

POSITIVE REPRESENTATIONS OF SPLIT REAL SIMPLY-LACED QUANTUM GROUPS

IVAN C.H. IP

Dedicated to Igor Frenkel on his 60th birthday

ABSTRACT. We construct the positive principal series representations for $U_q(\mathfrak{g}_{\mathbb{R}})$ where \mathfrak{g} is of simply-laced type, parametrized by \mathbb{R}^r where r is the rank of \mathfrak{g} . In particular, the positivity of the operators and the transcendental relations between the generators of the modular double are shown. We define the modified quantum group $\mathbf{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ of the modular double and show that the representation of both parts of the modular double commute with each other, there is an embedding into the q -tori polynomials, and the commutant is the Langlands dual. We write down explicitly the action for type A_n, D_n and give the details of calculations for type E_6, E_7 and E_8 .

1. INTRODUCTION

In this paper, we give the construction of the positive principal series representation for the modular double $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ for simply-laced \mathfrak{g} , generalizing our recent work [2] where the positive representations for the modular double $U_{q\tilde{q}}(\mathfrak{sl}(n, \mathbb{R}))$ is constructed. This result strengthens the perspectives discussed in [2] for a new direction of representation theory of split real quantum groups since the discovery of the concept of modular double for quantum groups [1], and in the case for $U_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ the representations by Ponsot and Teschner [8].

Let E_i, F_i, K_i be the generators of $U_q(\mathfrak{g}_{\mathbb{R}})$ with the standard quantum relations, where $q = e^{\pi i b^2}$, $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ and $0 < b < 1$. Similarly let $\tilde{E}_i, \tilde{F}_i, \tilde{K}_i$ be the generators of $U_{\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ by replacing b with b^{-1} , where $\tilde{q} = e^{\pi i b^{-2}}$. Then using the rescaled variables

$$e_i = 2 \sin(\pi b^2) E_i, \quad f_i = 2 \sin(\pi b^2) F_i$$

and similarly for \tilde{e}_i and \tilde{f}_i with b replaced by b^{-1} , the positive representations has the following remarkable properties:

- (i) the generators $e_i, f_i, K_i^{\pm 1}$ and $\tilde{e}_i, \tilde{f}_i, \tilde{K}_i^{\pm 1}$ are represented by positive essentially self-adjoint operators,
- (ii) the generators satisfy the transcendental relations

$$e_i^{\frac{1}{b^2}} = \tilde{e}_i, \quad f_i^{\frac{1}{b^2}} = \tilde{f}_i, \quad K_i^{\frac{1}{b^2}} = \tilde{K}_i. \quad (1.1)$$

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Furthermore, by modifying the definition of $e_i, f_i, K_i^{\pm 1}$ and the tilde variables with certain factors of K 's, we also obtain the compatibility with the modular double $\mathbf{U}_{\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$:

(iii) the generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i^{\pm 1}$ commute with $\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{f}}_i, \widetilde{\mathbf{K}}_i^{\pm 1}$.

In the case of $SL(n, \mathbb{R})$, there are two natural coordinate systems on the positive unipotent semi-subgroup $U_{>0}^+$. These are the Lusztig's data parametrized by a given choice of reduced expression of the longest element w_0 , and the cluster coordinates given by the determinants of the square sub-matrices. In this paper, we choose the Lusztig's data as the coordinate of the positive unipotent subspace, since the cluster coordinate for arbitrary type \mathfrak{g} is not very explicit. Another advantage is that the transformation between the coordinates corresponding to different reduced expression of the longest element w_0 can be written explicitly in the Lusztig's coordinate. The main results of the paper are the following:

Main Theorem. *There exists a family of positive principal series representation for $U_q(\mathfrak{g}_{\mathbb{R}})$ and its (modified) modular double $\mathbf{U}_{\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$, parametrized by $\lambda \in \mathbb{R}^r$ where $r = \text{rank}(\mathfrak{g})$, satisfying properties (i)-(iii) above.*

More precisely, for every reduced expression for w_0 , we can construct explicitly the positive representations. For each change of words of w_0 , we establish the following unitary transformation, so that in particular the family of positive representation is *independent of choice* of reduced expression of w_0 .

Theorem 1.1. *The transformation of the operators of $U_q(\mathfrak{g}_{\mathbb{R}})$ corresponding to the change of words*

$$x_i(u)x_j(v)x_i(w) \longleftrightarrow x_j(u')x_i(v')x_j(w')$$

is given by

$$X \mapsto \Phi X \Phi^{-1}, \quad (1.2)$$

where

$$\Phi = T \circ g_b(e^{\pi b(2p_w - 2p_u + u - v + w)}) g_b^*(e^{\pi b(2p_w - 2p_u - u + v - w)}), \quad (1.3)$$

is a unitary transformation, where g_b the quantum dilogarithm function and g_b^* its complex conjugate, while T is a linear transformation of determinant one.

Since the transformation is unitary, it suffices to show the commutation relations, the positivity and the transcendental relations for a specific reduced expression of w_0 . In particular, by choosing a "good" reduced expression for w_0 , the above properties follow immediately.

On the other hand, by choosing the expression for w_0 in a particular way, we have

Theorem 1.2. *The positive representations for type $D_n, n \geq 4$ and E_6, E_7, E_8 are constructed explicitly. In particular the principal series representation for the classical $U(\mathfrak{g}_{\mathbb{R}})$ in terms of finite difference operators can be read off from the expressions.*

We have written down the explicit general expression for D_n in Theorem 8.1, while we simply state the numerical results for type E_n in Theorem 9.1, where the explicit expressions can be found in the appendix of [5].

Furthermore, as in the type A_n case, by using the modified version $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ of the modular double, we have the following important properties.

Theorem 1.3. *We have an embedding*

$$U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) \hookrightarrow \mathbb{C}[\mathbb{T}_{q\tilde{q}}^N]. \quad (1.4)$$

of the modified modular double into the Laurent polynomials generated by N q -tori, where $N = l(w_0) = \dim(U_{>0}^+)$.

Theorem 1.4. *The commutant of $U_q(\mathfrak{g}_{\mathbb{R}})$ is the Langlands dual group $U_{\tilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$.*

Finally, in the positive representations for $U_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$, it is shown in [9] that the representations corresponding to the parameters λ and $-\lambda$ are equivalent under certain transformations involving multiplications by the quantum dilogarithms. In the general case, there is a natural action of the Weyl group on the parameters $\lambda \in \mathbb{R}^r$, where r is the rank of \mathfrak{g} . Then we have the following result:

Theorem 1.5. *The positive representations corresponding to the parameters λ and $w(\lambda)$ where $w \in W, \lambda \in \mathbb{R}^r$ are unitary equivalent. In particular the positive representations are parametrized by $\lambda \in \mathbb{R}_{\geq 0}^r$.*

There are still several problems yet to be answered. A natural question is whether these representations can be generalized to $U_q(\mathfrak{g}_{\mathbb{R}})$ of arbitrary type. In a subsequent work [6], we will construct its positive representations, where it turns out that the transcendental relations play a crucial role relating to its Langlands dual.

In [3] it is shown that under the left and right regular representations of $U_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$, the space $L^2(SL_q^+(2, \mathbb{R}))$, suitably defined, decomposes into a direct integral of tensor products of the positive representations \mathcal{P}_λ . In general, the Haar functional needed to construct the L^2 space structure $L^2(G_q)$ of (the modular double of) the quantized function space $F_{q\tilde{q}}(G)$ is suggested in [4]. Together with the remark after Theorem 12.1, one can ask the following

Conjecture 1.6. *The space $L^2(G_q)$ is decomposed into (a direct integral) of tensor products of the positive representations under the left and right regular representations of $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$.*

Finally, the class of positive representations \mathcal{P}_λ for $U_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ is closed under the tensor product [9]. In particular the positivity and transcendental relations are preserved under the tensor product. A natural question for general \mathfrak{g} is then the following

Conjecture 1.7. *The positivity and transcendental relations completely characterize the class of positive representations for $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$. In particular the class of positive representations is closed under tensor product (as a direct integral).*

The paper is organized as follows. In Section 2 we recall the definition of Lusztig's parametrization for positive unipotent semigroup $U_{>0}^+$, and the transformation between the coordinate corresponding to the change of words for w_0 . In Section 3 we give the construction for $U_q(\mathfrak{g}_{\mathbb{R}})$ on a specific choice of reduced expression of w_0 . In Section 4 we recall the definition and properties of the quantum dilogarithm function needed in

Section 5, where we prove Theorem 1.1 by defining the unitary transformation bringing together the action of $U_q(\mathfrak{g}_{\mathbb{R}})$ for any choice of expression for w_0 . In Section 6 we prove the commutation relations between the generators for simply-laced type $U_q(\mathfrak{g}_{\mathbb{R}})$. In Section 7 and 8 we write explicitly the action for $U_q(\mathfrak{sl}(n, \mathbb{R}))$ and $U_q(\mathfrak{so}(n, n))$ in Lusztig's coordinate. In Section 9, we give explicit results of the calculations of type E_6, E_7 and E_8 . In Section 10 we recall the modified quantum group $\mathbf{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ defined in [2] and state the main theorems about the positive representations of the modular double, its embedding into the q -tori, and the Langlands dual as the commutant. In Section 11, we prove the unitary equivalence between positive representations with parameters related by Weyl actions. Finally in Section 12, we give some remarks on the possible approaches to the conjectures stated in the introduction.

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2. LUSZTIG'S DATA AND TRANSFORMATION

The following are described in detail in [7]. Recall that for any simple root $\alpha_i \in \Delta$ there exists a homomorphism $SL_2(\mathbb{R}) \rightarrow G$ denoted by

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a) \in U_i^+, \quad (2.1)$$

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \mapsto \chi_i(b) \in T, \quad (2.2)$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mapsto y_i(c) \in U_i^-, \quad (2.3)$$

called the *pinning* of G , where T is the split real maximal torus of G , and U_i^+ and U_i^- are the simple root subgroups of U^+ and U^- respectively. Then the positive unipotent semigroup is defined by the image of the map $\mathbb{R}_{>0}^m \rightarrow U^+$ given by

$$(a_1, a_2, \dots, a_m) \mapsto x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_n}(a_m), \quad (2.4)$$

where $s_{i_1}s_{i_2}\dots s_{i_m}$ is a reduced expression for the longest element w_0 of the Weyl group W .

Lemma 2.1. *The map $\mathbb{R}_{>0}^m \rightarrow U^+$ is injective: if*

$$x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_n}(a_m) = x_{i_1}(a'_1)x_{i_2}(a'_2)\dots x_{i_n}(a'_m)$$

then $a_i = a'_i$ for every i .

This follows from the Bruhat decomposition of the group.

Lemma 2.2. *We have the following identities:*

$$\chi_i(b)x_i(a) = x_i(b^2a)\chi_i(b), \quad (2.5)$$

$$x_i(a)y_j(c) = y_j(c)x_i(a) \quad \text{if } i \neq j, \quad (2.6)$$

$$x_i(a)\chi_i(b)y_i(c) = y_i\left(\frac{c}{ac+b^2}\right)\chi_i\left(\frac{ac+b^2}{b}\right)x_i\left(\frac{a}{ac+b^2}\right). \quad (2.7)$$

Assume the roots α_i and α_j are joined by an edge in the Dynkin diagram. Then we have

$$\chi_i(b)x_j(a) = x_j(b^{-1}a)\chi_i(b), \quad (2.8)$$

$$x_i(a)x_j(b)x_i(c) = x_j\left(\frac{bc}{a+c}\right)x_i(a+c)x_j\left(\frac{ab}{a+c}\right). \quad (2.9)$$

3. CONSTRUCTION OF THE REPRESENTATIONS

We will construct the action of the generators E_i, F_i, K_i of $U_q(\mathfrak{g}_{\mathbb{R}})$ on $L^2(\mathbb{R}^N)$ for a particular i using a specific realization of the longest element w_0 of the Weyl group W depending on i . Here $N = l(w_0)$ is the dimension of the positive unipotent subgroup $U_{>0}^+$. Then using Theorem 5.2 we can find the actions corresponding to arbitrary choice of reduced expression of w_0 by a unitary transformation, hence in particular the positivity and the transcendental relations are preserved once we prove it for a particular realization.

First let us recall the classical construction.

Proposition 3.1. *The minimal principal series representation for $\mathcal{U}(\mathfrak{g}_{\mathbb{R}})$ can be realized as the infinitesimal action of $g \in G_{\mathbb{R}}$ acting on $\mathbb{C}[U_{>0}^+]$ by*

$$g \cdot f(h) = \chi_{\lambda}(hg)f([hg]_+). \quad (3.1)$$

Here we write the Gauss decomposition of g as

$$g = g_- g_0 g_+ \in U_{>0}^- T_{>0} U_{>0}^+, \quad (3.2)$$

so that $[g]_+ = g_+$ is the projection of g onto $U_{>0}^+$, and $\chi_{\lambda}(g)$ is the character function defined by

$$\chi_{\lambda}(g) = \prod_{i=1}^r u_i^{2\lambda_i}, \quad (3.3)$$

where r is the rank of $\mathfrak{g}_{\mathbb{R}}$, $\lambda = (\lambda_i) \in \mathbb{C}^r$ and $u_i = \chi_i^{-1}(g_0) \in T_{>0}$.

Let $q = e^{\pi i b^2}$ with $b^2 \in \mathbb{R} \setminus \mathbb{Q}$, $0 < b < 1$,

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$$

and $Q = b + b^{-1}$. Let us also denote the Lusztig's coordinates of $U_{>0}^+$ given in the previous section by x_i^k , where i is the corresponding root index, and k denotes the sequence this root is appearing in w_0 from the right.

Example 3.2. *The coordinates of $U_{>0}^+$ for A_3 corresponding to $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ is given by*

$$(x_3^3, x_2^2, x_1^1, x_3^2, x_2^1, x_3^1)$$

Following the work in [2], we apply the formal Mellin transformation of the form

$$f(u) := \int F(x)x^u dx$$

on each variable, which transforms differential operators into finite difference operators. Then we introduce the following quantization method.

Definition 3.3. *Given a finite difference operator of the form*

$$(1 + \alpha + P(\mathbf{u}))f(\mathbf{u} + \mathbf{e}),$$

in the classical action, where α is a constant, $P(\mathbf{u})$ is linear in the coordinate vector $\mathbf{u} = (u_i^k)$ of the Mellin transformed space of $\mathbb{C}[U_{>0}^+]$, and \mathbf{e} is a constant vector, we define the corresponding positive quantized action by

$$\left[\frac{Q}{2b} + \frac{i}{b}\alpha - \frac{i}{b}P(\mathbf{u}) \right]_q e^{-2\pi b p_e} \quad (3.4)$$

where p_e is such that $e^{-2\pi b p_e} : f(\mathbf{u}) \mapsto f(\mathbf{u} + i\mathbf{e})$. This expression is positive whenever

$$[p_e, P(\mathbf{u})] = \frac{1}{2\pi i}.$$

Proposition 3.4. *Choose the reduced expression for $w_0 = w_{l-1}s_i$ where w_{l-1} is the reduced expression for ws_i . Then the action of $\exp(tE_i)$ on $U_{>0}^+$ is simply given by $x_i^1 \mapsto x_i^1 + t$, hence the action of E_i is given by*

$$E_i = \frac{\partial}{\partial x_i^1}, \quad (3.5)$$

where x_i^1 is the rightmost coordinate of the parametrization of $U_{>0}^+$. Under the Mellin transform the action is given by

$$E_i : f(u_i) \mapsto (u_i^1 + 1)f(u_i^1 + 1), \quad (3.6)$$

and the positive quantized action is given by

$$E_i = \left[\frac{Q}{2b} - \frac{i}{b}u_i^1 \right]_q e^{-2\pi b p_i^1} \quad (3.7)$$

$$= \frac{i}{q - q^{-1}} (e^{\pi b(u_i^1 - 2p_i^1)} + e^{\pi b(-u_i - 2p_i^1)}). \quad (3.8)$$

The action of F_i is constructed using the commutation relation (2.7) repeatedly, and only depends on the number of terms of s_i and s_j appearing in w_0 where j is the index of any root that is joined by an edge to the root with index i in the Dynkin diagram. Specifically, let $x_j^{k,m}$ be the variables corresponding to s_j appearing in between x_i^k and x_i^{k-1} .

Proposition 3.5. *We have*

$$\begin{aligned} & \exp(tF_i) : \\ & f(\dots, x_j^{n,m}, \dots, x_i^n, \dots, x_j^{n-1,m}, \dots, x_i^{n-1}, \dots, x_i^2, \dots, x_j^{1,m}, \dots, x_i^1, \dots) \\ \mapsto & D_n^{2\lambda_i} f(\dots, \hat{x}_j^{n,m}, \dots, \hat{x}_i^n, \dots, \hat{x}_j^{n-1,m}, \dots, \hat{x}_i^{n-1}, \dots, \hat{x}_i^2, \dots, \hat{x}_j^{1,m}, \dots, \hat{x}_i^1, \dots), \end{aligned}$$

where

$$D_k := 1 + x_i^1 t + \dots + x_i^k t, \quad D_0 := 1 \quad (3.9)$$

$$\hat{x}_i^k = \frac{x_i^k}{D_k D_{k-1}}, \quad (3.10)$$

$$\hat{x}_j^{k,m} = x_j^{k,m} D_k, \quad (3.11)$$

and $\lambda_i \in \mathbb{C}$ are complex parameters. Then the action of F_i is given on the coordinates by

$$\begin{aligned} F_i &= \sum_{k=1}^n -(x_i^k + 2x_i^{k-1} + \dots + 2x_i^1) x_i^k f_{x_i^k} \\ &\quad + \sum_{k=1}^n (x_i^k + x_i^{k-1} + \dots + x_i^1) \sum_{j,m} (x_j^{k,m} f_{x_j^{k,m}}) + 2\lambda_i (x_i^1 + \dots + x_i^n) \\ &= \sum_{k=1}^n x_i^k \left(\sum_{l=k}^n \left(\sum_{j,m} x_j^{l,m} f_{x_j^{l,m}} - 2x_i^l f_{x_i^l} \right) + x_i^k f_{x_i^k} + 2\lambda_i \right), \end{aligned} \quad (3.12)$$

where n is the number of s_i appearing in w_0 , the sum over (j, m) are over all variables $x_j^{l,m}$ with root j adjacent to root i and lying in between x_i^{l+1} and x_i^l . Therefore we can construct the Mellin transformed action:

$$F_i = \sum_{k=1}^n \left(1 + \sum_{l=k}^n \left(\sum_j^m u_j^{l,m} - 2u_i^l \right) + u_i^k + 2\lambda_i \right) f(u_i^k - 1). \quad (3.13)$$

The positive quantized action is then given by (with $\lambda_i \in \mathbb{R}$):

$$\begin{aligned} F_i &= \sum_{k=1}^n \left[\frac{Q}{2b} + \frac{i}{b} \left(\sum_{l=k}^n (2u_i^l - \sum_j^m u_j^{l,m}) - u_i^k + 2\lambda_i \right) \right]_q e^{2\pi b p_i^k} \\ &= \frac{i}{q - q^{-1}} \sum_{k=1}^n \left(e^{\pi b (\sum_{l=k}^n (2u_i^l - \sum_j^m u_j^{l,m}) - u_i^k + 2\lambda_i + 2p_i^k)} \right. \\ &\quad \left. + e^{-\pi b (\sum_{l=k}^n (2u_i^l - \sum_j^m u_j^{l,m}) - u_i^k + 2\lambda_i - 2p_i^k)} \right). \end{aligned} \quad (3.14)$$

Proof. We apply (2.7) repeatedly by moving $y_i(t)$ from the left to the right, picking up the denominators D_i on the way. Then we apply (2.5) on x_i and (2.8) on x_j . and move the diagonal terms to the right as well. Finally we cut off the lower triangular terms, while the diagonal terms, which is $\chi_i(D_n)$, is taken to the power of $2\lambda_i$. The expressions can be found using induction. \square

Proposition 3.6. *The action of $\exp(tH_i)$ is given by*

$$\exp(tH_i) : f \mapsto e^{2\lambda_i t} f(\dots, e^{-a_{i,r(k)} t} x_k, \dots), \quad (3.15)$$

where $r(k)$ is the root index corresponding to x_k , and $a_{i,j}$ is the Cartan matrix, so that

$$H_i = \sum_k -a_{i,r(k)} x_k f_k + 2\lambda_i. \quad (3.16)$$

The Mellin transformed action is simply multiplication by

$$H_i = \sum_k -a_{i,r(k)} u_k + 2\lambda_i, \quad (3.17)$$

and the quantized action (with the corresponding rescaling) is given by

$$K_i = e^{-\pi b(\sum_k a_{i,r(k)} u_k + 2\lambda_i)}. \quad (3.18)$$

4. QUANTUM DILOGARITHM

Let us briefly recall the definition of the quantum dilogarithm functions, first introduced by Faddeev [1], and its properties that are needed in the next section. References can be found e.g. in [3] and [9]. Let $q = e^{\pi i b^2}$ and $Q = b + b^{-1}$ where $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ and $0 < b < 1$.

Definition 4.1. *The quantum dilogarithm function $G_b(x)$ is defined on $0 \leq \operatorname{Re}(z) \leq Q$ by*

$$G_b(x) = \bar{\zeta}_b \exp\left(-\int_{\Omega} \frac{e^{\pi t z}}{(e^{\pi b t} - 1)(e^{\pi b^{-1} t} - 1)} \frac{dt}{t}\right), \quad (4.1)$$

where

$$\zeta_b = e^{\frac{\pi i}{2}(\frac{b^2 + b^{-2}}{6} + \frac{1}{2})}, \quad (4.2)$$

and the contour goes along \mathbb{R} with a small semicircle going above the pole at $t = 0$. This can be extended meromorphically to the whole complex plane.

Definition 4.2. *The function $g_b(x)$ is defined by*

$$g_b(x) = \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} + \frac{\log x}{2\pi i b})}. \quad (4.3)$$

where \log takes the principal branch of x .

We will need the following properties:

Lemma 4.3. *We have*

$$G_b(x)G_b(Q-x) = e^{\pi i x(x-Q)}. \quad (4.4)$$

Furthermore $g_b(x)$ is unitary when $x \in \mathbb{R}_+$, hence in particular $g_b(X)$ is a unitary operator for any positive operator X .

Lemma 4.4 (q -binomial theorem). *For positive self-adjoint variables u, v with $uv = q^2vu$, we have:*

$$(u+v)^{ib^{-1}t} = \int_C \binom{it}{i\tau}_b u^{ib^{-1}(t-\tau)} v^{ib^{-1}\tau} d\tau, \quad (4.5)$$

where the q -beta function (or q -binomial coefficient) is given by

$$\binom{t}{\tau}_b = \frac{G_b(-\tau)G_b(\tau-t)}{G_b(-t)}, \quad (4.6)$$

and C is the contour along \mathbb{R} that goes above the pole at $\tau = 0$ and below the pole at $\tau = t$.

Lemma 4.5. [tau-beta theorem] We have

$$\int_C e^{-2\pi\tau\beta} \frac{G_b(\alpha+i\tau)}{G_b(Q+i\tau)} d\tau = \frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha+\beta)}, \quad (4.7)$$

where the contour C goes along \mathbb{R} and goes above the poles of $G_b(Q+i\tau)$ and below those of $G_b(\alpha+i\tau)$. By the asymptotic properties of G_b , the integral converges for $\operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha+\beta) < Q$.

Lemma 4.6. If $uv = q^2vu$, then

$$g_b(u)^*vg_b(u) = q^{-1}uv + v, \quad (4.8)$$

$$g_b(v)ug_b(v)^* = u + q^{-1}uv. \quad (4.9)$$

5. TRANSFORMATIONS OF THE REPRESENTATIONS

In this section we derive the transformations of the actions corresponding to the change of wordings of the longest element w_0 . First let us consider the quantum analogue of Lusztig's decomposition and its transformation.

Let (a, b, c) and (a', b', c') be positive q -commuting variables such that

$$\begin{array}{ccc} & b & \\ \nearrow & & \\ a & \longleftarrow & c \end{array} \quad \text{and} \quad \begin{array}{ccc} a' & \longleftarrow & c' \\ & \nearrow & \\ & b' & \end{array}$$

where $u \rightarrow v$ means $uv = q^2vu$.

Then

$$x_2(a)x_1(b)x_2(c) = x_1(a')x_2(b')x_1(c')$$

forms a copy of $U_{>0}^+$ of the Gauss Decomposition of $GL_q^+(3, \mathbb{R})$ (cf. [4]), where

$$\begin{aligned} a' &= (a+c)^{-1}cb = bc(a+c)^{-1}, \\ b' &= a+c, \\ c' &= (a+c)^{-1}ab = ba(a+c)^{-1}, \end{aligned}$$

and this map

$$\phi : (a, b, c) \mapsto (a', b', c') \quad (5.1)$$

is an involution between $(a, b, c) \longleftrightarrow (a', b', c')$.

The map ϕ extends to the decomposition of general type, where given an orientation

$$\cdots \circ_i \longrightarrow \circ_j \cdots$$

of the Dynkin diagram, we let the quantum variables be related by

$$\begin{array}{ccc} & & a_i^k \\ & \nearrow & \\ a_j^m & \longleftarrow & a_j^n \end{array}$$

whenever a_j^m, a_i^k, a_j^n appears in this order in the longest element decomposition (2.4), so that in particular the triplet forms a copy of $U_{>0}^+$ of $GL_q^+(3, \mathbb{R})$. Here as before a_j^k refers to the coordinate corresponding to the j -th root, and appears the k -th time from the right.

Then the transformation ϕ preserves these assignments of q -commutation relations. Furthermore, Lemma 2.1 still holds in the quantum case, so that the above transformations of variables are consistent with different choices of reduced expression for w_0 .

Proposition 5.1. *The transformation Φ of the corresponding Mellin transformed map for ϕ on the coordinate functions of symmetric type acts as follows:*

$$\begin{aligned} \phi & : \iiint f(u, v, w) a^{-ib^{-1}u/2} b^{-ib^{-1}v} c^{-ib^{-1}w} a^{-ib^{-1}u/2} dudvdw \\ & \mapsto \iiint F(u, v, w) c'^{-ib^{-1}w/2} a'^{-ib^{-1}u} b'^{-ib^{-1}v} c'^{ib^{-1}u/2} dudvdw, \end{aligned}$$

such that $F = \Phi f$ is given by

$$\Phi = T \circ g_b(e^{\pi b(2p_w - 2p_u + u - v + w)}) g_b^*(e^{\pi b(2p_w - 2p_u - u + v - w)}), \quad (5.2)$$

where T is the composition of the following unitary transformations:

$$T = (u \longleftrightarrow v) \circ (v \longleftrightarrow w) \circ (u \mapsto u - w) \circ (v \mapsto v + w), \quad (5.3)$$

or simply the transformation matrix of determinant one:

$$T \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad (5.4)$$

In particular, it is an involution, $\Phi^2 = 1$.

Proof. Here symmetric type means that the order of the variables will not be changed when taking complex conjugate, hence the order is canonical. In other words complex conjugation gives the involution $f(u, v, w) \mapsto f(-u, -v, -w)$. Note that we have previously rescaled the variables by ib , hence the $-ib^{-1}$ factor.

After substituting (a', b', c') with (a, b, c) , we can use the q -binomial formula to expand

$$(a + c)^{ib^{-1}(u-v+w)} = \int_C c^{ib^{-1}(u-v+w-\tau)} a^{ib^{-1}\tau} \frac{G_b(-i\tau)G_b(-iu + iv - iw + i\tau)}{G_b(-iu + iv - iw)} d\tau.$$

After grouping up the terms with the corresponding q -factors, and with some change of variables, we arrive at the transformation on the coordinates:

$$f(u, v, w) \mapsto \int_C f(v-u-\tau, u+w, u+\tau) e^{\pi i \tau (\tau+u-v+w)} \frac{G_b(-i\tau)G_b(i\tau+iu-iv+iw)}{G_b(iu-iv+iw)} d\tau,$$

which can be written using the Tau-Beta Theorem as

$$\begin{aligned} & \int_C e^{\pi i \tau (2p_u - 2p_w + \tau + u - v + w)} \frac{G_b(-i\tau)G_b(i\tau+iu-iv+iw)}{G_b(iu-iv+iw)} d\tau \circ T \\ &= \frac{G_b(\frac{Q}{2} - ip_u + ip_w - \frac{1}{2}(iu-iv+iw))G_b(iu-iv+iw)}{G_b(\frac{Q}{2} - ip_u + ip_w + \frac{1}{2}(iu-iv+iw))G_b(iu-iv+iw)} \circ T \\ &= \frac{g_b(e^{\pi b(2p_u - 2p_w - u + v - w)})}{g_b(e^{\pi b(2p_u - 2p_w + u - v + w)})} \circ T \\ &= T \circ \frac{g_b(e^{\pi b(2p_w - 2p_u + u - v + w)})}{g_b(e^{\pi b(2p_w - 2p_u - u + v - w)})}. \end{aligned}$$

Finally one proceeds by similar calculations that the map going the other way is exactly the same. Techniques of calculations above involving the Heisenberg type variable p, u can be found e.g. in [3]. \square

Theorem 5.2. *Let $w_0 = \dots s_j s_k \dots = \dots s_k s_j \dots$ be a change of wordings for the longest element w_0 , where j and k are not adjacent in the Dynkin diagram, and the corresponding coordinates are given by (u, v) . Then the change of coordinate is simply given by $u \longleftrightarrow v$.*

On the other hand, let $w_0 = \dots s_j s_k s_j \dots = \dots s_k s_j s_k \dots$ be a change of wordings with j, k adjacent in the Dynkin diagram. We define the action of E_i by the following unitary transformation

$$E_i \mapsto \Phi E_i \Phi^{-1}, \quad (5.5)$$

where Φ is given in Proposition 5.1 for the corresponding coordinates.

Then the action of F_i and H_i constructed in Proposition 3.5 and 3.6 are also related in the same way.

Proof. The first statement is trivial.

For the second case, the transformation of E_i is consistent by the remarks before Proposition 5.1, while to check that Φ maps correctly F_i and H_i , it suffices to check their transformed action using Lemma 4.6 on the coordinate system of the form

$$x_i(a)x_j(b)x_i(c) \longleftrightarrow x_j(a')x_i(b')x_j(c')$$

and

$$x_k(a)x_j(b)x_k(c) \longleftrightarrow x_j(a')x_k(b')x_j(c'),$$

where j is adjacent to i and k , and (i, k) are not joined together. \square

This proves Theorem 1.1 in the introduction. In particular, we obtain the following

Corollary 5.3. *The representations are realized by positive self adjoint operators, and they satisfy the transcendental relations.*

Proof. Obviously the transcendental relations hold for $K_i = q^{H_i}$. The proof for E_i and F_i also follows from the argument in [2]. Both E_i and F_i can be written in the form

$$\frac{i}{q - q^{-1}} \sum_k (A_k^+ + A_k^-), \quad (5.6)$$

where A_k^\pm are q^2 -commuting terms, in the exact same way as (3.33) in Theorem 3.4 of [2]. This can be seen using the simple expression of E_i defined in (3.8) and the explicit expression of F_i from Proposition 3.5. Hence the transcendental relations follow. \square

Corollary 5.4. *We can read off the action of the classical $U(\mathfrak{g}_{\mathbb{R}})$ from the quantized action, where we mean identification of the form*

$$\left[\frac{Q}{2b} + \frac{i}{b} P(\mathbf{u}) \right]_q e^{2\pi b p} \longrightarrow \left(\frac{1}{2} + iP(\mathbf{u}) \right) f(\mathbf{u} - i\mathbf{e}).$$

Hence this gives the unitary classical principal representation of $U(\mathfrak{g}_{\mathbb{R}})$ for any choice of reduced expression of w_0 as finite difference operators.

More precisely, using Theorem 12.1 in the last section, we observe that under the shifting of the variables u_k by $\frac{iQ}{2}\beta_k$, each weight factor $\left[\frac{Q}{2b} + \frac{i}{b} P(\mathbf{u}) \right]_q$ becomes the standard quantum numbers $[P(\mathbf{u})]_q$ and does not depend on $\frac{1}{b}$. In particular since the action of each H_i under this shifting only changes by a constant, any invariant subspace will go under the classical limit to an invariant subspace of the classical principal series representations. Since we know that the classical principal series representations are irreducible, we conclude that

Proposition 5.5. *The positive representations of $U_q(\mathfrak{g}_{\mathbb{R}})$ is irreducible for all $\lambda_i \in \mathbb{R}$.*

6. PROOF OF COMMUTATION RELATIONS

In order to prove the commutation relations, we just need to pick a reduced expression for w_0 such that the action is simple enough for us to check the relations, since Theorem 5.2 says that the commutation relations do not depend on the choice of reduced expression of w_0 .

Proposition 6.1. *We have $[E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}} = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$.*

Proof. Choose w_0 such that it is of the form $w_{l-1}s_i$. Rename the coordinates so that u_k is the k -th coordinate from the right. Then

$$E_i = \left[\frac{Q}{2b} - \frac{i}{b} u_1 \right]_q e^{-2\pi b p_1}$$

while the only term in F_i not commuting with E_i is the first term:

$$F_i = \left[\frac{Q}{2b} + \frac{i}{b} \left(\sum_{k=2} a_{i,r(k)} u_k + u_1 + 2\lambda_i \right) \right]_q e^{2\pi b p_1} + \text{commuting with } E_i,$$

and H_i is of the form

$$\frac{i}{b} \left(\sum_{k=2} a_{i,r(k)} u_k + 2u_1 + 2\lambda_i \right)$$

where $r(k)$ denotes the root index corresponding to u_k , and $a_{i,j}$ is the Cartan matrix.

Then the commutation relation follows directly from the identity

$$[A]_q[B-1]_q - [A-1]_q[B]_q = [B-A]_q. \quad (6.1)$$

□

Proposition 6.2. *We have $[E_i, F_j] = 0$ if $i \neq j$.*

Proof. Again take w_0 such that it is of the form $w_{l-1}s_i$. Then it is obvious since F_j does not involve any shifting in the variables of u_i . □

Proposition 6.3. *We have $K_i E_i = q^2 E_i K_i$ and $K_i F_i = q^{-2} F_i K_i$.*

Proof. For E_i , choose w_0 of the form $w_{l-1}s_i$. Then it is immediate from the expression of the action E_i in (3.8) and K_i in (3.18). For F_i it follows easily from the expression for F_i in (3.14) since it only involves shifting in variables corresponding to root i , and H_i acts by -2 on those roots. □

Proposition 6.4. *We have $[E_i, E_j] = [F_i, F_j] = 0$ if the roots α_i and α_j are not adjacent.*

Proof. The case for E_i follows by choosing w_0 of the form $w_{l-2}s_i s_j$. The case for F_i follows immediately from its expression in Prop 3.5, since the shifting of the variables do not appear in one another. □

Proposition 6.5. *We have the Serre relations for E_i .*

Proof. We choose w_0 of the form $w_{l-3}s_i s_j s_i$. This is possible because $l(w_{l-3}) = l(w) - l(s_i s_j s_i) = l - 3$. Now the action of E_i and E_j only depends on the first 3 variables, hence it follows immediately from the corresponding reduced expression for w_0 in the type A_2 case which is easy to check. □

Proposition 6.6. *We have the Serre relations for F_i .*

Proof. Using the expression in Proposition 3.5, let us write the action of F_i as

$$F_i \cdot f(\mathbf{u}) = \sum_k F_i^k(\mathbf{u}) f(\mathbf{u} + \mathbf{e}_i^k) = \sum_k F_i^k(\mathbf{u}) e^{2\pi b p_{i,k}} f(\mathbf{u}), \quad (6.2)$$

where \mathbf{e}_i^k denote the standard vector with entries $-ib$ in the coordinate $u_{i,k}$ and 0 otherwise. Then the expression for the commutation factor $B_q(a, b, c)$ for the Serre relation is exactly the same as (3.10) in the proof of Theorem 3.1 in [2] with the same variables a, b, c defined by comparing the shifts in e_i^k with $F_i^{k'}(\mathbf{u})$ for various k and k' . Hence the Serre relations for F_i follows. □

Therefore we have constructed the positive representations satisfying properties (i) and (ii) in the Main Theorem of the introduction. In order for it to be compatible with the modular double, we will give the definition of the modified quantum group $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ in Section 10. But first let us write down explicitly the positive representations for quantum groups of type A_n , D_n and discuss some calculations involving E_n .

7. POSITIVE REPRESENTATIONS FOR TYPE A_n

In [2], we studied the positive representations for $U_q(\mathfrak{sl}(n, \mathbb{R}))$ for the standard expression of w_0 using the “cluster coordinates” of $U_{>0}^+$. These are the initial minors of $U_{>0}^+ \subset SL^+(n+1, \mathbb{R})$, which are the determinants of the square sub-matrices that start from the top row. Let $X_{i,j}$ denote the initial minors of the square sub-matrix with lower right entry at (i, j) . We choose the standard reduced expression

$$w_0 = s_n s_{n-1} \dots s_3 s_2 s_1 s_n s_{n-1} \dots s_3 s_2 s_n s_{n-1} \dots s_3 \dots s_n, \quad (7.1)$$

where we label the Dynkin diagram by

$$\circ_1 \text{---} \circ_2 \text{---} \circ_3 \text{---} \dots \text{---} \circ_n .$$

We recall from Section 3 the convention that the coordinate x_i^k corresponds to the root index i appearing in the k -th position from the right. Then we have the following relations between $X_{i,j}$ and the Lusztig’s data x_i^k .

Proposition 7.1. *The cluster coordinates $X_{i,j}$ and the Lusztig’s data x_i^k are related by*

$$x_i^j = \frac{X_{j,i+1} X_{j-1,i-1}}{X_{j,i} X_{j-1,i}}, \quad (7.2)$$

$$X_{i,i+j} = \prod_{m=1}^j \prod_{n=1}^i x_{m+n-1}^n. \quad (7.3)$$

Here we denote by $X_{i,i} = X_{i,0} = X_{0,j} = 1$.

Since these just become linear transformations in the Mellin transformed variables, we can write down explicitly the action of [2] in terms of the Lusztig’s coordinates.

To simplify notation, let us introduce

Definition 7.2. *We denote by*

$$[u] := \left[\frac{Q}{2b} - \frac{i}{b} u \right]_q, \quad e(p) := e^{2\pi b p}. \quad (7.4)$$

Then if $[p, u] = \frac{1}{2\pi i}$ we have

$$[u]e(-p) = \left[\frac{Q}{2b} - \frac{i}{b} u \right]_q e^{-2\pi b p} = \left(\frac{i}{q - q^{-1}} \right) (e^{\pi b(u-2p)} + e^{\pi b(-u-2p)}). \quad (7.5)$$

which is positive self-adjoint. Note that $[u]e(-p) = e(-p)[-u]$.

Theorem 7.3. *The action of E_i, F_i, K_i is given by*

$$\begin{aligned} E_i &= \sum_{k=1}^{n-i+1} [u_{i+k-1}^k - u_{i+k}^k] e \left(\sum_{l=1}^k (p_{i+l-1}^{l-1} - p_{i+l-1}^l) \right), \\ F_i &= \sum_{k=1}^i \left[u_i^k - \sum_{l=k}^i (2u_i^l - u_{i-1}^l - u_{i+1}^{l+1}) - 2\lambda_i \right] e(p_i^k), \\ K_i &= e^{\pi b(\sum_{k=1}^i (u_{i-1}^k + u_{i+1}^k - 2u_i^k) + 2\lambda_i)}, \end{aligned}$$

where $u_i^k = p_i^k = 0$ if the variable does not exist.

8. POSITIVE REPRESENTATIONS FOR TYPE D_n

For each choice of reduced expression for w_0 , we have a positive representation. In general, however, there exists the “best” choice of expression so that the representation is “minimal”, in the sense that the total number of terms is minimized.

With the help of a computer program, we discovered that the best w_0 can be written as the form:

$$w_0 = w_{t_1} w_{t_2} \dots w_{t_n} \quad (8.1)$$

so that $w^k := w_{t_1} \dots w_{t_k}$ gives the longest element of the group corresponding to a connected Dynkin sub-diagram with nodes t_1, \dots, t_k , with $\text{length}(w^k)$ as long as possible.

Let the labeling of the Dynkin diagram for D_n be

$$\begin{array}{ccccccc} \circ_1 & - & \circ_2 & - & \circ_3 & - & \dots & - & \circ_{n-1} \\ & & | & & & & & & \\ & & \circ_0 & & & & & & \end{array}$$

For type D_n , we the corresponding sequence is

$$(t_1, \dots, t_n) = (2, 1, 0, 3, 4, 5, 6, 7, \dots, n-1)$$

Then w_0 of length $n(n-1)$ is of the form

$$w_0 = (2 \ 12 \ 012 \ 320123 \ 43201234 \ \dots \ (n-1) \dots 43201234 \dots (n-1)) \quad (8.2)$$

where for simplicity we denoted by $k := s_k$.

However, we will transform 212012 in the beginning of w_0 to 012012 so that the expression can be made symmetric in E_0 and E_1 , but with slightly more terms.

The explicit formula for the positive representation can actually be carried out involving only the simple transformation rule:

$$\Phi : [w]e(-p_w) \mapsto [u]e(-p_u - p_v + p_w) + [v-w]e(-p_v) \quad (8.3)$$

using Proposition 5.1, and we obtain the following general formula by a series of inductions.

Theorem 8.1. *The positive representation for $U_q(\mathfrak{g}_{\mathbb{R}})$ of type D_n is given by:*

For $i = 0$ or 1 :

$$E_i = \sum_{k=1}^{n-1} [u_{k+i-1}^k - u_2^{2k-1}] e \left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k-2} (-1)^{l_2} p_2^{l_2} \right) \\ + \sum_{k=1}^{n-2} [u_2^{2k} - u_{k+i}^k] e \left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k} (-1)^{l_2} p_2^{l_2} \right)$$

and for $i \geq 2$,

$$E_i = \sum_{k=1}^{2n-2i-1} [(-1)^k (u_{i+1}^k - u_i^k)] e \left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{i+1}^{l_1} \right),$$

where $\bar{k} := k \pmod{2} \in \{0, 1\}$, and

$$s_1(k) := 2 \left\lfloor \frac{k}{2} \right\rfloor - 1 \quad , \quad s_2(k) := 2 \left\lfloor \frac{k}{2} \right\rfloor .$$

The actions of F_i and K_i are given as before by Proposition 3.5 and 3.6, and as before we ignore the variables that do not exist (see Table 2 in the next section for the restrictions of m for u_i^m).

9. POSITIVE REPRESENTATIONS FOR TYPE E_n

We follow the same strategy as in type D_n , by choosing a “good” expression for w_0 coming from adding nodes successively to the Dynkin diagram:

$$w_0 = w_{t_1} w_{t_2} \dots w_{t_n} .$$

It turns out that the best case is obtained by the following embedding:

$$D_4 \subset D_5 \subset E_6 \subset E_7 \subset E_8$$

where we label the Dynkin diagram by the following index:

$$\begin{array}{cccccccc} \circ_1 & - & \circ_2 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 \\ & & & & | & & & & & & & & \\ & & & & \circ_0 & & & & & & & & \end{array}$$

The reduced expressions used, together with the sequence (t_1, t_2, \dots, t_n) are given below:

$$\begin{aligned} w_0(E_6) &= 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321 \\ (t_1, \dots, t_6) &= (4, 3, 0, 2, 1, 5) \\ w_0(E_7) &= 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321\ 654320345612345034230123456 \\ (t_1, \dots, t_7) &= (4, 3, 0, 2, 1, 5, 6) \\ w_0(E_8) &= 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321\ 654320345612345034230123456 \\ &\quad 765432103243546503423012345676543203456123450342301234567 \\ (t_1, \dots, t_8) &= (4, 3, 0, 2, 1, 5, 6, 7) \end{aligned}$$

The explicit actions of E_i can be found in the appendix of [5]. Here we summarize the numerical results in the following

Theorem 9.1. *We have constructed the positive representation for simply-laced quantum group, where the number of terms of the form $[\mathbf{u}]e(\mathbf{p})$ in the action of E_i, F_i are summarized in the table below:*

	A_n	D_n	E_6	E_7	E_8
E_0		$2n - 3$	9	15	27
E_1	n	$2n - 3$	1	11	23
E_2	$n - 1$	$2n - 5$	11	13	25
E_3	$n - 2$	$2n - 7$	10	16	28
E_4	\vdots	$2n - 9$	7	17	29
E_5	\vdots	$2n - 11$	5	7	19
E_6	\vdots	\vdots		1	23
E_7	\vdots	\vdots			1
\vdots	\vdots	\vdots			
E_k	$n - k$	$2n - 2k - 1$			
\vdots	\vdots	\vdots			
E_{n-1}	2	1			
E_n	1				
<i>Total :</i>	$\frac{n(n+1)}{2}$	$n^2 - 2$	43	80	175

Table 1. Number of terms for the action of E_i

The number of terms in F_i corresponds to the number of s_i appearing in w_0 :

	A_n	D_n	E_6	E_7	E_8
F_0		$n - 1$	5	8	14
F_1	1	$n - 1$	4	6	10
F_2	2	$2n - 4$	7	11	19
F_3	3	$2n - 6$	10	16	28
F_4	\vdots	$2n - 8$	8	13	23
F_5	\vdots	$2n - 10$	2	6	14
F_6	\vdots	\vdots		3	9
F_7	\vdots	\vdots			3
\vdots	\vdots	\vdots			
F_k	$k + 1$	$2n - 2k + 2$			
\vdots	\vdots	\vdots			
F_{n-1}	$n - 1$	2			
F_n	n				
<i>Total :</i>	$\frac{n(n+1)}{2}$	$n(n - 1)$	36	63	120

Table 2. Number of terms for the action of F_i

Using Corollary 5.4, these explicit expressions of positive representations for $U_q(\mathfrak{g}_{\mathbb{R}})$ give explicitly the classical principal series representations for $U(\mathfrak{g}_{\mathbb{R}})$.

Remark 9.2. *In the above choice for w_0 , only the simple transformation rule (8.3) is involved. In general, however, things can get complicated. In particular, we can have for example:*

$$\Phi : [w-u]e(p_v-p_w) \mapsto [v-2w]e(p_u-p_w) + [2]_q[u-w]e(p_w-p_v) + [2u-v]e(2p_w-p_u-p_v).$$

Therefore we see that the quantization rule $(2u) \mapsto [2u]$ is not always true in general.

On the other hand, the number of terms can go really bad if we choose w_0 badly, due to the transformation rule which produces new terms exponentially. For example, if we choose w_0 of the form

$$w_0 = w' w_{A_n} s_0$$

so that the action of E_i are all simple except the action of E_2 for Type D_n or E_3 for type E_n , then compared with the table above, we obtain for example, 1043 terms for E_6 , 77565 terms for E_7 , and over 1 million terms for E_8 for the action of E_3 .

10. MODIFIED QUANTUM GROUP

Finally as in [2], we modify the generators with powers of K_i in order to take care of the commutation relations between the generators and the other part of the modular double. The definition is slightly modified to fit subsequent work on non simply-laced group [6].

Proposition 10.1. *For each node i in the Dynkin diagram, we assign a weight $n_i \in \{0, 1\}$ such that $|n_i - n_j| = 1$ if i, j are connected in the diagram, so that n_i alternates along the edges.*

We define $q := q^2 = e^{2\pi i b^2}$ and

$$q_i := \begin{cases} q^{-1} & \text{if } n_i = 0, \\ q & \text{if } n_i = 1, \end{cases} \quad (10.1)$$

and define the modified quantum generators as

$$\begin{aligned} \mathbf{E}_i &:= q^{n_i} E_i K_i^{n_i}, \\ \mathbf{F}_i &:= q^{1-n_i} F_i K_i^{n_i-1}, \\ \mathbf{K}_i &:= q_i^{H_i} = \begin{cases} K_i^{-2} & \text{if } n_i = 0, \\ K_i^2 & \text{if } n_i = 1, \end{cases} \end{aligned}$$

Then the variables are positive self-adjoint. Let

$$[A, B]_q = AB - q^{-1}BA \quad (10.2)$$

be the quantum commutator. Then the quantum relations in the new variables become:

$$\mathbf{K}_i \mathbf{E}_j = q_i^{a_{ij}} \mathbf{E}_j \mathbf{K}_i, \quad (10.3)$$

$$\mathbf{K}_i \mathbf{F}_j = q_i^{-a_{ij}} \mathbf{F}_j \mathbf{K}_i, \quad (10.4)$$

$$\mathbf{E}_i \mathbf{F}_j = \mathbf{F}_j \mathbf{E}_i \quad \text{if } i \neq j, \quad (10.5)$$

$$[\mathbf{E}_i, \mathbf{F}_i]_{q_i} = \frac{1 - \mathbf{K}_i}{1 - q_i}, \quad (10.6)$$

and the quantum Serre relations become

$$[[\mathbf{E}_j, \mathbf{E}_i]_{q_i}, \mathbf{E}_i] = 0 = [[\mathbf{F}_j, \mathbf{F}_i]_{q_i}, \mathbf{F}_i]. \quad (10.7)$$

We denote the modified quantum group by $U_q(\mathfrak{g}_{\mathbb{R}})$.

Hence we can now state the main theorem from the introduction for $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$:

Theorem 10.2. *Let $\tilde{q} := \tilde{q}^2 = e^{2\pi i b^{-2}}$. We define the tilde part of the modified modular double by representing the generators $\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{F}}_i, \widetilde{\mathbf{K}}_i$ using the formulas above with all the terms replaced by tilde. Then all the relations with tilde replaced hold.*

Furthermore the properties of positive representations are satisfied:

- (i) *the operators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i$ and their tilde counterparts are represented by positive essentially self-adjoint operators,*
- (ii) *we have the transcendental relations:*

$$(\mathbf{e}_i)^{\frac{1}{b^2}} = \widetilde{\mathbf{e}}_i, \quad (10.8)$$

$$(\mathbf{f}_i)^{\frac{1}{b^2}} = \widetilde{\mathbf{f}}_i, \quad (10.9)$$

$$(\mathbf{K}_i)^{\frac{1}{b^2}} = \widetilde{\mathbf{K}}_i, \quad (10.10)$$

- (iii) *all the generators $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i$ commute with all $\widetilde{\mathbf{E}}_j, \widetilde{\mathbf{F}}_j, \widetilde{\mathbf{K}}_j$.*

Here $\mathbf{e}_i = 2 \sin(\pi b^2) \mathbf{E}_i$ and $\mathbf{f}_i = 2 \sin(\pi b^2) \mathbf{F}_i$ and similarly for $\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{f}}_i$ with b replaced by b^{-1} in all the formulas. Therefore we have constructed the positive principal series representations of the modular double $U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ where \mathfrak{g} is of simply-laced type, parametrized by rank(\mathfrak{g}) numbers $\lambda_i \in \mathbb{R}$.

Let $\mathbb{C}[\mathbb{T}_{q\tilde{q}}^N] := \mathbb{C}[\mathbf{u}_i^{\pm 1}, \mathbf{v}_i^{\pm 1}, \widetilde{\mathbf{u}}_i^{\pm 1}, \widetilde{\mathbf{v}}_i^{\pm 1}]_{i=1}^N$, be the Laurent polynomials in the quantum tori variables that are positive self adjoint and satisfy

$$\mathbf{u}_i \mathbf{v}_i = q \mathbf{v}_i \mathbf{u}_i, \quad \widetilde{\mathbf{u}}_i \widetilde{\mathbf{v}}_i = \tilde{q} \widetilde{\mathbf{v}}_i \widetilde{\mathbf{u}}_i, \quad (10.11)$$

which can be realized by

$$\mathbf{u}_i = e^{2\pi b u_i}, \quad \mathbf{v}_i = e^{2\pi b p_i}, \quad (10.12)$$

and similarly for $\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{v}}_i$ with b replaced by b^{-1} . Then we have Theorem 1.3 of the introduction generalizing the results of type A_n in [2]:

Theorem 10.3. *We have an embedding*

$$U_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) \hookrightarrow \mathbb{C}[\mathbb{T}_{q\tilde{q}}^N]. \quad (10.13)$$

where $N = l(w_0)$ is the dimension of $U_{>0}^+$.

Proof. By the explicit expressions for the operators of \mathbf{E}_i and \mathbf{F}_i constructed in the previous sections (modified by the K factors), we note that all operators are sums of terms that q^2 commute with each other. Therefore there exists a unitary transformation such that we can diagonalize the symplectic form corresponding to these q^2 commuting terms and obtain a realization in terms of the standard tori. Since the representation

on $L^2(\mathbb{R}^N)$ is irreducible, the dimension follows. Explicitly, it can be obtained from multiplication by the unitary functions (using the notations from Proposition 3.5)

$$\begin{aligned} e^{\pi i u_i^n u_j^{k,m}} &: 2p_i^n \mapsto 2p_i^n + u_j^{k,m}, \quad p_j^{k,m} \mapsto 2p_j^{k,m} + u_i^n \\ e^{\frac{1}{2}\pi i (u_i^n)^2} &: 2p_i^n \mapsto 2p_i^n + u_i^n, \end{aligned}$$

whenever $n > k$, $n_i = 0$ and i, j are adjacent in the Dynkin diagram. \square

Finally we also relate the commutant of the representations with the Langlands dual, as described in Theorem 1.4 of the introduction. Recall that the Langlands dual is obtained by switching root lattice with coroot lattice. More explicitly, let $\mathbf{b}_k = \sum_j b_j^k \alpha_j$ be the weight so that it is dual to the root α_k :

$$(\mathbf{b}_k, \alpha_i) = \delta_{ik}.$$

This means that

$$\begin{aligned} \sum_j b_j^k (\alpha_j, \alpha_i) &= \delta_{ik} \\ \iff \sum_j b_j^k a_{ij} &= \delta_{ik} \\ \iff A\mathbf{b}_k &= \mathbf{e}_k \end{aligned}$$

where $A = (a_{ij})$ is the Cartan matrix, and \mathbf{e}_k is the standard unit vector.

Theorem 10.4. *The commutant for the positive representation of $U_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$ where \mathfrak{g} is simply-laced is generated by $\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{F}}_i$ and elements of the form*

$$\widetilde{\mathbf{K}}_0^{b_0} \widetilde{\mathbf{K}}_1^{b_1} \cdots \widetilde{\mathbf{K}}_{n-1}^{b_{n-1}}, \quad (10.14)$$

for $k = 0, \dots, n-1$, where the vector $\mathbf{b}_k = (b_0^k, b_1^k, \dots, b_{n-1}^k)^T$ satisfies

$$A\mathbf{b}_k = \mathbf{e}_k \quad (10.15)$$

where $A = (a_{ij})$ is the Cartan matrix, and \mathbf{e}_k is the standard unit vector. In particular, this means that the commutant of $U_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$ is precisely its Langlands dual quantum group $U_{\widetilde{\mathfrak{q}}}(L\mathfrak{g}_{\mathbb{R}})$.

Proof. $\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{F}}_i$ does not commute with $\mathbf{E}_j, \mathbf{F}_j$ in the strong sense, any fractional powers of $\widetilde{\mathbf{E}}_i$ and $\widetilde{\mathbf{F}}_i$ will not commute simultaneously with each individual terms of the form $[\mathbf{u}]e(\mathbf{p})$. For the $\widetilde{\mathbf{K}}_i$ generators, it suffices to choose a reduced expression w_0 for each i and calculate the commutant with \mathbf{E}_i . Let us choose $w_0 = w_{l-1}s_i$. Then

$$[\widetilde{\mathbf{K}}_0^{b_0} \widetilde{\mathbf{K}}_1^{b_1} \cdots \widetilde{\mathbf{K}}_{n-1}^{b_{n-1}}, \mathbf{E}_i] = 0$$

implies

$$\sum_{j=0}^{n-1} a_{ij} b_j = (-1)^{n_i+1} k_i$$

for some integer k_i . Combining for every i , we obtain the condition

$$A\mathbf{b} = (k_0, k_1, \dots, k_{n-1})^T$$

for integers $k_i \in \mathbb{Z}$, hence the statement. \square

11. UNITARY EQUIVALENCE $\mathcal{P}_\lambda \simeq \mathcal{P}_{w(\lambda)}$

So far we have constructed the positive principal series representations for the parameter $\lambda \in \mathbb{R}^r$ where r is the rank of \mathfrak{g} . We know that in the compact case, the finite dimensional representations are parametrized by the cone of the positive weights $P^+ \subset \mathfrak{h}_\mathbb{R}^*$, where $\mathfrak{h}_\mathbb{R}$ is the real form of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Below we will show Theorem 1.5 from the introduction that the situation for $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ is exactly the same.

Theorem 11.1. *Let \mathcal{P}_λ denote the positive principal series representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ corresponding to the parameter $\lambda = (\lambda_i)_{i=1}^r$ where $r = \text{rank}(\mathfrak{g})$. Then*

$$\mathcal{P}_\lambda \simeq \mathcal{P}_{w(\lambda)} \tag{11.1}$$

are unitary equivalent representations for any Weyl group element w acting on λ , namely for simple reflections,

$$s_i(\lambda_j) := \lambda_j - a_{ij}\lambda_i = \begin{cases} -\lambda_i & i = j, \\ \lambda_i + \lambda_j & i \text{ and } j \text{ are connected,} \\ \lambda_j & \text{otherwise,} \end{cases} \tag{11.2}$$

where a_{ij} is the Cartan matrix. In particular, the positive principal series representations are parametrized by $\lambda \in \mathbb{R}_{\geq 0}^r$.

Proof. Let us first fixed the root index i . Since representations corresponding to different reduced expression of w_0 amount to unitary transformation by Φ , we can take $w_0 = s_i w_{l-1}$, and call u the variable to the leftmost corresponding to s_i . Then by Proposition 3.5 only the action of F_i contains the term with p_u , namely, the term

$$F_i = [-2\lambda_i - u]e(p_u) + (\text{independent of } p_u) \dots$$

On the other hand, by the transformation rule (8.3), we see that the weight of the “leftmost” term $[u]$ is always preserved. Since we know from the explicit expressions constructed in the previous sections that there is a unique E_k with a single term involving the leftmost variable, this means that under the transformations Φ between different reduced expressions, there is a unique E_k with the term of the form

$$E_k = [u]e(-p_u + \dots) + (\text{independent of } u, p_u) \dots$$

and no other terms from the action of E 's contain the variable u . Now we can define our intertwiner B as

$$B = (u \mapsto u - \lambda_i) \circ G_{\lambda_i}(u) \circ (u \mapsto u - \lambda_i), \tag{11.3}$$

where

$$G_{\lambda_i}(u) = \frac{g_b(e^{2\pi b(u+\lambda_i)})}{g_b(e^{2\pi b(u-\lambda_i)})} e^{-2\pi i \lambda_i u} \tag{11.4}$$

is a unitary function which is essentially the same in the $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ case [9]. Note that if the action does not involve p_u , then it is just shifting by $u \mapsto u - 2\lambda_i$. One can check that this map preserves the E_k action, change the terms in F_i as

$$[-2\lambda_i - u]e(p_u) \mapsto [2\lambda_i - u]e(p_u),$$

$$[-2\lambda_i - 2u + \dots - w]e(p_w) \mapsto [2\lambda_i - 2u + \dots - w]e(p_w),$$

and for the action of F_j with j adjacent to i , all the terms change as

$$[-2\lambda_j + u + \dots - v]e(p_v) \mapsto [-2(\lambda_j + \lambda_i) + u + \dots - v]e(p_v).$$

□

12. REMARKS ABOUT THE CONJECTURES

Finally we would like to discuss possible approaches to the conjectures stated in the introduction. First let us generalize the result in [2, Thm 3.3], where there exists a unitary transformation on $L^2(\mathbb{R}^N)$ so that the actions of H_i do not depend on λ_i . In particular, this provides us a way to find the Haar functional in order to define an L^2 space structure for the harmonic analysis of $L^2(G_q)$.

Theorem 12.1. *There exists a linear transformation on the coordinates u_i^k so that the action of H_i do not depend on λ_i , while all the weights of E_i and F_i are of the form*

$$\left[\frac{Q}{2b} + \frac{i}{b} \left(\sum c_i^k u_i^k + \lambda'_i \right) \right]_q,$$

for r distinguished parameters λ'_i , where r is the rank of $\mathfrak{g}_{\mathbb{R}}$.

Proof. Let us rename the variables so that it starts from u_1 all the way up to $u_{l(w_0)}$ from the left. Then we can see that for each k , there exists a unique weight from F_i of the form

$$\left[\frac{Q}{2b} + \frac{i}{b} \left(\sum_{j=1}^k c_j u_j + 2\lambda_{r(k)} \right) \right]_q,$$

where $c_j \in \mathbb{Z}$ are some constants with $c_k = 1$, and $r(k)$ denotes the root index corresponding to u_k . Given a linear combinations $\lambda = \sum_j d_j \lambda_j$ of the parameters λ_i , we define $\beta(\lambda) = \sum_j d_j$. Then the transformation is given by the following.

- In the first step, we let $u_1 \mapsto u_1 - \lambda_{r(1)}$.
- In the k -th step, let the k -th weight be of the form

$$\left[\frac{Q}{2b} + \frac{i}{b} \left(\sum_{j=1}^k c_j u_j + \lambda'_{r(k)} \right) \right]_q$$

for some modified $\lambda'_{r(k)}$. Denote by $\beta_k = \beta(\lambda'_{r(k)}) - 1$. Then we let

$$u_k \mapsto u_k - \frac{\beta_k}{\beta_k + 1} \lambda'_{r(k)}.$$

Notice that β_k is always a positive integer. □

This result allows us to state Conjecture 1.6 in a more precise setting, namely to define an L^2 structure on (the modular double of) the quantized function space $F_{q\bar{q}}(G)$. If we ignore the factor $\frac{i}{b}$ and treat originally the parameters λ_i to have real part $-\frac{Q}{2}$, then the shifting in the theorem above tells us that $-\frac{Q}{2}\beta_k$ was the correct real part of u_k to make the representation positive, which reflects its original classical Haar measure, namely, the Haar measure on the original variables a_k is given by $a_k^{\beta_k-1} da_k$. Then one can apply the method in [4] to define the space $L^2(G_q)$ in general using the Gauss-Lusztig decomposition. By studying the Plancherel measure arising from the decomposition of $L^2(G_q)$, it may shed some light on Conjecture 1.7, where in the case of $U_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ it is found in [9] that the same measure shows up in the decomposition of the tensor product.

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DEPARTMENT OF MATHEMATICS,
 YALE UNIVERSITY,
 10 HILLHOUSE AVENUE,
 NEW HAVEN, CT 06511, U.S.A.

E-mail address: ivan.ip@yale.edu