$

where \mathcal{H} is a Hilbert space, $\Lambda : \mathcal{A} \longrightarrow \mathcal{H}$ is a linear map, and $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a representation of \mathcal{A} on \mathcal{H} such that $\Lambda(\mathcal{N})$ is dense in \mathcal{H} , and

$$\pi(a)\Lambda(b) = \Lambda(ab) \quad \forall a \in \mathcal{A}, b \in \mathcal{N}, \quad (2.18)$$

$$\langle \Lambda(a), \Lambda(b) \rangle = \phi(b^*a) \quad \forall a, b \in \mathcal{N}, \quad (2.19)$$

where $\mathcal{N} = \{a \in \mathcal{A} : \phi(a^*a) < \infty\}$.

The Tomita–Takesaki’s Theory [39,40] provides a detailed description of the GNS construction in the von Neumann algebraic setting:

Definition 2.10 Giving a weight ϕ that determines the Hilbert space structure by the GNS construction, the operator

$$T : x \longrightarrow x^* \quad (2.20)$$

is closable and has a polar decomposition as

$$T = J \nabla^{\frac{1}{2}}, \tag{2.21}$$

where J is called the modular conjugation, and ∇ is called the modular operator.

A very important property of J is the following

Theorem 2.11 (Murray-von Neumann) *Let M be the completion of $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ in the weak operator topology as a von Neumann algebra. Then, considering $M \subset \mathcal{B}(\mathcal{H})$, we have*

$$J M J = M', \tag{2.22}$$

where M' is the commutant of M .

Definition 2.12 For $x \in \mathcal{A}$, the operator

$$\sigma_t^\phi(x) := \nabla^{it} x \nabla^{-it} \tag{2.23}$$

is called the modular automorphism group.

We have $\phi(a^*b) = \phi(b\sigma_0^\phi(a^*))$

On the Hopf $*$ algebra level, we have the following properties:

Proposition 2.13 *The antipode S has a polar decomposition*

$$S = \tau_{-i/2} \circ R, \tag{2.24}$$

where $\tau_{-i/2}$ denotes the analytic generator of $(\tau_t)_{t \in \mathbb{R}}$, called the scaling group, which is a group of automorphisms of M , and R , called the unitary antipode, is an anti-automorphism of M .

We have $R^2 = 1$ and $S^2 = \tau_{-i}$.

Finally, some properties concerning the left and right invariant Haar functional:

Proposition 2.14 *The left invariant Haar functional ϕ and right invariant Haar functional ψ are related by $\psi = \phi \circ R$.*

We have

$$\phi \circ \tau_t = v^{-t} \phi, \quad \psi \circ \sigma_t^\phi = v^{-t} \psi, \tag{2.25}$$

where $v > 0$ is called the scaling constant.

Furthermore, there exists an element $\delta \in \mathcal{A}$ such that

$$\sigma_t^\phi(\delta) = v^t \delta,$$

where $\delta \in \mathcal{A}$ is called the modular element.

Proposition 2.15 *The map $\Lambda_R : \mathcal{A} \rightarrow \mathcal{H}$ defined by*

$$\Lambda_R(a) := \Lambda_L(a\delta^{-\frac{1}{2}}), \quad (2.26)$$

gives the GNS representation of \mathcal{A} with the right Haar functional on the same space \mathcal{H} , where δ is the modular element defined above.

2.4 The multiplicative unitary

Multiplicative unitaries are fundamental to the theory of quantum groups in the setting of C^* -algebras and von Neumann algebras, and to the generalization of Pontryagin duality. In particular, a multiplicative unitary encodes all the structure maps of a quantum group and its dual. A very good exposition is given in [40].

First, let us define the leg notation.

Definition 2.16 Let \mathcal{H} be a Hilbert space and $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a bounded operator. Then, we define $W_{ij} \in \mathcal{B}(\mathcal{H}^{\otimes k})$ by letting W acts on the factors at the position i, j . In particular, the operators $W_{12}, W_{23}, W_{13} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ are given by the formulas

$$T_{12} := T \otimes Id_H, \quad T_{23} := Id_H \otimes T, \quad (2.27)$$

$$T_{13} := \Sigma_{12} T_{23} \Sigma_{12} = \Sigma_{23} T_{12} \Sigma_{23}, \quad (2.28)$$

where $\Sigma \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denotes the flip $f \otimes g \mapsto g \otimes f$.

Definition 2.17 Let \mathcal{H} be a Hilbert space. A unitary operator $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is called a multiplicative unitary if it satisfies the pentagon equation

$$W_{23} W_{12} = W_{12} W_{13} W_{23}, \quad (2.29)$$

where the leg notation is used.

Given a GNS representation $(\mathcal{H}, \pi, \Lambda)$ of a locally compact quantum group \mathcal{A} , we can define a unitary operator

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)). \quad (2.30)$$

It is known that W is a multiplicative unitary [24, Thm 3.16, Thm 3.18], and the coproduct on \mathcal{A} defining it can be recovered from W :

Proposition 2.18 *Let $x \in \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ as operator. Then,*

$$W^*(1 \otimes x)W = \Delta(x) \quad (2.31)$$

as operators on $\mathcal{H} \otimes \mathcal{H}$.

Proof For $x, f, g \in \mathcal{A}$, we have $x \cdot \Lambda(f) = \Lambda(xf)$; hence,

$$\begin{aligned} W^*((1 \otimes x) \cdot (\Lambda(f) \otimes \Lambda(g))) &= W^*(\Lambda(f) \otimes \Lambda(xg)) \\ &= (\Lambda \otimes \Lambda)(\Delta(xg)(f \otimes 1)) \\ &= (\Lambda \otimes \Lambda)(\Delta(x)\Delta(g)(f \otimes 1)) \\ &= \Delta(x) \cdot (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1)) \\ &= \Delta(x)W^*(f \otimes g). \end{aligned}$$

□

As a motivation, let us consider an example involving classical group:

Example 2.19 [40, Ex 7.1.4][45] Let G be a locally compact group with right Haar measure λ . Then, the operator

$$(Wf)(x, y) := f(xy, y) \tag{2.32}$$

is a multiplicative unitary in $\mathcal{B}(L^2(G, \lambda) \otimes L^2(G, \lambda)) = \mathcal{B}(L^2(G \times G, \lambda \times \lambda))$. The pentagon equation for W is equivalent to the associativity of the multiplication in G . Indeed, for $f \in L^2(G \times G \times G, \lambda \times \lambda \times \lambda) \simeq L^2(G, \lambda)^{\otimes 3}$ and $x, y, z \in G$, we have

$$\begin{aligned} (W_{23}W_{12}f)(x, y, z) &= f(x(yz), yz, z), \\ (W_{12}W_{13}W_{23}f)(x, y, z) &= f((xy)z, yz, z). \end{aligned}$$

An important property of W is that it encodes the information of the dual quantum group $\widehat{\mathcal{A}}$. By definition, $\widehat{\mathcal{A}}$ is the closure of the linear span of

$$\{(\omega \otimes 1)W : \omega \in \mathcal{B}(\mathcal{H})^*\} \subset \mathcal{B}(\mathcal{H}), \tag{2.33}$$

and we actually have

$$W \in M(\mathcal{A} \otimes \widehat{\mathcal{A}}), \tag{2.34}$$

where M stands for the multiplier algebra.

Similarly, the multiplicative unitary for the dual $\widehat{\mathcal{A}}$ is given by $\widehat{W} := W_{21}^*$; hence, it is known from the Pontryagin duality that $\widehat{W} \in M(\widehat{\mathcal{A}} \otimes \mathcal{A})$ as well. From the pentagon equation, we then obtain:

Corollary 2.20 *As an element $W \in M(\mathcal{A} \otimes \widehat{\mathcal{A}})$, together with the pentagon equation, we have*

$$(\Delta \otimes 1)W = W_{13}W_{23}, \tag{2.35}$$

$$(1 \otimes \widehat{\Delta})W = W_{13}W_{12}. \tag{2.36}$$

Finally, let us define the notion of “manageability” introduced by Woronowicz [45,46] that describes a class of well-behaved multiplicative unitary. It is shown

that any manageable multiplicative unitary gives rise to a quantum group on the C^* -algebraic level.

Definition 2.21 A multiplicative unitary W is manageable if there exists a positive self-adjoint operator Q acting on \mathcal{H} and a unitary operator \tilde{W} acting on $\overline{\mathcal{H}} \otimes \mathcal{H}$ such that $\ker(Q) = \{0\}$,

$$(Q \otimes Q)W = W(Q \otimes Q), \quad (2.37)$$

and

$$\langle x \otimes u, W(z \otimes y) \rangle_{\mathcal{H} \otimes \mathcal{H}} = \langle \bar{z} \otimes Qu, \tilde{W}(\bar{x} \otimes Q^{-1}y) \rangle_{\overline{\mathcal{H}} \otimes \mathcal{H}} \quad (2.38)$$

for any $x, z \in \mathcal{H}$, $y \in D(Q^{-1})$, $u \in D(Q)$.

Here, $\overline{\mathcal{H}}$ denotes the complex conjugate of \mathcal{H} , so that the map $x \in \mathcal{H} \rightarrow \bar{x} \in \overline{\mathcal{H}}$ is an anti-unitary map. The inner product on $\overline{\mathcal{H}}$ is given by $\langle \bar{x}, \bar{y} \rangle_{\overline{\mathcal{H}}} := \langle y, x \rangle_{\mathcal{H}}$.

2.5 q -commuting operators

Throughout the paper, we will consider positive operators A, B satisfying $AB = q^2BA$ for $q = e^{\pi i b^2}$, $|q| = 1$.

We will realize the operators using the canonical pair $A = e^{2\pi b x}$, $B = e^{2\pi b p}$ where $p = \frac{1}{2\pi i} \frac{d}{dx}$.

Proposition 2.22 Both A, B are positive unbounded operators on $L^2(\mathbb{R})$ and they are essentially self-adjoint. The domain for A is given by

$$\mathcal{D}_A = \{f(x) \in L^2(\mathbb{R}) : e^{2\pi b x} f(x) \in L^2(\mathbb{R})\} \quad (2.39)$$

and the domain for B is given by the Fourier transform of \mathcal{D}_A .

Hence, we can apply functional calculus and obtain various functions in A and B . In particular, for any function defined on $x > 0 \in \mathbb{R}$ such that $|f(x)| = 1$, $f(A)$ will be a unitary operator.

In this paper, we will be using a dense subspace $\mathcal{W} \subset L^2(\mathbb{R})$ introduced in [12, 38].

Definition 2.23 The dense subspace $\mathcal{W} \subset L^2(\mathbb{R})$ is the linear span of functions of the form

$$e^{-\alpha x^2 + \beta x} P(x), \quad (2.40)$$

where $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, and $P(x)$ is a polynomial in x .

Then, it is known [38] that \mathcal{W} forms a core for both A and B and it is also stable under Fourier transform. It is obvious that these functions have analytic continuation to the whole complex plane, and they have rapid decay along the real direction.

All the operators in the remaining sections will first be defined on \mathcal{W} and extended by continuity to all of $L^2(\mathbb{R})$ or its natural domain if the operator is unbounded.

We will also use the notion

$$\mathcal{W} \widehat{\otimes} \mathcal{W} \subset L^2(\mathbb{R} \times \mathbb{R}, ds dt), \tag{2.41}$$

where the extra exponent $e^{\gamma st}$, $\gamma \in \mathbb{C}$ is allowed.

For convenience, we describe some computations involving A and B , and similarly for the operators \widehat{A} and \widehat{B} with the opposite commutation relations.

Lemma 2.24 *For $AB = q^2BA$, $\widehat{A}\widehat{B} = q^{-2}\widehat{B}\widehat{A}$, both $q^{-1}BA^{-1}$ and $q\widehat{B}\widehat{A}^{-1}$ are positive self-adjoint operators. By the Baker–Campbell–Hausdorff formula*

$$e^{2\pi bx} e^{2\pi bp} = e^{2\pi b(x+p)} e^{(2\pi b)^2[x,p]/2} = q e^{2\pi b(x+p)}, \tag{2.42}$$

we deduce for example

$$q^{-1}BA^{-1} = q^{-1}e^{2\pi bp}e^{-2\pi bx} = e^{2\pi b(p-x)}. \tag{2.43}$$

Hence, we can describe

$$(q^{-1}BA^{-1})^{ib^{-1}\tau} = e^{\pi i \tau^2} B^{ib^{-1}\tau} A^{-ib^{-1}\tau} = e^{-\pi i \tau^2} A^{-ib^{-1}\tau} B^{ib^{-1}\tau}, \tag{2.44}$$

$$(q\widehat{B}\widehat{A}^{-1})^{ib^{-1}\tau} = e^{\pi i \tau^2} \widehat{A}^{-ib^{-1}\tau} \widehat{B}^{ib^{-1}\tau} = e^{-\pi i \tau^2} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{-ib^{-1}\tau}. \tag{2.45}$$

Furthermore, we have commutation relations of the form

$$(B \otimes 1) e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} = e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} (B \otimes \widehat{A}^{-1}), \tag{2.46}$$

$$(1 \otimes \widehat{B}) e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} = e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} (A \otimes \widehat{B}), \tag{2.47}$$

and similar variants.

3 The quantum dilogarithm

We recall the definition of the quantum dilogarithm given in [14] (see also [2]), an important special function that will be used throughout the paper.

3.1 Definition and properties

Throughout this section, we let $q = e^{\pi i b^2}$ where $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ and $0 < b^2 < 1$, so that $|q| = 1$. We also denote $Q = b + b^{-1}$.

Let $\omega := (w_1, w_2) \in \mathbb{C}^2$.

Definition 3.1 The Double Zeta function is defined as

$$\zeta_2(s, z|\omega) := \sum_{m_1, m_2 \in \mathbb{Z}_{\geq 0}} (z + m_1 w_1 + m_2 w_2)^{-s}. \quad (3.1)$$

The Double Gamma function is defined as

$$\Gamma_2(z|\omega) := \exp\left(\frac{\partial}{\partial s} \zeta_2(s, z|\omega)|_{s=0}\right). \quad (3.2)$$

Let

$$\Gamma_b(x) := \Gamma_2(x|b, b^{-1}). \quad (3.3)$$

The quantum dilogarithm is defined as the function:

$$S_b(x) := \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}. \quad (3.4)$$

The following form is often useful and will be used throughout this paper:

$$G_b(x) := e^{\frac{\pi i}{2}x(x-Q)} S_b(x). \quad (3.5)$$

Let us also relate G_b to another well-known expression by Ruijsenaars [36]

Lemma 3.2

$$G_b(z) = G\left(b, b^{-1}; iz - \frac{iQ}{2}\right) e^{\frac{\pi i}{2}z(z-Q)}, \quad (3.6)$$

where

$$G(a_+, a_-; z) := \exp\left(i \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2 \sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y}\right)\right) \quad (3.7)$$

with $|\operatorname{Im}(z)| < (a_+ + a_-)/2$ and extends meromorphically to the whole complex plane.

The quantum dilogarithm satisfies the following properties:

Proposition 3.3 *Self-duality:*

$$S_b(x) = S_{b^{-1}}(x), \quad G_b(x) = G_{b^{-1}}(x); \quad (3.8)$$

Functional equations:

$$S_b(x + b^{\pm 1}) = 2 \sin(\pi b^{\pm 1} x) S_b(x), \quad G_b(x + b^{\pm 1}) = (1 - e^{2\pi i b^{\pm 1} x}) G_b(x); \quad (3.9)$$

Reflection property:

$$S_b(x)S_b(Q - x) = 1, \quad G_b(x)G_b(Q - x) = e^{\pi ix(x-Q)}; \tag{3.10}$$

Complex conjugation:

$$\overline{S_b(x)} = \frac{1}{S_b(Q - \bar{x})}, \quad \overline{G_b(x)} = \frac{1}{G_b(Q - \bar{x})}, \tag{3.11}$$

in particular

$$\left| S_b\left(\frac{Q}{2} + ix\right) \right| = \left| G_b\left(\frac{Q}{2} + ix\right) \right| = 1 \text{ for } x \in \mathbb{R}; \tag{3.12}$$

Analyticity:

$S_b(x)$ and $G_b(x)$ are meromorphic functions with poles at $x = -nb - mb^{-1}$ and zeros at $x = Q + nb + mb^{-1}$, for $n, m \in \mathbb{Z}_{\geq 0}$;

Asymptotic properties:

$$G_b(x) \sim \begin{cases} \bar{\zeta}_b & \text{Im}(x) \rightarrow +\infty \\ \zeta_b e^{\pi ix(x-Q)} & \text{Im}(x) \rightarrow -\infty \end{cases}, \tag{3.13}$$

where

$$\zeta_b = e^{\frac{\pi i}{4} + \frac{\pi i}{12}(b^2 + b^{-2})}; \tag{3.14}$$

Residues:

$$\lim_{x \rightarrow 0} xG_b(x) = \frac{1}{2\pi}, \tag{3.15}$$

or more generally,

$$\text{Res} \frac{1}{G_b(Q + z)} = -\frac{1}{2\pi} \prod_{k=1}^n (1 - q^{2k})^{-1} \prod_{l=1}^m (1 - \tilde{q}^{2l})^{-1} \tag{3.16}$$

at $z = nb + mb^{-1}$, $n, m \in \mathbb{Z}_{\geq 0}$ and $\tilde{q} = e^{\pi ib^{-2}}$.

From the asymptotic properties, we have the following useful corollary that is needed when we deal with interchanging of order of integrations:

Corollary 3.4 For $s, t \in \mathbb{C}$, $x \in \mathbb{R}$, the asymptotic behavior of the ratio is given by

$$\left| \frac{G_b(s + ix)}{G_b(t + ix)} \right| \sim \begin{cases} 1 & x \rightarrow +\infty \\ e^{2\pi x \text{Re}(t-s)} & x \rightarrow -\infty \end{cases}. \tag{3.17}$$

By analytic continuation in b , there exists a classical limit for $G_b(x)$ given in [14]:

Theorem 3.5 For $q = e^{\pi i b^2}$, by letting $b^2 \rightarrow i0^+$, we have

$$\lim_{b \rightarrow 0} \frac{G_b(bx)}{\sqrt{-i}|b|(1 - q^2)^{x-1}} = \Gamma(x), \tag{3.18}$$

where $\sqrt{-i} = e^{-\frac{\pi i}{4}}$ and $-\frac{\pi}{2} < \arg(1 - q^2) < \frac{\pi}{2}$. The limit converges uniformly for any compact set in \mathbb{C} .

We will also need another important variant of the quantum dilogarithm function:

$$g_b(x) := \frac{\bar{\zeta}_b}{G_b\left(\frac{Q}{2} + \frac{1}{2\pi i b} \log x\right)}, \tag{3.19}$$

where $|g_b(x)| = 1$ when $x \in \mathbb{R}_{>0}$ due to (3.12).

Lemma 3.6 Let u, v be self-adjoint operators with $uv = q^2vu$, $q = e^{\pi i b^2}$. Then,

$$g_b(u)g_b(v) = g_b(u + v), \tag{3.20}$$

$$g_b(v)g_b(u) = g_b(u)g_b(q^{-1}uv)g_b(v). \tag{3.21}$$

(3.20) and (3.21) are often referred to as the quantum exponential and the quantum pentagon relations, which follows from the other properties:

$$g_b(u)^*vg_b(u) = q^{-1}uv + v, \tag{3.22}$$

$$g_b(v)ug_b(v)^* = u + q^{-1}uv. \tag{3.23}$$

3.2 Integral transformations

Here, we describe several properties of the quantum dilogarithm involving integrations.

Lemma 3.7 [2, (3.31),(3.32)] We have the following Fourier transformation formula:

$$\int_{\mathbb{R}+i0} e^{2\pi itr} \frac{e^{-\pi it^2}}{G_b(Q + it)} dt = \frac{\bar{\zeta}_b}{G_b\left(\frac{Q}{2} - ir\right)} = g_b(e^{2\pi br}), \tag{3.24}$$

$$\int_{\mathbb{R}+i0} e^{2\pi itr} \frac{e^{-\pi Qt}}{G_b(Q + it)} dt = \zeta_b G_b\left(\frac{Q}{2} - ir\right) = \frac{1}{g_b(e^{2\pi br})} = g_b^*(e^{2\pi br}), \tag{3.25}$$

where the contour goes above the pole at $t = 0$.

Using the reflection properties, we also obtain

$$\int_{\mathbb{R}+i0} e^{2\pi itr} e^{\pi Qt} G_b(-it) dt = \frac{\bar{\zeta}_b}{G_b\left(\frac{Q}{2} - ir\right)}, \tag{3.26}$$

$$\int_{\mathbb{R}+i0} e^{2\pi i t r} e^{\pi i t^2} G_b(-it) dt = \zeta_b G_b\left(\frac{Q}{2} - ir\right), \tag{3.27}$$

where again the contour goes above the pole at $t = 0$.

Lemma 3.8 [2, B.4] *q-Binomial Theorem: For positive self-adjoint variables u, v with $uv = q^2vu$, we have:*

$$(u + v)^{ib^{-1}t} = \int_C \binom{it}{i\tau}_b u^{ib^{-1}(t-\tau)} v^{ib^{-1}\tau} d\tau, \tag{3.28}$$

where the q -beta function (or q -binomial coefficient) is given by

$$\binom{t}{\tau}_b = \frac{G_b(-\tau)G_b(\tau - t)}{G_b(-t)}, \tag{3.29}$$

and C is the contour along \mathbb{R} that goes above the pole at $\tau = 0$ and below the pole at $\tau = t$.

Lemma 3.9 [32, Lem 15] *We have the Tau–Beta Theorem:*

$$\int_C e^{-2\pi\tau\beta} \frac{G_b(\alpha + i\tau)}{G_b(Q + i\tau)} d\tau = \frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha + \beta)}, \tag{3.30}$$

where the contour C goes along \mathbb{R} and goes above the poles of $G_b(Q + i\tau)$ and below those of $G_b(\alpha + i\tau)$. By the asymptotic properties of G_b , the integral converges for $\text{Re}(\beta) > 0, \text{Re}(\alpha + \beta) < Q$.

Lemma 3.10 *Rewriting the integral transform in [44] in terms of G_b , we obtain the 4–5 relation given by:*

$$\int_C d\tau e^{-2\pi\gamma\tau} \frac{G_b(\alpha + i\tau)G_b(\beta + i\tau)}{G_b(\alpha + \beta + \gamma + i\tau)G_b(Q + i\tau)} = \frac{G_b(\alpha)G_b(\beta)G_b(\gamma)}{G_b(\alpha + \gamma)G_b(\beta + \gamma)}, \tag{3.31}$$

where the contour C goes along \mathbb{R} and goes above the poles of the denominator, and below the poles of the numerator. By the asymptotic properties of G_b , the integral converges for $\text{Re}(\gamma) > 0$.

An important corollary of Lemma 3.9 is the following:

Corollary 3.11 *Let $f(z) \in \mathcal{W}$ (cf. Definition 2.23), then we have*

$$\lim_{\epsilon \rightarrow 0} \int_C \frac{G_b(Q - \epsilon)G_b(iz)}{G_b(Q + iz - \epsilon)} f(z) dz = f(0), \tag{3.32}$$

where the contour goes above the poles of the denominator and below the poles of the numerator. Alternatively, we can shift the poles so that the expression is an integration over \mathbb{R} :

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{G_b(Q - 2\epsilon)G_b(iz + \epsilon)}{G_b(Q + iz - \epsilon)} f(z) dz = f(0). \quad (3.33)$$

Formally as distribution,

$$\frac{G_b(Q)G_b(ix)}{G_b(Q + ix)} = \delta(x), \quad x \in \mathbb{R}. \quad (3.34)$$

Proof By Lemma 3.9, we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{G_b(Q - 2\epsilon)G_b(iz + \epsilon)}{G_b(Q + iz - \epsilon)} f(z) dz \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \int_C e^{-2\pi\tau(iz+\epsilon)} \frac{G_b(Q - 2\epsilon + i\tau)}{G_b(Q + i\tau)} f(z) d\tau dz, \end{aligned}$$

where the contour C goes above the pole at $\tau = 0$.

Now, by Corollary 3.4, the integrand in τ has asymptotics

$$\left| e^{-2\pi\tau\epsilon} \frac{G_b(Q - 2\epsilon + i\tau)}{G_b(Q + i\tau)} \right| \sim \begin{cases} e^{-2\pi\tau\epsilon} & \tau \rightarrow \infty \\ e^{2\pi\tau\epsilon} & \tau \rightarrow -\infty \end{cases}.$$

Hence, the integrand is absolutely convergent in both τ and z . Hence, we can interchange the integration order of τ and z ,

$$\begin{aligned} &= \lim_{\epsilon \rightarrow 0} \int_C \int_{\mathbb{R}} e^{-2\pi\tau(iz+\epsilon)} \frac{G_b(Q - 2\epsilon + i\tau)}{G_b(Q + i\tau)} f(z) dz d\tau \\ &= \lim_{\epsilon \rightarrow 0} \int_C e^{-2\pi\tau\epsilon} \frac{G_b(Q - 2\epsilon + i\tau)}{G_b(Q + i\tau)} (\mathcal{F}f)(\tau) d\tau, \end{aligned}$$

where \mathcal{F} is the Fourier transform. Since $f \in \mathcal{W}$, $\mathcal{F}f \in \mathcal{W}$ as well, hence the integrand is absolutely convergent independent of ϵ . Hence, we can interchange the limit and finally obtain

$$= \int_{\mathbb{R}} \lim_{\epsilon \rightarrow 0} e^{-2\pi\tau(iz+\epsilon)} \frac{G_b(Q - 2\epsilon + i\tau)}{G_b(Q + i\tau)} (\mathcal{F}f)(\tau) d\tau = \int_{\mathbb{R}} (\mathcal{F}f)(\tau) d\tau = f(0),$$

where the last line follows from the properties of Fourier transform. \square

Remark 3.12 This is just the analogue of the delta function as a hyperfunction. Since the integrand is 0 unless z is close to 0, it suffices to notice that near $z = 0$, from the analytic properties for G_b , we have $G_b(z) \sim \frac{1}{G_b(Q-z)} \sim \frac{1}{2\pi z}$, so that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{G_b(Q - 2\epsilon)G_b(iz + \epsilon)}{G_b(Q + iz - \epsilon)} \\ & \sim \lim_{\epsilon \rightarrow 0} \frac{2\epsilon}{2\pi i(iz - \epsilon)(iz + \epsilon)} \\ & = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \left(\frac{1}{z - i\epsilon} - \frac{1}{z + i\epsilon} \right) \\ & = \delta(z). \end{aligned}$$

Intuitively, restricting to $x \in \mathbb{R}$, $\frac{G_b(Q)G_b(ix)}{G_b(Q+ix)}$ is always 0 unless $x = 0$, in which case, it is ∞ . The properties of G_b say that it gives the right normalization to be a delta function.

The other variants which we will also use include:

Corollary 3.13 *For $f(x) \in \mathcal{W}$, we have*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{G_b(\epsilon - ix)G_b(\epsilon + ix)}{G_b(2\epsilon)} f(x)dx = f(0), \tag{3.35} \\ & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{G_b(\epsilon + ix - b)G_b(Q + b - 2\epsilon)}{G_b(Q + ix - \epsilon)} f(x)dx = -q^2 f(0) + f(-ib), \tag{3.36} \end{aligned}$$

or formally

$$\begin{aligned} & \frac{G_b(-ix)G_b(ix)}{G_b(0)} = \delta(x), \tag{3.37} \\ & \frac{G_b(ix - b)G_b(Q + b)}{G_b(Q + ix)} = -q^2 \delta(x) + \delta(x + ib), \tag{3.38} \end{aligned}$$

which follows from the functional properties of $G_b(x)$ and similar arguments as above.

A particular important case in the study of modular double (see the next section) is proved in [2, Lem 3] and [44] which can also be obtained from the above arguments:

Corollary 3.14 *Let u, v be positive self-adjoint operators satisfying $uv = q^2vu$. By taking $t \rightarrow -ib^{-1}$ in Lemma 3.8, we obtain*

$$(u + v)^{1/b^2} = u^{1/b^2} + v^{1/b^2}. \tag{3.39}$$

4 The quantum plane

In this section, we will define the main object of the study; in this paper, the C^* -algebra \mathcal{A} of “functions vanishing at infinity” on the quantum plane. Its Haar functional is established, and we obtain the GNS construction of \mathcal{A} on a Hilbert space $\mathcal{H} \simeq L^2(\mathbb{R} \times \mathbb{R})$. Finally, we construct the multiplicative unitary W and show that it is in fact manageable.

Note that in analogy to the classical semigroup with inverses absent, in our context, we do not have a well-defined antipode S due to positivity. Hence, we may also call the object informally as a “quantum semigroup.” However, as we see in Definition 4.6, there is instead a well-defined unitary antipode R .

With the existence of the coproduct (Corollary 4.5), a left invariant faithful KMS Haar functional (Theorems 4.7, 4.9, Corollary 4.15), a right invariant Haar functional (Proposition 4.16) and the density condition (Theorem 5.8), the C^* -algebra \mathcal{A} can be put in the context of “locally compact quantum group” in the sense of [24, 25]. This allows us to apply the theory developed in those papers.

4.1 Motivations from classical $ax + b$ group

Recall that the classical $ax + b$ group is the group G of affine transformations on the real line \mathbb{R} , where $a > 0$ and $b \in \mathbb{R}$, and they can be represented by a matrix of the form (we use the transposed version):

$$g(a, b) = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \quad (4.1)$$

where the multiplication is given by

$$g(a_1, b_1)g(a_2, b_2) = \begin{pmatrix} a_1a_2 & 0 \\ b_1a_2 + b_2 & 1 \end{pmatrix}. \quad (4.2)$$

We can then talk about the space $C_\infty(G)$ of functions on G vanishing at infinity, and a dense subspace of functions \mathcal{W}_G of the form

$$\mathcal{W}_G = \{g(\log a) f(b) : f, g \in \mathcal{W} \subset L^2(\mathbb{R})\}, \quad (4.3)$$

where $f, g \in \mathcal{W}$ are the rapidly decaying functions defined in Definition 2.23. Then, $C_\infty(G)$ is the sup-norm closure of \mathcal{W}_G .

In the study of quantum plane, it is necessary to consider the semigroup G_+ of positive elements. Restricting $f(a, b) \in \mathcal{W}_G$ to $b > 0$, we can use the Mellin transform and express the functions as

$$f(a, b) = \int_{\mathbb{R}+i0} \int_{\mathbb{R}} F(s, t) a^{is} b^{it} ds dt. \quad (4.4)$$

Since the function is continuous at $b = 0$, it has at most $O(1)$ growth as $b \rightarrow 0^+$. Hence, from Theorem 2.2, we conclude that $F(s, t)$ is entire analytic with respect to s , and holomorphic on $\text{Im}(t) > 0$. Furthermore, it has rapid decay in s, t along the real direction and can be analytically continued to $\text{Im}(t) \leq 0$. Moreover, since $f(a, b)$ is analytic at $b = 0$, the analytic structure of $f(a, b)$ on b is given by $\sum_{k=0}^\infty A_k b^k$ for some constants A_k . Hence, $F(s, t)$ has possible simple poles at $t = -in, n = 0, 1, 2, \dots$

Finally, according to the Parseval’s Formula for Mellin transform, there is an L^2 norm on functions of G_+ given by

$$\|f(a, b)\|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left| F(s, t + \frac{1}{2}i) \right|^2 dt ds \tag{4.5}$$

induced from the Haar measure $\frac{da}{a} db$. However, it is no longer left invariant due to the lack of inverses.

4.2 The quantum plane algebra and its modular double

Throughout the section, we let $q = e^{\pi i b^2}, \tilde{q} = e^{\pi i b^{-2}}$ where $b^2 \in \mathbb{R} \setminus \mathbb{Q}, 0 < b^2 < 1$. We have $|q| = |\tilde{q}| = 1$.

The quantum plane \mathcal{A}_q is formally the Hopf algebra generated by positive self-adjoint operators A, B satisfying

$$AB = q^2 BA, \tag{4.6}$$

with coproduct

$$\Delta(A) = A \otimes A, \tag{4.7}$$

$$\Delta(B) = B \otimes A + 1 \otimes B. \tag{4.8}$$

Hence, it is clear from the previous section that the quantum plane is just the quantum analogue of the classical $ax + b$ group. Their relationships are studied in detail in [14]. By the following proposition, the coproducts are positive essentially self-adjoint; hence, they are well defined.

Proposition 4.1 *The operator $e^{2\pi b x} + e^{2\pi b p}$ is positive and essentially self-adjoint. In particular, the coproduct $\Delta(B) = B \otimes A + 1 \otimes B$ defined in (4.8) is positive and essentially self-adjoint operator on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.*

Proof It follows from (3.23) by applying $u = A, v = q^{-1}BA^{-1}$, and notice that $g_b(v)$ is unitary. Hence, this gives a unitary transformation that sends $e^{2\pi b x}$ to $e^{2\pi b x} + e^{2\pi b p}$. In particular, the functional analytic property carries over to the new operator. \square

Since we are working with positive operators, this important property allows us to avoid the analysis of the coproduct of B that is studied extensively in [35,46].

An interesting object in the study of quantum plane is the modular double element [5, 10, 22]. $\mathcal{A}_{\tilde{q}}$ is formally the Hopf algebra generated by positive self-adjoint elements

$$\tilde{A} = A^{1/b^2}, \quad \tilde{B} = B^{1/b^2}, \quad (4.9)$$

such that they satisfy

$$\tilde{A}\tilde{B} = \tilde{q}^2\tilde{B}\tilde{A}. \quad (4.10)$$

Then, according to the formula above, using Corollary 3.14, we conclude that they have the same bi-algebra structure:

$$\Delta(\tilde{A}) = \tilde{A} \otimes \tilde{A}, \quad (4.11)$$

$$\begin{aligned} \Delta(\tilde{B}) &= (B \otimes A + 1 \otimes B)^{1/b^2} \\ &= (B \otimes A)^{1/b^2} + (1 \otimes B)^{1/b^2} \\ &= \tilde{B} \otimes \tilde{A} + 1 \otimes \tilde{B}. \end{aligned} \quad (4.12)$$

Motivated from the definition given in the previous section for the classical $ax + b$ group, we define the C^* -algebra $C_\infty(\mathcal{A}_q)$ of “functions on the quantum plane vanishing at infinity” as follows.

Definition 4.2 $C_\infty(\mathcal{A}_q)$ is the space of all functions

$$C_\infty(\mathcal{A}_q) := \left\{ \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right\}^{\text{norm closure}} \quad (4.13)$$

where f is entire analytic in s , and meromorphic in t with possible simple poles at

$$t = -ibn - i\frac{m}{b}, \quad n, m = 0, 1, 2, \dots$$

and for fixed $v > 0$, the functions $f(s + iv, t)$ and $f(s, t + iv)$ are of rapid decay (faster than any exponential) in both s and t . To define the C^* norm, we realize $A^{ib^{-1}s} f(x) = e^{2\pi is} f(x)$ and $B^{ib^{-1}t} f(x) = e^{2\pi it} f(x) = f(x + 1)$ as unitary operators on $L^2(\mathbb{R})$ and take the operator norm.

For simplicity, we will denote the space simply by $\mathcal{A} := C_\infty(\mathcal{A}_q)$.

Remark 4.3 As in the classical case, by Mellin transform in s , we can consider the linear span of

$$g(\log A) \int_{\mathbb{R}+i0} f_2(t) B^{ib^{-1}t} dt$$

instead, where $g(x)$ is entire analytic and of rapid decay in $x \in \mathbb{R}$. This form will be useful later when we deal with the pairing between \mathcal{A} and its dual space $\widehat{\mathcal{A}}$. We can also conclude from the definition that $A, A^{-1}, \tilde{A}, \tilde{A}^{-1}, B$ and \tilde{B} are all elements of the multiplier algebra $M(\mathcal{A})$.

This definition differs from the classical case in certain aspects. First of all, there is a $b \longleftrightarrow b^{-1}$ duality in this definition. Indeed, it is obvious that we have

$$C_\infty(\mathcal{A}_q, b) = C_\infty(\mathcal{A}_{\bar{q}}, b^{-1}) \tag{4.14}$$

because $A^{ib^{-1}s} B^{ib^{-1}t} = \tilde{A}^{ibs} \tilde{B}^{ibt}$ and the analyticity of $f(s, t)$ has the $b \longleftrightarrow b^{-1}$ duality. Hence, it encodes the information of both \mathcal{A}_q and $\mathcal{A}_{\bar{q}}$. For example, we see by Theorem 2.2 that as a function of B , at $B = 0$, it admits a series representation

$$F(B) \sim \sum_{m,n \geq 0} \alpha_{mn} B^{m+n/b^2} = \sum_{m,n \geq 0} a_{mn} B^m \tilde{B}^n. \tag{4.15}$$

This choice of poles in the definition is needed in order for the coproduct $\Delta(\mathcal{A})$ to lie in $M(\mathcal{A} \otimes \mathcal{A})$, the multiplier algebra, because of the appearances of the quantum dilogarithm $G_b(-i\tau)$; hence, it is a ‘‘minimal’’ choice for all the calculations to work.

Proposition 4.4 *Let us denote simply by $f := \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt$. Then, the coproduct on \mathcal{A}_q can be naturally extended to \mathcal{A} by*

$$\begin{aligned} \Delta(f) &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) (A^{ib^{-1}s} \otimes A^{ib^{-1}t})(B \otimes A + 1 \otimes B)^{ib^{-1}t} ds dt \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \int_C f(s, t) \frac{G_b(-i\tau)G_b(i\tau - it)}{G_b(-it)} A^{ib^{-1}s} B^{ib^{-1}(t-\tau)} \\ &\quad \otimes A^{ib^{-1}(s+t-\tau)} B^{ib^{-1}\tau} d\tau ds dt \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \int_{\mathbb{R}+i0} f(s, t + \tau) \frac{G_b(-i\tau)G_b(-it)}{G_b(-i(t + \tau))} A^{ib^{-1}s} B^{ib^{-1}t} \\ &\quad \otimes A^{ib^{-1}(s+t)} B^{ib^{-1}\tau} d\tau ds dt, \end{aligned}$$

or formally

$$\Delta(f)(s_1, t_1, s_2, t_2) = f(s_1, t_1 + t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-i(t_1 + t_2))} \delta(s_2, s_1 + t_1). \tag{4.16}$$

Corollary 4.5 *The map Δ sends $\mathcal{A} \longrightarrow M(\mathcal{A} \otimes \mathcal{A})$ where $M(\mathcal{A})$ is the multiplier algebra for \mathcal{A} , and it is coassociative in the sense of multiplier Hopf algebra. Hence, Δ is indeed a coproduct on \mathcal{A} .*

Proof We need to show that $\Delta(f)(g \otimes h)$ lies in $\mathcal{A} \otimes \mathcal{A}$ for $f, g, h \in \mathcal{A}$. From the formula for Δ , we see that the poles for f are canceled by $G_b(-i(t + \tau))$ and two new set of poles at the specified locations are introduced by the numerator $G_b(-i\tau)G_b(-it)$. Hence, the integrand of the coproduct is a generalized function with the specified analytic properties. Since $C_\infty(\mathcal{A}_q)$ is closed under product, which comes from the

Mellin transform of certain series representation, the product of $\Delta(f)$ and any element $g \otimes h \in \mathcal{A} \otimes \mathcal{A}$ will lie in $\mathcal{A} \otimes \mathcal{A}$.

Coassociativity is then obvious from the construction. \square

Finally, let us describe the antipode S and the unitary antipode R . On the Hopf algebra level, the antipode is given by $S(A) = A^{-1}$, $S(B) = -BA^{-1}$. Formally, we extend it to \mathcal{A} by

$$\begin{aligned} S \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) \\ = \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) (e^{\pi i} B A^{-1})^{ib^{-1}t} A^{-ib^{-1}s} ds dt \right), \end{aligned}$$

and hence we have

Definition 4.6 We define the antipode to be

$$S(f) = f(-s - t, t) e^{-\pi Q t} e^{\pi i (2st + t^2)}. \quad (4.17)$$

Then, $S^2(f)(s, t) = f(s, t) e^{-2\pi Q t}$; hence, we conclude the scaling group to be

$$\tau_{-i}(A^{ib^{-1}s}) = A^{ib^{-1}s}, \quad \tau_{-i}(B^{ib^{-1}t}) = e^{-2\pi Q t} B^{ib^{-1}t}, \quad (4.18)$$

or more generally

$$\tau_t(A) = A, \quad \tau_t(B) = e^{-2\pi b Q t} B. \quad (4.19)$$

Then, the unitary antipode $R = \tau_{i/2} S$ is given by

$$R(A) = A^{-1}, \quad (4.20)$$

$$R(B) = -e^{-\pi i b Q} B A^{-1} = e^{-\pi i b^2 - \pi i} B A^{-1} = q^{-1} B A^{-1}. \quad (4.21)$$

We note that R sends positive operators to positive operators, which differs from the usual choice (see, e.g., [40, Thm 8.4.33]). Furthermore, when we later realize $f(s, t)$ as element in $\mathcal{H} := L^2(\mathbb{R}^2)$ in the von Neumann picture, the antipode S can be seen to be an unbounded operator on \mathcal{H} due to the factor $e^{-\pi Q t}$ coming from the negative sign, while R is unitary. Hence, R is better suit in this positive semigroup context.

It is known that as polynomial algebras \mathcal{A}_q and $\mathcal{A}_{\tilde{q}}$ are C^* -algebras of Type II_1 [5, 22]. However, by expressing $A = e^{2\pi b x}$, $B = e^{2\pi b p}$, the definition of \mathcal{A} states that it is nothing but the algebra of all bounded operators on $L^2(\mathbb{R})$ via the Weyl formula [5]; hence, it is a C^* -algebra of Type I. Therefore, \mathcal{A} is precisely the modular double $\mathcal{A}_q \otimes \mathcal{A}_{\tilde{q}}$, and it arises as a natural framework to the study of this subject.

Finally, we will also see in the GNS construction that a natural L^2 -norm is given by (cf. Theorem 4.9)

$$\|f(s, t)\|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left| f\left(s, t + \frac{iQ}{2}\right) \right|^2 ds dt, \tag{4.22}$$

which is an analogue of the classical Parseval’s formula.

4.3 The Haar functional

Recall from Definition 2.8 that a linear functional on \mathcal{A} is called a left invariant Haar functional if it satisfies

$$(1 \otimes h)(\Delta x) = h(x) \cdot 1_{M(\mathcal{A})}, \tag{4.23}$$

and a right invariant Haar functional if it satisfies

$$(h \otimes 1)(\Delta x) = h(x) \cdot 1_{M(\mathcal{A})}, \tag{4.24}$$

where $1_{M(\mathcal{A})}$ is the unital element in the multiplier algebra $M(\mathcal{A})$.

Theorem 4.7 *There exists a left invariant Haar functional on \mathcal{A} , given by*

$$h \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) = f(0, iQ). \tag{4.25}$$

Proof From Proposition 4.4, the coproduct is given by

$$\begin{aligned} & \Delta \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \int_C f(s, t) \frac{G_b(-i\tau)G_b(i\tau - it)}{G_b(-it)} A^{ib^{-1}s} B^{ib^{-1}(t-\tau)} \\ & \quad \otimes A^{ib^{-1}(s+t-\tau)} B^{ib^{-1}\tau} d\tau ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}-i0} \int_C f(s, t + \tau - s) \frac{G_b(-i\tau)G_b(is - it)}{G_b(is - it - i\tau)} A^{ib^{-1}s} B^{ib^{-1}(t-s)} \\ & \quad \otimes A^{ib^{-1}t} B^{ib^{-1}\tau} d\tau ds dt. \end{aligned}$$

In the last step, we push the contour of t back to the real line and the contour of τ to be the real line that goes above the pole at $\tau = 0$ only. This will force the contour of s to move below the pole at $s = t$.

Now, as a function of τ , the integrand is well defined for $0 < \text{Im}(\tau) < Q$. Therefore, to apply the Haar functional, we take the analytic continuation to $\tau = iQ$ by

$$\tau = \lim_{\epsilon \rightarrow 0} iQ - i\epsilon.$$

Using Corollary 3.11, we have

$$\begin{aligned} &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}-i0} f(s, iQ - s - i\epsilon) \frac{G_b(Q - \epsilon)G_b(is)}{G_b(Q + is - \epsilon)} A^{ib^{-1}s} B^{ib^{-1}(-s)} ds \\ &= f(0, iQ) \end{aligned}$$

as desired. □

Using exactly the same technique, we can derive the counit for \mathcal{A} :

Corollary 4.8 *The counit is given by*

$$\epsilon(f) = \lim_{t \rightarrow 0} G_b(Q + it) \int_{\mathbb{R}} f(s, t) ds \tag{4.26}$$

and satisfies

$$(1 \otimes \epsilon)\Delta = Id_{\mathcal{A}} = (\epsilon \otimes 1)\Delta. \tag{4.27}$$

4.4 The GNS description

The GNS representation enables us to bring the quantum plane into the Hilbert space level, where the algebra is realized as operators on certain Hilbert space. This Hilbert space is nothing but the space of “ L^2 functions on the quantum plane,” which gives the necessary background later to describe the L^2 functions on $GL_q^+(2, \mathbb{R})$ using the quantum double construction. This will allow us to state the main theorem (Theorem 8.15) on the decomposition of $L^2(GL_q^+(2, \mathbb{R}))$ into the fundamental principal series representations of $U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ with certain Plancherel measure.

Now, equipped with the left invariant Haar functional h , we can describe completely the GNS representation of \mathcal{A} using h as the weight.

Theorem 4.9 *Let $\mathcal{H} = L^2(\mathbb{R} \times \mathbb{R})$ be the completion of $L^2(\mathbb{R}) \otimes \mathcal{W}$ equipped with the inner product*

$$\langle f, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g\left(s, t + \frac{iQ}{2}\right)} f\left(s, t + \frac{iQ}{2}\right) ds dt, \tag{4.28}$$

and the L^2 norm

$$\|f\|^2 = \iint \left| f\left(s, t + \frac{iQ}{2}\right) \right|^2 ds dt. \tag{4.29}$$

Then, the GNS map is simply given by

$$\Lambda \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) = f(s, t). \quad (4.30)$$

In particular, the Haar functional is a faithful weight.

The action π is given by the multiplication of \mathcal{A} on itself:

$$\pi(a)\Lambda(b) := \Lambda(ab), \quad a, b \in \mathcal{A}. \quad (4.31)$$

First of all, let us note that \mathcal{A} is closed under $*$:

Proposition 4.10 *The conjugation map T is given by*

$$T(f) = \overline{f}(-s, -t)e^{2\pi ist}, \quad (4.32)$$

where we denote by $\overline{f}(z) := \overline{f(\overline{z})}$.

We observe that $\overline{f}(-s, -t)e^{2\pi ist}$ still satisfies all the analytic properties required to be an element of \mathcal{A} : it is entire analytic in s , meromorphic in t with possible poles at $t = -inb - imb^{-1}$, etc. see Definition 4.2.

Proof

$$\begin{aligned} \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right)^* &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{f(s, t)} B^{-ib^{-1}\overline{t}} A^{-ib^{-1}\overline{s}} d\overline{s} d\overline{t} \\ &= \int_{\mathbb{R}-i0} \int_{\mathbb{R}} \overline{f(\overline{s}, \overline{t})} B^{-ib^{-1}t} A^{-ib^{-1}s} ds dt \\ &= \int_{\mathbb{R}-i0} \int_{\mathbb{R}} \overline{f(\overline{s}, \overline{t})} e^{2\pi ist} A^{-ib^{-1}s} B^{-ib^{-1}t} ds dt \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{f(-\overline{s}, -\overline{t})} e^{2\pi ist} A^{ib^{-1}s} B^{ib^{-1}t} ds dt \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{f}(-s, -t) e^{2\pi ist} A^{ib^{-1}s} B^{ib^{-1}t} ds dt. \end{aligned}$$

□

Next, we describe the product in terms of $f(s, t)$:

Proposition 4.11 *The product of two elements in \mathcal{A} is given by:*

$$(f \cdot g)(s, t) = \int_C \int_{\mathbb{R}} f(s - s', t - t') g(s', t') e^{2\pi is'(t-t')} ds' dt', \quad (4.33)$$

i.e., a twisted convolution. Here, C is the contour that, with possible poles, goes below $t' = t$ and above $t' = 0$.

Proof

$$\begin{aligned} & \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} g(s', t') A^{ib^{-1}s'} B^{ib^{-1}t'} ds' dt' \right) \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) g(s', t') e^{2\pi i s' t} A^{ib^{-1}(s+s')} B^{ib^{-1}(t+t')} ds' dt' ds dt \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \left(\int_C \int_{\mathbb{R}} f(s-s', t-t') g(s', t') e^{2\pi i s'(t-t')} ds' dt' \right) A^{ib^{-1}s} B^{ib^{-1}t} ds dt, \end{aligned}$$

where in the last step, the shift in t will push the contour of t' to go below $t' = t$. \square

Proof of Theorem 4.9 Using the formula for the Haar functional, and the definition of the GNS inner product, we have

$$\begin{aligned} \langle f, g \rangle &= h(g^* f) = h(\overline{g}(-s, -t) e^{2\pi i s t} \cdot f(s, t)) \\ &= h \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{g}(s' - s, t' - t) e^{2\pi i (s-s')(t-t')} f(s', t') e^{2\pi i s'(t-t')} ds' dt' \right) \\ &= h \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{g}(s' - s, t' - t) f(s', t') e^{2\pi i s(t-t')} ds' dt' \right) \\ &= \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \overline{g}(s', t' - iQ) f(s', t') ds' dt' \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g} \left(s', t' - \frac{iQ}{2} \right) f \left(s', t' + \frac{iQ}{2} \right) ds' dt' \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} g \left(s', t' + \frac{iQ}{2} \right) \overline{f} \left(s', t' + \frac{iQ}{2} \right) ds' dt', \end{aligned}$$

where in the second to last line, we do a shift in the contour of t by $\frac{iQ}{2}$ by holomorphicity. \square

Finally, we read off the representation π of \mathcal{A} on the functions $f(s, t) \in \mathcal{H}$:

Proposition 4.12 *The representation $\pi : \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ is given by*

$$\pi(A) = e^{-2\pi b p_s}, \quad \pi(B) = e^{2\pi b s} e^{-2\pi b p_t}, \quad (4.34)$$

where $p_s = \frac{1}{2\pi i} \frac{\partial}{\partial s}$, so that $e^{2\pi b p_s} \cdot f(s) = f(s - ib)$. Similarly, for p_t .

Proof A acts on $f(s, t)$ by:

$$\begin{aligned} A^{ib^{-1}s'} \iint f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt &= \iint f(s, t) A^{ib^{-1}(s+s')} B^{ib^{-1}t} ds dt \\ &= \iint f(s - s', t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt. \end{aligned}$$

Hence, $A^{ib^{-1}s'} \cdot f(s, t) = f(s - s', t)$, or $A = e^{-2\pi b p_s}$.

Similarly,

$$\begin{aligned} B^{ib^{-1}t'} \iint f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt &= \iint f(s, t) e^{2\pi i s t'} A^{ib^{-1}s} B^{ib^{-1}(t+t')} ds dt \\ &= \iint f(s, t - t') e^{2\pi i s t'} A^{ib^{-1}s} B^{ib^{-1}t} ds dt. \end{aligned}$$

Hence, $B^{ib^{-1}t'} \cdot f(s, t) = e^{2\pi i s t'} f(s, t - t')$, or $B = e^{2\pi b s} e^{-2\pi b p_t}$. \square

We will take this Hilbert space \mathcal{H} as our canonical choice when we introduce right measure and the dual space.

4.5 Modular maps and the right picture

In this section, we describe the remaining objects defined in Sect. 2.3, as well as the description of the right invariant picture for the quantum plane.

From now on, let us restrict all operators on the dense subspace \mathcal{W} , and we will extend any bounded operators on \mathcal{W} to the whole space \mathcal{H} .

Proposition 4.13 *The conjugation map T (cf. (2.20)) is given by*

$$T(f) = \overline{f}(-s, -t) e^{2\pi i s t}, \quad (4.35)$$

T^* is given by

$$T^*(f) = \overline{f}(-s, -t) e^{2\pi i s t} e^{2\pi s Q}. \quad (4.36)$$

Hence, the modular operator $\nabla = T^*T$ is given by

$$\nabla(f) = f(s, t) e^{2\pi s Q}, \quad (4.37)$$

and the modular conjugation $J = T\nabla^{-\frac{1}{2}} = \nabla^{\frac{1}{2}}T$ is given by

$$J(f) = \overline{f}(-s, -t) e^{2\pi i s t} e^{\pi s Q}. \quad (4.38)$$

We have $J^2 = J^*J = Id$.

Proof It suffices to establish the map T^* . Since T is anti-linear, from the definition of the adjoint of anti-linear maps, we have:

$$\begin{aligned} \langle Tf, g \rangle &= \overline{\langle f, T^*g \rangle} \\ RHS &= \iint T^*g \left(\bar{s}, \bar{t} + \frac{iQ}{2} \right) \overline{f \left(s, t + \frac{iQ}{2} \right)} d\bar{s}d\bar{t} \\ &= \iint T^*g \left(s, t + \frac{iQ}{2} \right) \overline{f \left(\bar{s}, \bar{t} + \frac{iQ}{2} \right)} dsdt \\ LHS &= \iint g \left(\bar{s}, \bar{t} + \frac{iQ}{2} \right) \overline{f \left(-s, -t - \frac{iQ}{2} \right)} e^{2\pi i s \left(t + \frac{iQ}{2} \right)} dsdt \\ &= \iint \bar{g} \left(s, t - \frac{iQ}{2} \right) \overline{f \left(-\bar{s}, -\bar{t} + \frac{iQ}{2} \right)} e^{2\pi i s \left(t + \frac{iQ}{2} \right)} dsdt \\ &= \iint \bar{g} \left(-s, -t - \frac{iQ}{2} \right) \overline{f \left(\bar{s}, \bar{t} + \frac{iQ}{2} \right)} e^{2\pi i (-s) \left(-t + \frac{iQ}{2} \right)} dsdt. \end{aligned}$$

Hence, $T^*g = \bar{g}(-s, -t)e^{2\pi i st}e^{2\pi sQ}$. □

Corollary 4.14 The modular group $\sigma_t(x) = \nabla^{it} x \nabla^{-it}$ is given by

$$\sigma_t(A) = e^{2\pi bQt} A, \quad (4.39)$$

$$\sigma_t(B) = B. \quad (4.40)$$

or in other words

$$\sigma_{t'}(f) = f(s, t)e^{2\pi Qt's}. \quad (4.41)$$

With this expression of the modular group, it is now easy to see the following:

Corollary 4.15 The Haar weight is a KMS weight, in the sense of [24], that is,

$$h \circ \sigma_t = h \quad \text{for every } t \in \mathbb{R}, \quad (4.42)$$

$$h(a^*a) = h(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*) \quad \text{for every } a \in \mathcal{A}. \quad (4.43)$$

Proof The first equation is obvious from (4.41). For the second equation, it follows by combining (4.35), (4.41) and Proposition 4.11. □

Proposition 4.16 The scaling constant ν is given by

$$\nu = e^{-2\pi Q^2} > 0, \quad (4.44)$$

the modular element δ is given by

$$\delta = A^{-\frac{Q}{b}}, \quad (4.45)$$

and the right Haar weight is given by

$$\psi(f) = f(-iQ, iQ)e^{-\pi i Q^2}. \quad (4.46)$$

Proof

$$\begin{aligned} h(\tau_{t'}(f)) &= h\left(\iint f(s, t)e^{-2\pi b Q t' i b^{-1} t} A^{i b^{-1} s} B^{i b^{-1} t} ds dt\right) \\ &= h\left(\iint f(s, t)e^{-2\pi i Q t' t} A^{i b^{-1} s} B^{i b^{-1} t} ds dt\right) \\ &= f(0, iQ)e^{-2\pi i Q t' i Q} \\ &= f(0, iQ)e^{2\pi Q^2 t'}, \end{aligned}$$

hence $\nu = e^{-2\pi Q^2}$.

Obviously,

$$\sigma_t(\delta) = e^{2\pi b Q t(-\frac{Q}{b})}\delta = e^{-2\pi Q^2 t}\delta = \nu^t \delta. \quad (4.47)$$

Finally, for the right Haar weight, recall that

$$\begin{aligned} R(A) &= A^{-1}, \\ R(B) &= q^{-1} B A^{-1}, \end{aligned}$$

hence

$$\begin{aligned} \psi(f) &= h(R(f)) \\ &= h\left(\iint f(s, t)(q^{-1} B A^{-1})^{i b^{-1} t} A^{-i b^{-1} s} ds dt\right) \\ &= h\left(\iint f(s, t)q^{-i b^{-1} t} q^{-i b^{-1} t(i b^{-1} t - 1)} q^{-2b^{-2} t(s+t)} A^{-i b^{-1}(s+t)} B^{i b^{-1} t} ds dt\right) \\ &= h\left(\iint f(s, t)e^{-\pi i(2st+t^2)} A^{-i b^{-1}(s+t)} B^{i b^{-1} t} ds dt\right) \\ &= h\left(\iint f(-s-t, t)e^{\pi i(2st+t^2)} A^{i b^{-1} s} B^{i b^{-1} t} ds dt\right) \\ &= f(-iQ, iQ)e^{\pi i(iQ)^2} \\ &= f(-iQ, iQ)e^{-\pi i Q^2}. \end{aligned}$$

□

We have an isometry $\mathcal{H}_R \rightarrow \mathcal{H}_L$ between the Hilbert space associated with the right and left invariant Haar weights, respectively, given in Proposition 2.15 by

$$\Lambda_R(a) = \Lambda_L(a\delta^{\frac{1}{2}}) = \Lambda_L(aA^{-\frac{Q}{2b}}). \quad (4.48)$$

Explicitly, we can rewrite all the maps in the right picture. We list here for convenience.

Proposition 4.17 *We have*

$$\Lambda_R \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right) = f \left(s - \frac{iQ}{2}, t \right) e^{-\pi Q t}, \quad (4.49)$$

with inner product given by

$$\langle f, g \rangle_R = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g \left(s - \frac{iQ}{2}, t + \frac{iQ}{2} \right)} f \left(s - \frac{iQ}{2}, t + \frac{iQ}{2} \right) e^{-2\pi Q t} ds dt. \quad (4.50)$$

$$T_R f(s, t) = \overline{f}(-s, -t) e^{2\pi i s t} e^{-\pi Q t}, \quad (4.51)$$

$$T_R^* f(s, t) = \overline{f}(-s, -t) e^{2\pi i s t + \pi Q t + 2\pi Q s - \pi i Q^2}, \quad (4.52)$$

$$\nabla_R f(s, t) = f(s, t) e^{2\pi s Q + 2\pi t Q - \pi i Q^2}, \quad (4.53)$$

$$J_R f(s, t) = \overline{f}(-s, -t) e^{2\pi i s t} e^{\pi s Q} e^{-\frac{\pi i Q^2}{2}}. \quad (4.54)$$

In particular, we recover the relation [25]

$$J_R = v^{i/4} J. \quad (4.55)$$

4.6 The multiplicative unitary

In this section, we will describe the multiplicative unitary W explicitly. Recall (cf. (2.30)) that it is defined as a unitary operator on $\mathcal{H} \otimes \mathcal{H}$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)). \quad (4.56)$$

We will use extensively the following variants of Lemma 3.7:

Corollary 4.18 *For $AB = q^2 BA$, $\widehat{A}\widehat{B} = q^{-2}\widehat{B}\widehat{A}$, using Lemma 2.24, we have*

$$g_b(B \otimes q\widehat{B}\widehat{A}^{-1}) = \int_{\mathbb{R}+i0} \frac{B^{ib^{-1}\tau} \otimes \widehat{A}^{-ib^{-1}\tau} \widehat{B}^{ib^{-1}\tau}}{G_b(Q + i\tau)} d\tau, \quad (4.57)$$

$$g_b(q^{-1}BA^{-1} \otimes \widehat{B}) = \int_{\mathbb{R}+i0} \frac{B^{ib^{-1}\tau} A^{-ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau}}{G_b(Q + i\tau)} d\tau, \quad (4.58)$$

$$g_b^*(B \otimes q\widehat{B}\widehat{A}^{-1}) = \int_{\mathbb{R}+i0} B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau} \widehat{A}^{-ib^{-1}\tau} G_b(-i\tau) d\tau, \quad (4.59)$$

$$g_b^*(q^{-1}BA^{-1} \otimes \widehat{B}) = \int_{\mathbb{R}+i0} A^{-ib^{-1}\tau} B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau} G_b(-i\tau) d\tau. \tag{4.60}$$

Note that the arguments inside g_b are all essentially self-adjoint; hence, the expression is well defined.

Proposition 4.19 *The multiplicative unitary $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is given by*

$$W = e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} g_b(B \otimes q\widehat{B}\widehat{A}^{-1}) \tag{4.61}$$

$$= g_b(q^{-1}BA^{-1} \otimes \widehat{B}) e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}}, \tag{4.62}$$

$$W^* = e^{\frac{i}{2\pi b^2} \log A \otimes \log \widehat{A}} g_b^*(q^{-1}BA^{-1} \otimes \widehat{B}) \tag{4.63}$$

$$= g_b^*(B \otimes q\widehat{B}\widehat{A}^{-1}) e^{\frac{i}{2\pi b^2} \log A \otimes \log \widehat{A}}, \tag{4.64}$$

where

$$A = e^{-2\pi bp_s}, \quad B = e^{2\pi bs} e^{-2\pi bp_t} \tag{4.65}$$

as before, and

$$\widehat{A} = e^{2\pi bs}, \quad \widehat{B} = G_b(-it) \circ e^{2\pi bp_t - 2\pi bp_s} \circ G_b(-it)^{-1}. \tag{4.66}$$

Remark 4.20 Note that we actually have $W \in M(\mathcal{A} \otimes \widehat{\mathcal{B}}(\mathcal{H}))$. Moreover, in the next section, we will see that the hat operators are precisely the dual space elements, hence indeed $W \in M(\mathcal{A} \otimes \widehat{\mathcal{A}})$. Furthermore, the multiplicative unitary will be invariant under the exchange $b \rightarrow b^{-1}$ and $\mathcal{A}_q \rightarrow \mathcal{A}_{\widehat{q}}$.

Proof First of all, notice that \widehat{A} and \widehat{B} are positive operators on the subspace \mathcal{W} (with the measure shifted in t by $\frac{iQ}{2}$) since $G_b(\frac{Q}{2} - it)$ is unitary. Furthermore, we have $\widehat{A}\widehat{B} = q^{-2}\widehat{B}\widehat{A}$. Moreover, both $q^{-1}BA^{-1}$ and $q\widehat{B}\widehat{A}^{-1}$ are positive operators; hence, the expressions are well defined.

The formula for W follows from W^* by conjugation and using the relations (2.46), (2.47). For simplicity, we compute W^* formally using the definition:

$$\begin{aligned} W^*(f \otimes g) &= \Delta(g)(f \otimes 1) \\ &= \left(g(s_1, t_1 + t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-i(t_1 + t_2))} \delta(s_2, s_1 + t_1) \right) \cdot f(s_1, t_1) \\ &= \int_C \int_{\mathbb{R}} g(s_1 - s'_1, t_1 + t_2 - t'_1) \frac{G_b(it'_1 - it_1)G_b(-it_2)}{G_b(-i(t_1 + t_2 - t'_1))} \\ &\quad \delta(s_2, s_1 + t_1 - t'_1 - s'_1) f(s'_1, t'_1) e^{2\pi i s'_1(t - t'_1)} ds'_1 dt'_1, \end{aligned}$$

where C goes below $t'_1 = t_1$. Replacing $s'_1 = s_1 + t_1 - s_2 - t'_1$, we get

$$\begin{aligned} &= \int_C f(s_1 + t_1 - s_2 - t'_1, t'_1) g(t'_1 - t_1 + s_2, t_1 + t_2 - t'_1) \\ &\quad e^{2\pi i(s_1+t_1-s_2-t'_1)(t-t'_1)} \frac{G_b(it'_1 - it_1)G_b(-it_2)}{G_b(-i(t_1 + t_2 - t'_1))} dt'_1. \end{aligned}$$

Renaming $t'_1 = \tau$ and do a shift $\tau \mapsto t_1 - \tau$, we get

$$\begin{aligned} W^*(f \otimes g) &= \int_{\mathbb{R}+i0} f(s_1 - s_2 + \tau, t_1 - \tau) g(s_2 - \tau, t_2 + \tau) e^{2\pi i(s_1 - s_2 + \tau)\tau} \\ &\quad \times \frac{G_b(-i\tau)G_b(-it_2)}{G_b(-i\tau - it_2)} d\tau, \end{aligned}$$

where the contour of τ now goes above $\tau = 0$.

On the other hand, using Corollary 4.18, the formula we desired is

$$\begin{aligned} &e^{\frac{i}{2\pi b^2} \log A \otimes \log \hat{A}} g_b^*(q^{-1} B A^{-1} \otimes \hat{B})(f \otimes g) \\ &= e^{\frac{i}{2\pi b^2} \log A \otimes \log \hat{A}} \left(\int_{\mathbb{R}+i0} A^{-ib^{-1}\tau} B^{ib^{-1}\tau} \otimes \hat{B}^{ib^{-1}\tau} G_b(-i\tau) d\tau \right) (f \otimes g) \\ &= e^{-2\pi i p_{s_1} s_2} \left(\int_{\mathbb{R}+i0} f(s_1 + \tau, t_1 - \tau) g(s_2 - \tau, t_2 + \tau) e^{2\pi i(s_1 + \tau)\tau} \frac{G_b(-it_2)G_b(-i\tau)}{G_b(-it_2 - i\tau)} d\tau \right) \\ &= \int_{\mathbb{R}+i0} f(s_1 - s_2 + \tau, t_1 - \tau) g(s_2 - \tau, t_2 + \tau) e^{2\pi i(s_1 - s_2 + \tau)\tau} \frac{G_b(-i\tau)G_b(-it_2)}{G_b(-i\tau - it_2)} d\tau. \end{aligned}$$

For completeness, let us note that the action of W is given by

$$W(f \otimes g) = \int_{\mathbb{R}+i0} \frac{G_b(-it_2)e^{2\pi i\tau s_1}}{G_b(-it_2 - i\tau)G_b(Q+i\tau)} f(s_1 + s_2, t_1 - \tau) g(s_2 - \tau, t_2 + \tau) d\tau. \quad (4.67)$$

□

In [46], it is commented that manageability of the multiplicative unitary is the property that distinguish quantum groups from quantum semigroups. However, we obtain the following:

Theorem 4.21 *W defined above is manageable.*

Proof We need to look for the positive operator Q and the unitary \tilde{W} in Definition 2.21. It is known (e.g., see [24,40]) that Q can be expressed in terms of the scaling constant and the scaling group:

$$Q = P^{1/2}, \quad P^{it} \Lambda(x) = v^{1/2} \Lambda(\tau_t(a)). \tag{4.68}$$

From the expressions derived in the previous sections, we obtain

$$Q \cdot f(s, t) = e^{-\pi Q \left(t - \frac{iQ}{2}\right)} f(s, t). \tag{4.69}$$

Note that in fact Q is positive, under the inner product of \mathcal{H} (with a shift in $t \mapsto t + \frac{iQ}{2}$).

Then, we see that $Q \otimes Q$ commutes with $A \otimes 1$, $1 \otimes \hat{A}$ and $B \otimes \hat{B}$. Therefore, from the formula (4.61), we conclude that $Q \otimes Q$ commutes with W .

Next, we will express \tilde{W} from definition and show that it is a unitary operator. First of all, let us note that restricting to holomorphic functions, when a shift in contour is involved, the inner product actually reads

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(s, t + \frac{iQ}{2}\right) \overline{g\left(\bar{s}, \bar{t} + \frac{iQ}{2}\right)} ds dt, \tag{4.70}$$

see the derivation in the proof of Theorem 4.9.

By definition, we have

$$\langle f' \otimes g', W(f \otimes g) \rangle_{\mathcal{H} \otimes \mathcal{H}} = \langle \bar{f} \otimes Qg', \tilde{W}(\bar{f}' \otimes Q^{-1}g) \rangle_{\overline{\mathcal{H}} \otimes \mathcal{H}},$$

where $\bar{f} := \overline{f}(z) = \overline{f(\bar{z})} \in \overline{\mathcal{H}}$, and the inner product is given by

$$\begin{aligned} \langle \bar{f}, \bar{g} \rangle_{\overline{\mathcal{H}}} &:= \langle g, f \rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{f}\left(s, t - \frac{iQ}{2}\right) g\left(s, t + \frac{iQ}{2}\right) ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{f}\left(s, t - \frac{iQ}{2}\right) \overline{g\left(\bar{s}, \bar{t} - \frac{iQ}{2}\right)} ds dt. \end{aligned} \tag{4.71}$$

Now, using the action of W given by (4.67), we have

$$\begin{aligned}
\text{LHS} &= \iint \iint ds_1 dt_1 ds_2 dt_2 f' \left(s_1, t_1 + \frac{iQ}{2} \right) g' \left(s_2, t_2 + \frac{iQ}{2} \right) \cdot \\
&\quad \overline{\int_{\mathbb{R}+i0} \frac{G_b \left(\frac{Q}{2} - i\bar{t}_2 \right) e^{2\pi i \tau \bar{s}_1}}{G_b \left(\frac{Q}{2} - i\bar{t}_2 - i\tau \right) G_b(Q+i\tau)} f \left(\bar{s}_1 + \bar{s}_2, \bar{t}_1 - \tau + \frac{iQ}{2} \right) g \left(\bar{s}_2 - \tau, \bar{t}_2 + \tau + \frac{iQ}{2} \right) d\tau} \\
&= \iint \iint \int_{\mathbb{R}+i0} d\tau ds_1 dt_1 ds_2 dt_2 f' \left(s_1, t_1 + \frac{iQ}{2} \right) g' \left(s_2, t_2 + \frac{iQ}{2} \right) e^{-2\pi i \tau s_1} \cdot \\
&\quad \times \frac{G_b \left(\frac{Q}{2} - it_2 - i\tau \right) G_b(i\tau)}{G_b \left(\frac{Q}{2} - it_2 \right)} \bar{f} \left(s_1 + s_2, t_1 - \tau - \frac{iQ}{2} \right) \bar{g} \left(s_2 - \tau, t_2 + \tau - \frac{iQ}{2} \right) \\
&= \iint \iint \int_{\mathbb{R}+i0} d\tau ds_1 dt_1 ds_2 dt_2 \bar{f} \left(s_1, t_1 - \frac{iQ}{2} \right) g' \left(s_2, t_2 + \frac{iQ}{2} \right) e^{-2\pi i \tau (s_1 - s_2)} \cdot \\
&\quad \times \frac{G_b \left(\frac{Q}{2} - it_2 - i\tau \right) G_b(i\tau)}{G_b \left(\frac{Q}{2} - it_2 \right)} f' \left(s_1 - s_2, t_1 + \tau + \frac{iQ}{2} \right) \bar{g} \left(s_2 - \tau, t_2 + \tau - \frac{iQ}{2} \right) \\
\text{RHS} &= \iint \iint ds_1 dt_1 ds_2 dt_2 \bar{f} \left(s_1, t_1 - \frac{iQ}{2} \right) g' \left(s_2, t_2 + \frac{iQ}{2} \right) e^{-\pi Q t_2} \\
&\quad \overline{\tilde{W} \left(\bar{f}' \otimes Q^{-1} g \right) \left(\bar{s}_1, \bar{t}_1 - \frac{iQ}{2}, \bar{s}_2, \bar{t}_2 + \frac{iQ}{2} \right)}.
\end{aligned}$$

Comparing, we get

$$\begin{aligned}
\overline{\tilde{W} \left(\bar{f}' \otimes Q^{-1} g \right) \left(\bar{s}_1, \bar{t}_1 - \frac{iQ}{2}, \bar{s}_2, \bar{t}_2 + \frac{iQ}{2} \right)} &= \int_{\mathbb{R}+i0} e^{\pi Q t_2} e^{-2\pi i \tau (s_1 - s_2)} \frac{G_b \left(\frac{Q}{2} - it_2 - i\tau \right) G_b(i\tau)}{G_b \left(\frac{Q}{2} - it_2 \right)} \\
&\quad \times f' \left(s_1 - s_2, t_1 + \tau + \frac{iQ}{2} \right) \bar{g} \left(s_2 - \tau, t_2 + \tau - \frac{iQ}{2} \right) d\tau \\
\tilde{W} \left(\bar{f}' \otimes Q^{-1} g \right) \left(\bar{s}_1, \bar{t}_1 - \frac{iQ}{2}, \bar{s}_2, \bar{t}_2 + \frac{iQ}{2} \right) &= \int_{\mathbb{R}+i0} e^{\pi Q \bar{t}_2} e^{2\pi i \tau (\bar{s}_1 - \bar{s}_2)} \frac{G_b \left(\frac{Q}{2} - i\bar{t}_2 \right)}{G_b \left(\frac{Q}{2} - i\bar{t}_2 - i\tau \right) G_b(Q+i\tau)} \\
&\quad \times \bar{f}'(\bar{s}_1 - \bar{s}_2, \bar{t}_1 + \tau - \frac{iQ}{2}) g(\bar{s}_2 - \tau, \bar{t}_2 + \tau + \frac{iQ}{2}) d\tau
\end{aligned}$$

so that

$$\begin{aligned}
&\tilde{W}(\bar{f}' \otimes Q^{-1} g)(s_1, t_1, s_2, t_2) \\
&= \int_{\mathbb{R}+i0} \frac{e^{\pi Q(t_2 - \frac{iQ}{2})} e^{2\pi i \tau (s_1 - s_2)} G_b(-it_2)}{G_b(-it_2 - i\tau) G_b(Q+i\tau)} \bar{f}'(s_1 - s_2, t_1 + \tau) g(s_2 - \tau, t_2 + \tau) d\tau.
\end{aligned}$$

Now renaming $G = Q^{-1}g$, i.e. $G(s_2, t_2) = e^{\pi Q(t_2 - \frac{iQ}{2})}g(s_2, t_2)$, we have

$$g(s_2 - \tau, t_2 + \tau) = G(s_2 - \tau, t_2 + \tau)e^{-\pi Q(t_2 + \tau - \frac{iQ}{2})},$$

hence the action of \widetilde{W} becomes

$$\begin{aligned} \widetilde{W}(\bar{f} \otimes g) &= \int_{\mathbb{R}+i0} \frac{e^{-\pi Q\tau} e^{2\pi i\tau(s_1-s_2)} G_b(-it_2)}{G_b(-it_2 - i\tau)G_b(Q + i\tau)} \bar{f}(s_1 - s_2, t_1 + \tau)g(s_2 - \tau, t_2 + \tau)d\tau \\ &= \int_{\mathbb{R}+i0} \bar{f}(s_1 - s_2, t_1 + \tau)g(s_2 - \tau, t_2 + \tau)e^{2\pi i\tau(s_1-s_2+t_2+\tau-\frac{iQ}{2})} \\ &\quad \times \frac{G_b(Q + it_2 + i\tau)G_b(-i\tau)}{G_b(Q + it_2)}d\tau. \end{aligned}$$

Hence, we can conclude that

$$\begin{aligned} \widetilde{W} &= e^{-2\pi i s_2 p_{s_1}} \int_{\mathbb{R}+i0} (B^\#)^{ib^{-1}\tau} \otimes (\widehat{B}^\#)^{ib^{-1}\tau} (\widehat{A}^\#)^{ib^{-1}\tau} G_b(-i\tau)d\tau \\ &= e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} g_b^*(B^\# \otimes q\widehat{B}^\# \widehat{A}^\#), \end{aligned}$$

where

$$\begin{aligned} B^\# &= e^{2\pi b(s+p_t)}, \quad \widehat{A}^\# = e^{2\pi b(t-\frac{iQ}{2})}, \\ \widehat{B}^\# &= G_b(Q + it)^{-1} e^{2\pi b(p_t - p_s)} \circ G_b(Q + it), \end{aligned}$$

are all positive operators with respect to the measure in $\overline{\mathcal{H}} \otimes \mathcal{H}$, and we used an analogue of (4.59) with $\widehat{A}^\# \widehat{B}^\# = q^2 \widehat{B}^\# \widehat{A}^\#$. Hence, \widetilde{W} is a unitary operator. \square

Remark 4.22 This is a rather curious and striking result since our quantum plane has been restricted to $B > 0$ so that all the operators involved are positive essentially self-adjoint and nicely defined. In fact, the key difference is that in [46], the multiplicative unitary W lies in $\mathcal{A} \otimes \mathcal{A}$, while our W lies in $\mathcal{A} \otimes \widehat{\mathcal{A}}$, and the Hilbert space $\mathcal{H} \simeq L^2(\mathbb{R} \times \mathbb{R})$ is “twice” larger than the canonical space $L^2(\mathbb{R})$ for the action of \mathcal{A} only. For example, we see that the operator $\widehat{A}^\# = e^{2\pi b(t - \frac{iQ}{2})}$ does not lie in the action spanned by \mathcal{A} itself, which only contains actions on the variable s and the shifting operator $e^{2\pi b p_t}$. Therefore, it is worth studying a deeper meaning of manageability of W in this wider context of quantum semigroups. The difference in the definition of W mentioned above in fact produces a new transformation formula for G_b , see Sect. 9 for further detail.

5 The dual space

We have encountered the dual space elements in the expression for the multiplicative unitary W . Following the recipe in [24], in order to describe the GNS representation for the dual space $\widehat{\mathcal{A}}$, which is self-dual for quantum plane, we need to establish a non-degenerate pairing between \mathcal{A} and $\widehat{\mathcal{A}}$.

First of all, let us describe its own GNS representation $(\widehat{\mathcal{H}}, \widehat{\pi}, \widehat{\Lambda})$. After establishing the pairing, we can then relate them to the canonical space \mathcal{H} .

5.1 Definitions

First, let us describe the dual space on the Hopf algebra level.

Definition 5.1 The algebraic dual space $\mathcal{A}_q^* \simeq \mathcal{A}_q$ is generated by self-adjoint operators X, Y with $XY = q^2 YX$ and the same coproduct.

Following [10], occasionally we will use the notation \mathcal{B}_q to denote \mathcal{A}_q^* when we want to think about the dual space as the quantum *algebra* counterpart of the quantum *group* \mathcal{A}_q . This correspondence is discussed in [14]. Similarly, we use the notation $\mathcal{B}_{q\bar{q}}$ to denote the modular double $\mathcal{B}_q \otimes \mathcal{B}_{\bar{q}}$.

Definition 5.2 We define $\widehat{\mathcal{A}}_q = \mathcal{A}_q^{*op}$ to be the algebra generated by X, Y with $XY = q^2 YX$ and the opposite coproduct. Alternatively, by defining $\widehat{A} = X^{-1}, \widehat{B} = q^{-1} YX^{-1}$, we define $\widehat{\mathcal{A}}_q$ to be generated by positive elements \widehat{A}, \widehat{B} with $\widehat{A}\widehat{B} = q^{-2} \widehat{B}\widehat{A}$ and the same coproduct as A and B :

$$\Delta(\widehat{A}) = \widehat{A} \otimes \widehat{A}, \tag{5.1}$$

$$\Delta(\widehat{B}) = \widehat{B} \otimes \widehat{A} + 1 \otimes \widehat{B}. \tag{5.2}$$

Definition 5.3 Similar to the quantum plane, we define $\widehat{\mathcal{A}} := \mathcal{C}_\infty(\widehat{\mathcal{A}}_q)$ to be the closure of the linear span of elements of the form

$$\widehat{f} = \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt, \tag{5.3}$$

where $\widehat{f}(s, t)$ has the same analytic properties as those in \mathcal{A} .

Since the spaces are self-dual, we immediately have the following

Proposition 5.4 *The left and right Haar functionals are given as before:*

$$\widehat{h}_L(\widehat{f}) = \widehat{f}(0, iQ), \tag{5.4}$$

$$\widehat{h}_R(\widehat{f}) = \widehat{f}(-iQ, iQ) e^{-\pi i Q^2}. \tag{5.5}$$

The GNS representation on $\widehat{\mathcal{H}}$ associated with \widehat{h}_L is simply given by

$$\widehat{\Lambda}_L \left(\int_{\mathbb{R}+i0} \int_{\mathbb{R}} \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \right) = f(s, t), \tag{5.6}$$

and the inner product is given by (notice the extra exponent):

$$\langle \widehat{f}, \widehat{g} \rangle := \widehat{h}_L(\widehat{g}^* \widehat{f}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g} \left(s, t + \frac{iQ}{2} \right)} \widehat{f} \left(s, t + \frac{iQ}{2} \right) e^{2\pi Qs} ds dt. \tag{5.7}$$

Following the same method as in the previous section, we also establish the following maps:

Proposition 5.5 *The product of two elements is given by*

$$(\widehat{f} \cdot \widehat{g})(s, t) = \iint \widehat{f}(s - s', t - t') \widehat{g}(s', t') e^{2\pi i(s-s')t} ds' dt'. \tag{5.8}$$

The action of \widehat{A} and \widehat{B} is given by:

$$\widehat{\pi}(\widehat{A}) = e^{2\pi bt} e^{-2\pi bp_s}, \quad \widehat{\pi}(\widehat{B}) = e^{-2\pi bp_t}. \tag{5.9}$$

The antipode is given by

$$\widehat{S}(\widehat{f}) = \widehat{f}(-s - t, t) e^{\pi Qt} e^{\pi i t^2 + 2\pi i s t}. \tag{5.10}$$

The scaling group is given by

$$\widehat{\tau}_t(\widehat{A}) = \widehat{A} \quad \widehat{\tau}_t(\widehat{B}) = e^{2\pi bQt} \widehat{B}. \tag{5.11}$$

The unitary antipode is given by

$$\widehat{R}(\widehat{A}) = \widehat{A}^{-1} \quad \widehat{R}(\widehat{B}) = q \widehat{B} \widehat{A}^{-1}, \tag{5.12}$$

or explicitly by

$$\widehat{R}(\widehat{f}) = \widehat{f}(-s - t, t) e^{\pi i t^2 + 2\pi i s t}. \tag{5.13}$$

The scaling constant is given by

$$\widehat{v} = e^{2\pi Q^2} = v^{-1}. \tag{5.14}$$

Note that by general theory, for the dual space, we have instead $\widehat{S} = \widehat{R} \widehat{\tau}_{i/2}$.

Proposition 5.6 *Acting on the space $\widehat{\mathcal{H}}$, we have*

$$\widehat{T} : \widehat{f}(s, t) \mapsto \overline{\widehat{f}}(-s, -t)e^{2\pi i s t}, \quad (5.15)$$

$$\widehat{T}^* : \widehat{f}(s, t) \mapsto \overline{\widehat{f}}(-s, -t)e^{2\pi i s t}e^{-2\pi s Q}, \quad (5.16)$$

$$\widehat{V} : \widehat{f}(s, t) \mapsto \widehat{f}(s, t)e^{-2\pi s Q}, \quad (5.17)$$

$$\widehat{J} : \widehat{f}(s, t) \mapsto \overline{\widehat{f}}(-s, -t)e^{2\pi i s t}e^{-\pi s Q}. \quad (5.18)$$

5.2 The non-degenerate pairing

Recall that a Hopf pairing between two Hopf algebras \mathcal{A} and \mathcal{A}^* is a non-degenerate pairing such that

$$\langle a, xy \rangle = \langle \Delta(a), x \otimes y \rangle, \quad \langle ab, x \rangle = \langle a \otimes b, \Delta(x) \rangle. \quad (5.19)$$

for $a, b \in \mathcal{A}$, $x, y \in \mathcal{A}^*$. (In this paper, we will not consider counit and antipode in the pairing.)

In the current setting, we have the pairing between \mathcal{A}_q and \mathcal{A}_q^* from the compact case:

$$\begin{aligned} \langle A, X \rangle &= q^{-2}, & \langle B, X \rangle &= 0, \\ \langle A, Y \rangle &= 0, & \langle B, Y \rangle &= c \in \mathbb{C}, \end{aligned}$$

where c is any complex number.

The pairings between A, B and $\widehat{A} = X^{-1}$, $\widehat{B} = q^{-1}YX^{-1}$ are given by

$$\begin{aligned} \langle A, \widehat{A} \rangle &= q^2, & \langle B, \widehat{A} \rangle &= 0, \\ \langle A, \widehat{B} \rangle &= 0, & \langle B, \widehat{B} \rangle &= cq := c'. \end{aligned}$$

We will choose $c' = 1$. From this, it is extended to

$$\langle B^m A^n, \widehat{A} \rangle = q^{2n} \delta_{m0}, \quad \langle B^m A^n, \widehat{B} \rangle = \delta_{m1},$$

and then to

$$\langle B^m A^n, \widehat{B}^{m'} \widehat{A}^{n'} \rangle = q^{2n'(n+m)} [m]_q! \delta_{mm'},$$

where $[m]_q! := [m]_q [m-1]_q \dots [1]_q$ is the q -factorial, and $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$ is the q -number.

From the analogy between the quantum dilogarithm G_b and the Γ_q function established in [14], we prove the following theorem:

Theorem 5.7 *The Hopf pairing between \mathcal{A} and $\mathcal{A}^* \simeq \widehat{\mathcal{A}}^{cop}$ is given by*

$$\begin{aligned} & \left\langle \iint g(s, t) B^{ib^{-1}t} A^{ib^{-1}s} ds dt, \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \right\rangle \\ &= \iiint g(s, t) \widehat{f}(s', t) G_b(Q + it) e^{-2\pi i s'(s+t)} ds' ds dt, \end{aligned} \tag{5.20}$$

$$\begin{aligned} & \left\langle \iint g(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt, \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \right\rangle \\ &= \iiint g(s, t) \widehat{f}(s', t) G_b(Q + it) e^{-2\pi i s'(s+t)} e^{-2\pi i s t} ds' ds dt, \end{aligned} \tag{5.21}$$

or using the Mellin transformed picture (see Remark 4.3):

$$\begin{aligned} & \left\langle \int f(t) B^{ib^{-1}t} g(\log A) dt, \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \right\rangle \\ &= \iint g(-2\pi b s') \widehat{f}(s', t) f(t) G_b(Q + it) e^{-2\pi i t s'} ds' dt, \end{aligned} \tag{5.22}$$

$$\begin{aligned} & \left\langle \int g(\log A) f(t) B^{ib^{-1}t} dt, \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \right\rangle \\ &= \iint g(-2\pi b(s' + t)) \widehat{f}(s', t) f(t) G_b(Q + it) e^{-2\pi i t s'} ds' dt. \end{aligned} \tag{5.23}$$

Here, the Hopf pairing between two C^* -algebra is defined by naturally extending the pairing in the defining relations (5.19) to a pairing between the multiplier algebras $M(\mathcal{A} \otimes \mathcal{A})$ and $M(\mathcal{A}^* \otimes \mathcal{A}^*)$.

Proof It suffices to show that

$$\langle f, \widehat{g\widehat{h}} \rangle = \langle \Delta(f), \widehat{g} \otimes \widehat{h} \rangle. \tag{5.24}$$

The other relations are similar by duality. We will prove the first form (5.21). For simplicity, we omit all the integrations after the pairing. Every variable is to be integrated.

$$\begin{aligned} LHS &= \langle f(s, t), \iint \widehat{g}(s-s', t-t') \widehat{h}(s', t') e^{2\pi i(s-s')t'} ds' dt' \rangle \\ &= f(s, t) \widehat{g}(s''-s', t-t') \widehat{h}(s', t') e^{2\pi i(s''-s')t'} G_b(Q+it) e^{-2\pi i s''(s+t)} e^{-2\pi i s t} \\ &= f(s, t) \widehat{g}(s''-s', t-t') \widehat{h}(s', t') G_b(Q+it) e^{2\pi i(s''-s')t'} e^{-2\pi i s''(s+t)} e^{-2\pi i s t}, \\ RHS &= \langle f(s_1, t_1+t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-it_1-it_2)} \delta(s_2, s_1+t_1), \widehat{g}(s_1, t_1) \widehat{h}(s_2, t_2) \rangle \\ &= f(s_1, t_1+t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-it_1-it_2)} G_b(Q+it_1) G_b(Q+it_2) \delta(s_2, s_1+t_1) \\ &\quad \widehat{g}(s'_1, t_1) \widehat{h}(s'_2, t_2) e^{-2\pi i s'_1(s_1+t_1)} e^{-2\pi i s'_2(s_2+t_2)} e^{-2\pi i s_1 t_1} e^{-2\pi i s_2 t_2} \end{aligned}$$

$$= f(s_1, t_1 + t_2) \widehat{g}(s'_1, t_1) \widehat{h}(s'_2, t_2) G_b(Q + it_1 + it_2) e^{-2\pi i s'_2 t_2} e^{-2\pi i s_1 (t_1 + t_2)} e^{-2\pi i (s_1 + t_1)(s'_1 + s'_2)},$$

where we used the reflection properties (3.10) of the G_b function:

$$\frac{G_b(-is)G_b(-it)}{G_b(-is-it)} = \frac{G_b(Q+is+it)e^{2\pi ist}}{G_b(Q+is)G_b(Q+it)}. \quad (5.25)$$

Now, shifting $t_1 \mapsto t_1 - t_2$, $s'_1 \mapsto s'_1 - s'_2$, we obtain

$$\begin{aligned} &= f(s_1, t_1) \widehat{g}(s'_1 - s'_2, t_1 - t_2) \widehat{h}(s'_2, t_2) G_b(Q + it_1) e^{-2\pi i s'_2 t_2} e^{-2\pi i s_1 t_1} e^{-2\pi i s'_1 (s_1 + t_1 - t_2)} \\ &= f(s_1, t_1) \widehat{g}(s'_1 - s'_2, t_1 - t_2) \widehat{h}(s'_2, t_2) G_b(Q + it_1) e^{2\pi i (s'_1 - s'_2) t_2} e^{-2\pi i s'_1 (s_1 + t_1)} e^{-2\pi i s_1 t_1}. \end{aligned}$$

Hence, we see that the expression is equal on renaming $s_1 = s$, $t_1 = t$, $s'_1 = s''$, $s'_2 = s'$ and $t_2 = t'$. \square

With the pairing established, we can prove the density condition needed in order for \mathcal{A} to be a locally compact quantum group.

Theorem 5.8 *We have*

$$\mathcal{A} = \text{span}\{(\omega \otimes 1)\Delta(a) \mid \omega \in \mathcal{A}^*, a \in \mathcal{A}\}^{\text{closure}} \quad (5.26)$$

$$= \text{span}\{(1 \otimes \omega)\Delta(a) \mid \omega \in \mathcal{A}^*, a \in \mathcal{A}\}^{\text{closure}}. \quad (5.27)$$

Proof We will prove that $(\omega \otimes 1)\Delta(a)$ is dense in $1 \otimes \mathcal{A}$, while the other statement is similar. Let us write

$$\begin{aligned} \omega &= \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \in \mathcal{A}^* \\ a &= \iint g(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \in \mathcal{A}. \end{aligned}$$

Using Proposition 4.4, the required pairing is (here, we understood that it is an element in $1 \otimes \mathcal{A}$ with coordinates s_2, t_2 , and that all integrations converge absolutely with the appropriate contours):

$$\begin{aligned} &\left\langle g(s_1, t_1 + t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-i(t_1+t_2))} \delta(s_2, s_1 + t_1), \widehat{f}(s, t) \right\rangle \\ &= \iiint \widehat{f}(s', t_1) g(s_1, t_1 + t_2) \frac{G_b(-it_1)G_b(-it_2)}{G_b(-i(t_1+t_2))} G_b(Q + it_1) \delta(s_2, s_1 + t_1) \\ &\quad e^{-2\pi i s' (s_1 + t_1)} ds' ds_1 dt_1 \\ &= \iint \widehat{f}(s', t_1) g(s_2 - t_1, t_1 + t_2) \frac{G_b(-it_2)}{G_b(-i(t_1+t_2))} e^{-2\pi i s' s_2} e^{\pi i t_1^2 - \pi t_1 Q} ds' dt_1. \end{aligned}$$

Now, if we choose $\widehat{f}(s', t_1) = \widehat{f}_1(s')\widehat{f}_2(t_1)$, the integration over s' is just the Fourier transform $\widehat{\mathcal{F}}_1(s_2)$ of $\widehat{f}_1(s')$, and we get

$$= \int \widehat{\mathcal{F}}_1(s_2)\widehat{f}_2(t_1)g(s_2 - t_1, t_1 + t_2) \frac{G_b(-it_2)}{G_b(-i(t_1 + t_2))} e^{\pi i t_1^2 - \pi t_1 Q} dt_1.$$

Finally, we can choose any nice approximation of identity for $\widehat{f}_2(t_1) \rightarrow \delta(t_1)$; then, in the limit, we obtain simply

$$\rightarrow \widehat{\mathcal{F}}_1(s_2)g(s_2, t_2),$$

which is then obviously all of $1 \otimes \mathcal{A}$ for different choices of $\widehat{\mathcal{F}}$ and g . □

Corollary 5.9 *\mathcal{A} is a locally compact quantum group in the sense of [24, 25].*

5.3 The GNS description

Given our multiplicative unitary

$$W = e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} g_b(B \otimes q\widehat{B}\widehat{A}^{-1}) \in M(\mathcal{A} \otimes \mathcal{B}(\mathcal{H})), \tag{5.28}$$

the space $\widehat{\mathcal{A}} \subset \mathcal{B}(\mathcal{H})$ is originally defined to be

$$\widehat{\mathcal{A}} = \{(\omega \otimes 1)W : \omega \in \mathcal{A}^*\}^{\text{norm closure}}. \tag{5.29}$$

In order to find the GNS representation $\widehat{\Lambda}$ of $\widehat{\mathcal{A}}$ on the original space \mathcal{H}_L , we introduce the contraction map λ [24, (8.2)] which relates \mathcal{A}^* to the definition of $\widehat{\mathcal{A}}$ above using the multiplicative unitary, and the ξ map [24, Notation 8.4], which relates \mathcal{A}^* to the original GNS representation space \mathcal{H}_L of \mathcal{A} by the Riesz’s theorem for Hilbert spaces. Then, $\widehat{\Lambda}$ is just the composition of the two maps, which is naturally compatible with the previous GNS construction for \mathcal{A} .

Definition 5.10 For $\omega \in \mathcal{A}^*$, $x \in \mathcal{A}$, we define

$$\begin{aligned} \lambda : \mathcal{A}^* &\longrightarrow \widehat{\mathcal{A}} \\ \lambda(\omega) &= (\omega \otimes 1)W. \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} \xi : \mathcal{A}^* &\longrightarrow \mathcal{H}_L \\ \omega(x^*) &= \langle \xi(\omega), \Lambda(x) \rangle_L. \end{aligned} \tag{5.31}$$

Then, the GNS map $\widehat{\Lambda}$ is given by

$$\begin{aligned} \widehat{\Lambda} : \widehat{\mathcal{A}} &\longrightarrow \mathcal{H}_L \\ \widehat{\Lambda}(\lambda(\omega)) &= \xi(\omega). \end{aligned} \tag{5.32}$$

Proposition 5.11 λ is the identity map from \mathcal{A}^* to $\widehat{\mathcal{A}}$ as elements in $\mathcal{B}(\mathcal{H})$.

Proof Using the second form (4.62) for W , and the integral form (4.58) for g_b , we get

$$W = \left(\int_{\mathbb{R}+i0} \frac{B^{ib^{-1}\tau} A^{-ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau}}{G_b(Q+i\tau)} d\tau \right) e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}}.$$

By the pairing (5.21), we have

$$\begin{aligned} \lambda(\omega) &= \left(\iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt \otimes 1 \right) (W) \\ &= \iint \widehat{f}(s, t) \frac{(q^{2ib^{-1}s})^{-ib^{-1}t}}{G_b(Q+it)} G_b(Q+it) e^{-2\pi ist} \widehat{B}^{ib^{-1}t} \widehat{A}^{\frac{i}{2\pi b^2}(2\pi bs)} ds dt \\ &= \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt. \end{aligned}$$

□

Proposition 5.12 The map $\xi : \mathcal{A}^* \subset \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ is given by:

$$\xi : \widehat{f}(s, t) \mapsto F(s, t) := \int_{\mathbb{R}} \widehat{f}(s', iQ-t) G_b(-it) e^{2\pi is'(s+t-iQ)} ds', \quad (5.33)$$

it naturally extends to an invertible map $\xi : \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ with the inverse given by

$$\xi^{-1} : F(s, t) \mapsto \widehat{f}(s, t) := \int_{\mathbb{R}} \frac{F(s', iQ-t)}{G_b(Q+it)} e^{2\pi is(t-s')} ds'. \quad (5.34)$$

Proof Let $\omega = \iint \widehat{f}(s, t) \widehat{B}^{ib^{-1}t} \widehat{A}^{ib^{-1}s} ds dt$, $x = \iint g(s, t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt$. Then

$$\begin{aligned} \omega(x^*) &= \langle \xi(\omega), \Lambda(x) \rangle_L \\ \text{LHS} &= \iiint \overline{g(-s, -t)} e^{2\pi ist} F(s', t) G_b(Q+it) e^{-2\pi is's-2\pi is't-2\pi ist} ds' ds dt \\ &= \iiint \overline{g(-s, -t)} F(s', t) G_b(Q+it) e^{-2\pi is's-2\pi is't} ds' ds dt \\ \text{RHS} &= \iint \overline{g\left(\bar{s}, \bar{t} + \frac{iQ}{2}\right)} \xi(\omega)\left(s, t + \frac{iQ}{2}\right) ds dt \\ &= \iint \overline{g\left(s, t - \frac{iQ}{2}\right)} \xi(\omega)\left(s, t + \frac{iQ}{2}\right) ds dt \\ &= \iint \overline{g(-s, -t)} \xi(\omega)(-s, -t + iQ) ds dt. \end{aligned}$$

Hence,

$$\begin{aligned} \xi(\omega)(s, t) &= \int F(s', iQ - t)G_b(Q + i(iQ - t))e^{2\pi is's - 2\pi is'(iQ - t)} ds' \\ &= \int F(s', iQ - t)G_b(-it)e^{2\pi is'(s + t - iQ)} ds'. \end{aligned}$$

We observe that ξ can be realized as

$$\xi = \mathcal{F}_s \circ P_t \circ G_b(Q + it)e^{-2\pi ist},$$

where \mathcal{F}_s is Fourier transform on s , $P_t f(t) := f(iQ - t)$ and $G_b(Q + it)e^{-2\pi ist}$ is just multiplication by this function. Hence, we have

$$\xi^{-1} = G_b(Q + it)^{-1}e^{2\pi ist} \circ P_t^{-1} \circ \mathcal{F}_s^{-1},$$

or the formula desired. □

Proposition 5.13 ξ is an isometry, $\xi : \widehat{\mathcal{H}} \rightarrow \mathcal{H}$.

Proof We have

$$\begin{aligned} &(\xi(\widehat{f}), \xi(\widehat{g}))_L \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s'', iQ - \left(t + \frac{iQ}{2}\right)\right)} G_b\left(-i\left(t + \frac{iQ}{2}\right)\right) e^{2\pi is''\left(s + \left(t + \frac{iQ}{2}\right) - iQ\right)} \\ &\quad \times \widehat{f}\left(s', iQ - \left(t + \frac{iQ}{2}\right)\right) G_b\left(-i\left(t + \frac{iQ}{2}\right)\right) e^{2\pi is'\left(s + \left(t + \frac{iQ}{2}\right) - iQ\right)} ds' ds'' ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s'', -t + \frac{iQ}{2}\right)} G_b\left(\frac{Q}{2} - it\right) e^{-2\pi is''\left(s + t + \frac{iQ}{2}\right)} \\ &\quad \times \widehat{f}\left(s', -t + \frac{iQ}{2}\right) G_b\left(\frac{Q}{2} - it\right) e^{2\pi is'\left(s + t - \frac{iQ}{2}\right)} ds' ds'' ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s'', t + \frac{iQ}{2}\right)} \widehat{f}\left(s', t + \frac{iQ}{2}\right) e^{-2\pi is''\left(s - t + \frac{iQ}{2}\right)} e^{2\pi is'\left(s - t - \frac{iQ}{2}\right)} ds' ds'' ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s'', t + \frac{iQ}{2}\right)} \widehat{f}\left(s', t + \frac{iQ}{2}\right) e^{-2\pi is''\left(-t + \frac{iQ}{2}\right)} e^{2\pi is'\left(-t - \frac{iQ}{2}\right)} \\ &\quad e^{2\pi is(-s'' + s')} ds' ds'' ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s, t + \frac{iQ}{2}\right)} \widehat{f}\left(s, t + \frac{iQ}{2}\right) e^{-2\pi is\left(-t + \frac{iQ}{2}\right)} e^{2\pi is\left(-t - \frac{iQ}{2}\right)} ds dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s, t + \frac{iQ}{2}\right)} \widehat{f}\left(s, t + \frac{iQ}{2}\right) e^{2\pi Qs} ds dt \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{g}\left(s, t + \frac{iQ}{2}\right)} \widehat{f}\left(s, t + \frac{iQ}{2}\right) e^{2\pi Qs} ds dt \\
&= \langle \widehat{f}, \widehat{g} \rangle_{\widehat{\mathcal{L}}}.
\end{aligned}$$

□

Hence, we conclude that

Corollary 5.14 $\widehat{\Lambda}(\omega) := \xi(\omega)$ gives the GNS representation for \widehat{A} on \mathcal{H} as desired.

5.4 The multiplicative unitary and modular maps

Under the transformation ξ , we can now express all the operators defined earlier on $\widehat{\mathcal{H}}$ to \mathcal{H} . We have

Proposition 5.15 The action $\pi : \widehat{A} \longrightarrow \mathcal{H}$ is given by

$$\pi(\widehat{A}) = e^{-2\pi bs}, \quad (5.35)$$

$$\pi(\widehat{B}) = G_b(-it) \circ e^{2\pi b(p_t - p_s)} \circ G_b(-it)^{-1}. \quad (5.36)$$

Hence, in particular, W is indeed a genuine element in $M(\mathcal{A} \otimes \widehat{\mathcal{A}})$.

We also note that by general theory, the multiplicative unitary for $\widehat{\mathcal{A}}$ is given by

$$\widehat{W} = \Sigma W^* \Sigma, \quad (5.37)$$

where Σ is the permutation operator on the tensor product. The coproduct induces by \widehat{W} is precisely the one we defined earlier that \widehat{A} and \widehat{B} transformed as how A and B do.

Proposition 5.16 The modular maps act on \mathcal{H} by:

$$\widehat{T}(f) = \overline{f}(s + t - iQ, -t) e^{-\pi Qt} e^{-\pi it^2} \quad (5.38)$$

$$\widehat{T}^*(f) = \overline{f}(s + t, -t) e^{-\pi Qt} e^{-\pi it^2} \quad (5.39)$$

$$\widehat{V}(f) = f(s + iQ, t) \quad (5.40)$$

$$\widehat{J}(f) = \overline{f}\left(s + t - \frac{iQ}{2}, -t\right) e^{-\pi Qt} e^{-\pi it^2}. \quad (5.41)$$

From these actions, we can verify all the well-known properties between these maps on \mathcal{H} (see [25] Prop 2.1, 2.11, 2.12):

Proposition 5.17 For $x \in \mathcal{A}$, $y \in \widehat{\mathcal{A}}$, we have the properties:

$$\widehat{J}\Lambda_R(x) = \Lambda(R(x)^*), \quad (5.42)$$

$$\widehat{V}^{-\frac{1}{2}}\Lambda(x) = \Lambda_R(\tau_{i/2}(x)), \quad (5.43)$$

$$\widehat{T}^* \Lambda(x) = \Lambda(S^{-1}(x)^*), \tag{5.44}$$

$$\widehat{J}J = \nu^{i/4} J \widehat{J}, \tag{5.45}$$

$$R(x) = \widehat{J}x^* \widehat{J}, \tag{5.46}$$

$$\widehat{R}(y) = Jy^* J. \tag{5.47}$$

Furthermore, if we define on \mathcal{H} the operator G ([24] Prop 3.22):

$$G \Lambda((h_R \otimes 1)(\Delta(x^*)(y \otimes 1))) = \Lambda((h_R \otimes 1)(\Delta(y^*)(x \otimes 1))) \tag{5.48}$$

and its polar decomposition $G = IN^{1/2}$, then we have $\widehat{T}^* = G$, $\widehat{J} = I$ and $\widehat{V} = N^{-1}$.

6 Transformation to new Hilbert spaces

To prepare for the construction of the quantum double, we found it useful to introduce a transformation of the Hilbert space \mathcal{H} as well as $\widehat{\mathcal{H}}$, so that the action of \mathcal{A} and $\widehat{\mathcal{A}}$ acts as the canonical Weyl algebra. Furthermore, this transformation will bring the inner product to the canonical form for $L^2(\mathbb{R})$.

6.1 Unitary transformations

In this section, we list the transformations that will be used. They are all unitary transformations on $L^2(\mathbb{R} \times \mathbb{R})$ equipped with the standard Lebesgue measure. For an operator P , its action under any transformation \mathcal{T} is given by

$$P \mapsto \mathcal{T} \circ P \circ \mathcal{T}^{-1}. \tag{6.1}$$

Proposition 6.1 *The following lists the transformations and their effects on the multiplication and differential operators s and p_s .*

$$f(s, t) \mapsto :$$

$$f(-s, t) : s \longrightarrow -s, \quad p_s \longrightarrow -p_s, \tag{6.2}$$

$$f(s, -t) : t \longrightarrow -t, \quad p_t \longrightarrow -p_t, \tag{6.3}$$

$$f(t, s) : s \longleftrightarrow t, \quad p_s \longleftrightarrow p_t, \tag{6.4}$$

$$f(s, t)e^{\pm 2\pi i s t} : p_s \longrightarrow p_s \mp t, \quad p_t \longrightarrow p_t \mp s, \tag{6.5}$$

$$f(s, t)e^{\pm \pi i s^2} : p_s \longrightarrow p_s \mp s, \tag{6.6}$$

$$f(s, t)e^{\pm \pi i t^2} : p_t \longrightarrow p_t \mp t, \tag{6.7}$$

$$f(s \pm t, t) : s \longrightarrow s \pm t, \quad p_t \longrightarrow p_t \mp p_s, \tag{6.8}$$

$$f(s, t \pm s) : t \longrightarrow t \pm s, \quad p_s \longrightarrow p_s \mp p_t, \tag{6.9}$$

$$\mathcal{F}(f) = \int f(\tau)e^{-2\pi i \tau x} d\tau : x \longrightarrow -p_x, \quad p_x \longrightarrow x, \tag{6.10}$$

$$\mathcal{F}^{-1}(f) = \int f(\tau)e^{2\pi i \tau x} d\tau : x \longrightarrow p_x, \quad p_x \longrightarrow -x, \tag{6.11}$$

where \mathcal{F} is the Fourier transform with

$$\mathcal{F}^2 = -Id.$$

All the transformations above preserve the dense subspace $\mathcal{W} \widehat{\otimes} \mathcal{W}$ (cf. (2.41)).

6.2 The representation space

We introduce the transformations such that \mathcal{A} acts on \mathcal{H} canonically, and $\widehat{\mathcal{A}}$ acts on $\widehat{\mathcal{H}}$ canonically. We will use capital letter to denote the transformed function. All the new Hilbert spaces are $L^2(\mathbb{R} \times \mathbb{R})$.

Definition 6.2 We define $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}_{\text{rep}}$ by

$$\mathcal{T} : f(s, t) \mapsto \int_{\mathbb{R}} f\left(\alpha, t - s + \frac{iQ}{2}\right) e^{2\pi i \alpha s} d\alpha, \tag{6.12}$$

$$\mathcal{T}^{-1} : F(s, t) \mapsto \int_{\mathbb{R}} F\left(\alpha, t + \alpha - \frac{iQ}{2}\right) e^{-2\pi i \alpha s} d\alpha, \tag{6.13}$$

or simply $\mathcal{T} = (t \mapsto t - s) \circ \mathcal{F}_s^{-1} \circ (t \mapsto t + \frac{iQ}{2})$.

Similarly, we define $\widehat{\mathcal{T}} : \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}_{\text{rep}}$ by

$$\widehat{\mathcal{T}} : \widehat{f}(s, t) \mapsto \int \widehat{f}\left(\alpha, t - s + \frac{iQ}{2}\right) e^{-2\pi i \alpha t} e^{\pi Q \alpha} d\alpha, \tag{6.14}$$

$$\widehat{\mathcal{T}}^{-1} : \widehat{F}(s, t) \mapsto \int \widehat{F}\left(\alpha, t + \alpha - \frac{iQ}{2}\right) e^{2\pi i s(\alpha+t)} d\alpha, \tag{6.15}$$

or simply $\widehat{\mathcal{T}} = (t \mapsto t - s) \circ \mathcal{F}_s \circ (t \mapsto t + \frac{iQ}{2}) \circ e^{-2\pi i s t}$.

Proposition 6.3 Under the transformations, both the spaces \mathcal{H}_{rep} and $\widehat{\mathcal{H}}_{\text{rep}}$ carry the usual inner product with respect to the standard Lebesgue measure:

$$\langle F(s, t), G(s, t) \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{G(s, t)} F(s, t) ds dt. \tag{6.16}$$

Proposition 6.4 Under the transformations, the action of \mathcal{A} on \mathcal{H}_{rep} is given by

$$A = e^{2\pi b s}, \quad B = e^{2\pi b p_s}, \tag{6.17}$$

and the action of $\widehat{\mathcal{A}}$ on $\widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$\widehat{A} = e^{-2\pi b s}, \quad \widehat{B} = e^{2\pi b p_s}. \tag{6.18}$$

Proof Let us demonstrate the use of transformation rule for say, the operator B . The rest is similar. Recall that B acts on \mathcal{H} as $e^{2\pi bs} e^{-2\pi bp_t}$. Hence, under the transformation, it becomes

$$\begin{aligned} e^{2\pi bs} e^{-2\pi bp_t} &\mapsto_{t \mapsto t + \frac{iQ}{2}} e^{2\pi bs} e^{-2\pi bp_t} \\ &\mapsto_{\mathcal{F}_s^{-1}} e^{2\pi bp_s} e^{-2\pi bp_t} \\ &\mapsto_{t \mapsto t-s} e^{2\pi bp_s}. \end{aligned}$$

□

As a corollary, we have

Corollary 6.5 *As a representation of \mathcal{A} , we have*

$$\mathcal{H} \simeq \mathcal{H}_{\text{irr}} \otimes L^2(\mathbb{R}), \quad (6.19)$$

where \mathcal{H}_{irr} is the canonical irreducible representation (1.2) of \mathcal{A} on $L^2(\mathbb{R})$.

Let us also note that

Proposition 6.6 *The product of two elements in \mathcal{A} induces a twisted product on \mathcal{H} by*

$$F \cdot G = \mathcal{T}(\mathcal{T}^{-1}(F) \cdot \mathcal{T}^{-1}(G)) = \int_{\mathbb{R}} F(s, \tau) G(\tau, t) d\tau. \quad (6.20)$$

We can do the same analysis to the dual space:

Proposition 6.7 *Under the transformations, the action of $\widehat{\mathcal{A}}$ on \mathcal{H}_{rep} is given by*

$$\widehat{A} = e^{2\pi b(p_s + p_t)}, \quad \widehat{B} = G_b \left(\frac{Q}{2} + is - it \right) \circ e^{2\pi b(p_t + s)} \circ G_b \left(\frac{Q}{2} + is - it \right)^{-1}, \quad (6.21)$$

and the action of \mathcal{A} on $\widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$A = e^{2\pi b(p_s + p_t)}, \quad B = G_b \left(\frac{Q}{2} - is + it \right)^{-1} \circ e^{2\pi b(p_t - s)} \circ G_b \left(\frac{Q}{2} - is + it \right). \quad (6.22)$$

Remark 6.8 Note that the action of \mathcal{A} on $\widehat{\mathcal{H}}$ is obtained by the ξ transformation. Moreover, recall that $G_b(\frac{Q}{2} + is)$ is unitary; hence, the action above is still positive.

Finally, we will need to describe the action of the modular conjugation J and \widehat{J} that are crucial in the construction of the multiplicative unitary for the quantum double.

Proposition 6.9 *The action of J and \widehat{J} on \mathcal{H}_{rep} is given by*

$$J : F(s, t) \mapsto \overline{F}(t, s), \quad (6.23)$$

$$\widehat{J} : F(s, t) \mapsto \overline{F}(-s, -t)e^{\pi i s^2 - \pi i t^2} v^{i/8}, \quad (6.24)$$

where $v = e^{-2\pi Q^2}$.

Similarly, the action of \widehat{J} on $\widehat{\mathcal{H}}_{\text{rep}}$ is also given by

$$\widehat{J} : \widehat{F}(s, t) \mapsto \overline{\widehat{F}}(t, s), \quad (6.25)$$

while the action of J on $\widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$J : \widehat{F}(s, t) \mapsto \overline{\widehat{F}}(-s, -t)e^{\pi i t^2 - \pi i s^2} v^{-i/8}. \quad (6.26)$$

Proposition 6.10 *The action of AdJ on \mathcal{A} is given by:*

$$J(x)J = \widehat{R}_*(x^*), \quad (6.27)$$

where \widehat{R}_* is an anti-homomorphism given by

$$\widehat{R}_*(A) = \widehat{A}_*, \quad \widehat{R}_*(B) = \widehat{B}_*, \quad (6.28)$$

where

$$\widehat{A}_* = e^{2\pi bt}, \quad \widehat{B}_* = e^{-2\pi bp_t}. \quad (6.29)$$

The action of $Ad\widehat{J}$ on $\widehat{\mathcal{A}}$ is given by:

$$\widehat{J}(y)\widehat{J} = R_*(y^*), \quad (6.30)$$

where R_* is an anti-homomorphism given by

$$R_*(\widehat{A}) = A_*, \quad R_*(\widehat{B}) = B_*, \quad (6.31)$$

where

$$A_* = e^{-2\pi bt}, \quad B_* = e^{-2\pi bp_t}. \quad (6.32)$$

Corollary 6.11 *$J \circ \mathcal{A} \circ J$ commutes with \mathcal{A} as operators on \mathcal{H}_{rep} , and similarly $\widehat{J} \circ \widehat{\mathcal{A}} \circ \widehat{J}$ commutes with $\widehat{\mathcal{A}}$ as operators on $\widehat{\mathcal{H}}_{\text{rep}}$, which verifies Theorem 2.11.*

6.3 The corepresentation space

Recall that W satisfies the pentagon equation and the coaction axiom (2.35). In a more familiar form, we see that $W' = W_{21} := \Sigma W \Sigma \in M(\widehat{\mathcal{A}} \otimes \mathcal{A})$ satisfies

$$(1 \otimes \Delta)W' = W'_{12}W'_{13}. \tag{6.33}$$

Hence, if we treat W' as an element in $M(\mathcal{B}(\mathcal{H}) \otimes \mathcal{A})$ instead, we obtain the *left regular corepresentation*

$$\begin{aligned} \Pi : \mathcal{H} &\longrightarrow \mathcal{H} \otimes M(\mathcal{A}) \\ f &\mapsto W'(f \otimes 1) \end{aligned} \tag{6.34}$$

which satisfies

$$(1 \otimes \Delta) \circ \Pi = (\Pi \otimes 1) \circ \Pi. \tag{6.35}$$

This corepresentation picture is useful when we study the corepresentation of the quantum double. Therefore, we introduce the transformation from \mathcal{H} to $\mathcal{H}_{\text{corep}}$ so that $\widehat{\mathcal{A}}$ acts canonically. Similarly, by considering \widehat{W} , we also describe the transformation so that \mathcal{A} acts canonically on $\widehat{\mathcal{H}}$.

Definition 6.12 We define the transformation $\mathcal{T}_{co} : \mathcal{H} \longrightarrow \mathcal{H}_{\text{corep}}$ by

$$\mathcal{T}_{co} : f(s, t) \mapsto f\left(-s, s - t + \frac{iQ}{2}\right) G_b\left(\frac{Q}{2} - is + it\right)^{-1}, \tag{6.36}$$

$$\mathcal{T}_{co}^{-1} : f(s, t) \mapsto f\left(-s, -t - s + \frac{iQ}{2}\right) G_b(-it). \tag{6.37}$$

Similarly, we define the transformation $\widehat{\mathcal{T}}_{co} : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}_{\text{corep}}$ by

$$\widehat{\mathcal{T}}_{co} : \widehat{f}(s, t) \mapsto f\left(-s, s - t + \frac{iQ}{2}\right) G_b\left(\frac{Q}{2} + is - it\right) e^{2\pi is\left(s - t + \frac{iQ}{2}\right)}. \tag{6.38}$$

$$\widehat{\mathcal{T}}_{co}^{-1} : \widehat{f}(s, t) \mapsto f\left(-s, -t - s + \frac{iQ}{2}\right) e^{2\pi ist} G_b(Q + it)^{-1}. \tag{6.39}$$

Proposition 6.13 Under the transformation \mathcal{T}_{co} , the action of $\widehat{\mathcal{A}}$ on $\mathcal{H}_{\text{corep}}$ is given by

$$\widehat{A} = e^{-2\pi bs}, \quad \widehat{B} = e^{2\pi bp_s}. \tag{6.40}$$

Under the transformation $\widehat{\mathcal{T}}_{co}$, the action of \mathcal{A} on $\widehat{\mathcal{H}}_{\text{corep}}$ is given by

$$A = e^{2\pi bs}, \quad B = e^{2\pi bp_s}. \tag{6.41}$$

Furthermore, the L^2 measure on $\mathcal{H}_{\text{corep}}$ and $\widehat{\mathcal{H}}_{\text{corep}}$ becomes the standard Lebesgue measure.

Proof It suffices to see that the transformations can be written as

$$\mathcal{T}_{co} = (t \mapsto t - s) \circ (-s, -t) \circ \left(t \mapsto t + \frac{iQ}{2} \right) \circ G_b(-it)^{-1}, \quad (6.42)$$

$$\widehat{\mathcal{T}}_{co} = (t \mapsto t - s) \circ (-s, -t) \circ \left(t \mapsto t + \frac{iQ}{2} \right) \circ G_b(Q + it)e^{-2\pi ist}, \quad (6.43)$$

followed by applying the transformation rules. □

This choice allows us to reproduce the representation from the pairing between \mathcal{A} and $\widehat{\mathcal{A}}$ given by the corepresentation associated with the multiplicative unitary.

Proposition 6.14 *The left regular corepresentation associated with W' on \mathcal{A} is given by*

$$f(s, t) \mapsto \int_{\mathbb{R}+i0} \frac{f(s + \tau, t)}{G_b(Q + i\tau)} B^{ib^{-1}\tau} A^{ib^{-1}s} d\tau. \quad (6.44)$$

Remark 6.15 Note that the left regular corepresentation is related to the left “fundamental” representation

$$\int f(s)x^{b^{-1}(-\frac{Q}{2}+is)} ds \mapsto \left(\int f(s)(xA + B)^{b^{-1}(-\frac{Q}{2}+is)} ds \right) \delta^{-\frac{1}{2}}, \quad (6.45)$$

i.e.

$$\begin{aligned} f(s) &\mapsto \int f(s + \tau) \frac{G_b(-i\tau) G_b\left(\frac{Q}{2} - is\right)}{G_b\left(\frac{Q}{2} - i\tau - is\right)} A^{ib^{-1}s - \frac{Q}{2b}} B^{ib^{-1}\tau} A^{\frac{Q}{2b}} d\tau \\ &= \int f(s + \tau) \frac{G_b\left(\frac{Q}{2} + is + i\tau\right)}{G_b(Q + i\tau)G_b\left(\frac{Q}{2} + is\right)} B^{ib^{-1}\tau} A^{ib^{-1}s} d\tau \end{aligned} \quad (6.46)$$

by multiplication by $G_b(\frac{Q}{2} + is)^{-1}$. Here, $\delta = A^{-\frac{Q}{b}}$ is the modular element.

Proposition 6.16 *Under the pairing $\langle 1 \otimes \mathcal{A}_q^*, W'(f \otimes 1) \rangle$, the left regular representation of the quantum plane algebra $\mathcal{B}_q = \mathcal{A}_q^*$ is given by the canonical action*

$$X = e^{2\pi bs}, \quad Y = e^{2\pi b(p_s+s)}. \quad (6.47)$$

Similarly, we can define the action of its modular double counterpart

$$\widetilde{X} = e^{2\pi b^{-1}s}, \quad \widetilde{Y} = e^{2\pi b^{-1}(p_s+s)}, \quad (6.48)$$

so that it extends to a representation of the modular double $\mathcal{B}_{q\bar{q}}$. Under a unitary transform by multiplication by $e^{\pi is^2}$ on $\mathcal{H}_{\text{corep}}$, the action becomes the canonical action of $\mathcal{B}_{q\bar{q}}$.

Proof It follows directly from Proposition 5.11 since λ is the identity map; hence, the elements of \widehat{A} are sent to the corresponding actions. Now, the formula follows from the definition $X = \widehat{A}^{-1}$, $Y = q\widehat{B}\widehat{A}^{-1}$. \square

Similarly, the corepresentation associated with \widehat{W}' is given by

$$f(s) \mapsto \int f(s + \tau)G_b(-i\tau)\widehat{B}^{ib^{-1}\tau}\widehat{A}^{ib^{-1}s}d\tau. \tag{6.49}$$

For completeness, by composing \mathcal{T}_{co} with \mathcal{T}^{-1} , we obtain the transformation $\mathcal{S} : \mathcal{H}_{\text{rep}} \longrightarrow \mathcal{H}_{\text{corep}}$

Proposition 6.17 *The transformation $\mathcal{S} : \mathcal{H}_{\text{rep}} \longrightarrow \mathcal{H}_{\text{corep}}$ is given by*

$$\mathcal{S} : f(s, t) \mapsto \int_{\mathbb{R}} f(\alpha - s, \alpha - t)G_b\left(\frac{Q}{2} - is + it\right)^{-1} e^{2\pi i(\alpha-s)s}d\alpha, \tag{6.50}$$

$$\mathcal{S}^{-1} : f(s, t) \mapsto \int_{\mathbb{R}} f(\alpha - s, \alpha - t)G_b\left(\frac{Q}{2} + is - it\right) e^{-2\pi i(\alpha-s)s}d\alpha. \tag{6.51}$$

Similarly, we define the corresponding transformation for \mathcal{A} using $\widehat{\mathcal{S}} : \widehat{\mathcal{H}}_{\text{rep}} \longrightarrow \widehat{\mathcal{H}}_{\text{corep}}$ by

$$\widehat{\mathcal{S}} : \widehat{f}(s, t) \mapsto \int_{\mathbb{R}} f(\alpha - s, \alpha - t)G_b\left(\frac{Q}{2} + is - it\right) e^{-2\pi i(\alpha-s)s}d\alpha, \tag{6.52}$$

$$\widehat{\mathcal{S}}^{-1} : \widehat{f}(s, t) \mapsto \int_{\mathbb{R}} f(\alpha - s, \alpha - t)G_b\left(\frac{Q}{2} - is + it\right)^{-1} e^{2\pi i(\alpha-s)s}d\alpha. \tag{6.53}$$

7 The quantum double construction

7.1 Definitions

In this section, we will describe the quantum double group construction given by [30] (see also [34]) associated with the quantum plane and show that the object we obtain is exactly the quantum ‘‘semigroup’’ $GL_q^+(2, \mathbb{R})$, also called the split quantum

Minkowski spacetime, which is a generalization of the compact Minkowski spacetime introduced in [8,9].

Definition 7.1 We define the split quantum Minkowski Spacetime $\mathcal{M}_q^+(\mathbb{R})$ as the Hopf $*$ -algebra generated by positive self-adjoint operators z_{ij} , $i, j \in \{1, 2\}$ such that the following relations hold:

$$\begin{aligned} [z_{11}, z_{12}] &= 0, \\ [z_{21}, z_{22}] &= 0, \\ [z_{11}, z_{22}] &= [z_{12}, z_{21}], \\ z_{11}z_{21} &= q^2 z_{21}z_{11}, \\ z_{12}z_{22} &= q^2 z_{22}z_{12}, \\ z_{12}z_{21} &= q^2 z_{21}z_{12}, \end{aligned}$$

and the coproduct is given by

$$\Delta(z_{11}) = z_{11} \otimes z_{11} + z_{12} \otimes z_{21}, \quad (7.1)$$

$$\Delta(z_{12}) = z_{11} \otimes z_{12} + z_{12} \otimes z_{22}, \quad (7.2)$$

$$\Delta(z_{21}) = z_{21} \otimes z_{11} + z_{22} \otimes z_{21}, \quad (7.3)$$

$$\Delta(z_{22}) = z_{21} \otimes z_{12} + z_{22} \otimes z_{22}. \quad (7.4)$$

It can also be realized as $GL_q^+(2, \mathbb{R})$ in matrix form:

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \quad (7.5)$$

so that the coproduct is simply given by

$$\Delta\left(\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\right) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \otimes \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}. \quad (7.6)$$

The quantum determinant N is the positive self-adjoint operator defined by

$$N = z_{11}z_{22} - z_{12}z_{21} = z_{22}z_{11} - z_{21}z_{12}, \quad (7.7)$$

and we have

$$Nz_{11} = z_{11}N, \quad Nz_{12} = q^{-2}z_{12}N, \quad Nz_{21} = q^2z_{21}N, \quad Nz_{22} = z_{22}N. \quad (7.8)$$

Proposition 7.2 There is a projection map $\mathcal{P} : GL_q^+(2, \mathbb{R}) \longrightarrow SL_q^+(2, \mathbb{R})$ given by

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} N^{-1/2}z_{11} & q^{-1/2}N^{-1/2}z_{12} \\ q^{1/2}N^{-1/2}z_{21} & N^{-1/2}z_{22} \end{pmatrix}, \quad (7.9)$$

where a, b, c, d satisfies the usual relations for $SL_q(2, \mathbb{R})$:

$$\begin{aligned} ab = qba, \quad ac = qca, \quad ad = qda, \quad bd = qdb, \quad cd = qdc, \\ bc = cb, \quad ad - qbc = da - q^{-1}cb = 1. \end{aligned} \tag{7.10}$$

Proposition 7.3 *There is a Gauss decomposition for $GL_q^+(2, \mathbb{R})$ given by*

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 1 & \widehat{B} \\ 0 & \widehat{A} \end{pmatrix} = \begin{pmatrix} A & A\widehat{B} \\ B & B\widehat{B} + \widehat{A} \end{pmatrix}, \tag{7.11}$$

where $A, B, \widehat{A}, \widehat{B}$ are positive operators so that $\{A, B\}$ commutes with $\{\widehat{A}, \widehat{B}\}$, with $AB = q^2BA$ and $\widehat{A}\widehat{B} = q^{-2}\widehat{B}\widehat{A}$. Furthermore, we have $N = A\widehat{A}$.

Now, we will describe the quantum double group construction and show that the result is precisely $GL_q^+(2, \mathbb{R})$ together with the above Gauss decomposition.

Definition 7.4 The quantum double group $\mathcal{D}(\mathcal{A})$ is the Hopf algebra where as an algebra $\mathcal{D}(\mathcal{A}) \simeq \mathcal{A} \otimes \mathcal{A}^{*op} = \mathcal{A} \otimes \widehat{\mathcal{A}}$ with the usual tensor product algebra structure and with coproduct given by

$$\Delta_m(x \otimes \widehat{x}) = (1 \otimes \sigma m \otimes 1)(\Delta(x) \otimes \widehat{\Delta}(\widehat{x})), \tag{7.12}$$

where σ is the permutation of the tensor product, and $m : M(\mathcal{A} \otimes \widehat{\mathcal{A}}) \rightarrow M(\mathcal{A} \otimes \widehat{\mathcal{A}})$ is called the matching, defined by

$$m(x \otimes \widehat{x}) = W(x \otimes \widehat{x})W^*, \tag{7.13}$$

with $W \in M(\mathcal{A} \otimes \widehat{\mathcal{A}})$ the multiplicative unitary defined in (4.61).

Hence, a general element in $\mathcal{D}(\mathcal{A})$ can be written as

$$\iiint f(s_1, t_1)g(s_2, t_2)A^{ib^{-1}s_1}B^{ib^{-1}t_1}\widehat{B}^{ib^{-1}t_2}\widehat{A}^{ib^{-1}s_2}ds_1dt_1ds_2dt_2 \tag{7.14}$$

or simply $f(s_1, t_1)g(s_2, t_2)$. For simplicity, we write $A := A \otimes 1, \widehat{A} := 1 \otimes \widehat{A}$ and so on, and we will use this notation in the remaining of the paper.

Proposition 7.5 [30, Thm 4.1] Δ_m is coassociative, so it indeed defines a coproduct on $\mathcal{D}(\mathcal{A})$.

Proposition 7.6 [30, Thm 4.2] The Haar functional h on $\mathcal{D}(\mathcal{A})$ defined by

$$h = h_L \otimes \widehat{h}_R \tag{7.15}$$

is both left and right invariant:

$$(h \otimes 1 \otimes 1)\Delta_m(x \otimes \widehat{x}) = h(x \otimes \widehat{x})(1 \otimes 1), \tag{7.16}$$

$$(1 \otimes 1 \otimes h)\Delta_m(x \otimes \widehat{x}) = h(x \otimes \widehat{x})(1 \otimes 1). \tag{7.17}$$

In particular, the GNS representation of $\mathcal{D}(\mathcal{A})$ is given by $\Lambda_m := \Lambda \otimes \widehat{\Lambda}_R$ on $\mathcal{H} \otimes \widehat{\mathcal{H}}$.

Proof Although the theorem in [30] applies only to compact quantum groups, the calculations using the graphical method there can be adapted in this setting without any changes, since it only depends on the invariances for h_L, \widehat{h}_R and the relations between the matching m and the coproducts of \mathcal{A} and $\widehat{\mathcal{A}}$. \square

Remark 7.7 The Haar functional on a general element $f(s_1, t_1)g(s_2, t_2) \in \mathcal{H} \otimes \widehat{\mathcal{H}}$ is thus given by

$$h(f \otimes g) = f(0, iQ)g(-iQ, iQ)e^{-\pi i Q^2}. \tag{7.18}$$

If we parametrize the element instead as

$$f \otimes g := \iiint f(s_1, t_1)g(s_2, t_2)A^{ib^{-1}s_1}B^{ib^{-1}t_1}X^{ib^{-1}s_2}Y^{ib^{-1}t_2}ds_1dt_1ds_2dt_2, \tag{7.19}$$

where $X = \widehat{A}^{-1}, Y = q\widehat{B}\widehat{A}^{-1}$, then it takes a more symmetric form

$$h(f \otimes g) = f(0, iQ)g(0, iQ), \tag{7.20}$$

which means it only depends on the element BY . According to the Gauss decomposition, this is precisely

$$B(q\widehat{B}\widehat{A}^{-1}) = qB\widehat{B}AA^{-1}\widehat{A}^{-1} = qz_{21}z_{12}N^{-1}, \tag{7.21}$$

which corresponds under the projection to $SL_q^+(2, \mathbb{R})$ the hyperbolic element $\zeta := bc$ that is crucial in the study of the $SU_q(2)$ and $SU_q(1, 1)$ case in [27, 28].

Theorem 7.8 *By the Gauss decomposition, the Hopf algebra $GL_q^+(2, \mathbb{R})$ can be naturally put into the C^* -algebraic setting, so that it is identified with the quantum double $\mathcal{D}(\mathcal{A})$. Furthermore, the coproduct on $\mathcal{D}(\mathcal{A})$ induces the same coproduct on the generators z_{ij} .*

Proof By the Gauss decomposition, there is a one-to-one correspondence between the generators. Explicitly, the inverse is given by

$$\begin{aligned} A &= z_{11}, \\ B &= z_{21}, \\ \widehat{B} &= z_{12}z_{11}^{-1}, \\ \widehat{A} &= Nz_{11}^{-1}. \end{aligned}$$

We have to show that the coproduct is the same. The following calculations are very similar to the one given in [34]. \square

Lemma 7.9 *We have the following commutation relationships between W and $\mathcal{D}(A)$:*

$$\begin{aligned} W(A \otimes 1)W^* &= A \otimes 1 + B \otimes \widehat{B}, \\ W(A \otimes \widehat{A})W^* &= A \otimes \widehat{A}, \\ W(A \otimes \widehat{B})W^* &= 1 \otimes \widehat{B}, \\ W(B \otimes 1)W^* &= B \otimes \widehat{A}, \\ W^*(1 \otimes \widehat{A})W &= 1 \otimes \widehat{A} + B \otimes \widehat{B}. \end{aligned}$$

Proof These follow from the summation properties (3.22),(3.23) for g_b , as well as the commutation relations (2.46),(2.47) for the exponentials. \square

Now, we proceed to the calculation of the coproduct on the generators:

$$\begin{aligned} \Delta(z_{11}) &= \Delta(A \otimes 1) \\ &= A \otimes \sigma m(A \otimes 1) \otimes 1 \\ &= (A \otimes 1) \otimes (A \otimes 1) + (A \otimes \widehat{B}) \otimes (B \otimes 1) \\ &= z_{11} \otimes z_{11} + z_{12} \otimes z_{21}, \\ \Delta(z_{12}) &= \Delta(A \otimes \widehat{B}) \\ &= A \otimes \sigma m(A \otimes 1) \otimes \widehat{B} + A \otimes \sigma m(A \otimes \widehat{B}) \otimes \widehat{A} \\ &= A \otimes 1 \otimes A \otimes \widehat{B} + A \otimes \widehat{B} \otimes B \otimes \widehat{B} + A \otimes \widehat{B} \otimes 1 \otimes \widehat{A} \\ &= (A \otimes 1) \otimes (A \otimes \widehat{B}) + (A \otimes \widehat{B}) \otimes (B \otimes \widehat{B} + 1 \otimes \widehat{A}) \\ &= z_{11} \otimes z_{12} + z_{12} \otimes z_{22}, \\ \Delta(z_{21}) &= \Delta(B \otimes 1) \\ &= B \otimes \sigma m(A \otimes 1) \otimes 1 + 1 \otimes \sigma m(B \otimes 1) \otimes 1 \\ &= B \otimes 1 \otimes A \otimes 1 + B \otimes \widehat{B} \otimes B \otimes 1 + 1 \otimes \widehat{A} \otimes B \otimes 1 \\ &= (B \otimes 1) \otimes (A \otimes 1) + (B \otimes \widehat{B} + 1 \otimes \widehat{A}) \otimes (B \otimes 1) \\ &= z_{21} \otimes z_{11} + z_{22} \otimes z_{21}, \\ \Delta(z_{22}) &= \Delta(B \otimes \widehat{B} + 1 \otimes \widehat{A}) \\ &= B \otimes \sigma m(A \otimes 1) \otimes \widehat{B} + 1 \otimes \sigma m(B \otimes 1) \otimes \widehat{B} + B \otimes \sigma m(A \otimes \widehat{B}) \otimes \widehat{A} \\ &\quad + 1 \otimes \sigma m(B \otimes \widehat{B} + 1 \otimes \widehat{A}) \otimes \widehat{A}. \end{aligned}$$

Since $W^*(1 \otimes \widehat{A})W = 1 \otimes \widehat{A} + B \otimes \widehat{B}$, we have

$$\sigma m(B \otimes \widehat{B} + 1 \otimes \widehat{A}) = \widehat{A} \otimes 1.$$

Hence,

$$\begin{aligned} \Delta(z_{22}) &= B \otimes 1 \otimes A \otimes \widehat{B} + B \otimes \widehat{B} \otimes B \otimes \widehat{B} + 1 \otimes \widehat{A} \otimes B \otimes \widehat{B} \\ &\quad + B \otimes \widehat{B} \otimes 1 \otimes \widehat{A} + 1 \otimes \widehat{A} \otimes 1 \otimes \widehat{A} \\ &= (B \otimes 1) \otimes (A \otimes \widehat{B}) + (B \otimes \widehat{B} + 1 \otimes \widehat{A}) \otimes (B \otimes \widehat{B} + 1 \otimes \widehat{A}) \\ &= z_{21} \otimes z_{12} + z_{22} \otimes z_{22}. \end{aligned}$$

Finally, let us also derive the coproduct for the determinant N :

$$\begin{aligned} \Delta(N) &= \Delta(A \otimes \widehat{A}) \\ &= A \otimes \sigma m(A \otimes \widehat{A}) \otimes \widehat{A} \\ &= A \otimes \widehat{A} \otimes A \otimes \widehat{A} \\ &= N \otimes N. \end{aligned}$$

□

7.2 The matrix coefficients and the fundamental corepresentation

Recall that for $SL^+(2, \mathbb{R}) \subset SL(2, \mathbb{R})$ the positive semigroup, the class of principal series representation can be expressed by considering the actions on homogeneous monomials (see, e.g., [43]):

$$x^{l-i\mu}y^{l+i\mu} \mapsto (ax + cy)^{l-i\mu}(bx + dy)^{l+i\mu} := \int_{\mathbb{R}} t_{\mu\nu}^l(g)x^{l-i\nu}y^{l+i\nu}d\nu, \tag{7.22}$$

where $\mu \in \mathbb{R}, l \in -\frac{1}{2} + i\mathbb{R}$, so that the representation is unitary. In terms of coordinates, the representation acts on $L^2(\mathbb{R})$ by

$$g \cdot f(\mu) \mapsto \int_{\mathbb{R}} t_{\mu\nu}^l(g)f(\nu)d\nu. \tag{7.23}$$

In [8], it is noted that for commuting variables x, y ,

$$[xz_{11} + yz_{21}, xz_{12} + yz_{22}] = 0 \tag{7.24}$$

by the commutation relations. Hence, the following fundamental corepresentation for $\mathcal{D}(\mathcal{A}) = GL_q^+(2, \mathbb{R})$ is well defined:

Definition 7.10 The fundamental corepresentation of $GL_q^+(2, \mathbb{R})$ on $L^2(\mathbb{R})$ is defined by

$$T^\lambda : f(s) \mapsto \int_{\mathbb{R}} T_{s,\alpha}^\lambda(z)f(\alpha)d\alpha, \tag{7.25}$$

where $T_{s,\alpha}^\lambda(z)$ is formally defined by

$$(xz_{11} + yz_{21})^{b^{-1}(l-is)}(xz_{12} + yz_{22})^{b^{-1}(l+is)}N^{\frac{Q}{b}} = \left(\int_{\mathbb{R}} T_{s,\alpha}^\lambda(z)x^{b^{-1}(l-i\alpha)}y^{b^{-1}(l+i\alpha)}d\alpha \right) \tag{7.26}$$

with $l := -\frac{Q}{2} + i\lambda, \alpha, \lambda \in \mathbb{R}$ and $N = z_{11}z_{22} - z_{12}z_{21} = A\widehat{A}$.

More generally, we can introduce arbitrary character of the determinant and will consider the corepresentation $T^{\lambda,t}$ defined by

$$T_{s,\alpha}^{\lambda,t}(z) := N^{ib^{-1}(t-\lambda)/2} T_{s,\alpha}^\lambda(z) N^{ib^{-1}(t-\lambda)/2}. \tag{7.27}$$

Remark 7.11 The term $N^{\frac{Q}{2b}}$ is to make the corepresentation unitary. As seen from below, it is coming from the modular element $\delta^{-\frac{1}{2}}$ in the fundamental corepresentation (6.45) of the quantum plane. The factor chosen for N in $T^{\lambda,t}$ is for later convenience when we obtain the fundamental representation.

Remark 7.12 Using the Mellin transform, we can also write $T_{s,\alpha}^\lambda(z)$ as

$$T_{s,\alpha}^\lambda(z) = \frac{1}{2\pi b} \left(\int_0^\infty (xz_{11} + z_{21})^{b^{-1}(l-i\alpha)} (xz_{12} + z_{22})^{b^{-1}(l+i\alpha)} x^{b^{-1}(-l+i s)} \frac{dx}{x} \right) N^{\frac{Q}{2b}}. \tag{7.28}$$

This can be seen as the generalization of the matrix coefficients in the compact case, see, e.g., [11].

Proposition 7.13 *The matrix coefficient is given explicitly by*

$$\begin{aligned} T_{s,\alpha}^{\lambda,t}(z) &= \int_{\mathbb{R}+i0} \begin{pmatrix} l+i\alpha \\ i\tau+i\alpha \end{pmatrix}_b \begin{pmatrix} l+i\tau \\ is+i\tau \end{pmatrix}_b e^{\pi i(t-\lambda)(s+\alpha+2\tau)} e^{\pi Q(s+\tau)} \\ &\quad \times A^{ib^{-1}(t-s)} B^{ib^{-1}(s+\tau)} \widehat{B}^{ib^{-1}(\alpha+\tau)} \widehat{A}^{ib^{-1}(t-\tau)} d\tau \\ &= \int_{\mathbb{R}+i0} \frac{G_b(-i\tau-i\alpha)G_b\left(\frac{Q}{2}+i\tau-i\lambda\right)}{G_b\left(\frac{Q}{2}-i\alpha-i\lambda\right)} \frac{G_b(-i\tau-is)G_b\left(\frac{Q}{2}+is-i\lambda\right)}{G_b\left(\frac{Q}{2}-i\tau-i\lambda\right)} \\ &\quad \times e^{\pi i(t-\lambda)(s+\alpha+2\tau)} e^{\pi Q(s+\tau)} A^{ib^{-1}(t-s)} B^{ib^{-1}(s+\tau)} \widehat{B}^{ib^{-1}(\alpha+\tau)} \widehat{A}^{ib^{-1}(t-\tau)} d\tau, \end{aligned} \tag{7.29}$$

$$\tag{7.30}$$

where $l = -\frac{Q}{2} + i\lambda$ and $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_b = \frac{G_b(-\beta)G_b(\beta-\alpha)}{G_b(-\alpha)}$ is the q -binomial coefficient (cf. Lemma 3.8).

Hence, under some changes of variables, the corepresentation

$$f(s) \mapsto \int_{\mathbb{R}} T_{s,\alpha}^{\lambda,t}(z) f(\alpha) d\alpha$$

is given by

$$f(s) \mapsto \int_{\mathbb{R}+i0} \int_{\mathbb{R}+i0} f(s+\alpha-\tau) \frac{G_b(-i\alpha)}{G_b(Q+i\tau)} \frac{G_b(\frac{Q}{2}-is+i\lambda+i\tau)G_b(\frac{Q}{2}-is-i\lambda+i\tau)}{G_b(\frac{Q}{2}-is+i\lambda)G_b(\frac{Q}{2}-is-i\lambda-i\alpha+i\tau)} e^{\pi i\lambda(\tau-\alpha)} e^{-2\pi i\tau s} e^{\pi i\tau(\tau+\alpha)} A^{ib^{-1}(t-s)} B^{ib^{-1}\tau} \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{ib^{-1}(t+s-\tau)} d\alpha d\tau. \quad (7.31)$$

Proof We can make use of the Gauss decomposition (Proposition 7.3) and obtain

$$T_{s,\alpha}^\lambda(z) = \int t_{s,\tau}^\lambda(\mathcal{A}) \widehat{t}_{\tau,\alpha}^\lambda(\widehat{\mathcal{A}}) d\tau, \quad (7.32)$$

where the matrix coefficients correspond to the fundamental representation of the quantum plane (6.45):

$$\begin{aligned} t_{s,\alpha}^\lambda &: \int f(s) x^{b^{-1}(l-is)} y^{b^{-1}(l+is)} ds \\ &\mapsto \left(\int f(s) (xA + yB)^{b^{-1}(l-is)} y^{b^{-1}(l+is)} ds \right) A^{\frac{Q}{2b}} \\ &= \int f(s) \binom{l-is}{i\alpha}_b A^{-\frac{Q}{2b}+ib^{-1}(\lambda-s-\alpha)} B^{ib^{-1}\alpha} A^{\frac{Q}{2b}} x^{b^{-1}(l-is-i\alpha)} \\ &\quad \times y^{b^{-1}(l+is+i\alpha)} d\alpha ds \\ &= \int f(s) \binom{l-is}{i\alpha-is}_b e^{\pi Q(\alpha-s)} A^{ib^{-1}(\lambda-\alpha)} B^{ib^{-1}(\alpha-s)} x^{b^{-1}(l-i\alpha)} \\ &\quad \times y^{b^{-1}(l+i\alpha)} d\alpha ds. \end{aligned}$$

Hence, renaming $s \longleftrightarrow \alpha$, we see that the matrix coefficient is given by

$$t_{s,\alpha}^\lambda(\mathcal{A}) = \binom{l-i\alpha}{is-i\alpha}_b e^{\pi Q(s-\alpha)} A^{ib^{-1}(\lambda-s)} B^{ib^{-1}(s-\alpha)}. \quad (7.33)$$

A similar analysis using

$$\widehat{t}_{s,\alpha}^\lambda : \int f(s) x^{b^{-1}(l-is)} y^{b^{-1}(l+is)} ds \mapsto \left(\int f(s) x^{b^{-1}(l-is)} (x\widehat{B} + y\widehat{A})^{b^{-1}(l+is)} ds \right) \widehat{A}^{\frac{Q}{2b}}$$

shows that $\widehat{t}_{s,\alpha}^\lambda(\widehat{\mathcal{A}})$ is given by

$$\widehat{t}_{s,\alpha}^\lambda(\widehat{\mathcal{A}}) = \binom{-l+i\alpha}{-is+i\alpha}_b \widehat{B}^{ib^{-1}(\alpha-s)} \widehat{A}^{ib^{-1}(s+\lambda)}. \quad (7.34)$$

Hence, using (7.32) with the contour of τ separating the poles of $G_b(\cdot \pm i\tau)$, we obtain

$$\begin{aligned} T_{s,\alpha}^\lambda(z) &= \int_{\mathbb{R}-i0} \binom{l+i\alpha}{-i\tau+i\alpha}_b \binom{l-i\tau}{is-i\tau}_b e^{\pi Q(s-\tau)} A^{ib^{-1}(\lambda-s)} B^{ib^{-1}(s-\tau)} \widehat{B}^{ib^{-1}(\alpha-\tau)} \\ &\quad \times \widehat{A}^{ib^{-1}(\lambda+\tau)} d\tau \\ &= \int_{\mathbb{R}+i0} \binom{l+i\alpha}{i\tau+i\alpha}_b \binom{l+i\tau}{is+i\tau}_b e^{\pi Q(s+\tau)} A^{ib^{-1}(\lambda-s)} B^{ib^{-1}(s+\tau)} \widehat{B}^{ib^{-1}(\alpha+\tau)} \\ &\quad \times \widehat{A}^{ib^{-1}(\lambda-\tau)} d\tau. \end{aligned}$$

Finally, for $T_{s,\alpha}^{\lambda,t}(z)$, by commuting A and \widehat{A} of $N^{ib^{-1}(t-\lambda)/2} = (A\widehat{A})^{ib^{-1}(t-\lambda)/2}$ to the corresponding sides, we pick up the factor $e^{\pi i(t-\lambda)(s+\alpha+2\tau)}$ and obtain the desired formula. \square

Corollary 7.14 *The matrix coefficient $T_{s,\alpha}^\lambda(z)$ can also be expressed in closed form as*

$$\begin{aligned} T_{s,\alpha}^\lambda(z) &= A^{b^{-1}(l-is)} B^{b^{-1}(l+is)} \widehat{B}^{b^{-1}(l+i\alpha)} \binom{2l}{l+is}_b \\ &\quad \times F_b(-l-is, -l-i\alpha, -2l; -\zeta^{-1}) N^{\frac{Q}{2b}}, \end{aligned} \tag{7.35}$$

where $l = -\frac{Q}{2} + i\lambda$, $\zeta = qB\widehat{B}\widehat{A}^{-1}$ is the hyperbolic element defined in (7.21),

$$F_b(\alpha, \beta, \gamma; z) \doteq \frac{G_b(\gamma)}{G_b(\alpha)G_b(\beta)} \int_C (-z)^{ib^{-1}\tau} e^{\pi i\tau^2} \frac{G_b(\alpha+i\tau)G_b(\beta+i\tau)G_b(-i\tau)}{G_b(\gamma+i\tau)} d\tau \tag{7.36}$$

is the b -hypergeometric function (slightly modified from [32]), which is the quantum analogue of the classical ${}_2F_1(a, b, c; z)$. Hence, (7.35) is exactly the quantum analogue of the classical formula for $SL^+(2, \mathbb{R})$ given in [43, VII. 4.1(4)], where the hyperbolic element is $\zeta = \sinh^2 \theta$:

$$T_{mn}^l = \frac{1}{2\pi i} A^{l-m} B^{l+m} \widehat{B}^{l+n} \frac{\Gamma(-l-m)\Gamma(-l+m)}{\Gamma(-2l)} {}_2F_1(-l-m, -l-n, -2l; -\frac{1}{\sinh^2 \theta}), \tag{7.37}$$

where A, B, \widehat{B} are the corresponding variables in the Gauss decomposition for $SL_2^+(\mathbb{R})$.

7.3 The multiplicative unitary

Given two locally compact quantum groups (M_1, Δ_1) and (M_2, Δ_2) , with a matching $m : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2$, the multiplicative unitary for the double crossed product

is first constructed in [1]. The quantum double group construction is a special case given by $(M_1, \Delta_1) = (\mathcal{A}, \Delta^{op})$ and $(M_2, \Delta_2) = (\widehat{\mathcal{A}}, \widehat{\Delta})$, while the matching m is given by $m(x) = W(x)W^*$ as before [16]. Let us restrict to the quantum plane case and describe the main ingredients needed.

Proposition 7.15 *The multiplicative unitary operator is defined by*

$$W_m := W = W_{13}Z_{34}^*\widehat{W}_{24}Z_{34} \in \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}} \otimes \mathcal{H} \otimes \widehat{\mathcal{H}}), \tag{7.38}$$

where $Z \in \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}})$ is given by

$$Z = W(\widehat{J}_1 J_1 \otimes J_2 \widehat{J}_2)W^*(\widehat{J}_1 J_1 \otimes \widehat{J}_2 J_2) = W(\widehat{J} J \otimes \widehat{J} J)W^*(\widehat{J} J \otimes J \widehat{J}), \tag{7.39}$$

and $\widehat{W} = \Sigma W^* \Sigma = W_{21}^*$.

The coproduct is given by

$$\Delta_m = (\iota \otimes \sigma m \otimes \iota)(\Delta_1^{op} \otimes \Delta_2) = (\iota \otimes \sigma m \otimes \iota)(\Delta \otimes \widehat{\Delta}), \tag{7.40}$$

where $m(z) = ZzZ^*$, and

$$\Delta_m(z) = W^*(1 \otimes z)W \tag{7.41}$$

for $z \in \mathcal{D}(\mathcal{A})$.

The matching m satisfies

$$(\Delta^{op} \otimes 1)m = m_{23}m_{13}(\Delta^{op} \otimes 1), \tag{7.42}$$

$$(1 \otimes \widehat{\Delta})m = m_{13}m_{12}(1 \otimes \widehat{\Delta}). \tag{7.43}$$

First of all, we note the difference in the definition of the matching m . In fact, they are the same.

Proposition 7.16 *We actually have*

$$m(x) = ZxZ^* = WxW^* \tag{7.44}$$

for $x \in \mathcal{A} \otimes \widehat{\mathcal{A}}$. In particular, from the pentagon equation of W and $\widehat{W} = W_{21}^*$, the matching satisfies the property (7.42) and (7.43).

Proof Recall the conjugation properties from (5.46), (5.47):

$$(\widehat{J} \otimes J)W(\widehat{J} \otimes J) = W^*, \tag{7.45}$$

$$(J \otimes \widehat{J})\widehat{W}(J \otimes \widehat{J}) = \widehat{W}^*. \tag{7.46}$$

Using also (5.45), we have

$$\begin{aligned} W(\widehat{J}J \otimes \widehat{J}J)W^*(\widehat{J}J \otimes \widehat{J}J) &= v^{i/4}W(J \otimes \widehat{J})(\widehat{J} \otimes J)W^*(\widehat{J} \otimes J)(J \otimes \widehat{J}) \\ &= v^{i/4}W(J \otimes \widehat{J})W(J \otimes \widehat{J}). \end{aligned}$$

Hence, using the definition (7.39) of Z , expanding $ZxZ^* = WxW^*$, we have

$$W(J \otimes \widehat{J})W(J \otimes \widehat{J})x(J \otimes \widehat{J})W^*(J \otimes \widehat{J})W^* = WxW^*,$$

or

$$(J \otimes \widehat{J})W(J \otimes \widehat{J})x = x(J \otimes \widehat{J})W(J \otimes \widehat{J}),$$

that is, $(J \otimes \widehat{J})W(J \otimes \widehat{J})$ commutes with every $x \in \mathcal{A} \otimes \widehat{\mathcal{A}} \subset \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}})$ as operators. However, since $W \in \mathcal{A} \otimes \widehat{\mathcal{A}}$, from Corollary 6.11, it is clear that

$$(J \otimes \widehat{J})W(J \otimes \widehat{J}) = g_b^*(q\widehat{B}_*\widehat{A}_*^{-1} \otimes B_*)e^{\frac{i}{2\pi b^2} \log \widehat{A}_* \otimes \log A_*}$$

commutes with every $x \in \mathcal{A} \otimes \widehat{\mathcal{A}}$. □

For completeness, let us also reprove the coproduct formula.

Proof Recall

$$\begin{aligned} \mathbf{W} &= W_{13}Z_{34}^*W_{42}^*Z_{34}, \\ \Delta(x) &= W^*(1 \otimes x)W, \\ \widehat{\Delta}(y) &= \widehat{W}^*(1 \otimes y)\widehat{W} = W_{21}(1 \otimes y)W_{21}^*. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{W}^*(1 \otimes z)\mathbf{W} &= Z_{34}^*W_{42}Z_{34}W_{13}^*(1 \otimes z)W_{13}Z_{34}^*W_{42}^*Z_{34} \\ &= Z_{34}^*W_{42}Z_{34}(\Delta \otimes \iota)(z)_{134}Z_{34}^*W_{42}^*Z_{34} \\ &= Z_{34}^*W_{42}(\iota \otimes m)((\Delta \otimes \iota)(z))_{134}W_{42}^*Z_{34} \\ &= Z_{34}^*((\iota \otimes \iota \otimes \widehat{\Delta})(\iota \otimes m)(\Delta \otimes \iota)(z))_{1324}Z_{34} \\ &= Z_{34}^*(m_{24}m_{23}(\iota \otimes \iota \otimes \widehat{\Delta})(\Delta \otimes \iota)(z))_{1324}Z_{34} \\ &= Z_{34}^*(Z_{24}m_{23}(\iota \otimes \iota \otimes \widehat{\Delta})(\Delta \otimes \iota)(z)Z_{24}^*)_{1324}Z_{34} \\ &= Z_{34}^*Z_{34}m_{32}(\iota \otimes \iota \otimes \widehat{\Delta})(\Delta \otimes \iota)(z)Z_{34}^*Z_{34} \\ &= (\sigma m)_{23}(\iota \otimes \iota \otimes \widehat{\Delta})(\Delta \otimes \iota)(z) \\ &= \Delta_m(z). \end{aligned}$$

□

It is proved in [1] for general matching m that $\mathbf{W}_m := \mathbf{W}$ is a multiplicative unitary. Here, we present a direct proof using the fact that m is given by the multiplicative unitary W .

Theorem 7.17 *W is a multiplicative unitary, that is, it satisfies*

$$W_{3456}W_{1234} = W_{1234}W_{1256}W_{3456}.$$

Proof It suffices to check that

$$W_{3456}Z_{34}^*\widehat{W}_{24}Z_{34}W_{3456}^* = Z_{34}^*Z_{56}^*\widehat{W}_{24}\widehat{W}_{26}Z_{34}Z_{56}$$

and

$$W_{3456}W_{13}W_{3456}^* = W_{35}W_{13}W_{35}^* = W_{13}W_{15}.$$

The second equation follows directly from the definition and the pentagon equation for W .

Let us write $Z = VV'$ where $V \in \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}})$ and $V' = (J \otimes \widehat{J})V(J \otimes \widehat{J})$ are the copy of W based on different spaces. Note that V, V' commute. Also, V' commutes with entries in $\mathcal{A} \otimes \widehat{\mathcal{A}} \subset \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}})$.

From the pentagon equation for W , but with its legs sitting on different spaces, we have the relations (cf. Corollary 2.20):

$$V_{23}W_{12} = W_{12}V_{13}V_{23} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \widehat{\mathcal{H}}), \quad (7.47)$$

$$V_{13}\widehat{W}_{23} = \widehat{W}_{23}V_{13}V_{12} \in \mathcal{B}(\mathcal{H} \otimes \widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}}). \quad (7.48)$$

Using also (7.45) and (7.46), we derive the relations

$$\begin{aligned} V'_{56}W_{35}^* &= (J_5 \otimes \widehat{J}_6)V_{56}(J_5 \otimes \widehat{J}_6)(\widehat{J}_3 \otimes J_5)W_{35}(\widehat{J}_3 \otimes J_5) \\ &= (\widehat{J}_3 \otimes J_5 \otimes \widehat{J}_6)V_{56}W_{35}(\widehat{J}_3 \otimes J_5 \otimes \widehat{J}_6) \\ &= (\widehat{J}_3 \otimes J_5 \otimes \widehat{J}_6)W_{35}V_{36}V_{56}(\widehat{J}_3 \otimes J_5 \otimes \widehat{J}_6) \\ &= W_{35}^*(\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6)(J_5 \otimes \widehat{J}_6)V_{56}(J_5 \otimes \widehat{J}_6) \\ &= W_{35}^*(\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6)V'_{56}. \end{aligned}$$

$$\begin{aligned} \widehat{W}_{46}^*(\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6) &= (J_4 \otimes \widehat{J}_6)\widehat{W}_{46}(J_4 \otimes \widehat{J}_6)(\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6) \\ &= (\widehat{J}_3 \otimes J_4 \otimes \widehat{J}_6)\widehat{W}_{46}V_{36}(\widehat{J}_3 \otimes J_4 \otimes \widehat{J}_6) \\ &= (\widehat{J}_3 \otimes J_4 \otimes \widehat{J}_6)V_{36}\widehat{W}_{46}V_{34}^*(\widehat{J}_3 \otimes J_4 \otimes \widehat{J}_6) \\ &= (\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6)(J_4 \otimes \widehat{J}_6)\widehat{W}_{46}(J_4 \otimes \widehat{J}_6)(\widehat{J}_3 \otimes J_4)V_{34}^*(\widehat{J}_3 \otimes J_4) \\ &= (\widehat{J}_3 \otimes \widehat{J}_6)V_{36}(\widehat{J}_3 \otimes \widehat{J}_6)\widehat{W}_{46}^*V_{34}. \end{aligned}$$

Hence,

$$\begin{aligned} W_{3456}Z_{34}^*\widehat{W}_{24}Z_{34}W_{3456}^* &= W_{35}Z_{56}^*\widehat{W}_{46}Z_{56}Z_{34}^*\widehat{W}_{24}Z_{34}Z_{56}^*\widehat{W}_{46}^*Z_{56}W_{35}^* \\ &= (W_{35}V_{56}^*V'_{56}\widehat{W}_{46}V_{34}^*V'_{34})\widehat{W}_{24}(V_{34}V'_{34}\widehat{W}_{46}^*V_{56}V'_{56}W_{35}^*). \end{aligned}$$

Now, the right hand bracket is

$$\begin{aligned} V_{34}V'_{34}\widehat{W}_{46}^*V_{56}V'_{56}W_{35}^* &= V_{34}\widehat{W}_{46}^*V_{56}V'_{56}W_{35}^*V'_{34} \\ &= V_{36}^*\widehat{W}_{46}^*V_{36}V_{56}V'_{56}W_{35}^*V'_{34} \end{aligned}$$

$$\begin{aligned}
 &= V_{36}^* \widehat{W}_{46}^* V_{36} V_{56} W_{35}^* (\widehat{J}_3 \otimes \widehat{J}_6) V_{36} (\widehat{J}_3 \otimes \widehat{J}_6) V_{56}' V_{34}' \\
 &= V_{36}^* \widehat{W}_{46}^* W_{35}^* V_{56} (\widehat{J}_3 \otimes \widehat{J}_6) V_{36} (\widehat{J}_3 \otimes \widehat{J}_6) V_{56}' V_{34}' \\
 &= V_{36}^* W_{35}^* \widehat{W}_{46}^* (\widehat{J}_3 \otimes \widehat{J}_6) V_{36} (\widehat{J}_3 \otimes \widehat{J}_6) Z_{56} V_{34}' \\
 &= V_{36}^* W_{35}^* (\widehat{J}_3 \otimes \widehat{J}_6) V_{36} (\widehat{J}_3 \otimes \widehat{J}_6) \widehat{W}_{46}^* V_{34} Z_{56} V_{34}' \\
 &= V_{36}^* W_{35}^* (\widehat{J}_3 \otimes \widehat{J}_6) V_{36} (\widehat{J}_3 \otimes \widehat{J}_6) \widehat{W}_{46}^* Z_{56} Z_{34}.
 \end{aligned}$$

Combining, we see that the left most 3 terms will pass through \widehat{W}_{24} and cancel with the left hand inverses. Hence, we obtain

$$\begin{aligned}
 W_{3456} Z_{34} \widehat{W}_{24} Z_{34} W_{3456}^* &= Z_{34}^* Z_{56}^* \widehat{W}_{46} \widehat{W}_{24} \widehat{W}_{46}^* Z_{56} Z_{34} \\
 &= Z_{34}^* Z_{56}^* \widehat{W}_{24} \widehat{W}_{26} Z_{56} Z_{34}
 \end{aligned}$$

as desired. □

Note that \mathbf{W} is actually a composition of 6 dilogarithm functions. We can simplify the expression to just 4 g_b 's by applying the pentagon equation.

Proposition 7.18 *We can rewrite \mathbf{W} as*

$$\mathbf{W} = W_{13} V_{32}'' \widehat{W}_{24} V_{32}^*, \tag{7.49}$$

where

$$\begin{aligned}
 V'' &:= (J \otimes J) V (J \otimes J) \\
 &= e^{\frac{i}{2\pi b^2} \log \widehat{A}_* \otimes \log \widehat{A}^{-1}} g_b^*(\widehat{B}_* \otimes \widehat{B}) \\
 &= e^{\frac{i}{2\pi b^2} \log \widehat{A}_* \otimes \log \widehat{A}^{-1}} \int_{\mathbb{R}+i0} \widehat{B}_*^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau} e^{\pi i \tau^2} G_b(-i\tau) d\tau.
 \end{aligned}$$

Proof

$$\begin{aligned}
 \mathbf{W} &= W_{13} Z_{34}^* \widehat{W}_{24} Z_{34} \\
 &= W_{13} V_{34}' V_{34}^* \widehat{W}_{24} V_{34} V_{34}' \\
 &= W_{13} V_{34}' \widehat{W}_{24} V_{32}^* V_{34}' \\
 &= W_{13} V_{34}' \widehat{W}_{24} V_{34}' V_{32}^* \\
 &= W_{13} (J_2 \otimes J_3 \otimes \widehat{J}_4) V_{34}^* \widehat{W}_{24} V_{34} (J_2 \otimes J_3 \otimes \widehat{J}_4) V_{32}^* \\
 &= W_{13} (J_2 \otimes J_3) V_{32} (J_2 \otimes J_3) \widehat{W}_{24} V_{32}^*.
 \end{aligned}$$

The formula for V'' follows from the action of AdJ given in Proposition 5.17 on $\widehat{\mathcal{H}}$ and Proposition 6.10 on \mathcal{H} . □

Since \mathbf{W} is a unitary operator on $\mathcal{H} \otimes \widehat{\mathcal{H}} \otimes \mathcal{H} \otimes \widehat{\mathcal{H}}$, for completeness, let us describe its action.

Proposition 7.19 *The action of W on $\mathcal{H}_{\text{rep}} \otimes \mathcal{H}_{\text{rep}}$ is given by*

$$W = e^{\frac{i}{2\pi b^2} \log A^{-1} \otimes \log \widehat{A}} g_b(B \otimes q \widehat{B} \widehat{A}^{-1}),$$

$$W : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1 + \tau, t_1) G(s_2 - s_1 - \tau, t_2 - s_1) \frac{e^{2\pi i \tau (s_2 - s_1 - \tau)} G_b\left(\frac{Q}{2} + i s_2 - i t_2\right)}{G_b\left(\frac{Q}{2} + i s_2 - i t_2 - i \tau\right) G_b(Q + i \tau)} d\tau, \quad (7.50)$$

$$W^* : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1 + \tau, t_1) G(s_2 + s_1, t_2 + s_1 + \tau) \frac{e^{2\pi i s_2 \tau} G_b\left(\frac{Q}{2} + i s_2 - i t_2\right) G_b(-i \tau)}{G_b\left(\frac{Q}{2} + i s_2 - i t_2 - i \tau\right)} d\tau. \quad (7.51)$$

The action of \widehat{W} on $\widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$\widehat{W} = e^{\frac{i}{2\pi b^2} \log \widehat{A} \otimes \log A} g_b^*(\widehat{B} \otimes q^{-1} B A^{-1}),$$

$$\widehat{W}_{24} : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1 + \tau, t_1) G(s_2 - s_1 - \tau, t_2 - s_1) e^{2\pi i \tau (\tau + s_1 - s_2)} \frac{G_b\left(\frac{Q}{2} - i s_2 + i t_2 + i \tau\right) G_b(-i \tau)}{G_b\left(\frac{Q}{2} - i s_2 + i t_2\right)} d\tau, \quad (7.52)$$

$$\widehat{W}_{24}^* : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1 + \tau, t_1) G(s_2 + s_1, t_2 + s_1 + \tau) \frac{e^{-2\pi i \tau s_2} G_b\left(\frac{Q}{2} - i s_2 + i t_2 + i \tau\right)}{G_b\left(\frac{Q}{2} - i s_2 + i t_2\right) G_b(Q + i \tau)} d\tau. \quad (7.53)$$

The action of V on $\mathcal{H}_{\text{rep}} \otimes \widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$V : F \otimes G \mapsto \int F(s_1 + \tau, t_1) G(s_2 + \tau, t_2) \frac{e^{2\pi i s_2 (s_1 + \tau)}}{G_b(Q + i \tau)} d\tau, \quad (7.54)$$

$$V^* : F \otimes G \mapsto \int F(s_1 + \tau, t_1) G(s_2 + \tau, t_2) e^{-2\pi i s_1 (s_2 + \tau)} G_b(-i \tau) d\tau, \quad (7.55)$$

and the action of V'' on $\mathcal{H}_{\text{rep}} \otimes \widehat{\mathcal{H}}_{\text{rep}}$ is given by

$$V'' : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1, t_1 - \tau) G(s_2 + \tau, t_2) e^{2\pi i t_1 s_2} e^{\pi i \tau^2} G_b(-i \tau) d\tau, \quad (7.56)$$

$$V''^* : F \otimes G \mapsto \int_{\mathbb{R}+i0} F(s_1, t_1 - \tau) G(s_2 + \tau, t_2) \frac{e^{2\pi i \tau (s_2 - t_1)} e^{-2\pi i t_1 s_2} e^{\pi i \tau^2}}{G_b(Q + i \tau)} d\tau. \quad (7.57)$$

Hence, the action of \mathbf{W} and \mathbf{W}^* is the compositions of the corresponding operators.

Proof The actions follow directly from the closed form expression (Proposition 4.19) for W , using Corollary 4.18 for the integral representations for g_b , as well as the action described in Sect. 6.2. \square

7.4 The left regular corepresentation

With the construction of the multiplicative unitary \mathbf{W} , we can talk about the corepresentation induced by it. By the unitary transformations given in Sect. 6.3, we can choose the Hilbert space to be $\mathcal{H}_{\text{corep}} \otimes \widehat{\mathcal{H}}_{\text{corep}}$ so that the action can be nicely described.

Recall that the left regular corepresentation is given by

$$f \otimes g \mapsto \mathbf{W}'(f \otimes g \otimes 1 \otimes 1), \tag{7.58}$$

where $f \otimes g \in \mathcal{H}_{\text{corep}} \otimes \widehat{\mathcal{H}}_{\text{corep}}$ and $\mathbf{W}' = \mathbf{W}_{3412} \in \mathcal{B}(\mathcal{H}_{\text{corep}} \otimes \widehat{\mathcal{H}}_{\text{corep}}) \otimes \mathcal{A} \otimes \widehat{\mathcal{A}}$. Here, we realize $\mathcal{A} \otimes \widehat{\mathcal{A}}$ as operators in the 3rd and 4th components.

Theorem 7.20 *The left regular corepresentation is given by*

$$\begin{aligned} f(s_1, t_1)g(s_2, t_2) \mapsto & \int_{\mathbb{R}+i0} \int_{\mathbb{R}} \int_C f(s_1 - \alpha + \tau, t_1 - \sigma)g(s_2 + \sigma, t_2) \\ & \frac{G_b(-i\sigma)G_b(-i\alpha + i\sigma)G_b\left(\frac{Q}{2} + is_1 - it_1 + i\tau\right)G_b\left(\frac{Q}{2} + is_1 - is_2 - i\sigma + i\tau\right)}{G_b(Q + i\tau)G_b\left(\frac{Q}{2} + is_1 - it_1 - i\alpha + i\sigma + i\tau\right)G_b\left(\frac{Q}{2} + is_1 - is_2 - i\alpha + i\tau\right)} \\ & e^{2\pi i\sigma(t_1 - s_1 - \tau)}e^{2\pi is_1\tau}A^{ib^{-1}s_1}B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\alpha}\widehat{A}^{ib^{-1}(t_1 + s_2 - s_1 - \tau)}d\sigma d\alpha d\tau, \end{aligned}$$

where the contour C of σ goes above $\sigma = 0$ and below $\sigma = \alpha$.

Proof Since we have used a permutation, now \mathbf{W}' reads

$$\mathbf{W}' = \mathbf{W}_{3412} = W_{31}V''_{14}\widehat{W}_{42}V_{14}^*. \tag{7.59}$$

On the space $\mathcal{H}_{\text{corep}} \otimes \widehat{\mathcal{H}}_{\text{corep}}$, the coaction of the components of \mathbf{W} is given by:

$$\begin{aligned} W_{31} : f(s_1, t_1)g(s_2, t_2) & \mapsto \int_{\mathbb{R}+i0} f(s_1 + \tau, t_1)g(s_2, t_2) \frac{e^{2\pi is_1\tau}}{G_b(Q + i\tau)}A^{ib^{-1}s}B^{ib^{-1}\tau}d\tau, \\ \widehat{W}_{42} : f(s_1, t_1)g(s_2, t_2) & \mapsto \int_{\mathbb{R}+i0} f(s_1, t_1)g(s_2 + \sigma, t_2)G_b(-i\sigma)\widehat{B}^{ib^{-1}\sigma}\widehat{A}^{ib^{-1}s_2}d\sigma. \end{aligned}$$

The actions for V and V'' are harder to describe since they involve formally $A^{ib^{-1}p}$. Therefore, we use \mathcal{S} (cf. Proposition 6.17) to sent the first component back to \mathcal{H}_{rep} and obtain their actions.

$$\begin{aligned}
& SV_{14}^* S^{-1} : f \otimes g \\
& \mapsto \mathcal{S} \cdot \left(\int_{\mathbb{R}+i0} B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau} \widehat{A}^{-ib^{-1}\tau} G_b(-i\tau) d\tau \right) e^{\frac{i}{2\pi b^2} \log \widehat{A} \otimes \log A} . \\
& \int_{\mathbb{R}} f(\alpha - s_1, \alpha - t_1) g(s_2, t_2) G_b \left(\frac{Q}{2} - it_1 + is_1 \right) e^{-2\pi i(\alpha - s_1)s_1} d\alpha \\
& = \mathcal{S} \cdot \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(\alpha - s_1 - \tau, \alpha - t_1) g(s_2, t_2) G_b \left(\frac{Q}{2} - it_1 + is_1 + i\tau \right) G_b(-i\tau) \cdot \\
& \quad e^{-2\pi i(\alpha - s_1 - \tau)(s_1 + \tau)} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}\tau} d\alpha d\tau \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}+i0} f(\alpha - \beta + s_1 - \tau, \alpha - \beta + t_1) g(s_2, t_2) \frac{G_b \left(\frac{Q}{2} + it_1 - is_1 + i\tau \right) G_b(-i\tau)}{G_b \left(\frac{Q}{2} + it_1 - is_1 \right)} \cdot \\
& \quad e^{-2\pi i(\alpha - \beta + s_1 - \tau)(\beta - s_1 + \tau)} e^{2\pi i(\beta - s_1)s_1} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}(\beta - s_1)} d\alpha d\tau d\beta \\
& \quad \text{shifting } \alpha \mapsto \alpha + \beta + \tau, \beta \mapsto \beta + s_1 : \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s_1 + \alpha, t_1 + \alpha + \tau) g(s_2, t_2) \frac{G_b \left(\frac{Q}{2} + it_1 - is_1 + i\tau \right) G_b(-i\tau)}{G_b \left(\frac{Q}{2} + it_1 - is_1 \right)} \cdot \\
& \quad e^{-2\pi i(\alpha\beta + \alpha\tau + s_1\tau)} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}\beta} d\alpha d\tau d\beta,
\end{aligned}$$

and finally for V'' :

$$\begin{aligned}
& SV_{14}'' S^{-1} : f \otimes g \\
& \mapsto \mathcal{S} \cdot \left(e^{\frac{i}{2\pi b^2} \log \widehat{A}_* \otimes \log \widehat{A}^{-1}} \int_{\mathbb{R}+i0} \widehat{B}_*^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\tau} e^{\pi i \tau^2} G_b(-i\tau) d\tau \right) \cdot \\
& \int_{\mathbb{R}} f(\alpha - s_1, \alpha - t_1) g(s_2, t_2) G_b \left(\frac{Q}{2} - it_1 + is_1 \right) e^{-2\pi i(\alpha - s_1)s_1} d\alpha \\
& = \mathcal{S} \cdot \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(\alpha - s_1\alpha - t_1 + \tau) g(s_2, t_2) G_b \left(\frac{Q}{2} - it_1 + is_1 + i\tau \right) \\
& \quad e^{-2\pi i(\alpha - s_1)s_1} e^{\pi i \tau^2} G_b(-i\tau) \widehat{A}^{-ib^{-1}t_1} \widehat{B}^{ib^{-1}\tau} d\alpha d\tau \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(\alpha - \beta + s_1, \alpha - \beta + t_1 + \tau) g(s_2, t_2) \frac{G_b \left(\frac{Q}{2} + it_1 - is_1 + i\tau \right) G_b(-i\tau)}{G_b \left(\frac{Q}{2} + it_1 - is_1 \right)} \cdot \\
& \quad e^{-2\pi i(\alpha - \beta + s_1)(\beta - s_1)} e^{\pi i \tau^2} e^{2\pi i(\beta - s_1)s_1} e^{2\pi i\tau(t_1 - \beta)} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}(t_1 - \beta)} d\alpha d\tau d\beta \\
& \quad \text{shifting } \alpha \mapsto \alpha + \beta, \beta \mapsto t_1 - \beta : \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s_1 + \alpha, t_1 + \alpha + \tau) g(s_2, t_2) \frac{G_b \left(\frac{Q}{2} + it_1 - is_1 + i\tau \right) G_b(-i\tau)}{G_b \left(\frac{Q}{2} + it_1 - is_1 \right)} \cdot \\
& \quad e^{-2\pi i\alpha(t_1 - s_1)} e^{2\pi i\beta(\alpha + \tau)} e^{\pi i \tau^2} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}\beta} d\alpha d\tau d\beta.
\end{aligned}$$

Combining, we have

$$\begin{aligned}
 & f \otimes g \\
 \mapsto & V^* \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} d\alpha d\tau d\beta f(s_1 + \alpha, t_1 + \alpha + \tau) g(s_2, t_2) \cdot \\
 & \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau\right) G_b(-i\tau)}{G_b\left(\frac{Q}{2} + it_1 - is_1\right)} e^{-2\pi i(\alpha\beta + \alpha\tau + s_1\tau)} \widehat{B}^{ib^{-1}\tau} \widehat{A}^{ib^{-1}\beta} \\
 \mapsto & \widehat{W} \int_{\mathbb{R}+i\epsilon_\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} d\alpha d\tau d\beta d\sigma f(s_1 + \alpha, t_1 + \alpha + \tau) g(s_2 + \sigma, t_2) \cdot \\
 & \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau\right) G_b(-i\tau) G_b(-i\sigma)}{G_b\left(\frac{Q}{2} + it_1 - is_1\right)} e^{-2\pi i(\alpha\beta + \alpha\tau + s_1\tau)} \\
 & e^{2\pi is_2\tau} \widehat{B}^{ib^{-1}(\sigma+\tau)} \widehat{A}^{ib^{-1}(\beta+s_2)}.
 \end{aligned}$$

The integral for β and σ is independent; hence, we can interchange them. Furthermore, from the decay properties for $f \otimes g$ and the asymptotic properties for $G_b(-i\tau)G_b(-i\sigma)$, the integral for $\alpha, \tau,$ and σ is absolutely convergent; hence, we can interchange the order so that $d\sigma$ goes to the inner most layer.

$$\begin{aligned}
 \mapsto & V'' \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon'_\tau} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} d\sigma d\alpha d\tau d\beta d\alpha' d\tau' d\beta' \\
 & f(s_1 + \alpha + \alpha', t_1 + \alpha + \alpha' + \tau + \tau') g(s_2 + \sigma, t_2) \\
 & \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau + i\tau'\right) G_b(-i\tau) G_b(-i\sigma)}{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau'\right)} \\
 & \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau'\right) G_b(-i\tau')}{G_b\left(\frac{Q}{2} + it_1 - is_1\right)} \cdot e^{-2\pi i(\alpha\beta + \alpha\tau + (s_1 + \alpha')\tau)} \\
 & e^{2\pi is_2\tau} e^{-2\pi i\alpha'(t_1 - s_1)} e^{2\pi i\beta'(\alpha' + \tau')} e^{\pi i\tau'^2} e^{2\pi i\beta'(\sigma + \tau)} \\
 & \widehat{B}^{ib^{-1}(\sigma + \tau + \tau')} \widehat{A}^{ib^{-1}(\beta + \beta' + s_2)} \\
 \text{shifting } & \alpha \mapsto \alpha - \alpha', \beta \mapsto \beta - \beta', \tau \mapsto \tau - \tau' : \\
 = & \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon'_\tau} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau - i\epsilon'_\tau} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} d\sigma d\alpha d\tau d\beta d\alpha' d\tau' d\beta' f(s_1 + \alpha, t_1 + \alpha + \tau) \\
 & g(s_2 + \sigma, t_2) \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau\right) G_b(i\tau' - i\tau) G_b(-i\sigma) G_b(-i\tau')}{G_b\left(\frac{Q}{2} + it_1 - is_1\right)} \\
 & e^{2\pi i\beta'(\alpha + \sigma + \tau)} e^{2\pi i\alpha'(\beta + s_1 - t_1)} e^{2\pi i(\alpha + s_1 - s_2)(\tau' - \tau) - 2\pi i\alpha\beta + \pi i\tau'^2} \widehat{B}^{ib^{-1}(\sigma + \tau)} \widehat{A}^{ib^{-1}(\beta + s_2)}.
 \end{aligned}$$

The integral over $d\beta d\alpha'$ is just Fourier transforms; hence, we can integrate over α' to get $\beta = t_1 - s_1$:

$$\begin{aligned}
 &= \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon'_\tau} \int_{\mathbb{R}+i\epsilon_\tau-i\epsilon'_\tau} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} d\sigma d\alpha d\tau d\tau' d\beta' f(s_1 + \alpha, t_1 + \alpha + \tau) \\
 &g(s_2 + \sigma, t_2) \frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau\right) G_b(i\tau' - i\tau) G_b(-i\sigma) G_b(-i\tau')}{G_b\left(\frac{Q}{2} + it_1 - is_1\right)} \\
 &e^{2\pi i\beta'(\alpha+\sigma+\tau)} e^{2\pi i(\alpha+s_1-s_2)(\tau'-\tau)} e^{-2\pi i\alpha(t_1-s_1)+\pi i\tau^2} \widehat{B}^{ib^{-1}(\sigma+\tau)} \widehat{A}^{ib^{-1}(t_1+s_2-s_1)}.
 \end{aligned}$$

Next, from the decay properties for $G_b(i\tau' - i\tau)G_b(-i\tau')$, this is integrable over τ' . By shifting $\tau \mapsto \tau - \sigma - \alpha$, we see also that the integrations over σ, τ, α are absolutely convergent, and furthermore, integrations over α, σ do not depend on β' . Hence, we can interchange the order for $d\tau d\tau'$ and then $d\sigma d\alpha$ to get

$$\int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau-i\epsilon'_\tau} \int_{\mathbb{R}+i\epsilon'_\tau} d\tau' d\tau d\beta' d\sigma d\alpha.$$

Using the reflection properties, we bring $G_b(-i\tau')$ to the denominator. Notice that the contour goes above $\tau' = 0$ and below $\tau' = \tau$; hence, we can integrate over τ' using Lemma 3.9 to get

$$\begin{aligned}
 &= \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau-i\epsilon'_\tau} d\tau d\beta' d\sigma d\alpha f(s_1 + \alpha, t_1 + \alpha + \tau) g(s_2 + \sigma, t_2) \\
 &\frac{G_b\left(\frac{Q}{2} + it_1 - is_1 + i\tau\right) G_b(-i\sigma) G_b(-i\tau) G_b\left(\frac{Q}{2} - i\alpha + is_2 - is_1\right)}{G_b\left(\frac{Q}{2} + it_1 - is_1\right) G_b\left(\frac{Q}{2} - i\alpha - i\tau + is_2 - is_1\right)} \\
 &e^{2\pi i\beta'(\alpha+\sigma+\tau)} e^{-2\pi i(\alpha+s_1-s_2)\tau} e^{-2\pi i\alpha(t_1-s_1)} \widehat{B}^{ib^{-1}(\sigma+\tau)} \widehat{A}^{ib^{-1}(t_1+s_2-s_1)} \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}+i\epsilon_\tau-i\epsilon'_\tau} d\tau d\beta' d\sigma d\alpha f(s_1 - \alpha, t_1 - \alpha + \tau) g(s_2 + \sigma, t_2) \\
 &\frac{G_b\left(\frac{Q}{2} + is_1 - it_1\right) G_b(-i\sigma) G_b(-i\tau) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\alpha + i\tau\right)}{G_b\left(\frac{Q}{2} + is_1 - it_1 - i\tau\right) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\alpha\right)} \\
 &e^{2\pi i\beta'(\sigma+\tau-\alpha)} e^{2\pi i(t_1-s_1)(\tau-\alpha)} \widehat{B}^{ib^{-1}(\sigma+\tau)} \widehat{A}^{ib^{-1}(t_1+s_2-s_1)},
 \end{aligned}$$

where we used the reflection properties again and flipping $\alpha \mapsto -\alpha$. Now, we can integrate τ and β' by Fourier transform to get $\tau = \alpha - \sigma$. However, this should be interpreted as a function of σ over \mathbb{R} , and then analytic continued to $\mathbb{R}+i\epsilon_\sigma$, so that the contour for σ will be pushed under the pole for $\sigma = \alpha$ in $G_b(-i\tau) \mapsto G_b(i\sigma - i\alpha)$.

We get

$$\begin{aligned}
 &= \int_{\mathbb{R}} \int_C d\sigma d\alpha f(s_1 - \alpha, t_1 - \sigma) g(s_2 + \sigma, t_2) \\
 &\quad \frac{G_b\left(\frac{Q}{2} + is_1 - it_1\right) G_b(-i\sigma) G_b(i\sigma - i\alpha) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\sigma\right)}{G_b\left(\frac{Q}{2} + is_1 - it_1 - i\alpha + i\sigma\right) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\alpha\right)} \\
 &\quad e^{2\pi i(t_1-s_1)\sigma} \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{ib^{-1}(t_1+s_2-s_1)}.
 \end{aligned}$$

Finally, we apply W_{31} to get

$$\begin{aligned}
 &\mapsto_W \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} \int_C d\sigma d\alpha d\tau f(s_1 - \alpha + \tau, t_1 - \sigma) g(s_2 + \sigma, t_2) \\
 &\quad \frac{G_b(-i\sigma) G_b(-i\alpha + i\sigma) G_b\left(\frac{Q}{2} + is_1 - it_1 + i\tau\right) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\sigma + i\tau\right)}{G_b(Q + i\tau) G_b\left(\frac{Q}{2} + is_1 - it_1 - i\alpha + i\sigma + i\tau\right) G_b\left(\frac{Q}{2} + is_1 - is_2 - i\alpha + i\tau\right)} \\
 &\quad e^{2\pi i\sigma(t_1-s_1-\tau)} e^{2\pi is_1\tau} A^{ib^{-1}s_1} B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{ib^{-1}(t_1+s_2-s_1-\tau)}.
 \end{aligned}$$

□

We can further simplify the expression by the following unitary transformations, which will also be used in the right picture:

$$\begin{aligned}
 s_2 &\mapsto s_2 - t_1, \\
 s_1 &\mapsto s_1 + \frac{s_2}{2}, \\
 t_1 &\mapsto t_1 + \frac{s_2}{2},
 \end{aligned}$$

or equivalently

$$f \otimes g \mapsto f\left(s_1 + \frac{s_2}{2}, \frac{s_2}{2} - t_1\right) g\left(t_1 + \frac{s_2}{2}, t_2\right), \tag{7.60}$$

so that the expression becomes:

$$\begin{aligned}
 &= \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} \int_C d\sigma d\alpha d\tau f(s_1 - \alpha + \tau, t_1 - \sigma) g(s_2, t_2) \\
 &\quad \frac{G_b(-i\sigma) G_b(-i\alpha + i\sigma) G_b\left(\frac{Q}{2} + is_1 - it_1 + i\tau\right) G_b\left(\frac{Q}{2} + is_1 + it_1 - i\sigma + i\tau\right)}{G_b(Q + i\tau) G_b\left(\frac{Q}{2} + is_1 - it_1 - i\alpha + i\sigma + i\tau\right) G_b\left(\frac{Q}{2} + is_1 + it_1 - i\alpha + i\tau\right)} \\
 &\quad e^{2\pi i\sigma(t_1-s_1-\tau)} e^{2\pi is_1\tau} e^{\pi is_2\tau} A^{ib^{-1}(s_1+\frac{s_2}{2})} B^{ib^{-1}\tau} \otimes \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{ib^{-1}(\frac{s_2}{2}-s_1-\tau)}. \tag{7.61}
 \end{aligned}$$

7.5 The right regular corepresentation

In analogy to the left regular action, there is a notion of a right regular action. First, we consider the quantum plane.

Proposition 7.21 *Given W that defines the left regular corepresentation,*

$$W_R := (\widehat{J} \otimes \widehat{J})W_{21}^*(\widehat{J} \otimes \widehat{J}) \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A} \tag{7.62}$$

is also a multiplicative unitary and satisfies

$$(1 \otimes \Delta)W_R = W_{R,12}W_{R,13}, \tag{7.63}$$

i.e., it defines a corepresentation, called the right regular corepresentation.

Using the realization of $Ad\widehat{J}$, we see that W_R is given by

$$\begin{aligned} W_R &= e^{\frac{i}{2\pi b^2} \log A_* \otimes \log A} g_b(B_* \otimes B) \\ &= e^{\frac{i}{2\pi b^2} \log A_* \otimes \log A} \int_{\mathbb{R}+i0} B_*^{ib^{-1}\tau} \otimes B^{ib^{-1}\tau} \frac{e^{-\pi i \tau^2}}{G_b(Q+i\tau)} d\tau, \end{aligned} \tag{7.64}$$

where on the space $\mathcal{H}_{\text{corep}}$, we have $\widehat{J} = \overline{f}(t, s)$ so that

$$A_* = e^{-2\pi bt}, \quad B_* = e^{-2\pi bp_t}. \tag{7.65}$$

Therefore, we see that the corepresentation is acting on the t -coordinate by:

$$f(s, t) \mapsto \int_{\mathbb{R}+i0} f(s, t - \tau) \frac{e^{-\pi i \tau^2}}{G_b(Q+i\tau)} A^{-ib^{-1}t} B^{ib^{-1}\tau} d\tau, \tag{7.66}$$

in which under the pairing, we have:

$$X_r = e^{-2\pi bt}, \quad Y_r = e^{-2\pi bp_t}, \tag{7.67}$$

$$\widetilde{X}_r = e^{-2\pi b^{-1}t}, \quad \widetilde{Y}_r = e^{-2\pi b^{-1}p_t}. \tag{7.68}$$

In other words, under the transform $t \mapsto -t$, we conclude together with Proposition 6.16 that

Proposition 7.22 *We have the equivalence*

$$L^2(\mathcal{A}) \simeq \mathcal{H}_{\text{irr}} \otimes \mathcal{H}_{\text{irr}} \tag{7.69}$$

as representation of $\mathcal{B}_{q\bar{q},L} \otimes \mathcal{B}_{q\bar{q},R} \simeq \mathcal{B}_{q\bar{q}} \otimes \mathcal{B}_{q\bar{q}}$, where $\mathcal{H}_{\text{irr}} := L^2(\mathbb{R})$ is the canonical representation of $\mathcal{B}_{q\bar{q}}$ on $L^2(\mathbb{R})$. Hence, this is an analogue to the classical “Peter–Weyl” theorem on “functions on the quantum plane.”

Now, let us look at the case for $\mathcal{D}(\mathcal{A})$. To define the right regular corepresentation, we need the corresponding multiplicative unitary \mathbf{W}_R . It is known that \mathbf{W}_R is given by

$$\mathbf{W}_R = (\mathcal{U} \otimes 1)\mathbf{W}_{3412}(\mathcal{U}^* \otimes 1), \tag{7.70}$$

where

$$\mathcal{U} = \widehat{\mathbf{J}}_m \mathbf{J}_m = (J\widehat{J} \otimes \widehat{J}J)Z \in \mathcal{A} \otimes \widehat{\mathcal{A}}. \tag{7.71}$$

We simplify the expression and express \mathbf{W}_R in terms of 4 g_b 's as before.

Proposition 7.23 \mathbf{W}_R is given by (cf. (7.49))

$$\mathbf{W}_R = V_{32}W_{R,13}V_{32}''^*\widehat{W}_{R,24}, \tag{7.72}$$

where $V''^* = (\widehat{J} \otimes \widehat{J})V^*(\widehat{J} \otimes \widehat{J})$.

Proof From ([25] Prop 2.15), we have:

$$\begin{aligned} \widehat{W} &= W_{21}^*, \\ W_R &= (\widehat{J} \otimes \widehat{J})W_{21}^*(\widehat{J} \otimes \widehat{J}) \\ &= (\widehat{J} \otimes \widehat{J})(J \otimes \widehat{J})W_{21}(J \otimes \widehat{J})(\widehat{J} \otimes \widehat{J}) \\ &= (\widehat{J}J \otimes 1)W_{21}(J\widehat{J} \otimes 1), \\ \widehat{W}_R &= (J \otimes J)W(J \otimes J) \\ &= (J \otimes J)(\widehat{J} \otimes J)W^*(\widehat{J} \otimes J)(J \otimes J) \\ &= (J\widehat{J} \otimes 1)W^*(\widehat{J}J \otimes 1). \end{aligned}$$

Hence, we obtain (the indices indicate the legs in which the operators are acting):

$$\begin{aligned} \mathbf{W}_R &= ((J_1\widehat{J}_1\widehat{J}_2J_2)Z_{12})W_{31}Z_{12}^*\widehat{W}_{42}Z_{12}(Z_{12}^*(\widehat{J}_1J_1J_2\widehat{J}_2)) \\ &= (J_1\widehat{J}_1\widehat{J}_2J_2)Z_{12}W_{31}Z_{12}^*\widehat{W}_{42}(\widehat{J}_1J_1J_2\widehat{J}_2) \\ &= (J_1\widehat{J}_1\widehat{J}_2J_2)V_{12}V_{12}'W_{31}V_{12}''^*V_{12}''\widehat{W}_{42}(\widehat{J}_1J_1J_2\widehat{J}_2) \\ &= (J_1\widehat{J}_1\widehat{J}_2J_2)(\widehat{J}_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)V_{12}W_{31}V_{12}''\widehat{W}_{42}(\widehat{J}_1J_1J_2\widehat{J}_2) \\ &= (J_1\widehat{J}_1\widehat{J}_2J_2)(\widehat{J}_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)W_{31}V_{32}\widehat{W}_{42}(\widehat{J}_1J_1J_2\widehat{J}_2) \\ &= (J_1\widehat{J}_1\widehat{J}_2J_2)(\widehat{J}_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)(\widehat{J}_1J_1)W_{R,13}V_{32}(J_2\widehat{J}_2)\widehat{W}_{R,24} \\ &= (\widehat{J}_2J_2)(\widehat{J}_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)W_{R,13}V_{32}(J_2\widehat{J}_2)\widehat{W}_{R,24} \\ &= (J_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)W_{R,13}V_{32}(\widehat{J}_2J_2)\widehat{W}_{R,24} \\ &= V_{32}(J_2\widehat{J}_3)(\widehat{J}_2\widehat{J}_3)W_{R,13}V_{32}(\widehat{J}_2J_2)\widehat{W}_{R,24} \\ &= V_{32}W_{R,13}(J_2\widehat{J}_2)V_{32}(\widehat{J}_2J_2)\widehat{W}_{R,24} \\ &= V_{32}W_{R,13}(J_2\widehat{J}_2)(\widehat{J}_2\widehat{J}_3)V_{32}^*(J_2\widehat{J}_3)(\widehat{J}_2J_2)\widehat{W}_{R,24} \\ &= V_{32}W_{R,13}(\widehat{J}_2\widehat{J}_3)V_{32}^*(\widehat{J}_2\widehat{J}_3)\widehat{W}_{R,24}. \end{aligned}$$

□

Now, we can realize the action on $\mathcal{H}_{\text{corep}} \otimes \widehat{\mathcal{H}}_{\text{corep}}$ as in previous section:

Proposition 7.24 *The action of the components of W_R is given by*

$$\begin{aligned} \widehat{W}_{R,24} &= e^{-\frac{i}{2\pi b^2} \log \widehat{A}_* \otimes \log \widehat{A}} g_b^*(\widehat{B}_* \otimes \widehat{B}) \\ f(s_1, t_1)g(s_2, t_2) &\mapsto \int f(s_1, t_1)g(s_2, t_2 - \tau) e^{-2\pi i \tau t_2 + \pi i \tau^2} G_b(-i\tau) \widehat{B}^{ib^{-1}\tau} \widehat{A}^{-ib^{-1}t_2} d\tau, \\ W_{R,13} &= e^{\frac{i}{2\pi b^2} \log A_* \otimes \log A} g_b(B_* \otimes B) \\ f(s_1, t_1)g(s_2, t_2) &\mapsto \int f(s_1, t_1 - \tau)g(s_2, t_2) \frac{e^{-\pi i \tau^2}}{G_b(Q + i\tau)} A^{-ib^{-1}t_1} B^{ib^{-1}\tau} d\tau, \\ V_{32} &= e^{\frac{i}{2\pi b^2} \log \widehat{A}^{-1} \otimes \log A^{-1}} g_b(q\widehat{B}\widehat{A}^{-1} \otimes B) \\ f(s_1, t_1)g(s_2, t_2) &\mapsto \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} f(s_1, t_1)g(s_2 + \alpha, t_2 + \alpha + \tau) \\ &\quad \frac{e^{2\pi i(\alpha\beta + \alpha\tau + \beta\tau + s_2\tau)} G_b\left(\frac{Q}{2} + is_2 - it_2\right)}{G_b\left(\frac{Q}{2} + is_2 - it_2 - i\tau\right) G_b(Q + i\tau)} A^{ib^{-1}\beta} B^{ib^{-1}\tau} d\alpha d\tau d\beta, \\ V_{32}''^* &= e^{\frac{i}{2\pi b^2} \log A_* \otimes \log A} g_b(B_* \otimes B) \\ f(s_1, t_1)g(s_2, t_2) &\mapsto \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}} g(s_2 + \alpha, t_2 + \alpha + \tau) \\ &\quad \frac{e^{2\pi i\alpha(t_2 - s_2 - \beta)} e^{-\pi i \tau^2} G_b\left(\frac{Q}{2} + is_2 - it_2\right)}{G_b\left(\frac{Q}{2} + is_2 - it_2 - i\tau\right) G_b(Q + i\tau)} A^{ib^{-1}\beta} B^{ib^{-1}\tau} d\alpha d\tau d\beta. \end{aligned}$$

Theorem 7.25 *The right regular corepresentation is given by*

$$\begin{aligned} f(s_1, t_1)g(s_2, t_2) &\mapsto \int_{\mathbb{R}+i\epsilon_\tau} \int_{\mathbb{R}} \int_C d\sigma d\alpha d\tau f(s_1, t_1 - \sigma)g(s_2 + \sigma, t_2 + \tau - \alpha) \\ &\quad e^{\pi Q(-\alpha + \tau) + 2\pi i(-\sigma s_2 + \alpha\tau - \alpha t_2 + \sigma t_2)} \\ &\quad \frac{G_b(-i\sigma)G_b\left(\frac{Q}{2} + is_2 - it_2\right) G_b(i\sigma - i\tau)G_b\left(\frac{Q}{2} - i\sigma + it_1 - it_2\right)}{G_b\left(\frac{Q}{2} + is_2 - it_2 + i\sigma - i\tau\right) G_b(Q + i\alpha)G_b\left(\frac{Q}{2} - i\tau + it_1 - it_2\right)} \\ &\quad A^{ib^{-1}(t_2 - t_1 - s_2)} B^{ib^{-1}\tau} \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{-ib^{-1}(t_2 + \tau)} d\alpha d\tau d\sigma, \end{aligned}$$

where the contour C of σ goes above $\sigma = 0$ and below $\sigma = \tau$.

We have to follow the transformations given in the left regular picture. So we apply the transformations after Theorem 7.20 and an extra one that does not affect the previous action:

$$\begin{aligned} s_2 &\mapsto s_2 - t_1, \\ s_1 &\mapsto s_1 + \frac{s_2}{2}, \\ t_1 &\mapsto t_1 + \frac{s_2}{2}, \\ t_2 &\mapsto t_2 + \frac{s_2}{2}, \end{aligned}$$

and obtain

$$\begin{aligned} f \otimes g &\mapsto \int_{\mathbb{R}+i\epsilon\tau} \int_{\mathbb{R}} \int_{\mathbb{C}} d\sigma d\alpha d\tau f(s_1, t_1 - \sigma) g(s_2, t_2 + \tau - \alpha) \\ &\quad e^{\pi Q(-\alpha+\tau)+2\pi i(\sigma t_1+\alpha\tau-\alpha t_2+\sigma t_2)} e^{-\pi i\alpha s_2} \\ &\quad \frac{G_b(-i\sigma)G_b\left(\frac{Q}{2}-it_1-it_2\right)G_b(i\sigma-i\tau)G_b\left(\frac{Q}{2}-i\sigma+it_1-it_2\right)}{G_b\left(\frac{Q}{2}-it_1-it_2+i\sigma-i\tau\right)G_b(Q+i\alpha)G_b\left(\frac{Q}{2}-i\tau+it_1-it_2\right)} \\ &\quad A^{ib^{-1}(t_2-\frac{s_2}{2})} B^{ib^{-1}\tau} \widehat{B}^{ib^{-1}\alpha} \widehat{A}^{ib^{-1}(-t_2-\tau-\frac{s_2}{2})} d\alpha d\tau d\sigma. \end{aligned} \tag{7.73}$$

8 Regular representation

To obtain a representation of $U_q(\mathfrak{sl}(2, \mathbb{R}))$ on the space $L^2(GL_q^+(2, \mathbb{R}))$, we need to describe the non-degenerate pairing between them. Then, we apply the pairing to the corepresentations constructed above and obtain the desired representations for $U_q(\mathfrak{gl}(2, \mathbb{R}))$.

8.1 Pairing with $U_q(\mathfrak{gl}(2, \mathbb{R}))$

Let us first recall the following definition of $U_q(\mathfrak{gl}(2, \mathbb{R}))$ serving as a dual space for $\mathcal{M}_q = GL_q^+(2, \mathbb{R})$ that is observed by Frenkel [8]:

Definition 8.1 As a Hopf *-algebra, $U_q(\mathfrak{gl}(2, \mathbb{R}))$ is generated by positive self-adjoint operators E, F, K, K_0 such that

$$\begin{aligned} KE &= qEK, \\ KF &= q^{-1}FK, \\ EF - FE &= \frac{K^2 - K^{-2}}{q - q^{-1}}, \\ \Delta(K) &= K \otimes K, \\ \Delta(K_0) &= K_0 \otimes K_0, \\ \Delta(E) &= K_0^{-1}K^{-1} \otimes E + E \otimes K_0K, \\ \Delta(F) &= K_0K^{-1} \otimes F + F \otimes K_0^{-1}K, \end{aligned}$$

and K_0 commutes strongly with E, F, K .

In particular, we see that by setting $K_0 = 1$, we obtain the usual definition of $U_q(\mathfrak{sl}(2, \mathbb{R}))$. Now, we can similarly bring $U_q(\mathfrak{gl}(2, \mathbb{R}))$ into C^* -algebraic level by introducing continuous parameters. It is the algebra

$$\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \int_{\mathbb{R}+i0} f(r_0, r, s, t) K_0^{ib^{-1}r_0} K^{ib^{-1}r} E^{ib^{-1}s} F^{ib^{-1}t} ds dt dr_0 dr \right\}, \tag{8.1}$$

where $f(r_0, r, s, t)$ has simple poles at $s, t = -inb - imb^{-1}, n, m \in \mathbb{Z}_{\geq 0}$ and has similar decays as \mathcal{A} along the real direction in the variables r_0, r, s, t . This is to ensure that the coproduct lies in the multiplier algebra as argued before. The C^* -norm can be introduced once we obtain its representation as certain unitary operators on $L^2(\mathbb{R} \times \mathbb{R})$ under the pairing. By abuse of notation, we still denote it by $U_q(\mathfrak{gl}(2, \mathbb{R}))$, and a similar version with K_0 omitted by $U_q(\mathfrak{sl}(2, \mathbb{R}))$.

Let us ignore the $*$ -structure and consider only the polynomial algebra. Then, there exists a non-degenerate pairing defined on the generators:

Proposition 8.2 *The pairing between E, F, K, K_0 and the z -variable $z_{ij}, i = 1, 2$ is given by*

$$\begin{aligned} \langle E, z_{21} \rangle &= c, & \langle F, z_{12} \rangle &= c^{-1}, \\ \langle K, z_{11} \rangle &= q^{-1/2}, & \langle K, z_{22} \rangle &= q^{1/2}, \\ \langle K_0, z_{11} \rangle &= q^{-1/2}, & \langle K_0, z_{22} \rangle &= q^{-1/2}, \\ \langle E, N \rangle &= 0, & \langle F, N \rangle &= 0, \\ \langle K, N \rangle &= 1, & \langle K_0, N \rangle &= q^{-1}, \end{aligned}$$

and zero otherwise, where $c \in \mathbb{C}$ is any constant. The pairing is then extended to any monomial by the coproduct and induction.

Corollary 8.3 *The pairing between E, F, K, K_0 and the $\mathcal{D}(\mathcal{A})$ generators is given by*

$$\begin{aligned} \langle K, A \rangle &= q^{-\frac{1}{2}}, & \langle K, \widehat{A} \rangle &= q^{\frac{1}{2}}, \\ \langle K_0, A \rangle &= q^{-\frac{1}{2}}, & \langle K_0, \widehat{A} \rangle &= q^{-\frac{1}{2}}, \\ \langle E, B \rangle &= c, & \langle F, \widehat{B} \rangle &= c^{-1}, \end{aligned}$$

and zero otherwise, where $c \in \mathbb{C}$ is any constant. The pairing is then extended to any monomial by the coproduct and induction.

By induction, we obtain

Proposition 8.4 *The pairing between any monomials is given by*

$$\langle K^l E^m F^n, z_{11}^L z_{21}^{m'} z_{12}^{n'} N^{L'} \rangle = c^{m^2 - n^2} q^{l(m-n-L)/2 + mL + (n-m)L' - 2nm} [n]_q! [m]_q! \delta_{nn'} \delta_{mm'}, \tag{8.2}$$

or

$$\left\langle K^l E^m F^n, A^L B^{m'} \widehat{B}^{n'} \widehat{A}^{L'} \right\rangle = c^{m^2-n^2} q^{l(m+L'-L)/2+mL+nL'-nm} [n]_q! [m]_q! \delta_{nn'} \delta_{mm'}. \tag{8.3}$$

Remark 8.5 Restricting to $U_q(\mathfrak{sl}(2, \mathbb{R}))$, the pairing is essentially the same as the one given in [27], with the role of E and F interchanged. Hence, this provides a more elegant formula for the pairing of a general monomial.

Hence, as in Sect. 5.2, we introduce G_b for the expression in the general pairing with elements represented by integrations. It turns out that we need to replace $[n]_q!$ by

$$[n]_q! \mapsto \frac{G_b(Q + i\tau)}{(1 - q^2)^{ib^{-1}\tau}} \tag{8.4}$$

so that the commutation relation $[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$ is satisfied. Here, we take $1 - q^2$ as a complex number with $-\frac{\pi}{2} < \arg(1 - q^2) < \frac{\pi}{2}$.

Therefore, the general pairing is given by

$$\begin{aligned} & \langle f(r_0, r, s, t), g(s_1, t_1)h(s_2, t_2) \rangle \\ &= \iint f(r_0, r, t_1, t_2)g(s_1, t_1)h(s_2, t_2)e^{-\pi i(r(t_1+s_2-s_1)/2-r_0(s_2+s_1)/2+t_1s_1+t_2s_2-t_1t_2)} \\ & \quad c^{b^{-2}(t_2^2-t_1^2)} \frac{G_b(Q + it_1)G_b(Q + it_2)}{(1 - q^2)^{ib^{-1}(t_1+t_2)}} ds_1 ds_2 dt_1 dt_2 dr_0 dr, \end{aligned} \tag{8.5}$$

where we denoted by

$$f(r_0, r, s, t) := \iiint f(r, s, t) K_0^{ib^{-1}r_0} K^{ib^{-1}r} E^{ib^{-1}s} F^{ib^{-1}t} ds dt dr_0 dr,$$

and

$$\begin{aligned} & g(s_1, t_1)h(s_2, t_2) \\ & := \iiint g(s_1, t_1)h(s_2, t_2) A^{ib^{-1}s_1} B^{ib^{-1}t_1} \widehat{B}^{ib^{-1}t_2} \widehat{A}^{ib^{-1}s_2} ds_1 ds_2 dt_1 dt_2. \end{aligned}$$

Finally, in order to obtain a positive representation in later sections, it turns out we need to choose the pairing constant to be $c = -iq^{1/2} = -q \frac{Q}{2b}$.

8.2 The principal series representation

By taking the pairing above with the fundamental corepresentation defined in Sect. 7.2, we obtain:

Theorem 8.6 *The fundamental corepresentation $T^{\lambda,t}$ corresponds to the representation $\mathcal{P}_{\lambda,t}$, called the principal series representation, given on $L^2(\mathbb{R}, ds)$ by:*

$$\begin{aligned} K_0 &= e^{\pi bt}, \\ K &= e^{-\pi bs}, \\ E &= \left[\frac{Q}{2b} - \frac{i}{b}(s - \lambda) \right]_q e^{-2\pi bp_s}, \\ F &= \left[\frac{Q}{2b} + \frac{i}{b}(s + \lambda) \right]_q e^{2\pi bp_s}, \end{aligned}$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The operators are all positive essentially self-adjoint.

Proof Recall that we choose $c = -iq^{1/2}$ in the pairing. Recall also that we have to multiply the expressions by $\frac{G_b(Q+i\alpha)G_b(Q+i\tau)}{(1-q^2)^{ib^{-1}(\alpha+\tau)}}$. From the expression (7.31), we obtain for K_0 and K that $\tau = \alpha = 0$. The factor becomes $\lim_{\alpha \rightarrow 0} G_b(-i\alpha)G_b(Q + i\alpha) = 1$ and the rest becomes

$$\begin{aligned} K_0 : f(s) &\mapsto f(s) \cdot (q^{-1/2})^{ib^{-1}(t-s)} (q^{-1/2})^{ib^{-1}(t+s)} \\ &= e^{\pi bt} f(s), \\ K : f(s) &\mapsto f(s) \cdot (q^{-1/2})^{ib^{-1}(t-s)} (q^{1/2})^{ib^{-1}(t+s)} \\ &= e^{-\pi bs} f(s). \end{aligned}$$

For E , we have $\tau = -ib$, $\alpha = 0$ so that the factor $\lim_{\alpha \rightarrow 0} G_b(-i\alpha)G_b(Q + i\alpha) = 1$ as well, and we obtain

$$\begin{aligned} E : f(s) &\mapsto \frac{-iq^{1/2} f(s + ib)}{1 - q^2} \frac{G_b\left(\frac{Q}{2} - is + i\lambda + b\right)}{G_b\left(\frac{Q}{2} - is + i\lambda\right)} e^{\pi\lambda b - 2\pi bs + \pi bt} \cdot (q^{ib^{-1}(t-s)}) \\ &= \frac{iq^{-1/2}}{q - q^{-1}} (1 - e^{2\pi ib(\frac{Q}{2} - is + i\lambda)}) e^{\pi b(\lambda - s)} f(s + ib) \\ &= \frac{i}{q - q^{-1}} (q^{1/2} e^{\pi b(s - \lambda)} + q^{-1/2} e^{-\pi b(s - \lambda)}) f(s + ib) \\ &= \left[\frac{Q}{2b} - \frac{i}{b}(s - \lambda) \right]_q f(s + ib). \end{aligned}$$

Finally, for F , we have $\alpha = -ib$, $\tau = 0$, the factor $\lim_{\alpha \rightarrow -ib} G_b(-i\alpha)G_b(Q + i\alpha) = -q^2$, and we obtain

$$F : f(s) \mapsto \frac{iq^{-1/2}(-q^2)f(s - ib)}{1 - q^2} \frac{G_b\left(\frac{Q}{2} - is - i\lambda\right)}{G_b\left(\frac{Q}{2} - is - i\lambda - b\right)} e^{-\pi\lambda b + \pi bt} \cdot (q^{ib^{-1}(t+s)})$$

$$\begin{aligned}
 &= \frac{iq^{1/2}}{q - q^{-1}} \left(1 - e^{2\pi ib \left(\frac{Q}{2} - is - i\lambda - b \right)} \right) e^{-\pi b(\lambda+s)} f(s - ib) \\
 &= \frac{i}{q - q^{-1}} (q^{-1/2} e^{\pi b(s+\lambda)} + q^{1/2} e^{-\pi b(s+\lambda)}) f(s - ib) \\
 &= \left[\frac{Q}{2b} + \frac{i}{b}(s + \lambda) \right]_q f(s - ib).
 \end{aligned}$$

Since $\frac{i}{q - q^{-1}} = \frac{1}{2 \sin(\pi b^2)} > 0$, we immediately see that the operators are positive. We note that the expression for E can be rewritten as

$$E = \frac{i}{q - q^{-1}} (e^{\pi b(s-\lambda-p_s)} + e^{\pi b(\lambda-s-p_s)}), \tag{8.6}$$

where the summand q^2 -commutes. Hence, using Proposition 4.1, it is unitary equivalent to $e^{\pi b(s-\lambda-p_s)}$ which in turn is equivalent to $e^{-\pi b(p_s+\lambda)}$ by multiplication by $e^{-\pi is^2}$. Hence, it is essentially self-adjoint. Similar analysis applies to F ; hence, all operators are positive essentially self-adjoint. \square

Remark 8.7 We note that by restricting to $U_q(\mathfrak{sl}(2, \mathbb{R}))$, this is precisely (the Fourier transform of) the continuous series representation \mathcal{P}_s obtained in [2] given by

$$\begin{aligned}
 K \delta_k &= e^{-\pi bk} \delta_k, \\
 E_s \delta_k &= \left[\frac{Q}{2b} - \frac{i}{b}(k - s) \right]_q \delta_{k+ib}, \\
 F_s \delta_k &= \left[\frac{Q}{2b} + \frac{i}{b}(k + s) \right]_q \delta_{k-ib}
 \end{aligned}$$

under the correspondences $s \longleftrightarrow k, \lambda \longleftrightarrow s$.

8.3 Casimir operator

Next, we will study the action induced from the pairing with the multiplicative unitary \mathbf{W} . It turns out that the technical difficulty comes from the analysis of the following positive operator:

$$\mathbf{C} = e^{2\pi bx} + e^{-2\pi bx} + e^{-2\pi bp}, \tag{8.7}$$

which is studied in detail in [20] in a different context. This expression comes from the Casimir operator defined by

$$C = FE + \frac{qK^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2}. \tag{8.8}$$

More precisely, under the pairing given in the next section, the Casimir operator is given by

$$C = \left(\frac{i}{q - q^{-1}} \right)^2 (e^{2\pi bx} + e^{-2\pi bx} + e^{-2\pi bp} + 2). \quad (8.9)$$

Proposition 8.8 For $\lambda \in \mathbb{R}^+$,

$$\Phi_\lambda(s_2) := S_b(-ix + i\lambda)S_b(-ix - i\lambda) \quad (8.10)$$

is an eigenfunction for the operator \mathbf{C} with eigenvalue $e^{2\pi b\lambda} + e^{-2\pi b\lambda}$.

Proof We need to solve

$$\mathbf{C}\Phi_\lambda(x) = (e^{2\pi b\lambda} + e^{-2\pi b\lambda})\Phi_\lambda(x). \quad (8.11)$$

We consider

$$\begin{aligned} \mathbf{C} - (e^{2\pi b\lambda} + e^{-2\pi b\lambda}) &= \mathbf{C} - 2 \cos(2\pi i b\lambda) \\ &= 2 \cos(2\pi i b x) - 2 \cos(2\pi i b\lambda) + e^{2\pi b p s_2}. \end{aligned}$$

From the angle sum formula, we have

$$\begin{aligned} &2 \cos(2\pi i b x) - 2 \cos(2\pi i b\lambda) \\ &= -4 \sin(\pi i b(x - \lambda)) \sin(\pi i b(x + \lambda)), \end{aligned}$$

and from the functional equation (3.9)

$$S_b(x + b) = 2 \sin(\pi b x) S_b(x),$$

we immediately see that

$$\begin{aligned} &e^{2\pi b p x} \cdot S_b(-ix + i\lambda)S_b(-ix - i\lambda) \\ &= S_b(-ix + i\lambda + b)S_b(-ix - i\lambda + b) \\ &= 4 \sin(\pi i b(-x + \lambda)) \sin(\pi i b(-x - \lambda))S_b(-ix + i\lambda)S_b(-ix - i\lambda) \\ &= 4 \sin(\pi i b(x - \lambda)) \sin(\pi i b(x + \lambda))S_b(-ix + i\lambda)S_b(-ix - i\lambda). \end{aligned}$$

Hence, $\Phi_\lambda(x) = S_b(-ix + i\lambda)S_b(-ix - i\lambda)$ satisfies the above eigenvalue equation. \square

Therefore, we introduce the following transformation

Definition 8.9 We define the integral transformation

$$\begin{aligned} \Phi^{-1} : L^2(\mathbb{R}^+, d\mu(\lambda)) &\longrightarrow L^2(\mathbb{R}) \\ F(\lambda) \mapsto f(x) &:= \lim_{\epsilon \rightarrow 0} \int_0^\infty \Phi_\lambda(x + i\epsilon) e^{-2\pi x \epsilon} F(\lambda) d\mu(\lambda) \end{aligned} \tag{8.12}$$

where $d\mu(\lambda) = |S_b(Q + 2i\lambda)|^2 d\lambda$.

Theorem 8.10 Φ^{-1} is a unitary transformation that intertwines $e^{2\pi b\lambda} + e^{-2\pi b\lambda}$ and C .

Proof We compute the inner product for $f, g \in \mathcal{W}^+$ where \mathcal{W}^+ is the dense subspace

$$\mathcal{W}^+ := \{f(x) \in C_\infty(\mathbb{R}^+) : F(y) := f(e^y) \in \mathcal{W}\}.$$

Note that

$$\overline{\Phi_\lambda(x)} = \frac{1}{S_b(Q - ix - i\lambda)S_b(Q - ix + i\lambda)}.$$

We have

$$\begin{aligned} &\langle \Phi^{-1} f, \Phi^{-1} g \rangle \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \frac{S_b(\epsilon - ix - i\lambda)S_b(\epsilon - ix + i\lambda)e^{-4\pi x \epsilon}}{S_b(Q - \epsilon - ix - i\beta)S_b(Q - \epsilon - ix + i\beta)} \\ &\quad f(\lambda)\overline{g(\beta)}d\mu(\lambda)d\mu(\beta)dx \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \frac{G_b(\epsilon + ix - i\lambda)G_b(\epsilon + ix + i\lambda)e^{\pi i \lambda^2 - \pi i \beta^2} e^{-2\pi x(Q-4\epsilon)}}{G_b(Q - \epsilon + ix - i\beta)G_b(Q - \epsilon + ix + i\beta)} \\ &\quad f(\lambda)\overline{g(\beta)}d\mu(\lambda)d\mu(\beta)dx \end{aligned}$$

which means the contour for x separates the poles of the numerator and denominator.

Now, by Corollary 3.4, the integrand has asymptotics in x

$$\begin{aligned} &\left| \frac{G_b(\epsilon + ix - i\lambda)G_b(\epsilon + ix + i\lambda)e^{\pi i \lambda^2 - \pi i \beta^2} e^{-2\pi x(Q-4\epsilon)}}{G_b(Q - \epsilon + ix - i\beta)G_b(Q - \epsilon + ix + i\beta)} \right| \\ &= \begin{cases} e^{-2\pi x(Q-4\epsilon)} & x \longrightarrow \infty \\ e^{2\pi Qx} & x \longrightarrow -\infty \end{cases}. \end{aligned}$$

Hence, the integral is absolutely convergent in x, λ, β and we can interchange the order of integration to bring integration over x inside.

Shifting $x \mapsto x + \beta$, and using the 4-5 relation (Lemma 3.10), we obtain

$$\begin{aligned}
 &= \lim_{\epsilon \rightarrow 0} \int_0^\infty \int_0^\infty \frac{G_b(Q - 4\epsilon)G_b(2\epsilon + i\beta - i\lambda)G_b(2\epsilon + i\beta + i\lambda)}{G_b(Q - 2\epsilon + i\beta - i\lambda)G_b(Q - 2\epsilon + i\beta + i\lambda)} \\
 &\quad e^{\pi i \lambda^2 - \pi i \beta^2} e^{-2\pi Q\beta + 8\pi\beta\epsilon} f(\lambda)\overline{g(\beta)} |S_b(Q + 2i\lambda)|^2 |S_b(Q + 2i\beta)|^2 d\lambda d\beta.
 \end{aligned}$$

Note that the integration for β separates the poles of the numerator and denominator as well, and also since $\lambda, \beta > 0$, the case $\beta = -\lambda$ can be ignored; hence, we obtain $\delta(\lambda - \beta)$ using Lemma 3.11:

$$\begin{aligned}
 &= \int_0^\infty \frac{G_b(2i\lambda)e^{-2\pi Q\lambda}}{G_b(Q + 2i\lambda)} f(\lambda)\overline{g(\lambda)} |S_b(Q + 2i\lambda)|^4 d\lambda \\
 &= \int_0^\infty f(\lambda)\overline{g(\lambda)} |S_b(2i\lambda)|^2 |S_b(Q + 2i\lambda)|^4 d\lambda \\
 &= \int_0^\infty f(\lambda)\overline{g(\lambda)} |S_b(Q + 2i\lambda)|^2 d\lambda.
 \end{aligned}$$

This shows that the map is an isometry. The converse showing the eigenfunctions are also complete is more difficult and can be found in [20]. The technique involves writing the measure

$$|S_b(Q + 2i\lambda)|^2 = 4 \sinh(\pi b\lambda) \sinh(\pi b^{-1}\lambda)$$

as linear combinations of 4 exponentials and calculate the integral directly using hints from hyperfunctions.

From the proof, we therefore deduce the inverse map of the unitary transformation to be the complex conjugate

$$\begin{aligned}
 &\Phi : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^+, d\mu(\lambda)) \\
 &f(x) \mapsto F(\lambda) := \int_{\mathbb{R}-i0} \frac{f(x)}{S_b(Q - ix - i\lambda)S_b(Q - ix + i\lambda)} dx,
 \end{aligned}$$

where the contour goes below the poles.

Since $\Phi_\lambda(x)$ are the eigenfunctions satisfying $\mathbf{C}\Phi_\lambda(x) = (e^{2\pi b\lambda} + e^{-2\pi b\lambda})\Phi_\lambda(x)$, formally the integral transformation with $\Phi_\lambda(x)$ as kernel will intertwine the action. □

8.4 Left regular representation

Now, we apply the pairing to the left corepresentation (7.61).

Theorem 8.11 *The representation of $U_q(\mathfrak{gl}(2, \mathbb{R}))$ is equivalent to*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+}^{\oplus} \mathcal{P}_{\lambda, s_2/2} |S_b(Q + 2i\lambda)|^2 d\lambda ds_2 \otimes L^2(\mathbb{R}, dt_2),$$

where the representation space $\mathcal{P}_{\lambda, s_2/2} = L^2(\mathbb{R}, ds_1)$ is defined in Theorem 8.6.

Proof Recall that we have to multiply the expression by $\frac{G_b(Q+i\alpha)G_b(Q+i\tau)}{(1-q^2)^{ib^{-1}(\alpha+\tau)}}$ under the pairing. Also, since the resulting expression is analytic in α, τ on the lower half plane, by taking α, τ to $0, -ib$, we mean the analytic continuation from \mathbb{R} to the corresponding point, while respecting the contour of σ .

The pairing of K_0 and K is then given by $\tau = 0, \alpha = 0$. The factor

$$\frac{G_b(-i\sigma)G_b(-i\alpha + i\sigma)}{G_b(Q + i\tau)} G_b(Q + i\alpha)G_b(Q + i\tau)$$

becomes $\delta(\sigma)$ which means we also set $\sigma = 0$. Hence, we obtain

$$K_0 \cdot f(s_1, t_1)g(s_2, t_2) = f(s_1, t_1)g(s_2, t_2)e^{-\frac{\pi b}{2}s_2}, \tag{8.13}$$

$$K \cdot f(s_1, t_1)g(s_2, t_2) = f(s_1, t_1)g(s_2, t_2)e^{\pi bs_1}. \tag{8.14}$$

The pairing of E is given by $\tau = -ib$ and $\alpha = 0$, so that again $\sigma = 0$ and is given by

$$\begin{aligned} E \cdot f(s_1, t_1)g(s_2, t_2) &= -iq^{1/2} f(s_1 - ib, t_1)g(s_2, t_2) \frac{e^{2\pi bs_1 + \pi bs_2} e^{-\pi bs_1 - \frac{\pi b}{2}s_2}}{1 - q^2} \\ &= -iq^{1/2} f(s_1 - ib, t_1)g(s_2, t_2) \frac{e^{\pi bs_1 + \frac{\pi b}{2}s_2}}{1 - q^2}, \end{aligned}$$

or simply

$$E = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2}s_2} e^{\pi bs_1 + 2\pi bp_{s_1}}. \tag{8.15}$$

Finally, the pairing of F is given by $\tau = 0, \alpha = -ib$. By Corollary 3.11, we have

$$\begin{aligned} &\frac{G_b(-i\sigma)G_b(-i\alpha + i\sigma)}{G_b(Q + i\tau)} G_b(Q + i\alpha)G_b(Q + i\tau) \\ &= e^{-\pi i\sigma^2 - \pi Q\sigma} \frac{G_b(i\sigma - b)G_b(Q + b)}{G_b(Q + i\sigma)} \\ &= -q^2\delta(\sigma) - q^2\delta(\sigma + ib), \end{aligned}$$

so that we obtain

$$\begin{aligned}
 & F \cdot f(s_1, t_1)g(s_2, t_2) \\
 &= iq^{-1/2} \frac{-q^2}{1-q^2} e^{\pi bs_1 - \frac{\pi b}{2} s_2} \left(e^{2\pi b(t_1 - s_1)} f(s_1 + ib, t_1 + ib)g(s_2, t_2) \right. \\
 &\quad \left. + f(s_1 + ib, t_1)g(s_2, t_2) \frac{G_b\left(\frac{Q}{2} + is_1 - it_1\right) G_b\left(\frac{Q}{2} + is_1 + it_1\right)}{G_b\left(\frac{Q}{2} + is_1 - it_1 - b\right) G_b\left(\frac{Q}{2} + is_1 + it_1 - b\right)} \right).
 \end{aligned}$$

Using $\frac{G_b(ix)}{G_b(ix-b)} = (1 - e^{2\pi ib(ix-b)})$, we expand the expression:

$$\begin{aligned}
 F &= \frac{iq^{1/2} e^{-\frac{\pi b}{2} s_2}}{q - q^{-1}} (e^{-\pi b(s_1 - 2t_1)} e^{-2\pi b(p_{t_1} + p_{s_1})} \\
 &\quad + (1 + q^{-1} e^{-2\pi b(s_1 - t_1)})(1 + q^{-1} e^{2\pi b(-s_1 - t_1)}) e^{2\pi bs_1} e^{-2\pi bp_{s_1}}) \\
 &= \frac{ie^{-\frac{\pi b}{2} s_2}}{q - q^{-1}} (e^{-\pi b(2p_{s_1} - s_1)} + e^{-\pi b(2p_{s_1} + 3s_1)} \\
 &\quad + (e^{2\pi bt_1} + e^{-2\pi bt_1} + e^{-2\pi b(p_{t_1} - t_1)}) e^{-\pi b(2p_{s_1} + s_1)}),
 \end{aligned}$$

so that under multiplication by $e^{-\pi it_1^2}$ which changes $p_{t_1} \mapsto p_{t_1} + t_1$, we get

$$F = \frac{ie^{-\frac{\pi b}{2} s_2}}{q - q^{-1}} (e^{-\pi b(2p_{s_1} - s_1)} + e^{-\pi b(2p_{s_1} + 3s_1)} + \mathbf{C} e^{-\pi b(2p_{s_1} + s_1)}), \quad (8.16)$$

where \mathbf{C} depending only on t_1 is the operator defined in the previous section. Therefore, under the unitary transformation Φ defined previously, the representation is equivalent to

$$\begin{aligned}
 K_0 &= e^{\frac{\pi b}{2} s_2}, \\
 K &= e^{\pi bs_1}, \\
 E &= \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} e^{\pi bs_1 + 2\pi bp_{s_1}}, \\
 F &= \frac{ie^{-\frac{\pi b}{2} s_2}}{q - q^{-1}} (e^{-\pi b(2p_{s_1} - s_1)} + e^{-\pi b(2p_{s_1} + 3s_1)} + (e^{2\pi b\lambda} + e^{-2\pi b\lambda}) e^{-\pi b(2p_{s_1} + s_1)}),
 \end{aligned}$$

where the measure $d\mu(\lambda)$ is now $|S_b(Q + 2i\lambda)|^2 d\lambda$.

Motivated from the expression of the corepresentation, we multiply the space by $G_b(\frac{Q}{2} + is_1 - i\lambda)^{-1}$, and in effect changes the action of $e^{2\pi bp_{s_1}}$ to $(1 + qe^{-2\pi bs_1 + 2\pi b\lambda})e^{2\pi bp_{s_1}}$. The action of E and F then becomes:

$$E = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} (e^{\pi b s_1 + 2\pi b p_{s_1}} + e^{-\pi b s_1 + 2\pi b \lambda + 2\pi b p_{s_1}}),$$

$$F = \frac{i}{q - q^{-1}} e^{-\frac{\pi b}{2} s_2} (e^{\pi b s_1 - 2\pi b p_{s_1}} + e^{-\pi b s_1 - 2\pi b \lambda - 2\pi b p_{s_1}}).$$

Now, we make the following changes of variables (unitary transformations):

$$\begin{aligned} \circ s_1 \mapsto -s_1 : p_{s_1} &\mapsto -p_{s_1}, \\ \circ e^{\pi i s_1 \lambda} : p_{s_1} &\mapsto p_{s_1} - \frac{\lambda}{2}, \quad p_\lambda - \frac{s_1}{2}, \\ \circ e^{-\frac{\pi i}{2} s_1 s_2} : p_{s_1} &\mapsto p_{s_1} - \frac{s_2}{4}, \quad p_{s_2} \mapsto p_{s_2} - \frac{s_1}{4}. \end{aligned}$$

In other words, the transformation:

$$f(s_1, t_1)g(\lambda, t_2) \mapsto f(-s_1, t_1)g(\lambda, t_2)e^{\frac{\pi i}{2} s_1 (2\lambda - s_2)}.$$

Then, the representation (denoted by $\mathcal{P}_{\lambda, t_1}$) becomes:

$$\begin{aligned} K_0 &= e^{\frac{\pi b}{2} s_2}, \\ K &= e^{-\pi b s_1}, \\ E &= \frac{i}{q - q^{-1}} (e^{\pi b (-2p_{s_1} - s_1 + \lambda)} + e^{\pi b (-2p_{s_1} + s_1 - \lambda)}) \\ &= \frac{i}{q - q^{-1}} (q^{1/2} e^{\pi b (s_1 - \lambda)} + q^{-1/2} e^{-\pi b (s_1 - \lambda)}) e^{-2\pi b p_{s_1}} \\ &= \left[\frac{Q}{2b} - \frac{i}{b} (s_1 - \lambda) \right]_q e^{-2\pi b p_{s_1}}, \\ F &= \frac{i}{q - q^{-1}} (e^{\pi b (2p_{s_1} - s_1 - \lambda)} + e^{\pi b (2p_{s_1} + s_1 + \lambda)}) \\ &= \frac{i}{q - q^{-1}} (q^{-1/2} e^{\pi b (s_1 + \lambda)} + q^{1/2} e^{-\pi b (s_1 + \lambda)}) e^{2\pi b p_{s_1}} \\ &= \left[\frac{Q}{2b} + \frac{i}{b} (s_1 + \lambda) \right]_q e^{2\pi b p_{s_1}}. \end{aligned}$$

□

Remark 8.12 We note that there is a freedom of choice in multiplication by G_b in the above action. In fact, this precisely says that

$$\mathcal{P}_{\lambda, t} \simeq \mathcal{P}_{-\lambda, t} \tag{8.17}$$

$$f(x) \mapsto \frac{G_b \left(\frac{Q}{2} + i\lambda - ix \right)}{G_b \left(\frac{Q}{2} - i\lambda - ix \right)} f(x) \tag{8.18}$$

as observed in [32].

Since the pairing is non-degenerate, we obtain the following important realization of the multiplicative unitary \mathbf{W} :

Corollary 8.13 *As a corepresentation on $L^2(\mathbb{R})$, \mathbf{W} is equivalent to*

$$\mathbf{W} = \int_{\mathbb{R}} \int_{\mathbb{R}_+}^{\oplus} T^{\lambda, t/2} d\mu(\lambda) dt, \quad (8.19)$$

where $d\mu(\lambda) = |S_b(Q + 2i\lambda)|^2 d\lambda$.

This is a generalization of Podleś-Woronowicz's definition [30] of a multiplicative unitary for a compact group

$$u = \sum_{\alpha \in \widehat{G}}^{\oplus} u^\alpha, \quad (8.20)$$

where \widehat{G} is the set of all (finite dimensional) irreducible unitary representation of G , and

$$u^\alpha : V_\alpha \mapsto V_\alpha \otimes G$$

is the irreducible corepresentation on some finite dimensional vector space V_α . In particular, this defines the multiplicative unitary in a coordinate-free canonical way.

Furthermore, by applying the transformations that were done in the proof of Theorem 8.11 to the expression (7.61), and comparing the result with the expression of $T^{\lambda, t/2}$ given in (7.31), we recover the “6-9 relation” of G_b that is first observed by Volkov [44]. See Sect. 9 for further details.

8.5 Right regular representation

Similarly, by applying the pairing to the right regular corepresentation, we obtain

Theorem 8.14 *The right regular corepresentation is equivalent to*

$$L^2(\mathbb{R}, ds_1) \otimes \int_{\mathbb{R}} \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_{\lambda, -s_2/2} |S_b(Q + 2i\lambda)|^2 d\lambda ds_2.$$

Proof We apply the pairing to (7.73) and get

$$\begin{aligned} K_0 &= e^{-\pi b \frac{s_2^2}{2}}, \\ K &= e^{\pi b t_2}, \\ E &= \frac{(-iq^{1/2})(-q^2)e^{-\pi i Qb}}{1 - q^2} e^{\frac{\pi b}{2} s_2} e^{-\pi b t_2} \left(e^{\pi b(2t_1 + 2t_2)} e^{2\pi b(-p_{t_1} + p_{t_2})} \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{G_b\left(\frac{Q}{2} - it_1 - it_2\right) G_b\left(\frac{Q}{2} + it_1 - it_2\right)}{G_b\left(\frac{Q}{2} - it_1 - it_2 - b\right) G_b\left(\frac{Q}{2} + it_1 - it_2 - b\right)} e^{2\pi b p_{t_2}} \\
 & = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} e^{-\pi b t_2} \left(e^{\pi b(-2p_{t_1} + 2t_1 + 2p_{t_2} + t_2)} \right. \\
 & \quad \left. + q^{1/2} (1 + q^{-1} e^{2\pi b(t_1 + t_2)}) (1 + q^{-1} e^{2\pi b(-t_1 + t_2)}) e^{2\pi b p_{t_2}} \right) \\
 & = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} \left(e^{\pi b(2p_{t_2} - t_2)} + e^{\pi b(2p_{t_2} + 3t_2)} \right. \\
 & \quad \left. + \left(e^{2\pi b t_1} + e^{-2\pi b t_1} + e^{\pi b(-2p_{t_1} + 2t_1)} \right) e^{\pi b(2p_{t_2} + t_2)} \right), \\
 F & = \frac{i q^{-1/2} e^{\pi i Q b}}{1 - q^2} e^{\frac{\pi b}{2} s_2 + \pi b t_2} e^{-2\pi b t_2 - \pi b s_2} e^{-2\pi b p_{t_2}} \\
 & = \frac{i}{q - q^{-1}} e^{-\frac{\pi b}{2} s_2} e^{-\pi b(t_2 + 2p_{t_2})}.
 \end{aligned}$$

Again by shifting $p_{t_1} \mapsto p_{t_1} + t_1$ as in the left representation, the action of E becomes

$$E = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} (e^{\pi b(2p_{t_2} - t_2)} + e^{\pi b(2p_{t_2} + 3t_2)} + C e^{\pi b(2p_{t_2} - t_2)}).$$

Hence, under the unitary transformation Φ , E becomes

$$E = \frac{i}{q - q^{-1}} e^{\frac{\pi b}{2} s_2} (e^{\pi b(2p_{t_2} - t_2)} + e^{\pi b(2p_{t_2} + 3t_2)} + (e^{2\pi b \lambda} + e^{-2\pi b \lambda}) e^{\pi b(2p_{t_2} - t_2)}).$$

Finally, the remaining transformation used in the left regular representation does not change these actions. In exactly the same way if we apply:

$$\begin{aligned}
 & \text{multiplication by } G_b\left(\frac{Q}{2} + i\lambda - it_2\right)^{-1}, \\
 & \text{ot}_2 \mapsto -t_2 : t_2 \mapsto -t_2, p_{t_2} \mapsto -p_{t_2}, \\
 & \text{oe}^{-\pi i t_2 \lambda} : p_\lambda \mapsto p_\lambda - \frac{t_2}{2}, p_{t_2} \mapsto p_{t_2} - \frac{\lambda}{2}, \\
 & \text{oe}^{-\frac{\pi i}{2} s_2 t_2} : p_{s_2} \mapsto p_{s_2} + \frac{t_2}{4}, p_{t_2} \mapsto p_{t_2} + \frac{s_2}{4},
 \end{aligned}$$

that does not change the left representation either, it becomes:

$$\begin{aligned}
 K_0 & = e^{-\pi b \frac{s_2^2}{2}}, \\
 K & = e^{-\pi b t_2}, \\
 E & = \frac{i}{q - q^{-1}} (q^{\frac{1}{2}} e^{\pi b(t_2 - \lambda)} + q^{-\frac{1}{2}} e^{-\pi b(t_2 - \lambda)}) e^{-2\pi b p_{t_2}}
 \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{Q}{2b} - \frac{i}{b}(t_2 - \lambda) \right]_q e^{-2\pi b p_{t_2}}, \\
F &= \frac{i}{q - q^{-1}} (q^{-\frac{1}{2}} e^{\pi b(t_2 + \lambda)} + q^{\frac{1}{2}} e^{-\pi b(t_2 + \lambda)}) e^{2\pi b p_{t_2}} \\
&= \left[\frac{Q}{2b} + \frac{i}{b}(t_2 + \lambda) \right]_q e^{2\pi b p_{t_2}}
\end{aligned}$$

which is precisely the representation denoted by $\mathcal{P}_{\lambda, -s_2/2}$. \square

Combining Theorem 8.11 and 8.14, and note that all the transformations used are unitary, we conclude the main theorem:

Theorem 8.15 *As a representation of $U_q(\mathfrak{gl}(2, \mathbb{R}))_L \otimes U_q(\mathfrak{gl}(2, \mathbb{R}))_R$,*

$$L^2(GL_q^+(2, \mathbb{R})) \simeq \int_{\mathbb{R}} \int_{\mathbb{R}^+}^{\oplus \oplus} \mathcal{P}_{\lambda, s} \otimes \mathcal{P}_{\lambda, -s} |S_b(Q + 2i\lambda)|^2 d\lambda ds \quad (8.21)$$

and the equivalence is unitary.

In particular, we proved Ponsot–Teschner’s claim in [31] for $U_q(\mathfrak{sl}(2, \mathbb{R}))$ by setting $s_2 = 0$ which corresponds to the determinant element $N = A\tilde{A}$, as well as the action of the central element K_0 .

8.6 The modular double

It is known that the representation space $\mathcal{P}_{\lambda, t}$ is actually a nontrivial representation for the modular double $U_q(\mathfrak{gl}(2, \mathbb{R})) \otimes U_{\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$. In this section, we describe its action using the same multiplicative unitary \mathbf{W}_m .

The modular double counterpart $U_{\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ is defined in the same way with q replaced by \tilde{q} throughout. In particular, we can define the pairing between $\tilde{E}, \tilde{F}, \tilde{K}, \tilde{K}_0$ with $\tilde{A}, \tilde{B}, \tilde{\tilde{A}}, \tilde{\tilde{B}}$ as before. Recall that $\tilde{A} = A^{1/b^2}$ and similarly for the other quantum plane variables. Hence, by the $b \longleftrightarrow b^{-1}$ duality of \mathbf{W}_m , we can pair it with $U_{\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ and obtain:

Proposition 8.16 *The left regular action of $U_{\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ on $L^2(GL_q^+(2, \mathbb{R}))$ is given by replacing b with b^{-1} , i.e.,*

$$\begin{aligned}
\tilde{K}_0 &= e^{\frac{\pi b^{-1}}{2} s_2}, \\
\tilde{K} &= e^{-\pi b^{-1} s_1}, \\
\tilde{E} &= \frac{i}{\tilde{q} - \tilde{q}^{-1}} (\tilde{q}^{1/2} e^{\pi b^{-1}(s_1 - \lambda)} + \tilde{q}^{-1/2} e^{-\pi b^{-1}(s_1 - \lambda)}) e^{-2\pi b^{-1} p_{s_1}}, \\
\tilde{F} &= \frac{i}{\tilde{q} - \tilde{q}^{-1}} (\tilde{q}^{-1/2} e^{\pi b^{-1}(s_1 + \lambda)} + \tilde{q}^{1/2} e^{-\pi b^{-1}(s_1 + \lambda)}) e^{2\pi b^{-1} p_{s_1}}.
\end{aligned}$$

We note that \tilde{E} and \tilde{F} are positive operators only when $\frac{1}{2n+1} < b^2 < \frac{1}{2n}$ for $n \in \mathbb{N}$, and they are not necessarily essentially self-adjoint on the natural domain \mathcal{W} [36]. To obtain essential self-adjointness, we have to apply Proposition 4.1 and define the operators on the respective transformed domain $g_b \cdot \mathcal{W}$.

Ignoring the factors, we note from Theorem 8.11 that the summand of E and F are q -commuting; hence, we immediately obtain using Corollary 3.14 (see also [2, Cor 1]).

Proposition 8.17 *As positive self-adjoint operators, we have*

$$\begin{aligned} K_0^{1/b^2} &= \tilde{K}_0, \\ K^{1/b^2} &= \tilde{K}, \\ 2 \sin(\pi b^{-2}) \tilde{E} &= (2 \sin(\pi b^2) E)^{1/b^2}, \\ 2 \sin(\pi b^{-2}) \tilde{F} &= (2 \sin(\pi b^2) F)^{1/b^2}. \end{aligned}$$

Similar analysis works as well for the right regular representation; hence, Theorem 8.15 is actually an equivalence as a representation of the modular double

$$U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))_L \otimes U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))_R,$$

where $U_{q\tilde{q}}(\mathfrak{gl}(2, \mathbb{R})) := U_q(\mathfrak{gl}(2, \mathbb{R})) \otimes U_{\tilde{q}}(\mathfrak{gl}(2, \mathbb{R}))$ denotes the modular double.

9 Representation theoretic meaning for certain integral transforms

In this section, we state without proof certain integral transformations of G_b that arise in the calculations of certain representation relations.

Proposition 9.1 *The pentagon equation $W_{23}W_{12} = W_{12}W_{13}W_{23}$ is equivalent to the 4–5 relation (Lemma 3.10)*

$$\frac{G_b(\alpha)G_b(\beta)G_b(\gamma)}{G_b(\alpha + \gamma)G_b(\beta + \gamma)} = \int_C d\tau e^{-2\pi\gamma\tau} \frac{G_b(\alpha + i\tau)G_b(\beta + i\tau)}{G_b(\alpha + \beta + \gamma + i\tau)G_b(Q + i\tau)}. \tag{9.1}$$

By a change of variables and using the reflection formula, it can be rewritten as

$$\begin{aligned} &\frac{G_b(\alpha + \gamma)G_b(\beta + \gamma)G_b(\alpha)G_b(\beta)}{G_b(\alpha + \beta + \gamma)} \\ &= \int_C d\tau e^{-2\pi i(\beta+i\tau)(\alpha+i\tau)} G_b(\alpha + i\tau)G_b(\beta + i\tau)G_b(\gamma - i\tau)G_b(-i\tau). \end{aligned} \tag{9.2}$$

Scaling all the variables by b and taking the limit $b \rightarrow 0$ by applying Theorem 3.5, we recover precisely Barnes' first lemma:

$$\frac{\Gamma(a+c)\Gamma(b+c)\Gamma(a)\Gamma(b)}{\Gamma(a+b+c)} = \frac{1}{2\pi} \int_C \Gamma(a+i\tau)\Gamma(b+i\tau)\Gamma(c-i\tau)\Gamma(-i\tau) d\tau \tag{9.3}$$

in the special case for $d = 0$.

Next, as noted in Sect. 8.4, by comparing $\mathbf{W}_m = \iint^{\oplus} T^{\lambda,t/2} d\tau d\mu(\lambda)$ as corepresentation, we obtain the following relation that is first observed by Volkov [44]:

Proposition 9.2 *The 6–9 relation for $G_b(x)$ can be written as*

$$\begin{aligned} & \int_C e^{-2\pi\tau(\delta-i\tau)} \frac{G_b(\alpha+i\tau)G_b(\beta+i\tau)G_b(\gamma+i\tau)G_b(\delta-i\tau)G_b(-i\tau)}{G_b(\alpha+\beta+\gamma+\delta+i\tau)} d\tau \\ &= \frac{G_b(\alpha)G_b(\beta)G_b(\gamma)G_b(\alpha+\delta)G_b(\beta+\delta)G_b(\gamma+\delta)}{G_b(\alpha+\beta+\delta)G_b(\alpha+\gamma+\delta)G_b(\beta+\gamma+\delta)}, \end{aligned} \tag{9.4}$$

where the contour goes along \mathbb{R} and separates the increasing and decreasing sequence of poles. By the asymptotic properties of G_b , the integral converges for any choice of parameters.

Again by scaling all the variables by b and applying Theorem 3.5, we recover Barnes' second lemma:

$$\begin{aligned} & \int_C \frac{\Gamma(a+i\tau)\Gamma(b+i\tau)\Gamma(c+i\tau)\Gamma(d-i\tau)\Gamma(-i\tau)}{\Gamma(a+b+c+d+i\tau)} d\tau \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+d)\Gamma(a+c+d)\Gamma(b+c+d)}, \end{aligned} \tag{9.5}$$

which in turn is a generalization of Pfaff–Saalschütz's sum

$$\sum_{j \geq 0} \frac{(n+m+l+k-j)!}{(m-j)!(l-j)!(k-j)!(n+j)!j!} = \frac{(n+m+l)!(n+m+k)!(n+l+k)!}{m!l!(n+m)!(n+l)!(n+k)!}. \tag{9.6}$$

Finally, in [46], an alternative description of the multiplicative unitary lying instead in $\mathcal{A} \otimes \mathcal{A}$ is defined to be (slightly modified to fit our definition):

$$V = g_b(B^{-1} \otimes q^{-1}BA^{-1})e^{\frac{i}{2\pi b^2} \log(qAB^{-1}) \otimes \log A^{-1}}, \tag{9.7}$$

so that

$$V(x \otimes 1)V^* = \Delta(x). \quad (9.8)$$

On comparing

$$W^*(1 \otimes x)W = \Delta(x) = V(x \otimes 1)V^* \quad (9.9)$$

as operators on $\mathcal{H} \otimes \mathcal{H}$, we obtain a new relation involving $G_b(x)$, which is essentially the same as the relation in [42, Theorem 5.6.7] in the case $n = 1$:

Proposition 9.3 *The 3–2 relation is given by*

$$\int_C G_b(\alpha + i\tau)G_b(\beta - i\tau)G_b(\gamma - i\tau)e^{-2\pi i(\beta - i\tau)(\gamma - i\tau)}d\tau = G_b(\alpha + \gamma)G_b(\alpha + \beta), \quad (9.10)$$

where the contour goes along \mathbb{R} and separates the poles for $i\tau$ and $-i\tau$. By the asymptotic properties for G_b , the integral converges for $\operatorname{Re}(\alpha - \beta - \gamma) < \frac{Q}{2}$.

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