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THE CLASSICAL LIMIT OF REPRESENTATION THEORY OF THE QUANTUM PLANE

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We showed that there is a complete analogue of a representation of the quantum plane $\mathcal{B}_q$ where $|q| = 1$, with the classical $ax + b$ group. We showed that the Fourier transform of the representation of $\mathcal{B}_q$ on $\mathcal{H} = L^2(\mathbb{R})$ has a limit (in the dual corepresentation) toward the Mellin transform of the unitary representation of the $ax + b$ group, and furthermore the intertwiners of the tensor products representation has a limit toward the intertwiners of the Mellin transform of the classical $ax + b$ representation. We also wrote explicitly the multiplicative unitary defining the quantum $ax + b$ semigroup and showed that it defines the corepresentation that is dual to the representation of $\mathcal{B}_q$ above, and also correspond precisely to the classical family of unitary representation of the $ax + b$ group.

Keywords: Classical limit; quantum plane; affine transformations; quantum group; quantum Teichmüller space.

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1. Introduction

The $ax + b$ group is the group of affine transformations on the real line $\mathbb{R}$. Together with the three-dimensional Heisenberg group they can be viewed as the simplest examples of non-abelian non-compact Lie group. Various difficulties in studying higher-dimensional non-compact Lie group are reflected in these simple examples. For example, in the $ax + b$ group, the unitary irreducible representations are now infinite dimensional, and the Mellin transform is used to "diagonalize" the representation. The matrix coefficients in this case are realized as integral transformations, which can be viewed as the matrix elements with respect to a continuous basis of the representation space. These matrix elements are expressed in terms of the gamma function $\Gamma(x)$. We will see that in the quantum picture, its q-analogue, the q-gamma function $\Gamma_q(x)$, is closely related to the important quantum dilogarithm function $G_b(x)$. Furthermore, to deal with non-compactness, there is a need to introduce the language of multiplier C^* algebra to define a natural coproduct on the algebra of continuous functions vanishing at infinity, and also to construct the non-compact Haar measure [26]. Motivating from this, in the quantum picture we must deal with unbounded operators, and the theory of functional calculus for self-adjoint operators will be the main technical tool.

The quantum plane $\mathcal{B}_q$ is the Hopf *-algebra over $\mathbb{C}$ with *self-adjoint* generators A, B satisfying

$$AB = q^2 BA, \tag{1.1}$$

with the coproduct given by

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes A + 1 \otimes B. \tag{1.2}$$

It is known that this object is self-dual, so that they can be considered both as the quantum counterpart of $C(G)$, a certain algebra of functions on G, the "$ax + b$" group, or $U(\mathfrak{g})$, the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Classically for a Lie group G, $U(\mathfrak{g})$ and $C(G)$ are paired by treating $U(\mathfrak{g})$ as left invariant differential operators on G and evaluate the result at the identity. In such a way, representation of $U(\mathfrak{g})$ on a vector space $\mathcal{H}$ corresponds to corepresentation of the group algebra $C(G)$ on $\mathcal{H}$ by this pairing. Therefore in order to study the quantum counterpart of these representations, naturally we would like to study the representation of the quantum plane $\mathcal{B}_q$, and the corepresentation of its dual object, called $\mathcal{A}_q$ in this paper, under a natural pairing.

Recently in [6], Frenkel and Kim derived the quantum Teichmüller space, previously constructed by Kashaev [13] and by Fock and Chekhov [2], from tensor products of a single canonical representation of the modular double of the quantum plane $\mathcal{B}_q$. The representation is realized as positive unbounded self-adjoint operators acting on $\mathcal{H} = L^2(\mathbb{R})$, and the main ingredient in their construction of the quantum Teichmüller space is the decomposition of the tensor product of two $\mathcal{B}_q$-representations into a direct integral parametrized by a "multiplicity" module $M \simeq L^2(\mathbb{R})$, namely:

$$\mathcal{H} \otimes \mathcal{H} \simeq M \otimes \mathcal{H}. \tag{1.3}$$

The intertwiner of this decomposition is given by a certain kind of "quantum dilogarithm transform" (cf. Proposition 4.2), where the remarkable quantum dilogarithm function has been introduced by Faddeev and Kashaev [4].

On the other hand, in order to define a corepresentation on the dual object $\mathcal{A}_q$ with positive generators, the space of "continuous functions vanishing at infinity" for the quantum plane $C_\infty(\mathcal{A}_q)$ based on the functional calculus of self-adjoint operators is introduced. This coincides with Woronowicz's construction of the quantum "$ax+b$" group [29] using the theory of multiplicative unitaries, however restricted to the semigroup setting where we consider $B > 0$, so that we do not run into the difficulty of the self-adjointness of the coproduct. The multiplicative unitary involved produces the corepresentation of the quantum plane desired, and the corepresentation obtained in this way is shown to have certain classical limit toward the unitary representation for the classical group. Furthermore, a pairing between the dual space corresponds precisely to the canonical representation of $\mathcal{B}_q$ by unbounded self-adjoint operators defined in [6] mentioned above.

In the quantum torus setting, where generators of $\mathcal{B}_q$ are represented by unitary operators, the representation of $\mathcal{B}_q$ on $\mathcal{H} = L^2(\mathbb{R})$ only becomes algebraically irreducible when we consider also its modular double $\mathcal{B}_{q\tilde{q}} := \mathcal{B}_q \otimes \mathcal{B}_{\tilde{q}}$, so that it generates a von Neumann algebra of Type I factor, while representation of $\mathcal{B}_q$ itself generates Type II_1 factor which is more exotic [3]. Now taking the real structure into account, the modular double of the quantum plane also naturally arises in this setting, and what we are considering in this paper should be viewed as restriction of the representation on $\mathcal{H}$ to $\mathcal{B}_q \subset \mathcal{B}_{q\tilde{q}}$, especially useful in studying the classical limit. On the other hand, in the dual picture, quite interestingly the modular double elements are also involved in the definition of $C_\infty(\mathcal{A}_q)$ due to the analytic properties of the Mellin transform, see Remark 6.3.

The quantum dilogarithm function played a prominent role in this quantum theory. This function and its many variants are being studied [8, 21, 28] and applied to vast amount of different areas, for example the construction of the "$ax + b$" quantum group by Woronowicz *et al.* [19, 29], the harmonic analysis of the non-compact quantum group $U_q(\mathfrak{sl}(2,\mathbb{R}))$ and its modular double [1, 16, 17], the q-deformed Toda chains [14] and hyperbolic knot invariants [12]. One of the important properties of this function is its invariance under the duality $b \leftrightarrow b^{-1}$ that provides the basis for the definition of the modular double of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ first introduced by Faddeev [3], and also related, for example, to the self-duality of Liouville theory [16] that has no classical counterpart.

It is an interesting problem to find a classical limit to these quantum theories described by the quantum dilogarithm function. Due to the duality between $b \leftrightarrow b^{-1}$ and the appearance of the term $Q = b + b^{-1}$, there is no classical limit by directly taking $b \to 0$. In this paper, by utilizing the properties of the quantum dilogarithm function $G_b(x)$, we showed that under a suitable rescaling of parameters and a limiting process that takes $q \to 1$ from inside the unit circle in the complex plane, it is possible to obtain the classical gamma function. More precisely, by taking b away from the real axis, Theorem 3.11 states that the following limit holds for $b^2 = ir \to i0^+$:

$$\lim_{r\to 0^+} \frac{(2\pi b)G_b(bx)}{(-2\pi i b^2)^x} = \Gamma(x), \tag{1.4}$$

where $(-2\pi i b^2) > 0$, hence the denominator is well-defined. This gives another proof of a similar limit first observed in [20].

In this way, most properties of this special function reduce to its classical analogues. For example, the q-binomial theorem (Lemma 3.7) derived in [1]:

$$(u+v)^{it} = b\int_C d\tau \binom{t}{\tau}_b u^{i(t-\tau)}v^{i\tau} \tag{1.5}$$

is actually the q-analogue of the classical formula

$$(x+y)^{it} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(-is)\Gamma(-it+is)}{\Gamma(-it)} x^{is}y^{it-is}ds, \tag{1.6}$$

see Remark 3.9. In particular, the main results of this paper state that the intertwiners of the tensor product decomposition $\mathcal{H}\otimes\mathcal{H} \simeq \mathcal{M}\otimes\mathcal{H}$ of the representation of $\mathcal{B}_q$ given by [6] has a nice classical analogue, namely the intertwiners of the classical "$ax+b$" group representation under suitable transformation (Theorem 5.2):

$$b^2\mathcal{F}\begin{bmatrix} b\lambda & bt \\ bt_1 & bt_2 \end{bmatrix}_* \to \begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix}_{\text{classical}}, \tag{1.7}$$

$$b^2\mathcal{F}\begin{bmatrix} b\lambda & bt \\ bt_1 & bt_2 \end{bmatrix}_* \to \begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix}_{\text{classical}} \tag{1.8}$$

as $b^2 = ir \to i0^+$. Furthermore, the corepresentation constructed using the multiplicative unitary also has a classical limit toward the unitary representation R_+ of the classical $ax+b$ group (Theorem 6.13).

The study of the relationship between the quantum plane and the classical $ax+b$ group is important as it serves as building blocks toward higher quantum group. First of all, we choose to work with quantum semigroup (representing the generators by *positive* operators) since it induces the $b \leftrightarrow b^{-1}$ duality for $SL_q^+(2,\mathbb{R})$ as explained in [16], and it also provides an important results on the closure of tensor product of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ representations [17]. These observations are essential to the relationship between quantum Liouville theory and quantum geometry on Riemann surface [24]. Moreover, it is fundamental in the construction of $GL_q^+(2,\mathbb{R})$ by the Drinfeld's double construction proposed in [9, 18], an analogue of the classical Gauss decomposition, which provides an important first step leading to the research program of harmonic analysis and positive representations of split real quantum groups in the case $|q|=1$ proposed in [5, 10].

The present paper is organized as follows. In Sec. 2, we recall the definitions and facts about the classical "$ax+b$" group and its representations, and derive the tensor product decomposition of two irreducible representations. In Sec. 3, we recall some properties of the q-special functions, in particular a version of the quantum dilogarithm $G_b(x)$ introduced in [17], and derive a special limiting procedure that enables us to compare it with the classical gamma function. In Sec. 4, we recall the q-intertwiner for the representation of the quantum plane $\mathcal{B}_q$ that is obtained in [6] to deal with the quantization of Teichmüller space, and we showed in Sec. 5 that this intertwiner, under suitable modification, has a classical limit toward precisely the intertwiner of the $ax+b$ group. Finally, in Sec. 6, we introduce on the dual space $\mathcal{A}_q$ the space of continuous functions vanishing at infinity $C_\infty(\mathcal{A}_q)$, and starting from Woronowicz's multiplicative unitary of the quantum "$ax+b$" semigroup, we derive explicitly the corepresentation of the dual space $\mathcal{A}_q$. We showed that this corepresentation has a limit toward the classical $ax+b$ group representation, and on the other hand, it induces the same representation of $\mathcal{B}_q$ under a non-degenerate pairing.

2. Classical $ax + b$ Group

2.1. *Representation*

First let us recall the theory of representation of the $ax + b$ group. The classical $ax+b$ group is by definition, the group of affine transformations on the real line $\mathbb{R}$, where $a > 0$ and $b \in \mathbb{R}$, and they can be represented by a matrix of the form

$$g(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

with multiplication given by

$$g(a_1,b_1)g(a_2,b_2) = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

We will also consider the representation of the transpose group

$$g(a,c) = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}, \tag{2.3}$$

where the multiplication is given by

$$g(a_1,c_1)g(a_2,c_2) = \begin{pmatrix} a_1a_2 & 0 \\ c_1a_2 + c_2 & 1 \end{pmatrix}. \tag{2.4}$$

This corresponds to the coproduct of the quantum plane $\mathcal{B}_q$ introduced later on (cf. Sec. 4).

Theorem 2.1 (Gelfand [27, Ch.V.1]). *Every irreducible unitary representation of the $ax + b$ group is equivalent to one of the following (acting on the left):*

- $R_+ := R_{-i}$ *or* $R_- := R_i$ *where* R_λ *denote the representation of the* $ax + b$ *group on* $L^2(\mathbb{R}_+, \frac{dx}{x})$ *by*

$$R_\lambda(g) \cdot f(x) = e^{\lambda bx} f(ax); \tag{2.5}$$

- T_ρ, *the representation on* $\mathbb{C}$ *by multiplication by* $a^{i\rho}$.

Similarly, the left action of the transpose group is given by the action of the inverse element

$$g^{-1} = \begin{pmatrix} a^{-1} & -\frac{c}{a} \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

$$R_\lambda(g^T) \cdot f(x) = e^{-\lambda cx/a} f(a^{-1}x) = R_\lambda(g^{-1}) \cdot f(x). \tag{2.7}$$

Let us recall the method of Mellin transform, which gives us an explicit expression of the matrix coefficients in terms of the gamma function:

Theorem 2.2. *Let $f(x)$ be a continuous function on the half line $0 < x < \infty$. Then its Mellin transform is defined by*

$$\phi(s) := (\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)dx, \tag{2.8}$$

whenever the integration is absolutely convergent for $a < \mathrm{Re}(s) < b$. *By the Mellin inversion theorem,* $f(x)$ *is recovered from* $\phi(s)$ *by*

$$f(x) := (\mathcal{M}^{-1}\phi)(x) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\phi(s)ds, \tag{2.9}$$

where $c \in \mathbb{R}$ *is any value in between* a *and* b.

Here we also list some analytic properties for the Mellin transform. For further details see [15].

Proposition 2.3 (Strip of analyticity). *If* $f(x)$ *is a locally integrable function on* $(0, \infty)$ *such that it has decay property*:

$$f(x) = \begin{cases} O(x^{-a-\epsilon}) & x \to 0^+, \\ O(x^{-b+\epsilon}) & x \to +\infty, \end{cases} \tag{2.10}$$

for every $\epsilon > 0$ *and some* $a < b$, *then the Mellin transform defines an analytic function* $(\mathcal{M}f)(s)$ *in the strip*

$$a < \mathrm{Re}(s) < b.$$

(*Analytic continuation*) *Assume* $f(x)$ *behaves algebraically for* $x \to 0^+$, *i.e.*

$$f(x) \sim \sum_{k=0}^{\infty} A_k x^{a_k}, \tag{2.11}$$

where $\mathrm{Re}(a_k)$ *increases monotonically to* ∞ *as* $k \to \infty$. *Then the Mellin transform* $(\mathcal{M}f)(s)$ *can be analytically continued into* $\mathrm{Re}(s) \le a = -\mathrm{Re}(a_0)$ *as a meromorphic function with simple poles at the points* $s = -a_k$ *with residue* A_k.

A similar analytic property holds for the continuation to the right half plane.

(*Growth*) *Let* $f(x)$ *be a holomorphic function of the complex variable* x *in the sector* $-\alpha < \arg x < \beta$ *where* $0 < \alpha, \beta \le \pi$, *and satisfies the growth property* (2.10) *uniformly in any sector interior to the above sector.*

Then $(\mathcal{M}f)(s)$ *has exponential decay in* $a < \mathrm{Re}(s) < b$ *with*

$$(\mathcal{M}f)(s) = \begin{cases} O(e^{-(\beta-\epsilon)t}) & t \to +\infty, \\ O(e^{(\alpha-\epsilon)t}) & t \to -\infty, \end{cases} \tag{2.12}$$

for any $\epsilon > 0$ *uniformly in any strip interior to* $a < \mathrm{Re}(s) < b$.

(*Parseval's formula*) *We have*

$$\int_0^{\infty} f(x)g(x)x^{z-1}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}f(s)\mathcal{M}g(z-s)ds, \tag{2.13}$$

where $\mathrm{Re}(s) = c$ *lies in the common strip for* $\mathcal{M}f$ *and* $\mathcal{M}g$. *In particular the map*

$$f(x) \mapsto F(t) := (\mathcal{M}f)(it) = \int_0^{\infty} x^{it} f(x)\frac{dx}{x} \tag{2.14}$$

gives a unitary transformation between $L^2(\mathbb{R}_+, \frac{dx}{x})$ *and* $L^2(\mathbb{R}, dx)$:

$$\int_0^\infty |f(x)|^2 \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^\infty |F(t)|^2 dt. \tag{2.15}$$

This allows us to study the representation $R_\lambda(g)$ *in the space* $L^2(\mathbb{R})$ *instead, as Proposition* 2.6 *below shows. By abuse of notation, we will also denote this unitary transformation by* $\mathcal{M}$.

Throughout the paper, we will restrict to a special class of functions that is dense in $L^2(\mathbb{R})$.

Definition 2.4. Let $\mathcal{W}$ denote the finite $\mathbb{C}$-linear combinations of functions of the form

$$e^{-Ax^2+Bx}P(x), \tag{2.16}$$

where $P(x)$ is a polynomial in x, $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{C}$.

Proposition 2.5. *We have the following properties for* $\mathcal{W}$:

(1) *Every function* $f(z) \in \mathcal{W}$ *is entire analytic in* z, *and* $F_y(x) := f(x+iy)$ *is of rapid decay in* x.
(2) *The space* $\mathcal{W}$ *is closed under Fourier transform.*
(3) $\mathcal{W}$ *is dense in* $L^2(\mathbb{R})$.
(4) $\mathcal{W}$ *is a core for the unbounded operator* $e^{\alpha x}$ *and* $e^{\beta p}$ *on* $L^2(\mathbb{R})$ *where* $\alpha, \beta \in \mathbb{R}$ *and* $p = \frac{1}{2\pi i}\frac{d}{dx}$ [22, *Lemma* 7.2].

Under the Mellin transform, the representations R_λ can be expressed by the following integral operator:

Proposition 2.6 ([27]). *The action of the* $ax+b$ *group on* $\mathcal{W} \subset L^2(\mathbb{R})$ *is given by*

$$R_\lambda(g) \cdot F(w) = \int_{\mathbb{R}+i0} K(w,z;g)F(z)dz, \tag{2.17}$$

where the integral kernel is given by

$$K(w,z;g) = \frac{\Gamma(iw-iz)a^{-iw}}{2\pi}\left(-\frac{\lambda b}{a}\right)^{iz-iw}. \tag{2.18}$$

Similarly, the left action of the transposed group will be given by

$$R_\lambda(g^T) \cdot F(w) = \int_{\mathbb{R}+i0} K(w,z;g)F(z)dz, \tag{2.19}$$

where the integral kernel is given by

$$K(w,z;g) = \frac{\Gamma(iw-iz)a^{iw}}{2\pi}(\lambda b)^{iz-iw}. \tag{2.20}$$

Here the branch of the factor is chosen so that $|\arg(-\lambda b)| < \pi$ *and the contour of integration goes above the pole at* $z = w$.

2.2. *Tensor product decomposition*

Using the above expressions, we can construct explicit intertwiners for the tensor product decomposition of the irreducible representation R_+, R_- and T_ρ:

Theorem 2.7. *Recall that $R_\pm \simeq L^2(\mathbb{R}_+, \frac{dx}{x})$ and $T_\rho \simeq \mathbb{C}$ as Hilbert spaces.*

(a) *We have*

$$\psi : R_\pm \otimes R_\pm \simeq L^2\left(\mathbb{R}^+, \frac{d\alpha}{\alpha}\right) \otimes R_\pm, \tag{2.21}$$

$$f(x_1, x_2) \mapsto F(\alpha, x), \tag{2.22}$$

where the unitary equivalence is given by

$$F(\alpha, x) := f\left(\frac{\alpha x}{\alpha+1}, \frac{x}{\alpha+1}\right), \tag{2.23}$$

$$f(x_1, x_2) := F\left(\frac{x_1}{x_2}, x_1 + x_2\right). \tag{2.24}$$

(*This formula also holds for $R_\lambda \otimes R_\lambda$ for all $\lambda \in \mathbb{C}$.*)

(b) *We have*

$$\psi : R_\pm \otimes R_\mp \simeq \left(L^2\left(\mathbb{R}_{<1}, \frac{d\alpha}{\alpha}\right) \otimes R_\mp\right) \oplus \left(L^2\left(\mathbb{R}_{>1}, \frac{d\alpha}{\alpha}\right) \otimes R_\pm\right), \tag{2.25}$$

$$f(x_1, x_2) \mapsto F(\alpha, x), \tag{2.26}$$

where the unitary equivalence is given by

$$F(\alpha, x) := f\left(\frac{\alpha x}{|\alpha-1|}, \frac{x}{|\alpha-1|}\right), \tag{2.27}$$

$$f(x_1, x_2) := F\left(\frac{x_1}{x_2}, |x_1 - x_2|\right). \tag{2.28}$$

(c) *We have*

$$\psi : R_\pm \otimes T_\rho \simeq R_\pm, \tag{2.29}$$

$$f(x) \mapsto F(w), \tag{2.30}$$

where the unitary equivalence is given by

$$F(w) := f(w - \rho), \tag{2.31}$$

$$f(x) := F(x + \rho) \tag{2.32}$$

in the space of the Mellin transform of $R_\pm$.

Proof. Let us prove (a) for the case R_+, while the case for R_- is similar. First of all it is obvious that the maps given are inverse of each other. To check that they are intertwiners, we compare the actions on the two spaces:

$$\begin{aligned}
R_+(g)\cdot F(\alpha,x) &= R_+(g)\cdot f\left(\frac{\alpha x}{\alpha+1},\frac{x}{\alpha+1}\right)\\
&= e^{-ibx} f\left(\frac{\alpha a x}{\alpha+1},\frac{ax}{\alpha+1}\right)\\
&= e^{-ib(x_1+x_2)} f\left(\frac{(\frac{x_1}{x_2}a(x_1+x_2)}{\frac{x_1}{x_2}+1},\frac{a(x_1+x_2)}{\frac{x_1}{x_2}+1}\right)\\
&= e^{-ibx_1}e^{-ibx_2} f(ax_1,ax_2)\\
&= (R_+\otimes R_+)(g)\cdot f(x_1,x_2).
\end{aligned}$$

Finally to check that it is unitary, we compute the norm after transformation:

$$\begin{aligned}
\|F(\alpha,x)\|^2 &= \iint \left|f\left(\frac{\alpha x}{\alpha+1},\frac{x}{\alpha+1}\right)\right|^2 \frac{dx}{x}\frac{d\alpha}{\alpha}\\
&= \iint |f(\alpha x_2,x_2)|^2\frac{dx_2}{x_2}\frac{d\alpha}{\alpha}\\
&= \iint |f(x_1,x_2)|^2\frac{dx_2}{x_2}\frac{dx_1}{x_1}\\
&= \|f(x_1,x_2)\|^2.
\end{aligned}$$

For (b) the argument is similar, where we split into the case $\alpha<1$ and $\alpha>1$:

$$\begin{aligned}
(R_+\otimes R_-)(g)\cdot f(x_1,x_2) &= e^{-ibx_1}e^{ibx_2} f(ax_1,ax_2)\\
&= e^{-ibx_1}e^{ibx_2} F\left(\frac{x_1}{x_2},a|x_1-x_2|\right)\\
&= e^{-ib\frac{\alpha x}{|\alpha-1|}}e^{ib\frac{x}{|\alpha-1|}} F(\alpha,ax)\\
&= \begin{cases} e^{-ibx}F(\alpha,ax) & \alpha>1\\ e^{ibx}F(\alpha,ax) & \alpha<1\end{cases}
\end{aligned}$$

as required.

Finally for (c), we apply the Mellin transformed action (2.17) to obtain:

$$\begin{aligned}
(R_+\otimes T_\rho)(g)\cdot F(w) &= \frac{a^{i\rho}}{2\pi}\int_{-\infty}^{\infty}\Gamma(iw-iz)a^{-iw}\left(\frac{ib}{a}\right)^{iz-iw}F(z)dz\\
&= \frac{a^{i\rho}}{2\pi}\int_{-\infty}^{\infty}\Gamma(iw-iz-i\rho)a^{-iw}\left(\frac{ib}{a}\right)^{iz-iw+i\rho}F(z+\rho)dz\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(ix-iz)a^{-ix}\left(\frac{ib}{a}\right)^{iz-ix}f(z)dz\\
&= R_+(g)\cdot f(x).
\end{aligned}$$

□

We will focus mainly on the case $R_+ \otimes R_+$. Under the Mellin transform of the function spaces, we can rewrite the intertwiners above in terms of gamma functions as follows.

Proposition 2.8. *Let* $F(\lambda, t) \in \mathcal{W} \otimes \mathcal{W} \subset L^2(\mathbb{R}, d\lambda) \otimes R_+$ *and* $f(t_1, t_2) \in \mathcal{W} \otimes \mathcal{W} \subset R_+ \otimes R_+$ *where* $R_+ \simeq L^2(\mathbb{R}, dt)$ *in the Mellin transformed action. Then the isomorphism*

$$\Psi : R_+ \otimes R_+ \simeq L^2(\mathbb{R}, d\lambda) \otimes R_+, \tag{2.33}$$

$$f(t_1, t_2) \mapsto F(\lambda, t) \tag{2.34}$$

can be expressed as

$$F(\lambda, t) = \Psi f := \frac{1}{2\pi} \int_C \frac{\Gamma(it_2 - it + i\lambda)\Gamma(-it_2 - i\lambda)}{\Gamma(-it)} f(t - t_2, t_2) dt_2 \tag{2.35}$$

and its inverse is given by

$$f(t_1, t_2) = \Psi^{-1} F := \frac{1}{2\pi} \int_{C'} \frac{\Gamma(-i\lambda + it_1)\Gamma(i\lambda + it_2)}{\Gamma(it_1 + it_2)} F(\lambda, t_1 + t_2) d\lambda, \tag{2.36}$$

where C *is the contour going along* $\mathbb{R}$ *that goes above the poles of* $\Gamma(-it_2 - i\lambda)$ *and below the poles of* $\Gamma(it_2 - it + i\lambda)$, *and similarly* C' *is the contour along* $\mathbb{R}$ *that goes above the poles of* $\Gamma(-i\lambda + it_1)$ *and below the poles of* $\Gamma(i\lambda + it_2)$.

Hence formally we can write the above transformations as integral transformations

$$F(\lambda, t) = \iint_{\mathbb{R}^2} \begin{vmatrix} \lambda & t \\ t_1 & t_2 \end{vmatrix}\!\!\rfloor f(t_1, t_2) dt_1 dt_2, \tag{2.37}$$

$$f(t_1, t_2) = \iint_{\mathbb{R}^2} \begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix} F(\lambda, t) d\lambda dt, \tag{2.38}$$

where the integral kernels are given be

$$\left\lfloor \begin{matrix} \lambda & t \\ t_1 & t_2 \end{matrix} \right\rfloor = \frac{1}{2\pi} \delta(t_1 + t_2 - t) \frac{\Gamma(i\lambda - it_1)\Gamma(-it_2 - i\lambda)}{\Gamma(-it)}, \tag{2.39}$$

$$\left\lceil \begin{matrix} \lambda & t \\ t_1 & t_2 \end{matrix} \right\rceil = \frac{1}{2\pi} \delta(t - t_1 - t_2) \frac{\Gamma(-i\lambda + it_1)\Gamma(it_2 + i\lambda)}{\Gamma(it)}$$

$$= \overline{\left\lfloor \begin{matrix} \lambda & t \\ t_1 & t_2 \end{matrix} \right\rfloor}. \tag{2.40}$$

Proof. To calculate Ψ, it suffices to calculate $\mathcal{M} \circ \psi \circ \mathcal{M}^{-1}$, where

$$\mathcal{M} : L^2 \left(\mathbb{R}_+^2, \frac{dx_1}{x_1} \frac{dx_2}{x_2} \right) \to L^2(\mathbb{R}^2, dt_1 dt_2)$$

is the Mellin transform on both variables, and ψ is the unitary equivalence from Theorem 2.7. We will write using separation of variables

$$f(t_1, t_2) := f_1(t_1) f_2(t_2) \in \mathcal{W} \otimes \mathcal{W}.$$

First, we have

$$(\mathcal{M}^{-1}f)(x_1,x_2) = \int_{\mathbb{R}+ic_2}\int_{\mathbb{R}+ic_1} x_1^{-it_1}x_2^{-it_2} f_1(t_1)f_2(t_2)dt_1dt_2.$$

Next, applying ψ:

$$\to \int_{\mathbb{R}+ic_2}\int_{\mathbb{R}+ic_1} \left(\frac{\alpha x}{\alpha+1}\right)^{-it_1}\left(\frac{x}{\alpha+1}\right)^{-it_2} f_1(t_1)f_2(t_2)dt_1dt_2.$$

Finally, taking the Mellin transform on the (α, x) variables, we arrive at

$$\begin{aligned} F(\lambda,t) := \Psi f &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}+ic_2}\int_{\mathbb{R}+ic_1} x^{it-1}\alpha^{i\lambda-1}\left(\frac{\alpha x}{\alpha+1}\right)^{-it_1}\left(\frac{x}{\alpha+1}\right)^{-it_2} \\ &\quad \cdot f_1(t_1)f_2(t_2)dt_1dt_2dxd\alpha \\ &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}+ic_2} x^{it-1}\alpha^{i\lambda-1}\left(\frac{x}{\alpha+1}\right)^{-it_2}\mathcal{M}^{-1} \\ &\quad \cdot f_1\left(\frac{\alpha x}{\alpha+1}\right) f_2(t_2)dt_2dxd\alpha. \end{aligned} \tag{2.41}$$

From the Mellin transform properties (Proposition 2.5), $\mathcal{M}^{-1}f_1(\frac{\alpha x}{\alpha+1})$ is of rapid decay in x. Hence the integrand is absolutely convergent with respect to x and t_2 and we can interchange the order of integration in (2.41) to obtain

$$\begin{aligned} F(\lambda,t) &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}+ic_2}\int_{\mathbb{R}+ic_1} x^{it-1}\alpha^{i\lambda-1}\left(\frac{\alpha x}{\alpha+1}\right)^{-it_1}\left(\frac{x}{\alpha+1}\right)^{-it_2} \\ &\quad \cdot f_1(t_1)f_2(t_2)dt_1dxdt_2d\alpha \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}_+}\int_{\mathbb{R}+ic_2}\int_{\mathbb{R}_+}\int_{\mathbb{R}+ic_1} x^{it-it_1-it_2-1}\alpha^{i\lambda-it_1-1}(\alpha+1)^{it_1+it_2} \\ &\quad \cdot f_1(t_1)f_2(t_2)dt_1dxdt_2d\alpha \\ &= \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}+ic_2}\alpha^{i\lambda-it+it_2-1}(\alpha+1)^{it}f_1(t-t_2)f_2(t_2)dt_2d\alpha \end{aligned}$$

by the Mellin transform property.

Next from the gamma–beta integral [27, V.1.6(7)], we have

$$\frac{\Gamma(w+u)\Gamma(-u)}{\Gamma(w)} = \int_0^\infty t^{w+u-1}(1+t)^{-w}dt, \tag{2.42}$$

where $\mathrm{Re}(w+u) > 0, \mathrm{Re}(u) < 0$. Assuming $\lambda \in \mathbb{R}$, we see that the integrand is absolutely convergent in α when

$$\mathrm{Re}(it_2 - it) > 0, \quad \mathrm{Re}(it_2) < 0.$$

Hence for $c_2 > 0$ and $\mathrm{Im}(t) > c_2$, we can interchange the order of integration to obtain

$$\begin{aligned} F(\lambda,t) &= \frac{1}{2\pi}\int_{\mathbb{R}+ic_2}\int_0^\infty \alpha^{i\lambda-it+it_2-1}(\alpha+1)^{it}f_1(t-t_2)f_2(t_2)d\alpha dt_2 \\ &= \frac{1}{2\pi}\int_{\mathbb{R}+ic_2}\frac{\Gamma(it_2-it+i\lambda)\Gamma(-it_2-i\lambda)}{\Gamma(-it)}f(t-t_2,t_2)dt_2, \end{aligned} \tag{2.43}$$

which holds for $\mathrm{Im}(t) > c_2 > 0$. Finally we can deform the contour of t_2 so that it goes under $t_2 = t-\lambda$ and above $t_2 = -\lambda$. Then the above expression can be analytically extended to $\mathrm{Im}(t) = 0$, and we obtain our desired formula.

Similarly, to calculate $\Psi^{-1} = \mathcal{M}\circ\psi^{-1}\circ\mathcal{M}^{-1}$, we start with the Mellin transform

$$(\mathcal{M}^{-1}F)(\alpha,x) = \int_{\mathbb{R}+ic_\lambda}\int_{\mathbb{R}+ic_t}\alpha^{-i\lambda}x^{-it}F_\lambda(\lambda)F_t(t)dtd\lambda,$$

and transform using ψ^{-1} from Theorem 2.7 into

$$\to \int_{\mathbb{R}+ic_\lambda}\int_{\mathbb{R}+ic_t}\left(\frac{x_1}{x_2}\right)^{-i\lambda}(x_1+x_2)^{-it}F_\lambda(\lambda)F_t(t)dtd\lambda,$$

and finally taking the Mellin transform on the variables (x_1,x_2), we have:

$$\begin{aligned} f(t_1,t_2) = \Psi^{-1}F &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}+ic_\lambda}\int_{\mathbb{R}+ic_t} x_1^{it_1-1}x_2^{it_2-1}\left(\frac{x_1}{x_2}\right)^{-i\lambda}(x_1+x_2)^{-it} \\ &\quad \cdot F_\lambda(\lambda)F_t(t)dtd\lambda dx_2dx_1 \quad \text{(replacing } x_1 \text{ by } x_1x_2\text{:)} \\ &= \frac{1}{(2\pi)^2}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}+ic_\lambda}\int_{\mathbb{R}+ic_t} x_2^{it_1+it_2-it-1}x_1^{it_1-i\lambda-1}(x_1+1)^{-it} \\ &\quad \cdot F_\lambda(\lambda)F_t(t)dtd\lambda dx_2dx_1. \end{aligned} \tag{2.44}$$

By the same arguments, we can interchange the order of integration with respect to $d\lambda$ and dx_2, and involve the Mellin transform in x_2 and t, to obtain

$$= \frac{1}{2\pi}\int_0^\infty\int_{\mathbb{R}+ic_\lambda} x_1^{it_1-i\lambda-1}(x_1+1)^{-it-it_2}F_\lambda(\lambda)F_t(t_1+t_2)d\lambda dx_1.$$

Finally, assuming $\tau_1 \in \mathbb{R}$, the integrand is absolutely convergent when

$$\mathrm{Re}(-i\lambda) > 0, \quad \mathrm{Re}(-i\lambda-it_2) < 0.$$

Hence for $0 < c_\lambda < -\mathrm{Im}(t_2)$ we can interchange the order of integration, and obtain

$$f(t_1,t_2) = \frac{1}{2\pi}\int_{\mathbb{R}+ic_\lambda}\frac{\Gamma(-i\lambda+it_1)\Gamma(i\lambda+it_2)}{\Gamma(it_1+it_2)}F(\lambda,t_1+t_2)d\lambda. \tag{2.45}$$

Again by shifting the contours for λ so that it goes above $\lambda = -t_2$ and below $\lambda = t_1$, the expression can be analytically extended to $\mathrm{Im}(t_2) = 0$, and we obtain the desired formula. □

These expressions will play an important role in the comparison with the quantum case.

3. q-Special Functions

3.1. *Definitions*

Throughout this section, we let $q = e^{\pi i b^2}$ where $b \in \mathbb{R}\backslash\mathbb{Q}$ and $0 < b^2 < 1$, so that $|q| = 1$ is not a root of unity.

We will consider the quantum dilogarithm $G_b(x)$ defined in [16, 17] throughout the paper. The reason is that it admits a nice classical limit toward the gamma function, as will be shown in the next section, and a lot of classical formula has a straightforward q-analogue using $G_b(x)$, where the proofs are nearly identical. For its relationship with other special functions in the literature, see e.g. [11]. Here we recall its definition.

Let $\omega := (w_1, w_2) \in \mathbb{C}^2$.

Definition 3.1. The double zeta function is defined as

$$\zeta_2(s, z|\omega) := \sum_{m_1, m_2 \in \mathbb{Z}_{\geq 0}} (z + m_1 w_1 + m_2 w_2)^{-s}. \tag{3.1}$$

The double gamma function is defined as

$$\Gamma_2(z|\omega) := \exp\left(\frac{\partial}{\partial s}\zeta_2(s, z|\omega)|_{s=0}\right). \tag{3.2}$$

Let

$$\Gamma_b(x) := \Gamma_2(x|b, b^{-1}), \tag{3.3}$$

then the quantum dilogarithm is defined as the function:

$$S_b(x) := \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}, \tag{3.4}$$

where $Q = b + b^{-1}$. The following form is often useful, and will be used throughout this paper:

$$G_b(x) := e^{\frac{\pi i}{2}x(x-Q)} S_b(x). \tag{3.5}$$

Proposition 3.2. *The quantum dilogarithm satisfies the following properties*:

Self-duality:

$$S_b(x) = S_{b^{-1}}(x), \quad G_b(x) = G_{b^{-1}}(x). \tag{3.6}$$

Functional equations:

$$S_b(x + b^{\pm 1}) = 2\sin(\pi b^{\pm 1} x) S_b(x), \quad G_b(x + b) = (1 - e^{2\pi i b x}) G_b(x). \tag{3.7}$$

Reflection property:

$$S_b(x) S_b(Q - x) = 1, \quad G_b(x) G_b(Q - x) = e^{\pi i x(x-Q)}. \tag{3.8}$$

Complex conjugation:

$$\overline{G_b(x)} = e^{\pi i \bar{x}(Q-\bar{x})} G_b(\bar{x}) = \frac{1}{G_b(Q - \bar{x})}. \tag{3.9}$$

Analyticity:

$S_b(x)$ and $G_b(x)$ are meromorphic functions with poles at $x = -nb - mb^{-1}$ and zeros at $x = Q + nb + mb^{-1}$, for $n, m \in \mathbb{Z}_{\geq 0}$.

Asymptotic properties:

$$G_b(x) \sim \begin{cases} \bar{\zeta}_b & \mathrm{Im}(x) \to +\infty \\ \zeta_b e^{\pi i x(x-Q)} & \mathrm{Im}(x) \to -\infty, \end{cases} \tag{3.10}$$

where

$$\zeta_b = e^{\frac{\pi i}{4} + \frac{\pi i}{12}(b^2+b^{-2})}. \tag{3.11}$$

Residues:

$$\lim_{x\to 0} xG_b(x) = \frac{1}{2\pi}, \tag{3.12}$$

or more generally,

$$\mathrm{Res}\frac{1}{G_b(Q+z)} = -\frac{1}{2\pi}\prod_{k=1}^{n}(1-q^{2k})^{-1}\prod_{l=1}^{m}(1-\widetilde{q}^{-2l})^{-1} \tag{3.13}$$

at $z = nb + mb^{-1}, n, m \in \mathbb{Z}_{\geq 0}$ and $\widetilde{q} = e^{-\pi i b^{-2}}$.

Let us introduce another important variant of the quantum dilogarithm function:

$$g_b(x) := \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} + \frac{1}{2\pi i b}\log x)}. \tag{3.14}$$

Lemma 3.3. *Let u, v be positive self-adjoint operators with $uv = q^2 vu$, $q = e^{\pi i b^2}$. Then*

$$g_b(u)g_b(v) = g_b(u+v), \tag{3.15}$$

$$g_b(v)g_b(u) = g_b(u)g_b(q^{-1}uv)g_b(v). \tag{3.16}$$

Equations (3.15) *and* (3.16) *are often referred to as the quantum exponential and the quantum pentagon relations*, *respectively.*

We will also use the following useful lemma:

Lemma 3.4 ([23]). *For $\mathrm{Im}(b^2) > 0$, $G_b(x)$ admits an infinite product description given by*

$$G_b(x) = \bar{\zeta}_b \frac{\prod_{n=1}^{\infty}(1 - e^{2\pi i b^{-1}(x - nb^{-1})})}{\prod_{n=0}^{\infty}(1 - e^{2\pi i b(x+nb)})}. \tag{3.17}$$

Lemma 3.5 ([1]). *We have the following Fourier transformation formula*:

$$\int_{\mathbb{R}+i0} dt e^{2\pi i t r}\frac{e^{-\pi i t^2}}{G_b(Q+it)} = \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} - ir)} = g_b(e^{2\pi b r}), \tag{3.18}$$

$$\int_{\mathbb{R}+i0} dt e^{2\pi i t r}\frac{e^{-\pi Q t}}{G_b(Q+it)} = \zeta_b G_b\left(\frac{Q}{2} - ir\right) = \frac{1}{g_b(e^{2\pi b r})}, \tag{3.19}$$

where the contour goes above the pole at $t = 0$.

Using the reflection properties, we also obtain

$$\int_{\mathbb{R}-i0} dte^{-2\pi itr}e^{-\pi Qt}G_b(it) = \frac{\bar{\zeta}_b}{G_b\left(\frac{Q}{2}-ir\right)}, \tag{3.20}$$

$$\int_{\mathbb{R}-i0} dte^{-2\pi itr}e^{\pi it^2}G_b(it) = \zeta_b G_b\left(\frac{Q}{2}-ir\right), \tag{3.21}$$

where the contour goes below the pole at $t=0$.

Lemma 3.6 ([17]). *We have the tau–beta theorem*:

$$\int_C d\tau e^{-2\pi\tau\beta}\frac{G_b(\alpha+i\tau)}{G_b(Q+i\tau)} = \frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha+\beta)}, \tag{3.22}$$

where the contour C goes along $\mathbb{R}$ and goes above the poles of $G_b(Q+i\tau)$ and below those of $G_b(\alpha+i\tau)$.

Lemma 3.7 (q-binomial theorem [1]). *Let u,v be positive self-adjoint operators with $uv=q^2vu$. We have*:

$$(u+v)^{it} = b\int_C d\tau \binom{t}{\tau}_b u^{i(t-\tau)}v^{i\tau}, \tag{3.23}$$

where

$$\binom{t}{\tau}_b = \frac{e^{2\pi ib^2\tau(t-\tau)}G_b(Q+ibt)}{G_b(Q+ibt-ib\tau)G_b(Q+ib\tau)} = \frac{G_b(-ib\tau)G_b(ib\tau-ibt)}{G_b(-ibt)}, \tag{3.24}$$

and C is the contour along $\mathbb{R}$ that goes above the pole at $\tau=0$ and below the pole at $\tau=t$.

Similarly, for $uv=q^{-2}vu$, we have:

$$(u+v)^{it} = b\int_C d\tau \binom{t}{\tau}^b u^{i\tau}v^{i(t-\tau)}, \tag{3.25}$$

where

$$\binom{t}{\tau}^b = \frac{G_b(Q+ibt)}{G_b(Q+ibt-ib\tau)G_b(Q+ib\tau)}, \tag{3.26}$$

with the same contour C as above.

Remark 3.8. When t approaches $-in$ for positive integer n, by first shifting the contour along the poles at $\tau=t+ik$ for $0\le k\le n$, the integration vanishes and $n+1$ residues are left, which is precisely the terms in the usual q-binomial formula.

Remark 3.9. The q-binomial theorem is actually the q-analogue of the classical formula [15, (3.3.9)]:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(a-s)x^{-s}ds = \frac{\Gamma(a)}{(1+x)^a}, \tag{3.27}$$

when $0 < c < \mathrm{Re}(a)$. After a change of variables with x replaced by x/y, a by $-it$, s by $-is$ and a suitable shift of contour, we obtain

$$(x+y)^{it} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma(-is)\Gamma(-it+is)}{\Gamma(-it)} x^{is} y^{it-is} ds, \tag{3.28}$$

where the contour separates the poles of the two gamma functions. We can easily see that under the limiting process described in the next section, the q-binomial theorem reduces precisely to this classical formula.

3.2. *Limits of the quantum dilogarithm*

Recall that the b-hypergeometric function (slightly modified from [17]) is defined by:

$$F_b(\alpha,\beta,\gamma;z) := \frac{G_b(\gamma)}{G_b(\alpha)G_b(\beta)}\int_C (-z)^{ib^{-1}\tau} e^{\pi i\tau^2} \frac{G_b(\alpha+i\tau)G_b(\beta+i\tau)G_b(-i\tau)}{G_b(\gamma+i\tau)} d\tau, \tag{3.29}$$

where the contour along $\mathbb{R}$ separates the poles of $G_b(\alpha+i\tau)G_b(\beta+i\tau)$ from those of $\frac{G_b(-i\tau)}{G_b(\gamma+i\tau)}$.

In comparison with the classical formula:

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi}\int_C (-z)^{is}\frac{\Gamma(a+is)\Gamma(b+is)\Gamma(-is)}{\Gamma(c+is)} ds, \tag{3.30}$$

we see therefore that there is a strong analogy between the function $G_b(z)$ and the gamma function $\Gamma(x)$. However, we know that there is no direct classical limit $b \to 0$ because of the factor $Q = b + b^{-1}$ involved in the definitions. It turns out that we can still define certain kind of limit of the function $G_b(x)$ that enables one to compare it with the classical gamma function $\Gamma(x)$.

Recall that by Lemma 3.4, if $\mathrm{Im}(b^2) > 0$, then $G_b(x)$ can be expressed as a ratio of infinite product:

$$G_b(x) = \bar{\zeta}_b \frac{\prod_{n=1}^{\infty}(1 - e^{2\pi i b^{-1}(x - nb^{-1})})}{\prod_{n=0}^{\infty}(1 - e^{2\pi i b(x+nb)})}$$

or after scaling:

$$G_b(bx) = \bar{\zeta}_b \frac{\prod_{n=1}^{\infty}(1 - e^{2\pi i x}e^{-2\pi i n b^{-2}})}{\prod_{n=0}^{\infty}(1 - e^{2\pi i b^2(x+n)})}. \tag{3.31}$$

In order to take the limit, we let $b^2 = ir$ for real $r > 0$ (more generally for $\mathrm{Re}(r) > 0$). With respect to q, this means that we are going "inside the circle", and approach $q = 1$ from the interior of the unit disk.

Let $b^2 = ir$, then we can rewrite the above infinite product as

$$G_b(bx) = \bar{\zeta}_b \frac{\prod_{n=1}^{\infty}(1 - e^{2\pi i x}e^{-2\pi n/r})}{\prod_{n=0}^{\infty}(1 - e^{-2\pi r(x+n)})} = \bar{\zeta}_b \frac{(e^{2\pi i x - 2\pi/r}; e^{-2\pi/r})_\infty}{(q^{2x}; q^2)_\infty}.$$

Note that we also have

$$\bar{\zeta}_b = e^{-\frac{\pi i}{4} - \frac{\pi i}{12}(b^2+b^{-2})} = e^{-\frac{\pi i}{4} + \frac{\pi r - \pi/r}{12}}. \tag{3.32}$$

When $r \to 0^+$, the term

$$(e^{2\pi ix-2\pi/r}; e^{-2\pi/r})_\infty \to 1.$$

On the other hand, the denominator resembles that of the q-gamma function:

$$\Gamma_q(x) := \frac{(q^2;q^2)_\infty}{(q^{2x};q^2)_\infty}(1-q^2)^{-x+1}, \tag{3.33}$$

which is known to converge uniformly to $\Gamma(x)$ as $q \to 1$ for every compact subset in $\mathbb{C}$ [7].

For the ratio $\frac{\bar{\zeta}_b}{(q^2;q^2)_\infty}$, we have the following observation:

Lemma 3.10. *We have the limit*

$$\lim_{r\to 0^+} \frac{\bar{\zeta}_b}{\sqrt{-i}|b|(q^2;q^2)_\infty} = \lim_{r\to 0^+} e^{-\frac{\pi i}{4}} \frac{e^{\frac{\pi r-\pi/r}{12}}}{\sqrt{-i}\sqrt{r}(q^2;q^2)_\infty} = 1, \tag{3.34}$$

where we denote $e^{-\frac{\pi i}{4}}$ *by* $\sqrt{-i}$.

Proof. Let

$$\eta(ir) := e^{-\frac{\pi r}{12}}(q^2;q^2)_\infty \tag{3.35}$$

be the Dedekind eta function. Then from the well-known functional equation:

$$\eta(-\tau^{-1}) = \sqrt{-i\tau}\,\eta(\tau), \tag{3.36}$$

substituting $\tau = ir$, we have:

$$\eta\left(\frac{i}{r}\right) = \sqrt{r}\eta(ir),$$

$$e^{-\frac{\pi}{12r}}(e^{-\frac{2\pi}{r}}; e^{-\frac{2\pi}{r}})_\infty = e^{-\frac{\pi r}{12}}\sqrt{r}(q^2;q^2)_\infty,$$

$$\frac{e^{\frac{\pi r-\pi/r}{12}}}{\sqrt{r}(q^2,q^2)_\infty} = (e^{-\frac{2\pi}{r}}; e^{-\frac{2\pi}{r}})_\infty^{-1}$$

and taking the limit $r \to 0^+$, we have

$$\lim_{r\to 0^+}(e^{-\frac{2\pi}{r}}; e^{-\frac{2\pi}{r}})_\infty = 1$$

as required. □

Finally, combining with the obvious limit:

$$\lim_{r\to 0^+}\frac{|b|^2}{1-q^2} = \lim_{r\to 0^+}\frac{r}{1-e^{-2\pi r}} = \frac{1}{2\pi}, \tag{3.37}$$

we have the following theorem.

Theorem 3.11. *The following limit holds for* $b^2 = ir \to i0^+$:

$$\lim_{r\to 0^+}\frac{(2\pi b)G_b(bx)}{(-2\pi i b^2)^x} = \Gamma(x), \tag{3.38}$$

where $(-2\pi ib^2) = 2\pi r > 0$, *hence the denominator is well-defined. By the properties of the convergence of* $\Gamma_q(x)$, *the limit converges uniformly for every compact subset in* $\mathbb{C}$. *This gives another proof of a similar limit first observed in* [20].

A similar analysis shows that

$$\lim_{r\to 0} \frac{(2\pi b)G_b(Q+bx)}{(-2\pi ib^2)^{x+1}} = (1-e^{2\pi ix})\Gamma(x+1). \tag{3.39}$$

Proposition 3.12. *The two limits* (3.38) *and* (3.39) *are compatible with the reciprocal relations*

$$G_b(x)G_b(Q-x) = e^{\pi ix(x-Q)},$$
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Proof. We have

$$\begin{aligned} 1 &= G_b(bx)G_b(Q-bx)e^{-\pi ibx(bx-Q)} \\ &= \left(\frac{(2\pi b)G_b(bx)}{(-2\pi ib^2)^x}\right)\left(\frac{(2\pi b)G_b(Q-bx)}{(-2\pi ib^2)^{-x+1}}\right)\frac{(-2\pi ib^2)}{(2\pi b)^2}e^{-\pi ibx(bx-Q)} \\ &\to \Gamma(x)\Gamma(1-x)(1-e^{2\pi ix})\frac{-i}{2\pi}e^{\pi ix} \\ &= \frac{\pi}{\sin(\pi x)}\frac{e^{\pi ix}-e^{-\pi ix}}{2\pi i} \\ &= 1, \end{aligned}$$

where we used

$$e^{-\pi ibx(bx-Q)} = e^{-\pi ix(b^2x-b^2-1)} = e^{-\pi ixr(x-1)}e^{\pi ix} \to e^{\pi ix}. \qquad \square$$

4. *q*-Intertwiners

We begin with the definition of the quantum plane that is used in [6].

Definition 4.1. The quantum plane $\mathcal{B}_q$ for $|q| = 1$ is generated by two positive self-adjoint operators X, Y such that

$$XY = q^2YX$$

in the sense of [22], i.e.

$$X^{is}Y^{it} = q^{-2st}Y^{it}X^{is}, \tag{4.1}$$

for every $s, t \in \mathbb{R}$ as relations between unitary operators. The coproduct is given by

$$\Delta X = X \otimes X, \tag{4.2}$$
$$\Delta Y = Y \otimes X + 1 \otimes Y. \tag{4.3}$$

In [6], this is realized by $X = e^{-2\pi bp}$ and $Y = e^{2\pi bx}$, where $p = \frac{1}{2\pi i}\frac{\partial}{\partial x}$, acting as unbounded positive self-adjoint operators on $\mathcal{H} = L^2(\mathbb{R})$, such that

$$X \cdot f(x) = f(x + ib), \tag{4.4}$$

$$Y \cdot f(x) = e^{2\pi bx} f(x), \tag{4.5}$$

which is well-defined for functions in the core $\mathcal{W} \subset L^2(\mathbb{R})$ (cf. Definition 2.4). We remark that $\mathcal{B}_q$ is "dual" to the quantum plane $\mathcal{A}_q$ generated by A, B defined in the Sec. 6, due to the different coproducts.

In the study of tensor products of representations, the operators act by the coproduct (4.2), (4.3). It was shown in [6] that there is a quantum dilogarithm transform that gives a unitary equivalence as representations of $\mathcal{B}_q$:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{M} \otimes \mathcal{H}, \tag{4.6}$$

where $\mathcal{M} = L^2(\mathbb{R})$ is the parametrization space (or the multiplicity module), and carries the trivial representation.

Proposition 4.2. *The quantum dilogarithm transform is defined on $f, \phi \in \mathcal{W} \otimes \mathcal{W}$ by*

$$\phi(\alpha, x) = \int_{\mathbb{R}} \int_{\mathbb{R}-i0} \left\lfloor \begin{matrix} \alpha & x \\ x_1 & x_2 \end{matrix} \right\rfloor f(x_1, x_2) dx_2 dx_1, \tag{4.7}$$

$$f(x_1, x_2) = \int_{\mathbb{R}-i0} \int_{\mathbb{R}} \left\lceil \begin{matrix} \alpha & x \\ x_1 & x_2 \end{matrix} \right\rceil \phi(\alpha, x) d\alpha dx. \tag{4.8}$$

Here the integration kernel is given by:

$$\left\lfloor \begin{matrix} \alpha & x \\ x_1 & x_2 \end{matrix} \right\rfloor = e^{2\pi i\alpha(x-x_1)} \mathcal{E}_R(x - x_1, x_2 - x_1), \tag{4.9}$$

$$\left\lceil \begin{matrix} \alpha & x \\ x_1 & x_2 \end{matrix} \right\rceil = e^{-2\pi i\alpha(x-x_1)} \mathcal{E}_L(x_2 - x_1, x - x_1), \tag{4.10}$$

where

$$\mathcal{E}_R(z, w) = e^{2\pi izw} S_R(z - w), \tag{4.11}$$

$$\mathcal{E}_L(z, w) = e^{-2\pi izw} S_L(z - w), \tag{4.12}$$

and

$$S_R(z) = G(z - ia) e^{i\chi + \frac{\pi}{2}(z-ia)^2}, \tag{4.13}$$

$$S_L(z) = G(z - ia) e^{-i\chi - \frac{\pi}{2}(z-ia)^2}, \tag{4.14}$$

where $\chi = \frac{\pi}{24}(b^2 + b^{-2})$. The contour for x_2 goes below the pole at $x_2 = x$, and the contour for x goes below the pole at $x = x_2$.

The integral transforms are unitary, hence they extend to the whole of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{M} \otimes \mathcal{H}$, respectively.

Here the function $G(z) = G(b, b^{-1}; z)$ is the Ruijsenaars's definition of the quantum dilogarithm [21], and is given by

$$G(z) = \exp\left(i\int_0^\infty \frac{dy}{y}\left(\frac{\sin(2yz)}{2\sinh(by)\sinh(b^{-1}y)} - \frac{z}{y}\right)\right). \tag{4.15}$$

The relation between $G(z)$ and $G_b(z)$ is given by (cf. [11])

$$G(b, b^{-1}, x) = e^{\pi i x^2/2} e^{\pi i Q^2/8} G_b\left(\frac{Q}{2} - ix\right). \tag{4.16}$$

Proposition 4.3. *In terms of* $G_b(x)$, *we have*:

$$\begin{aligned}\begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rfloor &= \bar{\zeta}_b e^{2\pi i(x-x_1)(x_2-x_1+\alpha)} e^{\pi i(x_2-x)^2} e^{\pi Q(x-x_2)} G_b(ix_2 - ix)\\ &= \bar{\zeta}_b \frac{e^{2\pi i(x-x_1)(x_2-x_1+\alpha)}}{G_b(Q + ix - ix_2)},\end{aligned} \tag{4.17}$$

$$\begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rceil = \zeta_b e^{-2\pi i(x-x_1)(x_2-x_1+\alpha)} G_b(ix - ix_2), \tag{4.18}$$

where

$$\zeta_b = e^{\frac{\pi i}{4} + \frac{\pi i}{12}(b^2+b^{-2})}, \quad \bar{\zeta}_b = e^{-\frac{\pi i}{4} - \frac{\pi i}{12}(b^2+b^{-2})}.$$

5. Classical Limit of *q*-Intertwiners

In this section, we will compare the quantum dilogarithm transformation defined in the previous section, and the classical $ax + b$ group intertwiners studied in Sec. 2.2, and show that they correspond to each other under the limiting procedures suggested in Sec. 3.2.

5.1. *Fourier transform of the q-intertwiners*

In order to compare with the classical case, we need to take the Fourier transform of the actions on both function spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{M} \otimes \mathcal{H}$. In order to do this correctly, it turns out that we need to modify the kernel by

$$\begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rfloor_* := \frac{\bar{\zeta}_b e^{-\pi i(x-x_1)^2}}{G_b(\frac{Q}{2} + i\alpha)} \begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rfloor, \tag{5.1}$$

and

$$\begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rceil_* := \zeta_b e^{\pi i(x-x_1)^2} G_b\left(\frac{Q}{2} + i\alpha\right) \begin{bmatrix}\alpha & x\\ x_1 & x_2\end{bmatrix}\!\!\rceil. \tag{5.2}$$

The extra factors depend only on α and $(x - x_1)$, hence the integral kernels are still intertwiners. Note that $G_b(\frac{Q}{2} + i\alpha)$ is unitary by the complex conjugation property, so that the intertwiners are still unitary operators.

Now consider the effects of these intertwining maps on the Fourier transformed functions:

$$\mathcal{F}:\phi(\alpha,x)\mapsto\phi(\lambda,t):=\iint_{\mathbb{R}^2}e^{2\pi i\alpha x}e^{2\pi itx}\phi(\alpha,x)d\alpha dx, \tag{5.3}$$

$$\mathcal{F}:f(x_1,x_2)\mapsto f(t_1,t_2):=\iint_{\mathbb{R}^2}e^{2\pi it_1x_1}e^{2\pi it_2x_2}f(x_1,x_2)dx_1dx_2. \tag{5.4}$$

We will use the same symbols ϕ, f to denote the Fourier transformed functions.

Theorem 5.1. *Under the Fourier transform, the intertwining maps defined in Proposition* 4.2 *on* $f,\phi\in\mathcal{W}\otimes\mathcal{W}$ *become*:

$$\mathcal{F}(\Phi)(f):=\phi(\lambda,t)=\int_C\frac{G_b(it_2-it+i\lambda)G_b(-it_2-i\lambda)}{G_b(-it)}\cdot e^{\pi i\lambda(\lambda-2t+2t_2)}f(t-t_2,t_2)dt_2, \tag{5.5}$$

$$\mathcal{F}(\Phi^{-1})(\phi):=f(t_1,t_2)=\int_{C'}\frac{G_b(-i\lambda+it_1)G_b(i\lambda+it_2)}{G_b(it)}\cdot e^{\pi i\lambda(\lambda+2t_2)}e^{-2\pi it_1t_2}\phi(\lambda,t_1+t_2)d\lambda, \tag{5.6}$$

where C is the contour going along $\mathbb{R}$ that goes above the poles of $\Gamma_b(-it_2-i\lambda)$ and below the poles of $\Gamma_b(it_2-it+i\lambda)$, and similarly C' is the contour along $\mathbb{R}$ that goes above the poles of $\Gamma_b(-i\lambda+it_1)$ and below the poles of $\Gamma_b(i\lambda+it_2)$.

Hence formally we can write the above transformations as integral transformations:

$$\phi(\lambda,t)=\iint\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_* f(t_1,t_2)dt_1dt_2, \tag{5.7}$$

$$f(t_1,t_2)=\iint\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_*\phi(\lambda,t)d\lambda dt, \tag{5.8}$$

where the kernels are expressed as

$$\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_*=\delta(t_1+t_2-t)\frac{G_b(-it_1+i\lambda)G_b(-it_2-i\lambda)}{G_b(-it)}e^{\pi i\lambda(\lambda-2t_1)}, \tag{5.9}$$

$$\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_*=\delta(t-t_1-t_2)\frac{G_b(-i\lambda+it_1)G_b(it_2+i\lambda)}{G_b(it)}e^{\pi i\lambda(\lambda+2t_2)}e^{-2\pi it_1t_2}. \tag{5.10}$$

They are still intertwiners with respect to the Fourier transformed quantum plane

$$\widehat{X}=e^{2\pi bx}\quad\widehat{Y}=e^{2\pi bp} \tag{5.11}$$

with the same coproduct.

Proof. The intertwining properties are clear, since $\widehat{Y}\otimes\widehat{X}$ are commutative with respect to t_1, t_2, and Fourier transformation is linear, hence it preserves the action of

$$\Delta\widehat{Y} = \widehat{Y}\otimes\widehat{X} + 1\otimes\widehat{Y}.$$

The delta distribution explains the intertwining property for $\Delta\widehat{X} = \widehat{X}\otimes\widehat{X}$ explicitly.

We will calculate the integral transform using the Fourier transform property (Lemma 3.5) and tau–beta integral (Lemma 3.6) repeatedly. Similar to the classical case, the Fourier transformed action is given by $\mathcal{F}\circ\Phi\circ\mathcal{F}^{-1}$ where Φ is the quantum dilogarithm transform defined in Proposition 4.2.

First we take the (inverse) Fourier transform of $f(t_1, t_2)$:

$$(\mathcal{F}^{-1}f)(x_1, x_2) = \iint_{\mathbb{R}^2} e^{-2\pi i t_1 x_1} e^{-2\pi i t_2 x_2} f(t_1, t_2) dt_2 dt_1,$$

applying the quantum dilogarithm transformation Φ:

$$\int_{\mathbb{R}}\int_{\mathbb{R}-i0}\iint_{\mathbb{R}^2} \frac{\bar{\zeta}_b^2 e^{-\pi i(x-x_1)^2}}{G_b(\frac{Q}{2}+i\alpha)} \frac{e^{2\pi i(x-x_1)(x_2-x_1+\alpha)}}{G_b(Q+ix-ix_2)} \cdot e^{-2\pi i t_1 x_1} e^{-2\pi i t_2 x_2} f(t_1, t_2) dt_2 dt_1 dx_2 dx_1,$$

and take the Fourier transform back to the target space $L^2(\mathbb{R}^2, d\lambda dt)$ to obtain

$$\phi(\lambda, t) = \iint_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}-0}\iint_{\mathbb{R}^2} \frac{\bar{\zeta}_b^2 e^{-\pi i(x-x_1)^2}}{G_b(\frac{Q}{2}+i\alpha)} \frac{e^{2\pi i(x-x_1)(x_2-x_1+\alpha)}}{G_b(Q+ix-ix_2)} \cdot e^{-2\pi i t_1 x_1} e^{-2\pi i t_2 x_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1, t_2) dt_2 dt_1 dx_2 dx_1 dx d\alpha. \quad (5.12)$$

The integrand is absolutely convergent in t_1 and t_2 because $f(t_1, t_2) \in \mathcal{W}\otimes\mathcal{W}$. With respect to x_2, using the asymptotic properties for G_b, we see that the absolute value of the integrand has the growth

$$\begin{cases} e^{2\pi \mathrm{Im}(t_2)x_2} & x_2 \to -\infty, \\ e^{-\pi Q x_2} e^{2\pi \mathrm{Im}(t_2)x_2} & x_2 \to +\infty. \end{cases}$$

Hence it is absolutely convergent for

$$0 < \mathrm{Im}(t_2) < \frac{Q}{2},$$

and we can interchange the order of integration to obtain

$$\phi(\lambda, t) = \int_{\mathbb{R}^5}\int_{\mathbb{R}-i0} \frac{\bar{\zeta}_b^2 e^{-\pi i(x-x_1)^2}}{G_b(\frac{Q}{2}+i\alpha)} \frac{e^{2\pi i(x-x_1)(x_2-x_1+\alpha)}}{G_b(Q+ix-ix_2)} \cdot e^{-2\pi i t_1 x_1} e^{-2\pi i t_2 x_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} dx_2 f(t_1, t_2) dt_2 dt_1 dx_1 dx d\alpha.$$

Now substitute x_2 by $x - x_2$, we obtain

$$= \int_{\mathbb{R}^5}\int_{\mathbb{R}+i0} \bar{\zeta}_b^2 \frac{e^{-\pi i(x-x_1)^2} e^{2\pi i(x-x_1)(x-x_2-x_1+\alpha)}}{G_b(\frac{Q}{2}+i\alpha) G_b(Q+ix_2)} \cdot e^{-2\pi i t_1 x_1} e^{-2\pi i(x-x_2)t_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1, t_2) dt_2 dt_1 dx_1 dx_2 dx d\alpha.$$

The relevant exponential with respect to x_2 is

$$e^{2\pi i x_2(x_1+t_2-x)},$$

hence using Lemma 3.5, integrating over x_2 with $r = x_1+t_2-x-iQ/2$, the integrand becomes

$$= \bar{\zeta}_b \frac{G_b(ix-it_2-ix_1)}{G_b(\frac{Q}{2}+i\alpha)} e^{-\pi i(x-x_1)^2} e^{2\pi i(x-x_1)(x-x_1+\alpha)} e^{-2\pi i t_1 x_1}$$
$$\cdot e^{-2\pi i x t_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1,t_2)$$
$$= \bar{\zeta}_b \frac{G_b(ix-it_2-ix_1)}{G_b(\frac{Q}{2}+i\alpha)} e^{\pi i(x-x_1)^2} e^{2\pi i(x-x_1)\alpha} e^{-2\pi i t_1 x_1} e^{-2\pi i x t_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1,t_2).$$

Now the absolute value of this integrand with respect to x_1 has asymptotics

$$\begin{cases} e^{2\pi \mathrm{Im}(t_1)x_1} & x_1 \to -\infty, \\ e^{-\pi Q x_1} e^{2\pi(\mathrm{Im}(t_1)+\mathrm{Im}(t_2))x_1} & x_1 \to +\infty. \end{cases}$$

Hence the integral with respect to x_1 is absolutely convergent when

$$\mathrm{Im}(t_1) > 0, \quad \mathrm{Im}(t_1+t_2) < \frac{Q}{2}.$$

So we now have

$$\phi(\lambda,t) = \int_{\mathbb{R}^5} \bar{\zeta}_b \frac{G_b(ix-it_2-ix_1)}{G_b(\frac{Q}{2}+i\alpha)} e^{\pi i(x-x_1)^2} e^{2\pi i(x-x_1)\alpha}$$
$$\cdot e^{-2\pi i t_1 x_1} e^{-2\pi i x t_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1,t_2) dx_1 dt_2 dt_1 dx d\alpha.$$

Substitute x_1 by $-x_1-t_2+x$, we obtain

$$= \int_{\mathbb{R}^4}\int_{\mathbb{R}-i\mathrm{Im}(t_2)} \bar{\zeta}_b \frac{G_b(ix_1)}{G_b(\frac{Q}{2}+i\alpha)} e^{\pi i(x_1+t_2)^2} e^{2\pi i(x_1+t_2)\alpha} e^{-2\pi i t_1(x-t_2-x_1)}$$
$$\cdot e^{-2\pi i x t_2} e^{2\pi i t x} e^{2\pi i\lambda\alpha} f(t_1,t_2) dx_1 dt_2 dt_1 dx d\alpha.$$

The relevant exponential with respect to x_1 is

$$e^{-2\pi i x_1(-t_1-t_2-\alpha)} e^{\pi i x_1^2},$$

hence using Lemma 3.5, integrating over x_1 (valid since $\mathrm{Im}(t_2) > 0$) with $r = -t_1-t_2-\alpha$, the integrand becomes:

$$\frac{G_b(\frac{Q}{2}+it_1+it_2+i\alpha)}{G_b(\frac{Q}{2}+i\alpha)} e^{\pi i t_2^2} e^{2\pi i t_2\alpha} e^{2\pi i t_1 t_2} e^{-2\pi i x(t_1+t_2-t)} e^{2\pi i\lambda\alpha} f_1(t_1) f_2(t_2).$$

Now we can simplify the integration with respect to t_1 and x using the factor $e^{-2\pi i x(t_1+t_2-t)}$, which is just a Fourier transform and its inverse, to obtain

$$\phi(\lambda,t) = \iint_{\mathbb{R}^2} \frac{G_b(\frac{Q}{2}+it+i\alpha)}{G_b(\frac{Q}{2}+i\alpha)} e^{\pi i t_2^2} e^{2\pi i t_2\alpha} e^{2\pi i(t-t_2)t_2} e^{2\pi i\lambda\alpha} f(t-t_2,t_2) dt_2 d\alpha.$$

Now the absolute value of the integrand has asymptotics

$$\begin{cases} e^{2\pi \mathrm{Im}(t)\alpha} & \alpha \to -\infty, \\ e^{-2\pi \mathrm{Im}(t_2)\alpha} & \alpha \to +\infty. \end{cases}$$

Hence it is absolutely convergent when $\mathrm{Im}(t) > 0$. We do the final interchange of order of integration and integrate with respect to α:

$$\phi(\lambda, t) = \iint_{\mathbb{R}^2} \frac{G_b(\frac{Q}{2} + it + i\alpha)}{G_b(\frac{Q}{2} + i\alpha)} e^{\pi i t_2^2} e^{2\pi i t_2 \alpha} e^{2\pi i (t-t_2)t_2} e^{2\pi i \lambda \alpha} f(t - t_2, t_2) d\alpha dt_2$$

Shifting the contour of α by $\alpha \to \alpha - i\frac{Q}{2}$ we get

$$= \int_{\mathbb{R}} \int_{\mathbb{R}+i0} \frac{G_b(Q + it + i\alpha)}{G_b(Q + i\alpha)} e^{\pi i t_2^2} e^{2\pi i t_2 \alpha} e^{\pi t_2 Q} e^{2\pi i (t-t_2)t_2} e^{2\pi i \lambda \alpha} e^{\pi \lambda Q} \cdot f(t - t_2, t_2) d\alpha dt_2.$$

The relevant exponential for α is

$$e^{-2\pi\alpha(-it_2 - i\lambda)},$$

therefore using the tau–beta integral (Lemma 3.6) again, the integrand becomes:

$$\frac{G_b(Q + it)G_b(-it_2 - i\lambda)}{G_b(Q + it - it_2 - i\lambda)} e^{\pi i t_2^2} e^{\pi t_2 Q} e^{2\pi i (t-t_2)t_2} e^{\pi \lambda Q} f(t - t_2, t_2).$$

Finally using the reflection property $G_b(x)G_b(Q - x) = e^{\pi i x(x-Q)}$, we obtain

$$\frac{G_b(it_2 - it + i\lambda)G_b(-it_2 - i\lambda)}{G_b(-it)} e^{\pi i \lambda(\lambda - 2t + 2t_2)} f(t - t_2, t_2).$$

Therefore, we have the expression

$$\phi(\lambda, t) = \int_{\mathbb{R}+ic_2} \frac{G_b(it_2 - it + i\lambda)G_b(-it_2 - i\lambda)}{G_b(-it)} e^{\pi i \lambda(\lambda - 2t + 2t_2)} f(t - t_2, t_2) dt_2,$$

valid for $0 < c_2 < \frac{Q}{2}$ and $\mathrm{Im}(t) > 0$.

By a shift of contour on t_2 so that it goes below the pole at $t_2 = t - \lambda$ and above the poles at $t_2 = -\lambda$, the expression can be analytically continued to $t \in \mathbb{R}$, hence we can rewrite the expression as

$$\phi(\lambda, t) = \int_C \frac{G_b(it_2 - it + i\lambda)G_b(-it_2 - i\lambda)}{G_b(-it)} \cdot e^{\pi i \lambda(\lambda - 2t + 2t_2)} f(t - t_2, t_2) dt_2 \in \mathcal{M} \otimes \mathcal{H}$$

with the desired contour.

Working formally, for the kernel $\mathcal{F}\begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix}_*$, the target space is $L^2(\mathbb{R}^2, d\lambda dt)$ and the domain space is $L^2(\mathbb{R}^2, dt_1 dt_2)$. Since Fourier transform of complex conjugation is the complex conjugation of the inverse Fourier transform, $\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_*$ is just the complex conjugation of $\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_*$. Hence we have

$$\begin{aligned}\mathcal{F}\begin{bmatrix}\lambda & t\\ t_1 & t_2\end{bmatrix}_* &= \delta(t_1+t_2-t)\frac{G_b(-i\lambda+it_1)G_b(it_2+i\lambda)}{G_b(it)}e^{-\pi i\lambda(\lambda-2t_1)}\\ &\quad\cdot e^{\pi i(-it_1+i\lambda)(Q+it_1-i\lambda)}e^{\pi i(it_2+i\lambda)(Q-it_2-i\lambda)}e^{-\pi i(it_1+it_2)(Q-it_1-it_2)}\\ &= \delta(t_1+t_2-t)\frac{G_b(-i\lambda+it_1)G_b(it_2+i\lambda)}{G_b(it)}e^{\pi i\lambda(\lambda+2t_2)}e^{-2\pi it_1t_2}.\end{aligned}$$

Alternatively, we can work through the integrations as in the proof above using similar techniques of interchanging orders of integration and shifting of contours. □

5.2. *Classical limit*

We are now ready to compare the quantum intertwiners from Theorem 5.1 with the classical intertwiners from Proposition 2.8.

Theorem 5.2. *Under a suitable rescaling, as $b^2 \to i0^+$, or more generally, as $q \to 1$ from inside the unit disk, the quantum intertwining operator has a limit toward the classical intertwining transformation given by Proposition* 2.8.

Proof. The contour of integration is the same for the quantum and the classical intertwining transform. Therefore it suffices to do the limit formally for the intertwiners. First of all we need to rescale the function space $\mathcal{H} = L^2(\mathbb{R})$ by b on all the variables (including the parameter λ). More precisely, before taking the limit, $b \in \mathbb{R}_{>0}$ and we apply the *unitary* transformation

$$\begin{aligned}B : L^2(\mathbb{R}, dx) &\to L^2(\mathbb{R}, bdx),\\ f(x) &\mapsto f(bx)\end{aligned} \tag{5.13}$$

on each variable. Hence the kernel under this transformation is now given by

$$\begin{aligned}b^2\mathcal{F}\begin{bmatrix}b\lambda & bt\\ bt_1 & bt_2\end{bmatrix}_* &= b^2\delta(b(t_1+t_2-t))\frac{G_b(-ibt_1+ib\lambda)G_b(-ibt_2-ib\lambda)}{G_b(-ibt)}e^{\pi ib^2\lambda(\lambda-2t_1)}\\ &= \frac{b\delta(t_1+t_2-t)}{2\pi b}\frac{(2\pi b)G_b(-ibt_1+ib\lambda)}{(-2\pi ib^2)^{-it_1+i\lambda}}\frac{(2\pi b)G_b(-ibt_2-ib\lambda)}{(-2\pi ib^2)^{-it_2-i\lambda}}\\ &\quad\cdot\frac{(-2\pi ib^2)^{-it_1-it_2}}{(2\pi b)G_b(-ibt_1-ibt_2)}e^{\pi ib^2\lambda(\lambda-2t_1)}.\end{aligned}$$

Now treating this integral transformation kernel formally depending on b, we take the limit using Theorem 3.11 and obtain

$$\begin{aligned}&\to \frac{\delta(t_1+t_2-t)}{2\pi}\frac{\Gamma(-it_1+i\lambda)\Gamma(-it_2-i\lambda)}{\Gamma(-it_1-it_2)}\\ &= \frac{1}{2\pi}\delta(t_1+t_2-t)\frac{\Gamma(-it_1+i\lambda)\Gamma(-it_2-i\lambda)}{\Gamma(-it)},\end{aligned}$$

which is precisely the classical intertwiner $\begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix}_{\text{classical}}$.

Similarly, we have

$$b^2\mathcal{F}\begin{bmatrix} b\lambda & bt \\ bt_1 & bt_2 \end{bmatrix}_* \to \frac{1}{2\pi}\delta(t_1+t_2-t)\frac{\Gamma(-i\lambda+it_1)\Gamma(it_2+i\lambda)}{\Gamma(it)} = \begin{bmatrix} \lambda & t \\ t_1 & t_2 \end{bmatrix}_{\text{classical}}.$$

□

Therefore, we conclude that the quantum dilogarithm transform between tensor product representations of the quantum planes, is in a certain sense a quantized version of the intertwiners of the tensor product representations of the classical $ax+b$ group. This method of rescaling by the parameter b is essentially the key step in obtaining information of the classical counterpart from the quantum modular double, which does not have a direct classical limit due to the dual number $Q = b+b^{-1}$ appearing in the transformations.

6. Corepresentation

In order to compare the classical representation of the $ax+b$ group, and shed light on what kind of intertwiners the above transforms are, as explained in the introduction we need to find a corepresentation of the quantum plane $\mathcal{A}_q$ generated by positive self-adjoint elements A, B with $AB = q^2BA$, $|q|$=1, dual to $\mathcal{B}_q$, with the same coproduct given by

$$\Delta(A) = A\otimes A, \quad \Delta(B) = B\otimes A + 1\otimes B.$$

The corepresentation should possess a limit that goes to the classical representation. Since the action of $\mathcal{B}_q$ above is a left action, we expect to obtain a right corepresentation of $\mathcal{A}_q$.

The basic idea is to define a C^*-algebra $C_\infty(\mathcal{A}_q)$ of "functions vanishing at infinity" of the quantum plane $\mathcal{A}_q$. The technical details are given in [9]. Here we will briefly recall the motivation and its construction.

6.1. *Algebra of continuous functions vanishing at infinity*

Before defining $C_\infty(\mathcal{A}_q)$, let's look at the classical $ax+b$ group again. Denote the group by G and the positive semigroup by $G_+ = \{(a,b)|a>0, b>0\}$.

Consider the restriction of a rapidly decreasing analytic function $f(a,b)$ of G, to the semigroup G_+. Then the function is continuous at $b=0$, hence it has at most $O(1)$ growth as $b\to 0^+$.

Hence using the Mellin transform, we can write

$$f(a,b) = \int_{-i\infty}^{i\infty}\int_{c-i\infty}^{c+i\infty} F(s,t)a^{-s}b^{-t}dtds, \tag{6.1}$$

where $c>0$ and

$$F(s,t) = \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\infty f(a,b)a^{s-1}b^{t-1}dadb \tag{6.2}$$

is entire analytic with respect to s, and holomorphic on $\mathrm{Im}(t) > 0$. According to Proposition 2.3, $F(s,t)$ has rapid decay in s,t in the imaginary direction, and can be analytically continued to $\mathrm{Im}(t) \leq 0$ such that it is meromorphic with simple poles. Since the function $f(a,b)$ is analytic at $b = 0$, the analytic structure of $f(a,b)$ on b is given by $\sum_{k=0}^{\infty} A_k b^k$ for some constant A_k, hence according to Proposition 2.3, $F(s,t)$ has possible simple poles at $t = -n$ for $n = 0, 1, 2, \ldots$.

Therefore (changing the integration to the real axis), we conclude the following proposition.

Proposition 6.1. *The continuous functions of G_+, continuous at $b = 0$ and vanishing at infinity, is given by*

$$C_\infty(G)|_{G_+} = \textit{sup norm closure of } \mathcal{A}^\infty(G_+),$$

where

$$\mathcal{A}^\infty(G_+) := \textit{Linear span of } \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}+i0} f_1(s) f_2(t) a^{is} b^{it} ds dt \right\} \tag{6.3}$$

for $f_1(s)$ entire analytic in s, $f_2(t)$ meromorphic in t with possible simple poles at $t \in -in$, $n = 0, 1, 2, \ldots$, and for fixed $v > 0$, both the function $f_1(s + iv)$ and $f_2(t + iv)$ is of rapid decay.

Note that this also coincides with

$$C_\infty(G)|_{G_+} = \text{sup norm closure of } \{g(\log a) f(b) | g \in C_\infty(\mathbb{R}); f \in C_\infty[0, \infty)\},$$

where C_∞ denote functions vanishing at infinity.

We can also introduce an L^2 norm on functions of G_+ given by

$$\|f(a,b)\|_2 = \int_{\mathbb{R}} \int_{\mathbb{R}+\frac{1}{2}i} |f_1(s) f_2(t)|^2 dt ds \tag{6.4}$$

according to the Parseval's formula for the Mellin transform.

Due to the appearance of the quantum dilogarithm function $G_b(iz)$ in the expression of the corepresentation in the next section, following the same line above, we define $C_\infty(\mathcal{A}_q)$ as follows.

Definition 6.2. The $C_\infty(\mathcal{A}_q)$ space is the (operator) norm closure of $\mathcal{A}^\infty(\mathcal{A}_q)$ where

$$\mathcal{A}^\infty(\mathcal{A}_q) := \text{ Linear span of } \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}+i0} f_1(s) f_2(t) A^{ib^{-1}s} B^{ib^{-1}t} ds dt \right\} \tag{6.5}$$

for $f_1(s)$ entire analytic in s, $f_2(t)$ meromorphic in t with possible simple poles at

$$t = -ibn - i\frac{m}{b}, \quad n, m = 0, 1, 2, \ldots$$

and for fixed $v > 0$, the function $f_1(s+iv)$ and $f_2(t+iv)$ is of rapid decay. To define the norm, we realize $A^{ib^{-1}s} f(x) = e^{2\pi i s} f(x)$ and $B^{ib^{-1}t} f(x) = e^{2\pi i p} f(x) = f(x+1)$ as unitary operators on $L^2(\mathbb{R})$, so that $C_\infty(\mathcal{A}_q)$ is generated by bounded operators.

Note that in the language of [29], the generators A and B are *affiliated* with $C_\infty(\mathcal{A}_q)$. As discussed in [9], we can also introduce an L^2-norm given by

$$\|f(A,B)\|_2 = \int_{\mathbb{R}} \int_{\mathbb{R}+\frac{iQ}{2}} |f_1(s)f_2(t)|^2 dtds, \tag{6.6}$$

where $Q = b + b^{-1}$. However, we will focus on the C^*-theory in the remaining sections.

Remark 6.3. The above space $\mathcal{A}^\infty(\mathcal{A}_q)$ can be rewritten, according to the Mellin transform, as

$$\mathcal{A}^\infty(\mathcal{A}_q) := \text{ Linear span of } \{g(\log A)f(B)\},$$

where $g(x)$ is entire analytic in x and for every fixed v, $g(x+iv)$ is of rapid decay in x; $f(y)$ is a smooth function in y of rapid decay such that it admits a Puiseux series representation

$$f(y) \sim \sum_{n,m=0}^{\infty} \alpha_{mn} y^{n+m/b^2} \tag{6.7}$$

at $y = 0$.

Recall that the modular double elements [3] are given by non-integral power

$$\widetilde{A} = A^{\frac{1}{b^2}} \quad \widetilde{B} = B^{\frac{1}{b^2}}.$$

Together with the fact that $g(x)$ is entire analytic in $\log A$, it suggests that the space $\mathcal{A}^\infty(\mathcal{A}_q)$ actually includes "$\mathcal{A}^\infty$ functions" on the space of the modular double $\mathcal{A}_{q\widetilde{q}}$ as well. See [9] for further details.

6.2. *Multiplicative unitary*

Given a C^*-algebra $\mathcal{A}$ considered as a subspace of bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, we will denote by

$$M(\mathcal{A}) = \{B \in \mathcal{B}(\mathcal{H}) | B\mathcal{A} \subset \mathcal{A}, \mathcal{A}B \subset \mathcal{A}\}$$

the multiplier algebra of $\mathcal{A}$ viewed as a subset of $\mathcal{B}(\mathcal{H})$, and we let $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ denotes the compact operators acting on $\mathcal{H}$.

Multiplicative unitaries are fundamental to the theory of quantum groups in the setting of C^*-algebras and von Neumann algebras. It is one single map that encodes all structure maps of a quantum group and of its generalized Pontryagin dual simultaneously [25]. In particular, we can construct out of the multiplicative unitary a coproduct as well as a corepresentation of the quantum group. Here, we recall the basic properties of the multiplicative unitary, and the construction of the multiplicative unitary defined in [29] on the $ax+b$ quantum group $\mathcal{A}$ (see also [19]).

Definition 6.4. A unitary element $W \in \mathcal{A} \otimes \mathcal{A}$ is called a multiplicative unitary if it satisfies the pentagon equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}. \tag{6.8}$$

A multiplicative unitary provides us with the coproduct of the *multiplier* Hopf algebra:

$$\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$$

given by

$$\Delta(c) = W(c \otimes 1)W^*, \quad c \in \mathcal{A}. \tag{6.9}$$

Proposition 6.5. *The pentagon equation* (6.8) *implies the coassociativity of the coproduct defined by* (6.9).

By representing the first copy of $\mathcal{A}$ in W as bounded operator on a Hilbert space $\mathcal{H}$, we obtain a unitary element $V \in M(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$ which represents a (right) corepresentation $\mathcal{H} \to \mathcal{H} \otimes M(\mathcal{A})$. More precisely, we have the following proposition.

Proposition 6.6. *The unitary element $V \in M(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$ satisfies*

$$(1 \otimes \Delta)V = V_{12}V_{13} \tag{6.10}$$

or formally

$$(1 \otimes \Delta) \circ \Pi = (\Pi \otimes 1) \circ \Pi, \tag{6.11}$$

where Δ is given by (6.9) *and $\Pi : \mathcal{H} \to \mathcal{H} \otimes M(\mathcal{A})$ is given by*

$$\Pi(v) := V(v \otimes 1). \tag{6.12}$$

We will now focus on the case where $\mathcal{A} = C_\infty(\mathcal{A}_q)$ is the quantum plane C^*-algebra. Using the notations from [29], we have the following.

Proposition 6.7 ([29]). *Consider the quantum plane $C_\infty(\mathcal{A}_q)$ generated by positive self-adjoint elements A, B affiliated with $\mathcal{A}$, with $AB = q^2BA$ in the sense of Definition* 6.2, *with coproduct defined on the generators*

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes A + 1 \otimes B. \tag{6.13}$$

Then the multiplicative unitary W is given by:

$$W = V_\theta(\log(\widehat{B} \otimes sq^{-1}BA^{-1}))^* e^{\frac{i}{\hbar}\log \widehat{A} \otimes \log A^{-1}} \in C_\infty(\mathcal{A}_q) \otimes C_\infty(\mathcal{A}_q), \tag{6.14}$$

where $q = e^{-i\hbar}, \theta = \frac{2\pi}{\hbar}$, the admissible pair $\widehat{B} := B^{-1}$ and $\widehat{A} := qAB^{-1}$, and $s \in \mathbb{R}_{>0}$ is a constant. Note that in our case $\hbar = 2\pi b^2$.

Here the special function $V_\theta(z)$ is defined as

$$V_\theta(z) = \exp\left\{\frac{1}{2\pi i}\int_0^\infty \log(1 + a^{-\theta})\frac{da}{a + e^{-z}}\right\}. \tag{6.15}$$

Lemma 6.8. *$V_\theta(z)$ and $G_b(z)$ are related by the following formula*:

$$V_{1/b^2}(z) = \zeta_b G_b\left(\frac{Q}{2} - \frac{iz}{2\pi b}\right) = \frac{1}{g_b(e^z)}, \tag{6.16}$$

and the complex conjugation is given by

$$V_{1/b^2}(z)^* = \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} - \frac{iz}{2\pi b})} = g_b(e^z), \tag{6.17}$$

where we recall $\zeta_b = e^{\frac{\pi i}{4} + \frac{\pi i}{12}(b^2+b^{-2})}$.

Proof. In order to rewrite $V_\theta(z)$ in terms of $G_b(z)$, we pass to Ruijsenaars's more general hyperbolic gamma function (4.15). From [21, (A.18)], we have

$$V_\theta(z) = G(2\pi, 2\pi/\theta; z)\exp\left(-i\theta z^2/8\pi - \frac{\pi i}{24}\left(\theta + \frac{1}{\theta}\right)\right)$$

with $\theta = \frac{2\pi}{\hbar} = \frac{1}{b^2}$.

Also using

$$G(a_+; a_-; z) = G\left(1, \frac{a_+}{a_-}; \frac{z}{a_-}\right)$$

and (4.16):

$$G(b, b^{-1}, z) = e^{\pi i z^2/2} e^{\pi i Q^2/8} G_b\left(\frac{Q}{2} - iz\right)$$

we obtain

$$V_{1/b^2}(z) = \zeta_b G_b\left(\frac{Q}{2} - \frac{iz}{2\pi b}\right)$$

and the complex conjugation

$$V_{1/b^2}(z)^* = \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} - \frac{iz}{2\pi b})}. \qquad \square$$

Remark 6.9. Since we are using the "transpose" of $\mathcal{A}$ in [29], our W is related to that in [29] by

$$A = a^{-1}, \quad B = -qba^{-1},$$

i.e. they are related by the antipode associated to $\mathcal{A}$. Furthermore, the choice of the multiplicative unitary is different from [9], in which there we used instead the GNS representations to obtain the canonical W. In particular, W is not *manageable* in the current setting as pointed out in [29], and furthermore the result from Proposition 6.26 below is different from that of [9]. It turns out that this discrepancy leads to a new functional relation between the quantum dilogarithm function $G_b(x)$ discussed in the last section of [9].

6.3. *Corepresentation of $C_\infty(\mathcal{A}_q)$*

We can now define the coaction of the quantum space $C_\infty(\mathcal{A}_q)$:

Theorem 6.10. *For the choice* $s = 2\sin\pi b^2 \in \mathbb{R}_+$, *the multiplicative unitary* W *defined in* (6.14) *induces a unitary* (*right*) *coaction of the quantum space* $C_\infty(\mathcal{A}_q)$

on $\mathcal{H} = L^2(\mathbb{R})$ by

$$\Pi : \mathcal{H} \to \mathcal{H} \otimes M(C_\infty(\mathcal{A}_q)),$$
$$f(t) \mapsto F(x) := \int_{\mathbb{R}+i0} f(t) e^{\pi Q(t-x)} \frac{G_b(ix - it)}{(2\sin \pi b^2)^{ib^{-1}(x-t)}} \cdot A^{ib^{-1}x} B^{ib^{-1}(t-x)} dt, \tag{6.18}$$

where $f(z) \in \mathcal{W}$, and extends to $\mathcal{H}$ by density.

Remark 6.11. The choice of s is made so that we will obtain classical limit from G_b, as well as the necessary pairing in order to get the representation of $\mathcal{B}_q$ in the next subsection.

Proof. The element W can be reinterpreted as an element

$$V \in M(\mathcal{K}(\mathcal{H}) \otimes C_\infty(\mathcal{A}_q)) \tag{6.19}$$

by letting $\widehat{A}, \widehat{B}$ act on $\mathcal{H} = L^2(\mathbb{R})$, hence giving rise to a corepresentation of $C_\infty(\mathcal{A}_q)$. We start with $A = e^{2\pi bx}, B = e^{2\pi bp}$, so that the action is given by

$$\widehat{A} = qAB^{-1} = qe^{2\pi bx}e^{-2\pi bp} = e^{2\pi b(x-p)}, \tag{6.20}$$
$$\widehat{B} = B^{-1} = e^{-2\pi bp}. \tag{6.21}$$

However, the action is nontrivial in the factor

$$e^{\frac{i}{2\pi b^2} \log \widehat{A} \otimes \log A}.$$

Hence, we introduce a change of variables (of order 3) on $L^2(\mathbb{R})$ given by Kashaev [13, 6]:

$$\widetilde{\mathbf{A}} : f(\alpha) \mapsto F(\beta) = \int_{\mathbb{R}} e^{2\pi i\alpha\beta} e^{\pi i\beta^2 - \pi i/12} f(\alpha) d\alpha \tag{6.22}$$

such that

$$\widetilde{\mathbf{A}}^{-1} x \widetilde{\mathbf{A}} = -p,$$
$$\widetilde{\mathbf{A}}^{-1} p \widetilde{\mathbf{A}} = x - p.$$

Then the operator $\widehat{A}$ and $\widehat{B}$ becomes:

$$\widetilde{\mathbf{A}}^{-1} \widehat{A} \widetilde{\mathbf{A}} = e^{-2\pi bx}, \tag{6.23}$$
$$\widetilde{\mathbf{A}}^{-1} \widehat{B} \widetilde{\mathbf{A}} = e^{2\pi b(-x+p)} = qe^{-2\pi bx}e^{2\pi bp}. \tag{6.24}$$

Hence given a function $f(x) \in L^2(\mathbb{R})$, we have

$$e^{\frac{i}{2\pi b^2} \log \widehat{A} \otimes \log A^{-1}} f(x) = e^{\frac{i}{2\pi b^2}(-2\pi bx) \log A^{-1}} f(x)$$
$$= f(x) A^{ib^{-1}x}.$$

Next we deal with the quantum dilogarithm function $V_\theta(z)$. From the Fourier transform formula (Lemma 3.5), we found from (6.17)

$$V_{1/b^2}(z)^* = \int_{\mathbb{R}+i0} e^{ib^{-1}tz} e^{\pi Qt} G_b(-it)dt. \tag{6.25}$$

Hence the operator V (6.19) acts as

$$\begin{aligned}
(Vf)(x) &= V_{1/b^2}(\log(\widehat{B} \otimes q^{-1}sBA^{-1}))^* \cdot (f(x)A^{ib^{-1}x}) \\
&= \left(\int_{\mathbb{R}+i0} (\widehat{B} \otimes (q^{-1}sBA^{-1}))^{ib^{-1}t} e^{\pi Qt} G_b(-it)dt\right) \cdot (f(x)A^{ib^{-1}x}) \\
&= \left(\int_{\mathbb{R}+i0} (\widehat{B}^{ib^{-1}t} \otimes (q^{-1}sBA^{-1})^{ib^{-1}t}) e^{\pi Qt} G_b(-it)dt\right) \cdot (f(x)A^{ib^{-1}x}).
\end{aligned}$$

Now $\widehat{B}$ formally acts as $qe^{-2\pi bx}f(x-ib)$, and by induction

$$\widehat{B}^n f(x) = q^{n^2} e^{-2\pi bnx} f(x-ibn).$$

Hence using functional calculus, $\widehat{B}^{ib^{-1}t}$ acts (as a unitary operator) by

$$\widehat{B}^{ib^{-1}t} \cdot f(x) = q^{-b^{-2}t^2} e^{-2\pi itx} f(x+t) = e^{-\pi it^2 - 2\pi itx} f(x+t).$$

Next $(sq^{-1}BA^{-1})^{ib^{-1}t}$ can be split using the relation

$$(BA^{-1})^n = q^{-n(n-1)} B^n A^{-n},$$

we have

$$\begin{aligned}
(sq^{-1}BA^{-1})^{ib^{-1}t} &= s^{ib^{-1}t} q^{-ib^{-1}t} q^{b^{-2}t^2 + ib^{-1}t} B^{ib^{-1}t} A^{-ib^{-1}t} \\
&= s^{ib^{-1}t} e^{\pi it^2} B^{ib^{-1}t} A^{-ib^{-1}t}.
\end{aligned}$$

Combining, we obtain

$$\begin{aligned}
(Vf)(x) &= \int_{\mathbb{R}+i0} e^{-\pi it^2 - 2\pi itx} e^{\pi Qt} q^{-2tx} G_b(-it) s^{ib^{-1}t} \\
&\quad \cdot e^{\pi it^2} B^{ib^{-1}t} A^{-ib^{-1}t} A^{ib^{-1}(x+t)} f(x+t)dt \\
&= \int_{\mathbb{R}+i0} e^{\pi Qt} e^{-2\pi itx} G_b(-it) s^{ib^{-1}t} B^{ib^{-1}t} f(x+t) A^{ib^{-1}x} dt \\
&= \int_{\mathbb{R}+i0} f(x+t) e^{\pi Qt} G_b(-it) s^{ib^{-1}t} A^{ib^{-1}x} B^{ib^{-1}t} dt \\
&= \int_{\mathbb{R}+i0} f(t) e^{\pi Q(t-x)} G_b(ix-it) s^{ib^{-1}(t-x)} A^{ib^{-1}x} B^{ib^{-1}(t-x)} dt.
\end{aligned}$$

Now by setting

$$s = 2\sin \pi b^2 = i(q^{-1} - q) \in \mathbb{R}_{>0}$$

we obtain

$$= \int_{\mathbb{R}+i0} f(t)e^{\pi Q(t-x)} \frac{G_b(ix-it)}{(2\sin\pi b^2)^{ib^{-1}(x-t)}} A^{ib^{-1}x} B^{ib^{-1}(t-x)} dt$$

as desired. We see that the integrand is bounded by the asymptotic properties of $G_b(ix)$. □

Starting from the coaction formula, we can also see that it is a corepresentation by manipulating the functional properties of the special function $G_b(x)$ directly.

Corollary 6.12. *The coaction satisfies*

$$(1 \otimes \Delta) \circ \Pi = (\Pi \otimes 1) \circ \Pi$$

as a map from $\mathcal{H}$ to $\mathcal{H} \otimes M(C_\infty(\mathcal{A}_q) \otimes C_\infty(\mathcal{A}_q))$, where we recall that Δ is the coproduct of $\mathcal{A}_q$ given by

$$\Delta(A) = A \otimes A,$$
$$\Delta(B) = B \otimes A + 1 \otimes B$$

and extend to the multiplier Hopf algebra $C_\infty(\mathcal{A}_q)$ by

$$\Delta\left(\int_{\mathbb{R}}\int_{\mathbb{R}+i0} F(s,t)A^{is}B^{it}dsdt\right) := \int_{\mathbb{R}}\int_{\mathbb{R}+i0} F(s,t)\Delta(A^{is}B^{it})dsdt.$$

Proof. We check the corepresentation axioms formally.

First note that since A, B are positive self-adjoint, the coproduct $\Delta(A)$ and $\Delta(B)$ is still positive essentially self-adjoint, hence it is well-defined. (We do not run into the problem of choosing self-adjoint extension as in [29] since our B is positive.)

For notational convenience, without loss of generality we scale $b^{-1}x$ and $b^{-1}z$ to x and z, respectively. We need to calculate the coproduct $\Delta(A^{ix}B^{iz-ix})$:

$$\begin{aligned}
\Delta(A^{ix}B^{iz-ix}) &= \Delta(A)^{ix}\Delta(B)^{iz-ix} \\
&= (A \otimes A)^{ix}(B \otimes A + 1 \otimes B)^{iz-ix} \\
&= (A^{ix} \otimes A^{ix})B\int_{\mathbb{R}} d\tau \begin{pmatrix} z-x \\ \tau \end{pmatrix}_b (B \otimes A)^{iz-ix-i\tau}(1 \otimes B)^{i\tau} \\
&= b\int_C d\tau \frac{G_b(ib\tau - ibz + ibx)G_b(-ib\tau)}{G_b(ibx-ibz)}(A^{ix}B^{iz-ix-i\tau}) \otimes (A^{iz-i\tau}B^{i\tau}) \\
&= b\int_C d\tau \frac{G_b(ib\tau + ibx)G_b(-ibz-ib\tau)}{G_b(ibx-ibz)}(A^{ix}B^{-ix-i\tau}) \otimes (A^{-i\tau}B^{i\tau+iz}),
\end{aligned}$$

where the contour C, as before, goes above the poles at $\tau = -z$ and below the poles at $\tau = -x$. Hence we have

$$\begin{aligned}
(1 \otimes \Delta) \circ \Pi f(x) &= b^2 \int_{\mathbb{R}+i0} \int_C f(z) \frac{G_b(ibx - ibz) e^{\pi Q b(z-x)}}{(2 \sin \pi b^2)^{ix-iz}} \\
&\quad \cdot \frac{G_b(ib\tau + ibx) G_b(-ibz - ib\tau)}{G_b(ibx - ibz)} (A^{ix} B^{-ix-i\tau}) \otimes (A^{-i\tau} B^{i\tau+iz}) d\tau dz \\
&= b^2 \int_{\mathbb{R}+i0} \int_C \frac{f(z) e^{\pi Q b(z-x)}}{(2 \sin \pi b^2)^{ix-iz}} G_b(ib\tau + ibx) G_b(-ibz - ib\tau) \\
&\quad \cdot (A^{ix} B^{-ix-i\tau}) \otimes (A^{-i\tau} B^{i\tau+iz}) d\tau dz \\
&= b^2 \int_{\mathbb{R}-i0} \int_{\mathbb{R}+i0} \frac{f(z) e^{\pi Q b(z-x)}}{(2 \sin \pi b^2)^{ix-iz}} G_b(ib\tau + ibx) G_b(-ibz - ib\tau) \\
&\quad \cdot (A^{ix} B^{-ix-i\tau}) \otimes (A^{-i\tau} B^{i\tau+iz}) dz d\tau \\
&= b^2 \int_{\mathbb{R}+i0} \int_{\mathbb{R}+i0} \frac{f(z) e^{\pi Q b(z-x)}}{(2 \sin \pi b^2)^{ix-iz}} G_b(ibx - ibw) G_b(ibw - ibz) \\
&\quad \cdot (A^{ix} B^{iw-ix}) \otimes (A^{iw} B^{iz-iw}) dz dw,
\end{aligned}$$

where in the change of order of integration, the contour is such that $\mathrm{Im}(z) > \mathrm{Im}(\tau)$ and $\mathrm{Im}(\tau) < \mathrm{Im}(x) = 0$, hence the contour of τ after interchanging is shifted to $\mathbb{R} - i0$. The decay properties of G_b on τ guarantee the change of order of integration.

Finally, we have

$$\begin{aligned}
(\Pi \otimes 1) \circ \Pi f(x) &= b^2 \int_{\mathbb{R}+i0} \int_{\mathbb{R}+i0} f(z) \frac{G_b(ibx - ibw) e^{\pi Q b(w-x)}}{(2 \sin \pi b^2)^{ix-iw}} \\
&\quad \cdot \frac{G_b(ibw - ibz) e^{\pi Q b(z-w)}}{(2 \sin \pi b^2)^{iw-iz}} (A^{ix} B^{iw-ix}) \otimes (A^{iw} B^{iz-iw}) dz dw \\
&= b^2 \int_{\mathbb{R}+i0} \int_{\mathbb{R}+i0} \frac{f(z) e^{\pi Q b(z-x)}}{(2 \sin \pi b^2)^{ix-iz}} G_b(ibx - ibw) G_b(ibw - ibz) \\
&\quad \cdot (A^{ix} B^{iw-ix}) \otimes (A^{iw} B^{iz-iw}) dz dw \\
&= (1 \otimes \Delta) \circ \Pi f(x).
\end{aligned}$$

□

After rewriting the coaction explicitly, the relationship between the quantum corepresentation and the classical $ax + b$ group representation becomes clear.

Theorem 6.13. *Under the scaling by $x \to bx$ in the sense of Theorem* 5.2, *the limit of the coaction* (6.19) *is precisely the representation R_+ of the $ax + b$ group. Similarly, the coaction corresponding to V^* is R_-.*

Proof. Under the scaling, the coaction becomes

$$f(x) \mapsto \int_{\mathbb{R}+i0} \frac{G_b(ibx-ibz)e^{\pi Qb(z-x)}}{(2\sin\pi b^2)^{ix-iz}} A^{ix}B^{iz-ix}f(z)dz.$$

Using the limit formula (3.38) for $G_b(ibx)$, we have:

$$\begin{aligned}
&= \int_{\mathbb{R}+i0} \frac{(2\pi b)G_b(ibx-ibz)e^{\pi b^2(z-x)}e^{\pi(z-x)}(e^{-\frac{\pi i}{2}})^{ix-iz}}{(-2i\sin\pi b^2)^{ix-iz}} A^{ix}B^{iz-ix}f(z)dz \\
&= \frac{1}{2\pi}\left(\frac{\pi b^2}{\sin\pi b^2}\right)^{ix-iz} \int_{\mathbb{R}+i0} \frac{(2\pi b)G_b(ibx-ibz)}{(-2i\pi b^2)^{ix-iz}} e^{\pi b^2(z-x)}A^{ix}(-iB)^{iz-ix}f(z)dz \\
&\to \frac{1}{2\pi}\int_{\mathbb{R}+i0} \Gamma(ix-iz)A^{ix}(-iB)^{iz-ix}f(z)dz \\
&= R_+f(x).
\end{aligned}$$

Taking the conjugate of the above formula and renaming the variables, we see that the coaction corresponding to V^* is precisely R_-. □

Proposition 6.14 ([29, (4.19)]). *The space $C_\infty(\mathcal{A}_q)$ can be recovered from the multiplicative unitary $V \in M(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A}_q)$ by*

$$C_\infty(\mathcal{A}_q) = \textit{norm closure of } \{(\omega\otimes 1)V + (\omega'\otimes 1)V^* \mid \omega,\omega' \in \mathcal{B}(\mathcal{H})^*\}. \tag{6.26}$$

Recall that V corresponds to the representation R_+ and similarly V^* corresponds to R_- under the classical limit. Therefore in the classical "$ax+b$" group, the above translates to the fact that the space of C_∞ functions on G_+ is spanned by matrix coefficients

$$\frac{1}{2\pi}\Gamma(-iz)a^{iw}(-ib)^{iz}, \quad \frac{1}{2\pi}\Gamma(-iz)a^{iw}(ib)^{iz} \tag{6.27}$$

corresponding to V and V^*.

In order to understand this more explicitly, note that for functions on G_+ of the form

$$F(a,b) := g(\log a)f(b),$$

where $g \in L^2(\mathbb{R}), f \in L^2([0,\infty))$ are analytic, we can write using Fourier transform as

$$F(a,b) = \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{g}(s)a^{is}\widehat{f}(x)e^{ibx}dxds,$$

and then using formally the Mellin transform for $x>0$:

$$e^{\pm ibx} = \int_{\mathbb{R}+i0} \Gamma(-it)(\pm ibx)^{it}dt, \tag{6.28}$$

we see that the function $F(a,b)$ can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}+i0}&\widehat{g}(s)\widetilde{f_+}(t)\Gamma(-it)e^{\frac{\pi t}{2}}a^{is}b^{it}dtds\\&+\int_{\mathbb{R}}\int_{\mathbb{R}+i0}\widehat{g}(s)\widetilde{f_-}(t)\Gamma(-it)e^{-\frac{\pi t}{2}}a^{is}b^{it}dtds,\end{aligned}\tag{6.29}$$

where

$$\widetilde{f_\pm}(t)=\int_0^\infty \widehat{f}(\pm x)x^{it}dx$$

is analytic in $0<\mathrm{Im}(t)<1$ and of rapid decay in this strip.

Therefore Proposition 6.26 can be interpreted as a form of "Peter–Weyl" theorem for the quantum group $\mathcal{A}_q$, which says that the space of C_∞ functions on $\mathcal{A}_q$ is spanned continuously by matrix coefficients of its unitary corepresentations.

Remark 6.15. One should compare this result with a similar one obtained in [9], where a different multiplicative unitary W constructed from certain GNS representation is used, so that only the canonical representations appear in the L^2-decomposition.

6.4. *Pairing and representation of $\mathcal{B}_q$*

Recall that given a non-degenerate Hopf pairing $\langle\,,\rangle$, from a corepresentation of a Hopf algebra $\mathcal{A}$, we can construct a corresponding representation of the dual Hopf algebra $\mathcal{B}$ by

$$\mathcal{B}\otimes\mathcal{H}\xrightarrow{1\otimes\Pi}\mathcal{B}\otimes(\mathcal{H}\otimes\mathcal{A})=(\mathcal{B}\otimes\mathcal{A})\otimes\mathcal{H}\xrightarrow{\langle\,,\rangle\otimes Id}\mathcal{H}.$$

Let us now define the pairing between the generators (A,B) of $\mathcal{A}_q$ and (X,Y) of $\mathcal{B}_q$ as follow:

Definition 6.16. We define

$$\langle A,X\rangle=q^{-2},\quad \langle A,Y\rangle=0,$$
$$\langle B,X\rangle=0,\quad \langle B,Y\rangle=-i.$$

Then they satisfy the coproduct relations with

$$\langle A^nB^m,X\rangle=\langle A,X\rangle^n\delta_{m0}=q^{-2n}\delta_{m0},$$
$$\langle A^nB^m,Y\rangle=\langle A^n,1\rangle\langle B^m,Y\rangle=-i\delta_{m1}.$$

From this pairing, we can formally extend the pairing to elements in the subclass of $M(C_\infty(\mathcal{A}_q))$. Let $\mathcal{D}$ denote the image of $\mathcal{W}$ under the corepresentation Π to $\mathcal{H}\otimes M(C_\infty(\mathcal{A}_q))$. Then

$$\mathcal{D}\subset BC(\mathbb{R})\otimes\mathcal{E}\subset BC(\mathbb{R})\otimes\mathcal{F},$$

where $BC(\mathbb{R})$ are bounded continuous functions on $\mathbb{R}$;

$$\mathcal{E} = \text{Linear span of } \left\{A^{is}\int_{\mathbb{R}+i0} F(t)B^{it}dt\right\},$$

where $F(t)$ is the same as in the definition of $\mathcal{A}_\infty(\mathcal{A}_q)$: meromorphic with possible poles at $t = -in - im/b^2$, and of rapid decay along imaginary direction;

$$\mathcal{F} = \text{Linear span of } \left\{g(\log A)\int_{\mathbb{R}+i0} F(t)B^{it}dt\right\},$$

where $F(t)$ is as above, and $g(s)$ is a bounded function on $\mathbb{R}$ that can be analytically extended to $\mathrm{Im}s = -2\pi ib^2$. Then we define the pairing with X and Y by formally extracting the zeroth and first power of B, respectively. More precisely, we have the following definition.

Definition 6.17. We define X, Y as elements in the dual space $\mathcal{F}^*$ by

$$\left\langle \frac{i}{2\pi} g(\log A)\int_{\mathbb{R}+i0} F(t)B^{it}dt, X\right\rangle = g(\log q^2)(\mathrm{Res}_{t=0}F(t)),$$

$$\left\langle \frac{i}{2\pi} g(\log A)\int_{\mathbb{R}+i0} F(t)B^{it}dt, Y\right\rangle = -i(\mathrm{Res}_{t=-i}F(t)).$$

Theorem 6.18. *The representation of $\mathcal{B}_q$ on $\mathcal{W}$ given by*

$$\mathcal{B}_q : \mathcal{W} \xrightarrow{\Pi} BC(\mathbb{R}) \otimes \mathcal{F} \xrightarrow{1\otimes\langle -,\mathcal{B}_q\rangle} BC(\mathbb{R}),$$

induced from the corepresentation (6.19) *under the above pairing is precisely*

$$X \cdot f(x) = e^{2\pi bx} f(x),$$
$$Y \cdot f(x) = f(x - ib) = e^{2\pi bp} f(x),$$

which is the Fourier transformed action of (4.4) *and* (4.5) *defined in* [6].

Note that the image of $\mathcal{W}$ is actually preserved in $\mathcal{W} \subset BC(\mathbb{R})$.

Proof. Applying the pairing, and introducing the scaling of b in dz, we obtain for any $f(x) \in \mathcal{W}$:

$$\langle \Pi f, 1 \otimes X\rangle = \left\langle \int_{\mathbb{R}+i0} f(z)\frac{G_b(ix - iz)e^{\pi Q(z-x)}}{(2\sin \pi b^2)^{ib^{-1}(x-z)}} A^{ib^{-1}x}B^{ib^{-1}(z-x)}dz, X\right\rangle$$

changing z to $bz + x$:

$$= \left\langle \int_{\mathbb{R}+i0} f(bz + x)\frac{bG_b(-ibz)e^{\pi Qbz}}{(2\sin \pi b^2)^{-iz}} A^{ib^{-1}x}B^{iz}dz, X\right\rangle$$

$$= (-2\pi i)f(x)q^{-2(ib^{-1}x)}b(\mathrm{Res}_{z=0}G_b(-ibz))$$

$$= e^{2\pi bx} f(x),$$

since $\lim_{x\to 0} xG_b(x) = \frac{1}{2\pi}$, hence $\mathrm{Res}_{z=0}G_b(-ibz) = \frac{1}{-2\pi ib}$.

So the action for X is

$$X \cdot f(x) = e^{2\pi bx} f(x).$$

For the action of Y we have

$$\begin{aligned}\langle \Pi f, 1 \otimes Y\rangle &= \left\langle \int_{\mathbb{R}+i0} f(z) \frac{G_b(ix - iz)e^{\pi Q(z-x)}}{(2\sin \pi b^2)^{ib^{-1}(x-z)}} A^{ib^{-1}x} B^{ib^{-1}(z-x)} dz, Y \right\rangle \\ &= \left\langle \int_{\mathbb{R}+i0} f(bz + x) \frac{bG_b(-ibz)e^{\pi z(1+b^2)}}{(2\sin \pi b^2)^{-iz}} A^{ib^{-1}x} B^{iz} dz, Y \right\rangle \\ &= (-i)(-2\pi i) f(x - ib) b(-q^{-1}) i(q^{-1} - q)(\mathrm{Res}_{z=-i} G_b(-ibz)) \\ &= f(x - ib),\end{aligned}$$

where

$$\begin{aligned}\mathrm{Res}_{z=-i} G_b(-ibz) &= \lim_{z\to -i} (z + i) G_b(-ibz) \\ &= \lim_{z\to 0} z G_b(-ibz - b) \\ &= \lim_{z\to 0} z \frac{G_b(-ibz)}{1 - e^{2\pi ib(-ibz-b)}} \\ &= \frac{1}{-2\pi ib} \frac{1}{1 - e^{-2\pi ib^2}} \\ &= \frac{1}{-2\pi ib} \frac{1}{1 - q^{-2}}.\end{aligned}$$

So the action for Y is

$$Y \cdot f(x) = f(x - ib)$$

or $Y = e^{2\pi bp}$. □

Remark 6.19. If we choose to work with R_-, then under the pairing we will get instead $X = e^{2\pi bx}$ and $Y = -e^{2\pi bp}$, another representation for $\mathcal{B}_q$ by negative operator Y.

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