I. B. Frenkel and I. C. H. Ip (2014) "Positive Representations of Split Real Quantum Groups and Future Perspectives,"
International Mathematics Research Notices, Vol. 2014, No. 8, pp. 2126–2164
Advance Access Publication January 11, 2013
doi:10.1093/imrn/rns288

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Positive Representations of Split Real Quantum Groups and Future Perspectives

Igor B. Frenkel[1] **and Ivan C. H. Ip**[2]

[1]Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06520, USA and [2]Kavli IPMU (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

Correspondence to be sent to: ivan.ip@ipmu.jp

We construct a special principal series representation for the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ of type A_r representing the generators by positive essentially self-adjoint operators satisfying the transcendental relations that also relate q and $\tilde{q}$. We use the cluster variables parameterization of the positive unipotent matrices to derive the formulas in the classical case. Then we quantize them after applying the Mellin transform. Our construction is inspired by the previous results for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2, \mathbb{R})$ and can be generalized to all other types of simple split real Lie algebra $\mathfrak{g}_{\mathbb{R}}$. We conjecture that our positive representations are closed under the tensor product and we discuss the future perspectives of the new representation theory following the parallel with the established developments of the finite-dimensional representation theory of quantum groups.

1 Introduction

In their foundational papers, Drinfeld [10] and Jimbo [22] have defined for any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ (and more generally for any Kac–Moody algebra) a remarkable Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ known as a quantum group. As the notation indicates the quantum group $\mathcal{U}_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra

Received September 21, 2012; Accepted December 3, 2012
Communicated by Prof. Michio Jimbo

© The Author(s) 2013. Published by Oxford University Press. All rights reserved. For permissions, please e-mail: journals.permissions@oup.com.

$\mathcal{U}(\mathfrak{g})$ for a nonzero complex parameter q. They were also able to deform the irreducible finite-dimensional representations of $\mathcal{U}(\mathfrak{g})$ to corresponding representations of $\mathcal{U}_q(\mathfrak{g})$, which stay irreducible when q is not a root of unity. These representations as in the classical case are parameterized by the cone of the positive weights $P^+ \subset \mathfrak{h}_{\mathbb{R}}^*$, where $\mathfrak{h}_{\mathbb{R}}$ is the real form of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. These representations have a Hermitian form compatible with the quantum counterpart of the canonical Hermitian conjugation on $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{g}_c \subset \mathfrak{g}$ be a compact real form fixed by the classical Hermitian conjugation, and let $\mathcal{U}_q(\mathfrak{g}_c)$ denote the quantum group $\mathcal{U}_q(\mathfrak{g})$ equipped with the corresponding Hermitian structure that is well-defined *-Hopf algebra for the real nonzero parameter q [42].

It is natural to consider other real forms of $\mathfrak{g}$, most notably the split real form $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$, and address the question about the q-deformation of its irreducible unitary representations. Since the works of Drinfeld and Jimbo, the q-deformation of various infinite-dimensional irreducible representations were found [26, 30, 37]. However, the general problem of the q-deformations of all unitary irreducible representations of $\mathfrak{g}$ seems to be too difficult and we have to be contented to consider special classes of representations. For the split real form $\mathfrak{g}_{\mathbb{R}}$ there is one distinguished series of irreducible unitary representations associated to the minimal parabolic or Borel subalgebra $\mathfrak{b}_{\mathbb{R}}$ parameterized by the $\mathbb{R}_+$-span $P_{\mathbb{R}}^+ \subset \mathfrak{h}_R^*$ of the discrete cone P^+. This series, usually called the minimal principal series, also constitutes the decomposition of $L^2(G_{\mathbb{R}}/K)$, where $G_{\mathbb{R}}$ is the Lie group corresponding to $\mathfrak{g}_{\mathbb{R}}$ and K is its maximal compact subgroup [16], and it can also be viewed as the most continuous series in the decomposition of $L^2(G_{\mathbb{R}})$ [17]. In this paper, we present a construction of what we view as the most canonical q-deformation of this distinguished series of unitary representations in the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(n, \mathbb{R})$. The generalization of the construction to the case of an arbitrary split real form $\mathfrak{g}_{\mathbb{R}}$ by the second author will appear in separate publications [20, 21].

In the case of the split real form the Hermitian conjugation on $\mathcal{U}_q(\mathfrak{g})$ is well defined for q on the unit circle [42], and let $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ denote again the quantum group with this extra structure. Here, we consider

$$q = e^{\pi \mathbf{i} b^2}, \quad \tilde{q} = e^{\pi \mathbf{i} b^{-2}}, \tag{1}$$

where $\mathbf{i} = \sqrt{-1}$, $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ such that q and $\tilde{q}$ are not roots of unity. The starting point of our construction was the work of Teschner et al. [5, 32, 33] who studied extensively a very special "q-deformation" of the principal series of representations of the quantum

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ in the space $L^2(\mathbb{R})$. Explicitly, the (Fourier transformed) action is given by

$$E = \left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}(\alpha - u)\right]_q \mathrm{e}^{-2\pi bp}, \tag{2}$$

$$F = \left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}(\alpha + u)\right]_q \mathrm{e}^{2\pi bp}, \tag{3}$$

$$K = \mathrm{e}^{2\pi bu}, \tag{4}$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $Q = b + b^{-1}$, and $p = \frac{1}{2\pi \mathbf{i}}\frac{\mathrm{d}}{\mathrm{d}u}$ so that $\mathrm{e}^{\pm 2\pi bp} f(u) = f(u \mp \mathbf{i}b)$. Although the formula is a q-deformation of the classical action, the classical parameter $1 + \lambda = \frac{1}{2} + \mathbf{i}\alpha$ that gives a unitary representation for $\mathrm{SL}(2,\mathbb{R})$ is also perturbed so that this special series admit the parameter

$$1 + \lambda = \frac{Q}{2b} + \mathbf{i}\alpha = \frac{1}{2} + \frac{1}{2b^2} + \mathbf{i}\alpha. \tag{5}$$

The parameter of the representation appears in formulas with the factor b so that its real part gives $\frac{Q}{2} = \frac{1}{2}(b + b^{-1})$, which has no classical limit when

$$b \longrightarrow 0, \quad q \longrightarrow 1, \tag{6}$$

nor the limit corresponding to $b \longrightarrow \infty$, $\tilde{q} \longrightarrow 1$. However, what we have gained is that the action acquires a duality between $b \leftrightarrow b^{-1}$, and the operators become positive self-adjoint, and one can discuss its functional calculus. Details on the functional analysis of unbounded operators, the self-adjointness, as well as the important Lemma 3.2 can be found for example in [18, 36]. In particular, these representations are naturally extended to the modular double

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R})) = \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R})) \tag{7}$$

of the quantum group first introduced by Faddeev [11, 12]. The modular double has two sets of mutually commuting generators $\{E, F, K^{\pm 1}\}$ and $\{\tilde{E}, \tilde{F}, \tilde{K}^{\pm 1}\}$ satisfying the quantum group relations

$$KE = q^2 EK, \tag{8}$$

$$KF = q^{-2} FK, \tag{9}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \tag{10}$$

and similarly for the second set with tildes. To formulate some special additional properties of these representations it is convenient to introduce the rescaled generators

$$e = 2\sin(\pi b^2)E, \quad f = 2\sin(\pi b^2)F, \tag{11}$$

and similarly for the tilde set. Then the representations of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ in $L^2(\mathbb{R})$ possess the following properties:

(i) the generators e, f, $K^{\pm 1}$ and $\tilde{e}$, $\tilde{f}$, $\tilde{K}^{\pm 1}$ are represented by positive essentially self-adjoint operators,

(ii) the generators satisfy the transcendental relations

$$\mathrm{e}^{\frac{1}{b^2}} = \tilde{e}, \quad f^{\frac{1}{b^2}} = \tilde{f}, \quad K^{\frac{1}{b^2}} = \tilde{K}. \tag{12}$$

Our generalization of the Teschner et al. construction to the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ of the quantum group associated to Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of type A_r of rank r and dimension $r + 2N$ is completely analogous to the $\mathfrak{sl}(2,\mathbb{R})$ case. In particular, the functional analysis can be reduced to the $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case. However, in higher rank, there appears new algebraic features. Specifically, the extra relations

$$K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \tag{13}$$

where (a_{ij}) is the Cartan matrix, and the quantum Serre relations do not allow adjacent variables $\{E_i, F_i, K_i\}$ and $\{\tilde{E}_j, \tilde{F}_j, \tilde{K}_j\}$ to commute whenever $|i - j| = 1$. To remedy this, we have to introduce a slightly modified version of the quantum group so that the tilde variables commute with the original variables. We define

$$\mathfrak{q} := q^2 = \mathrm{e}^{2\pi \mathrm{i} b^2}, \quad \tilde{\mathfrak{q}} := \tilde{q}^2 = \mathrm{e}^{2\pi \mathrm{i} b^{-2}}, \tag{14}$$

and let $\mathfrak{q}_i = \mathfrak{q}$ if i is even, $\mathfrak{q}_i = \mathfrak{q}^{-1}$ if i is odd. Also define the $\mathfrak{q}$-commutator

$$[A, B]_{\mathfrak{q}} = AB - \mathfrak{q}^{-1} BA. \tag{15}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Let the quantum tori $\mathbb{T}_{\mathfrak{q}\tilde{\mathfrak{q}}}^{n(n-1)/2}$ be the algebra of Laurent polynomials generated by positive self-adjoint operators $\mathbf{u}_{ij}, \mathbf{v}_{ij}, \tilde{\mathbf{u}}_{ij}$, and $\tilde{\mathbf{v}}_{ij}$ for $1 \le i < j \le n$ such that

$$\mathbf{u}_{ij}\mathbf{v}_{ij} = \mathfrak{q}\mathbf{v}_{ij}\mathbf{u}_{ij}, \quad \tilde{\mathbf{u}}_{ij}\tilde{\mathbf{v}}_{ij} = \tilde{\mathfrak{q}}\tilde{\mathbf{v}}_{ij}\tilde{\mathbf{u}}_{ij}. \tag{16}$$

Main Theorem. Let $\{\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm 1}\}_{i=1}^r$ and $\{\tilde{\mathbf{E}}_i, \tilde{\mathbf{F}}_i, \tilde{\mathbf{K}}_i^{\pm 1}\}_{i=1}^r$ be two sets of mutually commuting generators of the modified modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ where $\mathfrak{g}_{\mathbb{R}}$ is of type A_r, satisfying the relations

$$\mathbf{K}_i\mathbf{E}_j = \mathfrak{q}_i^{a_{ij}}\mathbf{E}_j\mathbf{K}_i, \quad \mathbf{K}_i\mathbf{F}_j = \mathfrak{q}_i^{-a_{ij}}\mathbf{F}_j\mathbf{K}_i, \tag{17}$$

relations

$$[\mathbf{E}_i, \mathbf{F}_i]_{\mathfrak{q}_i} = \frac{1-\mathbf{K}_i}{1-\mathfrak{q}_i}, \tag{18}$$

as well as the modified quantum Serre relations

$$[[\mathbf{E}_j, \mathbf{E}_i]_{\mathfrak{q}_i}, \mathbf{E}_i] = 0 = [[\mathbf{F}_j, \mathbf{F}_i]_{\mathfrak{q}_i}, \mathbf{F}_i] \tag{19}$$

and similarly for the second set with tildes. Then there exist a family of irreducible representations of $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ parameterized by $\lambda \in P_{\mathbb{R}}^+$ on the space $L^2(\mathbb{R}^N)$ with the additional properties (i) and (ii) for $\{\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm 1}\}_{i=1}^r$. Moreover, there is an embedding

$$\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{sl}(n,\mathbb{R})) \hookrightarrow \mathbb{T}_{\mathfrak{q}\tilde{\mathfrak{q}}}^{n(n-1)/2}. \tag{20}$$

□

In our proof of the theorem, we are able to present explicit expressions for the generators and verify directly all the relations and properties. Our verification of the commutation relations (CRs), both in the classical and quantum case, is based on a new pictorial method, which we believe presents an independent interest. We also provide a derivation of our formulas in the classical case using a parameterization of the positive unipotent matrices by the cluster variables associated to the canonical orientation of the A_r quiver

$$\circ_1 \longrightarrow \circ_2 \longrightarrow \cdots \circ_{r-1} \longrightarrow \circ_r \tag{21}$$

or its opposite. Then using the positivity properties, we rewrite our formulas by applying the Mellin transform. Finally, using the rules of the q-deformation inspired by the

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$\mathfrak{sl}(2,\mathbb{R})$ case studied by Teschner et al., we obtain the desired representations of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(n,\mathbb{R}))$ and its modification $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{sl}(n,\mathbb{R}))$. Note that for different orientations of a quiver different generators of the quantum group admit especially simple expressions. In particular, for the canonical orientation (respectively, its opposite) of the A_r quiver as shown above, the generators E_r and F_1 (respectively, E_1 and F_r) contain only one shifting operators.

Though in our paper, we construct representations of the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ by a certain deformation of the representations of the classical Lie algebra, the actual relation between the quantum and classical cases is rather mysterious. Although there is a formal classical limit in the nonperturbed case when we consider a fixed generic complex parameter λ, there is no straightforward classical limit $b\longrightarrow 0$, $q\longrightarrow 1$ when we pass to the positive setting since $\tilde{q}$ "blows up" as we discussed earlier. It is an interesting problem how to "extract" the classical theory from its quantum counterpart.

The class of representations of the modular double considered in this paper also plays an important role for the deformation of the space of functions on the split real group $G_{\mathbb{R}}$. Since we always impose the requirement of positive definiteness of quantum generators it is more natural in our setting to consider the deformation of the space of functions on the positive semigroup $G^+_{\mathbb{R}}\subset G_{\mathbb{R}}$, which we denote by $F_{q\tilde{q}}(G^+_{\mathbb{R}})$. The construction of this space using quantum cluster variables is proposed in [4]. In the case when $G_{\mathbb{R}}=\mathrm{SL}(2,\mathbb{R})$ it was conjectured by Teschner [32] and proved by the second author [18] that the space $F_{q\tilde{q}}(G^+_{\mathbb{R}})$ under the regular representation of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$, and a suitable choice of L^2 structure, is decomposed into a direct integral of irreducible representations precisely given by our theorem. It is natural to conjecture that it is also true for the higher rank case, where the space $F_{q\tilde{q}}(\mathrm{GL}^+_q(n,\mathbb{R}))$ equipped with a suitable L^2 norm is constructed by the second author explicitly in [19]. Again it would be interesting to compare the classical and quantum cases by characterizing the restriction of the most continuous component of $L^2(G_{\mathbb{R}})$ to $G^+_{\mathbb{R}}$.

Since the positivity properties of generators in our representations of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$, as well as $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$, play a crucial role we call them *positive principal series representations* or just *positive representations*. Note that there are other ways to deform the principal series of representations even associated with the same minimal parabolic subalgebra $\mathfrak{b}^+_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$. For example, a class of representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(n,\mathbb{R}))$ has been constructed in [15] but the generators do not seem to be represented by positive self-adjoint operators. Another example of a principal series representation for $\mathcal{U}_q(\mathfrak{sl}(n,\mathbb{R}))$ is constructed in [2] using q-difference operators. However, the variables the authors are using come from the standard coordinates of U^+ of the Gauss

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

decomposition, which do not admit the construction of action by positive operators, and hence do not extend to a representation of the modular double. Finally, one more important example of a representation of $U_q(\mathfrak{sl}(n))$ (without any parameters) has been given in [24]. The authors did not study a particular split real case but they did represent the generators in terms of the algebra of q-tori only. Since this representation uses n^2 variables rather than $\frac{n(n-1)}{2}$ as in our construction, it cannot be irreducible. They also work with formal power series and use q-exponential function instead of quantum dilogarithm. It would be interesting to find more explicit relations of these earlier works to our present construction of the positive representations.

The plan of the paper is as follows. In Section 2, we construct the minimal principal series representation for $\mathcal{U}(\mathfrak{sl}(n))$ using the parameterization of totally positive matrix by cluster variables. Then we perform the Mellin transform and obtain a realization of the action using shifting operators. To motivate the calculations of the quantum case, we introduce the CR diagrams for the actions and prove directly all the Lie algebra relations including the Serre relations. In Section 3, we quantize the formulas obtained above, and generalizing the rank 1 case, we construct the positive principal series representations such that the action of $\mathcal{U}_q(\mathfrak{sl}(n,\mathbb{R}))$ is realized by positive essentially self-adjoint operators. We show that our construction is naturally extended to the modular double with the desired transcendental relations (12). In Section 4, we introduce the modified quantum generators to obtain the commutativity between the modular double variables, and present our main theorem. Finally, in Section 5, we discuss various future perspectives of the current program.

2 Principal Series Representations of $\mathcal{U}(\mathfrak{sl}(n))$

2.1 Total positivity and cluster variables

Total positivity for general reductive group is considered by Lusztig [29]. In the case of $G = \mathrm{GL}(n,\mathbb{R})$, a matrix is totally positive if all its entries and the determinant of the minors are positive. Furthermore, the positive monoid admits the Gauss decomposition $\mathrm{GL}^+(n,\mathbb{R}) = U^-_{>0}T_{>0}U^+_{>0}$, where $U^{\pm}_{>0}$ are totally positive upper/lower triangular matrices (considered only for the upper/lower triangular minors), and $T_{>0}$ are diagonal matrices with positive entries.

In [3], another parameterization using cluster variables are studied. These are given by the "initial minors" that are determinants of the square sub-matrices which start from either the top row or the leftmost column. Restricted to the upper triangular

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

unipotent $U_{>0}^+$, the cluster variables are $x_{i,j}$, $1 \le i < j \le n$, where $x_{i,j}$ is the initial minor

$$x_{i,j} = \det \begin{pmatrix} z_{1,j-i+1} & \cdots & z_{1,j} \\ \vdots & \ddots & \vdots \\ z_{i,j-i+1} & \cdots & z_{i,j} \end{pmatrix}. \tag{22}$$

This parameterization corresponds to the canonical decomposition of the maximal Weyl group element w_0 as

$$w_0 = s_{n-1}s_{n-2}\cdots s_2 s_1 s_{n-1}s_{n-2}\cdots s_2 s_{n-1}s_{n-2}\cdots s_3\cdots s_{n-1}, \tag{23}$$

where $s_k = (k, k+1)$ are the standard transpositions, so that

$$U_{>0}^+ = \left\{ \prod_{l=1}^{n-1}\prod_{k=1}^{n-l} s_{n-k}(a_{n-k,n-k-l+1}) \mid a_{ij} > 0 \text{ for } 1 \le j \le i \le n-1 \right\}, \tag{24}$$

with

$$s_i(t) = I_n + tE_{i,i+1}, \tag{25}$$

and $E_{i,j}$ is the standard matrix with 1 at the entry (i, j) and 0 otherwise. Then there is a 1–1 correspondence between a_{ij} and x_{ij} given by the following proposition.

Proposition 2.1. We have

$$a_{i,j} = \frac{x_{j,i+1}x_{j-1,i-1}}{x_{j,i}x_{j-1,i}}, \tag{26}$$

$$x_{i,i+j} = \prod_{m=1}^{j}\prod_{n=1}^{i} a_{m+n-1,n}. \tag{27}$$

Here, we denote by $x_{i,i} = x_{i,0} = x_{0,j} = 1$. □

Furthermore, by calculating the Jacobian of the change of variables from the standard coordinates z_{ij} to the cluster variables x_{ij}, we have the following proposition.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Proposition 2.2. The Haar measure on $U^+_{>0}$ induced by $\prod_{1\le i<j\le n} dz_{ij}$ on U^+ is given by

$$\prod_{1\le i<j\le n-1} \frac{dx_{ij}}{x_{ij}} \prod_{i=1}^{n-1} dx_{in}. \tag{28}$$

□

2.2 Infinitesimal action

The minimal principal series representation for $\mathcal{U}(\mathfrak{sl}(n,\mathbb{R}))$ can be realized on the totally positive matrices as the infinitesimal action of $g\in \mathrm{SL}^+(n,\mathbb{R})$ acting on $\mathbb{C}[U^+_{>0}]$ by

$$g\cdot f(g_+) = \chi_\lambda(g_+g) f([g_+g]_+). \tag{29}$$

Here, we write the Gauss decomposition of g as

$$g = g_-g_0g_+, \tag{30}$$

so that $[g]_+ = g_+$ is the projection of g onto $U^+_{>0}$, and $\chi_\lambda(g)$ is the character function defined by

$$\chi_\lambda(g) = \prod_{i=1}^{n} u_i^{2\lambda_i}, \tag{31}$$

where $\lambda = (\lambda_i)\in\mathbb{C}^n$ and u_i is the entry of the diagonal part $g_0\in T_{>0}$.

For a general matrix, the projection onto U^+ of the Gauss decomposition is given by the following lemma.

Lemma 2.3. The entry z_{ij} of $[g]_+$ is given by

$$\frac{\det N_i^j}{\det N_i}, \tag{32}$$

where N_i is the $i\times i$ determinant of the main diagonal submatrix of g, and N_i^j is the minor N_i with the last column replaced by the jth column $\langle g_{1j},\ldots,g_{ij}\rangle^{\mathrm{T}}$. □

Now, we can find the action of $\exp(tX)\in \mathrm{SL}(n,\mathbb{R})$ and hence $X\in\mathfrak{sl}(n,\mathbb{R})$ by infinitesimal method.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

First, we consider $X = E_i$. The elementary matrix

$$\exp(tE_i) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} = I + te_{i,i+1} \tag{33}$$

only modifies the $(i+1)$th column. Therefore, we can immediately read off its action:

Proposition 2.4. For $1 \leq j < k \leq n$, one has

$$\exp(tE_i) \cdot x_{jk} = \begin{cases} x_{jk} & \text{if } k - j \neq i, \\ x_{jk} + N_{i;j} t & \text{if } k - j = i, \end{cases} \tag{34}$$

where $N_{i;j}$ is the determinant of the original $j \times (j+1)$ block matrix from $z_{1,i}$ to $z_{j,i+j}$ with the second column removed:

$$N_{i;j} = \det \begin{pmatrix} z_{1,i} & z_{1,i+2} & \cdots & z_{1,i+j} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j,i} & z_{j,i+2} & \cdots & z_{j,i+j} \end{pmatrix}. \tag{35}$$

In particular, $N_{1;1} = 1$ and $N_{i;1} = x_{1i}$ for $i > 1$. □

Next, we consider $X = F_i$. The elementary matrix

$$\exp(tF_i) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} = I + te_{i+1,i} \tag{36}$$

only modifies the ith column.

Since F_i is lower triangular, the action will induce lower triangular terms where only a single entry is off. Therefore, applying the projection formulas as above, the entries can be easily determined:

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Lemma 2.5. The projection g_+ of g under the action of F_i is given by

$$([\exp(tF_i)\cdot g]_+)_{jk} := \begin{cases} z_{jk} & \text{if } j<i \text{ and } k\neq i,\\ z_{ji}+z_{j,i+1}t & \text{if } j<i \text{ and } k=i,\\ \dfrac{z_{jk}}{1+z_{i,i+1}t} & \text{if } j=i,\\ z_{jk}+t\det\begin{pmatrix} z_{i,i+1} & z_{i,k}\\ 1 & z_{i+1,k}\end{pmatrix} & \text{if } j=i+1,\\ z_{jk} & \text{if } j>i+1.\end{cases} \tag{37}$$

□

Proof. We note that the denominator for the projection formula is 1 unless $j=i$, which induces the factor $1+z_{i,i+1}t$ in the diagonal part. Therefore, the formula follows from a simple determinant calculation. ■

The diagonal factor $1+z_{i,i+1}t$ can be combined with the character function, and we obtain the following proposition.

Proposition 2.6. For $1\le j<k\le n$, one has

$$\exp(tF_i)\cdot x_{jk} = \begin{cases} x_{jk} & \text{if } j<i \text{ and } k\neq i,\\ x_{jk}+N^{i;j}t & \text{if } j<i \text{ and } k=i,\\ x_{ik}(1+z_{i,i+1}t)^{2\lambda_i-1} & \text{if } j=i,\\ x_{jk} & \text{if } j>i,\end{cases} \tag{38}$$

where $N^{i;j}$ is the determinant of the original $j\times(j+1)$ block matrix from $z_{1,i-j+1}$ to $z_{j,i+1}$ with the second to last column removed:

$$N^{i;j} = \det\begin{pmatrix} z_{1,i-j+1} & \cdots & z_{1,i-1} & z_{1,i+1}\\ \vdots & \ddots & \vdots & \vdots\\ z_{j,i-j+1} & \cdots & z_{j,i-1} & z_{j,i+1}\end{pmatrix}. \tag{39}$$

In particular, $N^{i;1}=x_{1,i+1}$. □

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Finally, the action of

$$\exp(tH_i) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} \tag{40}$$

can also be easily found:

Proposition 2.7. The action of $\exp(tH_i)$ is given by

$$\exp(tH_i) \cdot x_{jk} = \begin{cases} \mathrm{e}^{2\lambda_i t}\, \mathrm{e}^t x_{jk} & \text{if } k = i, \\ \mathrm{e}^{2\lambda_i t}\, \mathrm{e}^{-t} x_{jk} & \text{if } j = i \text{ or } k - j = i, \text{ but } k \neq 2i, \\ \mathrm{e}^{2\lambda_i t}\, \mathrm{e}^{-2t} x_{jk} & \text{if } j = i, k = 2i, \\ \mathrm{e}^{2\lambda_i t} x_{jk} & \text{otherwise.} \end{cases} \tag{41}$$

□

Combining the above propositions, we obtain the action of the Lie algebra generators:

Theorem 2.8. The action of E_i, F_i, and H_i are given by

$$E_i \cdot f = \sum_{j=1}^{n-i} N_{i;j} f_{j,i+j}, \tag{42}$$

$$F_i \cdot f = -\sum_{k=i+1}^{n} x_{ik} z_{i,i+1} f_{i,k} + \sum_{j=1}^{i-1} N^{i;j} f_{j,i} + 2z_{i,i+1}\lambda_1, \tag{43}$$

$$H_i \cdot f = \sum_{j=1}^{i-1} x_{ji} f_{j,i} - \sum_{k=i+1}^{n} x_{ik} f_{i,k} - \sum_{j=1}^{n-i} x_{j,i+j} f_{j,i+j} + 2\lambda_i. \tag{44}$$

□

Following the techniques from [3], the axillary terms $N_{i;j}$, $N^{i;j}$, and $z_{i,i+1}$ can actually be expressed in terms of x_{ij}:

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Proposition 2.9. We have the following expressions:

$$a_{i,j} = \frac{x_{j-1,i-1}x_{j,i+1}}{x_{j-1,i}x_{j,i}}, \tag{45}$$

$$z_{i,i+1} = \sum_{j=1}^{i} a_{i,j} = \sum_{j=1}^{i} \frac{x_{j,i+1}x_{j-1,i-1}}{x_{j,i}x_{j-1,i}}, \tag{46}$$

$$N_{i;j} = x_{j,i+j} \sum_{k=1}^{j} \frac{x_{k-1,i+k}x_{k,i+k-1}}{x_{k-1,i+k-1}x_{k,i+k}}, \tag{47}$$

$$N^{i;j} = x_{j,i} \sum_{k=1}^{j} a_{i,k} = x_{j,i} \sum_{k=1}^{j} \frac{x_{k-1,i-1}x_{k,i+1}}{x_{k-1,i}x_{k,i}}, \tag{48}$$

where $x_{0,j} = x_{k,k} = 1$. □

2.3 Mellin transformed action

In the classical theory of $\mathrm{SL}(2,\mathbb{R})$, the Mellin transform is used to study the matrix coefficients; see, for example, [43]. This transformation is valid because we are working with positive variables, and it enables us to express differential operators in terms of shifting operators.

Using the formal Mellin transform, we can look at the action of $U(\mathfrak{sl}(n,\mathbb{R}))$ as shifting operators with scalar weights:

$$\frac{\partial}{\partial x}\int f(\mu)x^u\, \mathrm{d}u = \int (u)\, f(u)x^{u-1}\, \mathrm{d}\mu = \int (u+1)\, f(u+1)x^u\, \mathrm{d}u$$

or

$$\frac{\partial}{\partial x} : f(u) \mapsto (u+1)\, f(u+1). \tag{49}$$

Similarly,

$$x : f(u) \mapsto f(u-1), \tag{50}$$

$$x\frac{\partial}{\partial x} : f(u) \mapsto uf(u), \ldots \tag{51}$$

Therefore, according to the expression of $z_{i,i+1}$, $N_{i;j}$, $N^{i;j}$, each monomial term in the action of E_i and F_i involves at most four shifting operators. We write explicitly the action under this transform as follows.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Theorem 2.10. The action of E_i, F_i, and H_i are given by

$$E_i \cdot f(\mathbf{u}) = \sum_{k=1}^{n-i}\left(1+\sum_{j=k}^{n-i} u_{j,i+j}\right) f(u_{k-1,i+k-1}+1, u_{k-1,i+k}-1, u_{k,i+k-1}-1, u_{k,i+k}+1), \tag{52}$$

$$F_i \cdot f(\mathbf{u}) = \sum_{k=1}^{i}\left(1+\sum_{j=k}^{i} u_{j,i} - \sum_{j=i+1}^{n} u_{i,j} + 2\lambda_i\right) f(u_{k-1,i-1}-1, u_{k-1,i}+1, u_{k,i}+1, u_{k,i+1}-1), \tag{53}$$

$$H_i \cdot f(\mathbf{u}) = \left(\sum_{j=1}^{i-1} u_{j,i} - \sum_{j=i+1}^{n} u_{i,j} - \sum_{j=1}^{n-i} u_{j,i+j} + 2\lambda_i\right) f(\mathbf{u}), \tag{54}$$

where $u_{0,i} = u_{k,k} = 0$ and the shifting operators at these indices are nonexistent. □

Formally this formula is nothing but the shifting operator induced by polynomials in x_{ij}. However, when u_{ij} has the appropriate real and imaginary part, Mellin transform can be carried out in the $L^2(\mathbb{R})$ sense, and these operators will become positive self-adjoint operators. These observations will be studied in Section 3.3.

2.4 The Lie algebra relations

The Lie algebra axioms are automatically satisfied for the action in Theorem 2.10, since they are coming from the standard infinitesimal action for $SL(n,\mathbb{R})$. However, the relations will not be guaranteed anymore when we try to quantize the above action. Hence, we first directly verify these relations in this classical setting, and we will observe that the quantum case is completely analogous.

Let us introduce the following notation for the action in Theorem 2.10:

$$E_i \cdot f(\mathbf{u}) = \sum_{k=1}^{n-i} E_i^k(\mathbf{u}) f(\mathbf{u}+\mathbf{e}_{E_i}^k), \tag{55}$$

$$F_i \cdot f(\mathbf{u}) = \sum_{k=1}^{i} F_i^k(\mathbf{u}) f(\mathbf{u}+\mathbf{e}_{F_i}^k), \tag{56}$$

$$H_i \cdot f(\mathbf{u}) = H_i(\mathbf{u}) f(\mathbf{u}). \tag{57}$$

In order to calculate the CR, it is useful to introduce the CR diagrams for E_i^k, F_i^k, and H_i (see Figures 1 and 2).

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

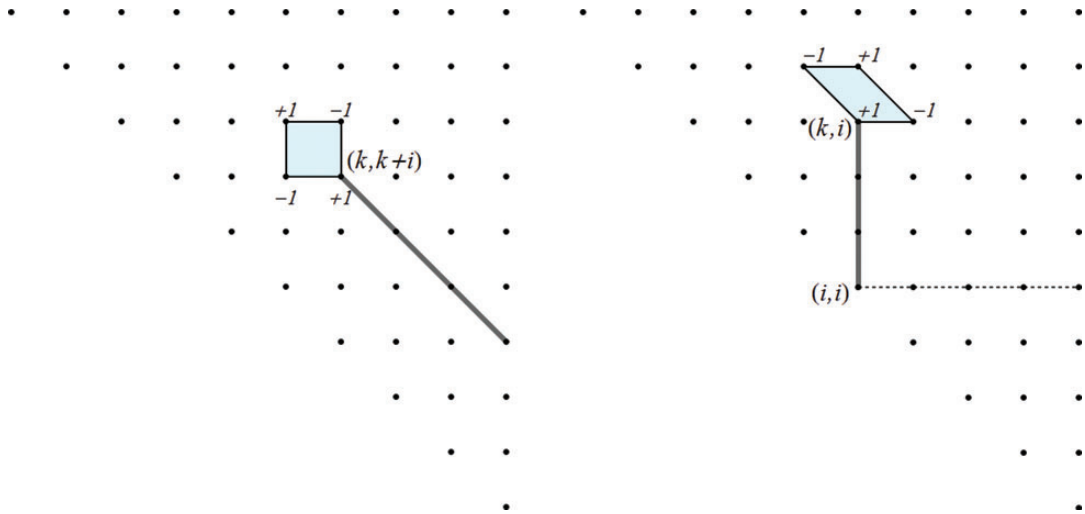

Fig. 1. The CR diagrams for E_i^k and F_i^k.

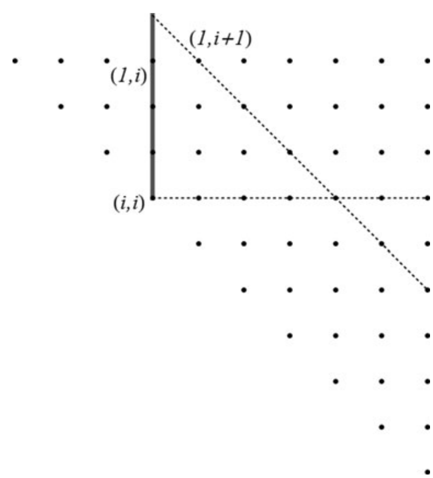

Fig. 2. The CR diagram for H_i.

Here, the quadrilateral encodes the shifting of the operator, and always lies within the grids, and also the "0th row" when $k=1$. So, for example, the operator E_i^k involves shifting in $u_{k,k+i}, u_{k-1,k+i-1}$ by $+1$, and $u_{k-1,k+i}, u_{k,k+i-1}$ by -1. The straight lines encode the multiplication of weights, where the solid lines indicate positive combinations, while the dashed lines indicate negative combinations. So, for example, the dashed line of F_i^k means $-\sum_{j=i}^n u_{ij}$. Note that the weight is 0 (hence the term is actually not there) where the solid and dashed lines meet.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Now, we can compute the CR by the following equation:

Lemma 2.11. Let P_i, where $i=1,2$, be the operators

$$P_i \cdot f(\mathbf{u}) = P_i(\mathbf{u}) f(\mathbf{u}+\mathbf{e}_i), \tag{58}$$

where $P_i(\mathbf{u})$ are linear functions. Then

$$[P_1, P_2] \cdot f(\mathbf{u}) = (P_1(\mathbf{u})P_2'(\mathbf{e}_1) - P_2(\mathbf{u})P_1'(\mathbf{e}_2)) f(\mathbf{u}+\mathbf{e}_1+\mathbf{e}_2), \tag{59}$$

where $P_i'(\mathbf{e}_j) = P_i(\mathbf{e}_j) - P_i(\beta 0)$, that is, it ignores the constant term in the expression of $P_i(\mathbf{u})$. □

Proof. Follows from linearity of $P_i(\mathbf{u})$. ■

Note that in our case, the expression $P_i'(\mathbf{e}_j)$ can be found by composing the quadrilateral from the CR diagram of P_j to the solid-dashed lines from the CR diagram of P_i, and summing up all the weights. So, for example, $F_i^k(\mathbf{e}_{E_{i'}}^{k'})$ is given by composing the square from $E_{i'}^{k'}$ onto the lines for F_i^k, and it will only pick up a nonzero sum of weights when $k=k'$ and $k+i=i'$.

Lemma 2.12. We have the following values:

$$E'^k_i(\mathbf{e}^{k'}_{F_{i'}}) = \begin{cases} +1 & \text{if } (k', i') = (k, i+k), \\ -1 & \text{if } (k', i') = (k, i+k-1), \\ 0 & \text{otherwise}, \end{cases} \tag{60}$$

$$F'^{k'}_{i'}(\mathbf{e}^{k}_{E_i}) = \begin{cases} +1 & \text{if } (k, k+i) = (k', i'), \\ -1 & \text{if } (k, k+i) = (k', i'+1), \\ 0 & \text{otherwise}, \end{cases} \tag{61}$$

$$H'_i(\mathbf{e}^{k}_{E_{i'}}) = \begin{cases} +1 & \text{if } |i-i'| = 1, \\ -2 & \text{if } i = i', \\ 0 & \text{otherwise}, \end{cases} \tag{62}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$H_i'(\mathbf{e}^k_{F_{i'}}) = \begin{cases} -1 & \text{if } |i-i'|=1, \\ 2 & \text{if } i=i', \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

□

Proof. Follows from a direct inspection of the CR diagrams. ■

Proposition 2.13. We have

$$[H_i, E_j] = a_{ij} E_j, \tag{64}$$

$$[H_i, F_j] = -a_{ij} F_j, \tag{65}$$

where

$$a_{ij} = \begin{cases} 2 & i=j, \\ -1 & |i-j|=1, \\ 0 & \text{otherwise} \end{cases}$$

is the Cartan matrix. □

Proof. Since H_i does not have a shift, by Lemma 2.11, we have

$$[H_i, E^k_{i'}] = -E^k_{i'}(\mathbf{u}) H_i'(\mathbf{e}^k_{E_{i'}}) f(\mathbf{u}+\mathbf{e}_{E_{i'}}) = -H_i'(\mathbf{e}^k_{E_{i'}}) E^k_{i'},$$

for all k. Hence, summing up k gives the required relation by Lemma 2.12. Similarly for F_i. ■

Proposition 2.14. We have

$$[E_i, F_{i'}] = \delta_{ii'} H_i. \tag{66}$$

□

Proof. By Lemma 2.12, we observe that $E^k_i(\mathbf{e}^{k'}_{F_{i'}}) = F^{k'}_{i'}(\mathbf{e}^k_{E_i})$ identically for all i, i', k, k'. Hence by Lemma 2.11, the relation reduces to

$$[E^k_i, F^{k'}_{i'}] \cdot f(\mathbf{u}) = (E^k_i(\mathbf{u}) - F^{k'}_{i'}(\mathbf{u})) E^k_i(\mathbf{e}^{k'}_{F_{i'}}) f(\mathbf{u} + \mathbf{e}^k_{E_i} + \mathbf{e}^{k'}_{F_{i'}}).$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

When $i > i'$, it is clear that $E_i^k(\mathbf{e}_{F_{i'}}^{k'}) = 0$ because $k+i > i$, so the nonzero cases can never be satisfied. When $i < i'$, only $k = k'$ gives nonzero value of $E_i^k(\mathbf{e}_{F_{i'}}^{k'}) = \delta_{i',k+i} - \delta_{i',k+i-1}$. Hence, we just need to consider the two terms for $k = i' - i$ and $k = i' - i + 1$. Next, we note that

$$\mathbf{e} = \mathbf{e}_{E_i}^{i'-i} + \mathbf{e}_{F_{i'}}^{i'-i} = \mathbf{e}_{E_i}^{i'-i+1} + \mathbf{e}_{F_{i'}}^{i'-i+1},$$

$$f(\mathbf{u} + \mathbf{e}) = f(u_{i'-i,i'-1} - 1, u_{i'-i,i'} + 2, u_{i'-i,i'+1} - 1),$$

therefore all we need to take care of is the factor. We have

$$\begin{aligned}
E_i^{i'-i}(\mathbf{u}) - F_{i'}^{i'-i}(\mathbf{u}) &= \sum_{j=i'-i}^{n-i} u_{j,i+j} - \sum_{j'=i'-i}^{i'} u_{j'i'} + \sum_{j''=i'+1}^{n} u_{i'j''} + 2\lambda_{i'} \\
&= \sum_{j=i'-i+1}^{n-i} u_{j,i+j} - \sum_{j'=i'-i+1}^{i'} u_{j'i'} + \sum_{j''=i'+1}^{n} u_{i'j''} + 2\lambda_{i'} \\
&= E_i^{i'-i+1}(\mathbf{u}) - F_{i'}^{i'-i+1}(\mathbf{u}),
\end{aligned}$$

hence we conclude that the factor equals:

$$\begin{aligned}
\sum_{k=i'-i}^{i'-i+1} (E_i^k(\mathbf{u}) - F_{i'}^k(\mathbf{u})) E_i^k(\mathbf{e}_{F_{i'}}^k) &= \sum_{k=i'-i}^{i'-i+1} (E_i^k(\mathbf{u}) - F_{i'}^k(\mathbf{u}))(\delta_{i',k+i} - \delta_{i',k+i-1}) \\
&= (E_i^{i'-i}(\mathbf{u}) - F_{i'}^{i'-i}(\mathbf{u})) - (E_i^{i'-i+1}(\mathbf{u}) - F_{i'}^{i'-i+1}(\mathbf{u})) \\
&= 0.
\end{aligned}$$

Finally, when $i = i'$, $E_i^k(\mathbf{e}_{F_i}^{k'}) = 0$ unless $k = k'$ and $i = i + k - 1$, that is, $k = k' = 1$ which gives $E_i^1(\mathbf{e}_{F_i}^1) = -1$. Furthermore, $\mathbf{e}_{E_i}^1 + \mathbf{e}_{F_i}^1 = \beta 0$. Hence, we just need to calculate

$$\begin{aligned}
[E_i^1, F_{i'}^1] &= (E_i^1(\mathbf{u}) - F_i^1(\mathbf{u}))(-1) \\
&= \sum_{j=1}^{i-1} u_{ji} - \sum_{j=i+1}^{n} u_{ij} - \sum_{j=1}^{n-i} u_{j,i+j} + 2\lambda_i \\
&= H_i. \qquad \blacksquare
\end{aligned}$$

Next, we verify the Serre relations. Since they involve relations in E only, or F only, we calculate the $P'(\mathbf{e})$ functions with respect to the shifts within E_i and F_i, respectively.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Lemma 2.15. We have the following values:

We have for E_i:

$$E_i^k(\mathbf{e}_{E_i'}^{k'}) = \begin{cases} 0 & \text{if } k > k', \\ 1 & \text{if } k = k', \\ 2 & \text{if } k < k', \end{cases} \tag{67}$$

$$E_i^k(\mathbf{e}_{E_{i+1}}^{k'}) = \begin{cases} 0 & \text{if } k > k', \\ -1 & \text{if } k \leq k', \end{cases} \tag{68}$$

$$E_{i+1}^k(\mathbf{e}_{E_i}^{k'}) = \begin{cases} 0 & \text{if } k \geq k', \\ -1 & \text{if } k < k'. \end{cases} \tag{69}$$

We have similarly for F_i:

$$F_i^k(\mathbf{e}_{F_i}^{k'}) = \begin{cases} 0 & \text{if } k > k', \\ 1 & \text{if } k = k', \\ 2 & \text{if } k < k', \end{cases} \tag{70}$$

$$F_i^k(\mathbf{e}_{F_{i+1}}^{k'}) = \begin{cases} 0 & \text{if } k \geq k', \\ -1 & \text{if } k < k', \end{cases} \tag{71}$$

$$F_{i+1}^k(\mathbf{e}_{F_i}^{k'}) = \begin{cases} 0 & \text{if } k > k' \\ -1 & \text{if } k \leq k'. \end{cases} \tag{72}$$

□

Proof. This again follows from a direct inspection of the diagrams: how square superimposes on the diagonal line, and how parallelogram superimposes on the solid-dashed lines. ■

Corollary 2.16. When $|i-j| \geq 2$, $[E_i, E_j] = [F_i, F_j] = 0$. □

Proof. The square or parallelogram never touches the lines, so all the $P_i(\mathbf{e}_j)$'s are zero in the CR (59). ■

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Lemma 2.17. We have the Serre relations

$$\begin{aligned} E_iE_iE_{i+1} - 2E_iE_{i+1}E_i + E_{i+1}E_iE_i &= 0, \\ F_iF_iF_{i+1} - 2F_iF_{i+1}F_i + F_{i+1}F_iF_i &= 0, \end{aligned} \tag{73}$$

$$\begin{aligned} E_{i+1}E_{i+1}E_i - 2E_{i+1}E_iE_{i+1} + E_iE_{i+1}E_{i+1} &= 0, \\ F_{i+1}F_{i+1}F_i - 2F_{i+1}F_iF_{i+1} + F_iF_{i+1}F_{i+1} &= 0. \end{aligned} \tag{74}$$

□

Proof. First, we observe that for both $P = E$ or F,

$$P_i^k(\mathbf{e}_{P_i}^{k'}) + P_i^{k'}(\mathbf{e}_{P_i}^k) = 2,$$

$$P_i^k(\mathbf{e}_{P_i}^{k''}) + P_{i+1}^{k''}(\mathbf{e}_{P_i}^k) = -1.$$

Hence, we let $a = P_i^{k'}(\mathbf{e}_{P_i}^k)$, $b = P_{i+1}^{k''}(\mathbf{e}_{P_i}^k)$, $c = P_{i+1}^{k''}(\mathbf{e}_{P_i}^k)$. Note that a can only take values $0, 1, 2$, while b, c can only take $0, -1$. Then the Serre relations amount to the vanishing of $B(a, b, c)$ given by

$$\begin{aligned} B(a,b,c) = \sum_{\mathrm{sym}\{k,k'\}} & (P_i^k(\mathbf{u})(P_i^{k'}(\mathbf{u}) + a)(P_{i+1}^{k''}(\mathbf{u}) + b + c) \\ & - 2P_i^k(\mathbf{u})(P_{i+1}^{k''}(\mathbf{u}) + b)(P_i^{k'}(\mathbf{u}) + a - c - 1) \\ & + P_{i+1}^{k''}(\mathbf{u})(P_i^k(\mathbf{u}) - b - 1)(P_i^{k'}(\mathbf{u}) + a - c - 1). \end{aligned} \tag{75}$$

Here, the sum is a symmetrized sum where in the second set k and k' are interchanged, b and c are interchanged, and $a \longrightarrow 2 - a$.

After expanding and simplifying, it becomes

$$B(a,b,c) = (P_i^k(\mathbf{u}) + P_i^{k'}(\mathbf{u}) + P_{i+1}^{k''}(\mathbf{u}))(b(2 - a + c) + (a + b)c). \tag{76}$$

Hence, the Serre relations amount to the equation

$$b(2 - a + c) + (a + b)c = 0. \tag{77}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

It is easy to see that $B(a,0,0)=B(a,-1,-1)=0$ for all a, and $B(0,0,-1)=B(2,-1,0)=0$. Therefore, it remains to check that the parameter must fall into these cases.

Indeed, if $a=0$, then $k'>k$, hence $(0,-1,0)$ cannot happen because otherwise we have $k'<k''<k$. Similarly $(2,0,-1)$ cannot happen. If $a=1$, then $k'=k$ and we must have $k''<k=k'$ or $k''>k=k'$ giving $(1,0,0)$ or $(1,-1,-1)$.

This proves the first Serre relations for both E_i and F_i. The second Serre relations are exactly the same with i replaced by $i+1$, and keeping a, b, and c in the same order. ■

3 Principal Series Representations of $\mathcal{U}_q(\mathfrak{sl}(n))$

3.1 Quantization

Assured by the CR diagrams, we can now quantize the action and repeat the same proofs for all the quantum group relations. The procedure is just to change the weight into its quantum number as follows:

Theorem 3.1. We have the action of $\mathcal{U}_q(\mathfrak{sl}(n))$ given by

$$E_i f(\mathbf{u}) = \sum_{k=1}^{n-i} \left[1+\sum_{j=k}^{n-i} u_{j,i+j}\right]_q f(u_{k-1,i+k-1}+1, u_{k-1,i+k}-1, u_{k,i+k-1}-1, u_{k,i+k}+1), \tag{78}$$

$$F_i f(\mathbf{u}) = \sum_{k=1}^{i} \left[1+\sum_{j=k}^{i} u_{ji} - \sum_{j=i+1}^{n} u_{ij} + 2\lambda_i\right]_q f(u_{k-1,i-1}-1, u_{k-1,i}+1, u_{k,i}+1, u_{k,i+1}-1), \tag{79}$$

$$K_i f(\mathbf{u}) = q^{\left(\sum_{j=1}^{i-1} u_{ji} - \sum_{j=i+1}^{n} u_{ij} - \sum_{j=1}^{n-i} u_{j,i+j} + 2\lambda_i\right)} f(\mathbf{u}), \tag{80}$$

where $[n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}$. They satisfy all the quantum CR (82)–(86). □

Proof. We need to check all the quantum relations. Let us denote by $E_i^k(\mathbf{u})$ and $F_i^k(\mathbf{u})$ as before, so that for $P=E$ and F,

$$P_i \cdot f(\mathbf{u}) = \sum_k [P_i^k(\mathbf{u})]_q f(\mathbf{u}+\mathbf{e}_i^k).$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

First note that

$$[n]_q = n \quad \text{for } n = 0, 1, -1. \tag{81}$$

The relations with K_i:

$$K_i E_j = q^{a_{ij}} E_j K_i, \tag{82}$$

$$K_i F_j = q^{-a_{ij}} F_j K_i \tag{83}$$

follow from the classical calculations because $K_i = q^{H_i}$, where H_i is just the classical action.

The relations

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \tag{84}$$

follow from the fact that when $E_i^k(\mathbf{e}_{F_{i'}}^{k'}) = F_{i'}^{k'}(\mathbf{e}_i^k)$ as in the classical case, the CR factor becomes

$$[E_i^k(\mathbf{u})]_q [F_{i'}^{k'}(\mathbf{u} + \mathbf{e}_{E_i}^k)]_q - [F_{i'}^{k'}(\mathbf{u})]_q [E_i^k(\mathbf{u} + \mathbf{e}_{F_{i'}}^{k'})]_q = [E_i^k(\mathbf{u}) - F_{i'}^{k'}(\mathbf{u})]_q [E_i^k(\mathbf{e}_{F_{i'}}^{k'})]_q.$$

Hence, all the classical calculations still hold, including the case $i = i'$, where we obtain

$$[E_i^1(\mathbf{u}) - F_i^1(\mathbf{u})]_q = [H_i]_q = \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

Finally, the quantum Serre relations

$$\begin{aligned} E_i E_i E_{i+1} - [2]_q E_i E_{i+1} E_i + E_{i+1} E_i E_i = 0, \\ F_i F_i F_{i+1} - [2]_q F_i F_{i+1} F_i + F_{i+1} F_i F_i = 0, \end{aligned} \tag{85}$$

$$\begin{aligned} E_{i+1} E_{i+1} E_i - [2]_q E_{i+1} E_i E_{i+1} + E_i E_{i+1} E_{i+1} = 0, \\ F_{i+1} F_{i+1} F_i - [2]_q F_{i+1} F_i F_{i+1} + F_i F_{i+1} F_{i+1} = 0, \end{aligned} \tag{86}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

where $[2]_q = q + q^{-1}$, are equivalent to the vanishing of the commutation factor (with $P = E$ or F)

$$B_q(a, b, c) = \sum_{\mathrm{sym}\{k,k'\}} ([P_i^k(\mathbf{u})]_q [P_i^{k'}(\mathbf{u}) + a]_q [P_{i+1}^{k''}(\mathbf{u}) + b + c]_q$$
$$- [2]_q [P_i^k(\mathbf{u})]_q [P_{i+1}^{k''}(\mathbf{u}) + b]_q [P_i^{k'}(\mathbf{u}) + a - c - 1]_q$$
$$+ [P_{i+1}^{k''}(\mathbf{u})]_q [P_i^k(\mathbf{u}) - b - 1]_q [P_i^{k'}(\mathbf{u}) + a - c - 1]_q)$$

with the same a, b, and c and the symmetrized sum as before (cf. (75)).

It turns out that this can also be simplified by expanding, and we obtain

$$B_q(a, b, c) = [P_i^k(\mathbf{u}) + P_i^{k'}(\mathbf{u}) + P_{i+1}^{k''}(\mathbf{u})]_q ([b]_q [2 - a + c]_q + [a + b]_q [c]_q), \tag{87}$$

completely analogous to the classical case. Hence, the quantum Serre relation amounts to the equation

$$[b]_q [2 - a + c]_q + [a + b]_q [c]_q = 0, \tag{88}$$

which is equivalent to the classical equation (77) using (81). ■

3.2 Positive representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$

Let us motivate our construction of the positive principal series representations by considering first the transition from $\mathcal{U}(\mathfrak{sl}(2))$ to $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$. For $\mathcal{U}(\mathfrak{sl}(2))$, the action of the generators is given by

$$E \cdot f(u) = (u + 1) f(u + 1),$$
$$F \cdot f(u) = (1 - u + 2\lambda) f(u - 1),$$
$$H \cdot f(u) = (-2u + 2\lambda) f(u).$$

Now, note that for $\mathrm{SL}_2^+(\mathbb{R})$, the Haar measure on $U_{>0}^+$ is given by du, hence when we apply the Mellin transform, we actually want $\mathrm{Re}(u) = -\frac{1}{2}$ in order for the L^2 structure to be preserved. Hence, if we make the translation $u \longrightarrow -\mathbf{i}u + \lambda$ with $\lambda = -\frac{1}{2} + \mathbf{i}\alpha$, $\alpha \in \mathbb{R}$,

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

we obtain

$$E \cdot f(u) = (\tfrac{1}{2} + \mathbf{i}\alpha - \mathbf{i}u) f(u + \mathbf{i}),$$

$$F \cdot f(u) = (\tfrac{1}{2} + \mathbf{i}\alpha + \mathbf{i}u) f(u - \mathbf{i}),$$

$$H \cdot f(u) = 2\mathbf{i}uf(u),$$

which can then be seen to be anti self-adjoint, unbounded operators, so that their exponentials are unitary operators.

In [33], the (Fourier transformed) action for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, where $q = e^{\pi \mathbf{i} b^2}$ for $0 < b < 1$, is constructed by scaling u by b and replacing $\frac{1}{2} + \mathbf{i}\alpha$ by $\frac{Q}{2b} + \mathbf{i}\frac{\alpha}{b}$ in the quantized formula, where $Q = b + b^{-1}$, so that the action becomes

$$E = \left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}(\alpha - u)\right]_q e^{-2\pi bp}, \tag{89}$$

$$F = \left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}(\alpha + u)\right]_q e^{2\pi bp}, \tag{90}$$

$$K = e^{2\pi bu}, \tag{91}$$

where $p = \frac{1}{2\pi \mathbf{i}}\frac{d}{du}$ so that $e^{\pm 2\pi bp} f(u) = f(u \mp \mathbf{i}b)$.

This representation, called the self-dual principal series in [33], has two remarkable properties. First, the action given is positive essentially self-adjoint. Owing to the factor $\frac{Q}{2b} = \frac{1}{2} + \frac{1}{2b^2}$, the expression for E and F is actually

$$E = \left(\frac{\mathbf{i}}{q - q^{-1}}\right)(e^{\pi b(\alpha - u - 2p)} + e^{-\pi b(\alpha - u + 2p)}), \tag{92}$$

$$F = \left(\frac{\mathbf{i}}{q - q^{-1}}\right)(e^{\pi b(\alpha + u + 2p)} + e^{-\pi b(\alpha + u - 2p)}), \tag{93}$$

which is positive essentially self-adjoint, as shown in [18]. Note that the factor

$$\left(\frac{\mathbf{i}}{q - q^{-1}}\right) = (2\sin(\pi b^2))^{-1} \tag{94}$$

is positive for $0 < b < 1$.

Secondly, it is self-dual under the change $b \longleftrightarrow b^{-1}$ in the following sense. This change gives the action $(\tilde{E}, \tilde{F}, \tilde{K})$ of its modular double counterpart $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ where

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$\tilde{q} = e^{\pi \mathbf{i} b^{-2}}$, which commute with (E, F, K) weakly (i.e., the spectrum does not commute). Furthermore, if we let

$$e = 2\sin(\pi b^2)E, \quad f = 2\sin(\pi b^2)F, \tag{95}$$

$$\tilde{e} = 2\sin(\pi b^{-2})\tilde{E}, \quad \tilde{f} = 2\sin(\pi b^{-2})\tilde{F}, \tag{96}$$

then the following transcendental relations are valid:

$$e^{\frac{1}{b^2}} = \tilde{e}, \quad f^{\frac{1}{b^2}} = \tilde{f}, \quad K^{\frac{1}{b^2}} = \tilde{K}. \tag{97}$$

The proof is based on the following lemma by Volkov [44] (see also [5, 18]), which is also repeatedly used in our construction.

Lemma 3.2. If u and v are essentially self-adjoint and $uv = q^2 vu$, then $u + v$ is essential self-adjoint, and

$$(u+v)^{1/b^2} = u^{1/b^2} + v^{1/b^2}. \tag{98}$$

□

3.3 Positive representations of $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{R}))$

In order to construct a positive representation, we need to shift our (pure imaginary) variables u_{ij} with appropriate real part such that the Mellin transform preserves the Haar measure, and moreover each expression in the quantum weight has the factor $\frac{1}{2} + \mathbf{i}\alpha_k$ where $\lambda_k = -\frac{1}{2} + \mathbf{i}\alpha_k$ with $\alpha_k \in \mathbb{R}$.

Theorem 3.3. There is a unique shift in $u_{i,j} \longrightarrow -\mathbf{i}u_{i,j} + c_{i,j}$ such that the action of $\mathcal{U}(\mathfrak{sl}(n, \mathbb{R}))$ takes the form

$$E_i f(\mathbf{u}) = \sum_{k=1}^{n-i} \left(\frac{1}{2} + \mathbf{i}\alpha'_k - \mathbf{i} \sum_{j=k}^{n-i} u_{j,i+j} \right) \cdot f(u_{k-1,i-1} + \mathbf{i}, u_{k-1,i} - \mathbf{i}, u_{k,i} - \mathbf{i}, u_{k,i+1} + \mathbf{i}), \tag{99}$$

$$F_i f(\mathbf{u}) = \sum_{k=1}^{i} \left(\frac{1}{2} + \mathbf{i}\alpha'_k - \mathbf{i} \sum_{j=k}^{i} u_{ji} + i \sum_{j=i+1}^{n} u_{ij} \right) \cdot f(u_{k-1,i-1} + \mathbf{i}, u_{k-1,i} - \mathbf{i}, u_{k,i} - \mathbf{i}, u_{k,i+1} + \mathbf{i}), \tag{100}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$H_i f(\mathbf{u}) = -\mathbf{i}\left(\sum_{j=1}^{i-1} u_{ji} - \sum_{j=i+1}^{n} u_{ij} - \sum_{j=1}^{n-i} u_{j,i+j}\right) f(\mathbf{u}). \tag{101}$$

Here, the new λ'_k is related to the old λ_k by

$$\lambda'_k := \frac{\sum_{m=1}^{n-k} m\lambda_{n-m}}{\sum_{m=1}^{n-k} m}, \tag{102}$$

and hence, in particular, $\mathrm{Re}(\lambda'_k) = -\frac{1}{2}$ and we set $\lambda'_k := -\frac{1}{2} + \mathbf{i}\alpha'_k$ with $\alpha'_k \in \mathbb{R}$. Furthermore, the shifts $c_{i,j}$ obey the Haar measure on $U^+_{>0}$ (cf. Proposition 2.2), namely

$$\mathrm{Re}(c_{i,j}) = \begin{cases} -\frac{1}{2} & \text{if } j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{103}$$

□

Proof. This is an exercise in linear algebra where we require the constant in H_i to disappear, and for each fixed k, the constant in E_i^k and $E_{i'}^k$ match for every i, i'. This gives $n(n-1)/2$ relations in the possible $n(n-1)/2$ constants $c_{i,j}$. An elementary reduction shows that this reduces to sets of simultaneous equations of the form

$$(k-1)x_{lk} + \sum_{m=1}^{k} 2x_{lm} = 2\lambda'_{l-k+1}, \quad k = 1, \ldots, l,$$

for $1 \le l \le n-1$, where $x_{lm} = c_{l-m+1,n-m+1}$, which obviously has a unique solution and can be solved explicitly. ■

Therefore following the quantization procedure of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, we obtain the following theorem.

Theorem 3.4. The following action of the generators gives the positive principal series representation for $\mathcal{U}_q(\mathfrak{sl}(n,\mathbb{R}))$:

$$E_i f(\mathbf{u}) = \sum_{k=1}^{n-i}\left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}\left(\alpha'_k - \sum_{j=k}^{n-i} u_{j,i+j}\right)\right]_q \cdot e^{2\pi b(-p_{k-1,i+k-1} + p_{k-1,i+k} + p_{k,i+k-1} - p_{k,i+k})}, \tag{104}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$F_i f(\mathbf{u}) = \sum_{k=1}^{i}\left[\frac{Q}{2b} + \frac{\mathbf{i}}{b}\left(\alpha'_k - \sum_{j=k}^{i} u_{ji} + \sum_{j=i+1}^{n} u_{ij}\right)\right]_q \cdot e^{2\pi b(p_{k-1,i-1} - p_{k-1,i} - p_{k,i} + p_{k,i+1})}, \tag{105}$$

$$K_i f(\mathbf{u}) = e^{\pi b(\sum_{j=1}^{i-1} u_{ji} - \sum_{j=i+1}^{n} u_{ij} - \sum_{j=1}^{n-i} u_{j,i+j})} f(\mathbf{u}), \tag{106}$$

where as usual $q = e^{\pi \mathbf{i} b^2}$, $Q = b + b^{-1}$, and $e^{\pm 2\pi b p_{ij}}$ is shift in u_{ij} by $\mp \mathbf{i}b$. These operators are all positive essentially self-adjoint and satisfy the transcendental relations

$$(e_i)^{1/b^2} = \tilde{e}_i, \tag{107}$$

$$(f_i)^{1/b^2} = \tilde{f}_i, \tag{108}$$

$$K_i^{1/b^2} = \tilde{K}_i, \tag{109}$$

where as before, $e_i = 2\sin(\pi b^2)E_i$ and $f_i = 2\sin(\pi b^2)F_i$ and similarly for $\tilde{e}_i$ and $\tilde{f}_i$ with b replaced by b^{-1} in all the formulas. $\square$

Proof. From the CR diagram, we know that only one shifts and one weight index coincide for both E and F. Hence the constant $\frac{Q}{2b}$ gives the positivity of the operator, using the CR of p and x:

$$q^{\frac{Q}{2b}+x} e^{2\pi b p} = \mathbf{i} e^{\pi b(x+2p)},$$

so that for c, c' commuting with x,

$$\left[\frac{Q}{2b} + x + c\right]_q e^{2\pi b(p+c')} = \left(\frac{\mathbf{i}}{q-q^{-1}}\right)(e^{\pi b(x+2p+c+2c')} + e^{\pi b(-x+2p+c+2c')})$$

is positive. Furthermore, the two factors q^2 commute.

Hence, let us write the operators (both E_i and F_i) in the form

$$\frac{\mathbf{i}}{q-q^{-1}} \sum_k (A_k^+ + A_k^-), \tag{110}$$

where $A_k^+ A_k^- = q^2 A_k^- A_k^+$. Moreover, by looking at the CR diagram, we can actually see that:

$$A_k^+ A_{k'}^{\pm} = q^2 A_{k'}^{\pm} A_k^+,$$

$$A_k^- A_{k'}^{\pm} = q^{-2} A_{k'}^{\pm} A_k^-,$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

whenever $k<k'$, so that, if we rearrange the summation as

$$A_1^+ + A_2^+ + \cdots + A_s^+ + A_s^- + A_{s-1}^- + \cdots + A_1^-,$$

then each term q^2 commute with the terms after that. Hence, using the fact that the operators

$$e^{\pi b(\sum \alpha_{ij}u_{ij} + \sum \beta_{ij}p_{ij})}$$

are essentially self-adjoint, by applying Lemma 3.2 and by induction, we immediately get the required conditions, as well as the transcendental relations. ■

We note that it is actually impossible for the operators (E_i, F_i, K_i) and $(\tilde{E}_j, \tilde{F}_j, \tilde{K}_j)$ to commute in general, for example, due to relations such as

$$K_iE_{i+1} = q^{-1}E_{i+1}K_i,$$

since

$$\begin{aligned} K_i\tilde{E}_{i+1} &= K_iE_{i+1}^{\frac{1}{b^2}} \\ &= q^{-\frac{1}{b^2}}E_{i+1}^{\frac{1}{b^2}}K_i \\ &= -\tilde{E}_{i+1}K_i. \end{aligned}$$

However, we do have the following proposition.

Proposition 3.5. The operators (E_i, F_i, K_i) commute with the generators $(\tilde{E}_j, \tilde{F}_j, \tilde{K}_j)$ up to a sign. □

Proof. The operators commute whenever by imposing the CR diagrams, there are even numbers of (x, p) pair, so that the commuting factor is of the form $q^{2\pi \mathbf{i}k} = 1$ for $k \in \mathbb{Z}$. Otherwise for odd numbers of (x, p) pair we pick up $e^{\pi \mathbf{i}(2k+1)} = -1$. Looking at the CR diagrams, it is then clear that we have:

$$E_i\tilde{E}_j = -\tilde{E}_jE_i \quad \text{if } |i-j| = 1,$$

$$F_i\tilde{F}_j = -\tilde{F}_jF_i \quad \text{if } |i-j| = 1,$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$E_i\tilde{K}_j = -\tilde{K}_j E_i \quad \text{if } |i-j| = 1,$$

$$K_i\tilde{E}_j = -\tilde{E}_j K_i \quad \text{if } |i-j| = 1,$$

$$F_i\tilde{K}_j = -\tilde{K}_j F_i \quad \text{if } |i-j| = 1,$$

$$K_i\tilde{F}_j = -\tilde{F}_j K_i \quad \text{if } |i-j| = 1,$$

and the variables commute otherwise. ■

In order to get commutativity, we have to modify the quantum group which will be considered in the next section. We conclude this section with several fundamental properties of these representations.

Proposition 3.6. The positive representations are irreducible. □

Proof. First, we note that we can recover the classical action from the representation defined in Section 3.1 by a formal classical limit $b \longrightarrow 0$ given by the limit on the quantum numbers $[n]_q$. Furthermore, we know that for any fixed generic parameter λ the classical action is irreducible. Since the positive representation is obtained by first rescaling the function space by b and shifting of the parameter, followed by specifying the real part of $1+\lambda_i$ to be $\frac{1}{2}+\frac{1}{2b^2}$ which is generic since b^2 is irrational, it follows that the quantum representation is also irreducible. ■

Remark 3.7. In the classical case, there is a family of intertwiners corresponding to the Weyl group elements $w \in W$ between representations of principal series parameterized by $\mathfrak{h}_{\mathbb{R}}^*$; see, for example, [27] and references therein. In the quantum case, the intertwiners are explicitly given by ratios of quantum dilogarithm function [20, 21]. Therefore, we obtain parameterization of the inequivalent positive representations by the positive cone $P_{\mathbb{R}}^+ \simeq \mathfrak{h}_{\mathbb{R}}^*/W$. Thus, we can restrict the values of the parameters to $\alpha_k' \geq 0$. □

Finally, we observe that the coproducts also satisfy the criterion of a positive representation:

Proposition 3.8. The coproducts

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \tag{111}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \tag{112}$$

$$\Delta(K_i) = K_i \otimes K_i \tag{113}$$

are positive essentially self-adjoint operators, and satisfy the transcendental relation

$$(\Delta e_i)^{\frac{1}{b^2}} = \Delta \tilde{e}_i, \tag{114}$$

$$(\Delta f_i)^{\frac{1}{b^2}} = \Delta \tilde{f}_i, \tag{115}$$

$$(\Delta K_i)^{\frac{1}{b^2}} = \Delta \tilde{K}_i. \tag{116}$$

□

Proof. It follows from the fact that the two summands of the coproducts for E_i and F_i are positive self-adjoint and q^2-commute, hence we can apply Lemma 3.2 and the transcendental relations (107)–(109) of the generators. ■

4 Main Theorem

4.1 Modified quantum group $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{sl}(n,\mathbb{R}))$ and its positive representations

In order to obtain a representation of the modular double, we would like to have generators corresponding to the two parts of the modular double commute with each other. We therefore introduce the following modified quantum generators:

Definition 4.1. Let

$$n_i = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \tag{117}$$

We define $\mathfrak{q} := q^2 = e^{2\pi i b^2}$ and

$$\mathfrak{q}_i := \begin{cases} \mathfrak{q}^{-1} & \text{if } n_i = 0, \\ \mathfrak{q} & \text{if } n_i = 1, \end{cases} \tag{118}$$

and define the modified quantum generators as

$$\mathbf{E}_i := q^{n_i} E_i K_i^{n_i},$$

$$\mathbf{F}_i := q^{1-n_i} F_i K_i^{n_i-1},$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$\mathbf{K}_i := \mathfrak{q}_i^{H_i} = \begin{cases} K_i^{-2} & \text{if } n_i = 0, \\ K_i^2 & \text{if } n_i = 1. \end{cases}$$

Then the variables are positive self-adjoint. Let

$$[A, B]_{\mathfrak{q}} = AB - \mathfrak{q}^{-1} BA \tag{119}$$

be the quantum commutator. Then the quantum relations in the new variables become

$$\mathbf{K}_i \mathbf{E}_j = \mathfrak{q}_i{}^{a_{ij}} \mathbf{E}_j \mathbf{K}_i, \tag{120}$$

$$\mathbf{K}_i \mathbf{F}_j = \mathfrak{q}_i{}^{-a_{ij}} \mathbf{F}_j \mathbf{K}_i, \tag{121}$$

$$\mathbf{E}_i \mathbf{F}_j = \mathbf{F}_j \mathbf{E}_i \quad \text{if } i \neq j, \tag{122}$$

$$[\mathbf{E}_i, \mathbf{F}_i]_{\mathfrak{q}_i} = \frac{1 - \mathbf{K}_i}{1 - \mathfrak{q}_i}, \tag{123}$$

and the quantum Serre relations for $|i - j| = 1$ become

$$[[\mathbf{E}_j, \mathbf{E}_i]_{\mathfrak{q}_i}, \mathbf{E}_i] = 0 = [[\mathbf{F}_j, \mathbf{F}_i]_{\mathfrak{q}_i}, \mathbf{F}_i]. \tag{124}$$

We denote the modified quantum group by $\mathbf{U}_{\mathfrak{q}}(\mathfrak{sl}(n, \mathbb{R}))$. □

First, we note that our construction implies the following proposition.

Proposition 4.2. Let $\tilde{\mathfrak{q}} := \tilde{q}^2 = e^{2\pi i b^{-2}} = \mathfrak{q}^{\frac{1}{b^2}}$. We define the tilde part of the modified modular double by representing the generators $\tilde{\mathbf{E}}_i, \tilde{\mathbf{F}}_i$, and $\tilde{\mathbf{K}}_i$ using the formulas above with all the terms replaced by tilde. Then all the relations with tilde replaced hold. □

Then we can now state our main theorem:

Theorem 4.3. The operators $\mathbf{e}_i$, $\mathbf{f}_i$, and $\mathbf{K}_i$ and their tilded counterparts are represented by positive essentially self-adjoint operators, and we have the transcendental relations:

$$(\mathbf{e}_i)^{\frac{1}{b^2}} = \tilde{\mathbf{e}}_i, \tag{125}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

$$(\mathbf{f}_i)^{\frac{1}{b^2}} = \tilde{\mathbf{f}}_i, \tag{126}$$

$$(\mathbf{K}_i)^{\frac{1}{b^2}} = \tilde{\mathbf{K}}_i. \tag{127}$$

Besides, all the variables $\mathbf{E}_i$, $\mathbf{F}_i$, and $\mathbf{K}_i$ commute with all $\tilde{\mathbf{E}}_j$, $\tilde{\mathbf{F}}_j$, and $\tilde{\mathbf{K}}_j$. Here, as before, $\mathbf{e}_i = 2\sin(\pi b^2)\mathbf{E}_i$ and $\mathbf{f}_i = 2\sin(\pi b^2)\mathbf{F}_i$ and similarly for $\tilde{\mathbf{e}}_i$ and $\tilde{\mathbf{f}}_i$ with b replaced by b^{-1} in all the formulas. □

Therefore, we obtain the positive representations of $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{sl}(n,\mathbb{R}))$.

Proof. The new quantum relations follow simply by substitution and commuting K_i to the same side and cancel out. For the transcendental relations, we observe that $\mathbf{E}_i^{\frac{1}{b^2}} = c\tilde{\mathbf{E}}_i$, where c is a constant of the form q^n where n is real. Since both operators are positive self-adjoint, $c = 1$. Similarly, analysis hold for $\mathbf{F}_i$. The case for $\mathbf{K}_i$ is trivial.

For the construction, we assume

$$\mathbf{E}_i = q^{c_i} E_i K_i^{c_i},$$

$$\mathbf{F}_i = q^{-d_i} F_i K_i^{d_i},$$

$$\mathbf{K}_i = \mathfrak{q}_i^{H_i}.$$

The relations (122) forced us to have $c_i + d_{i+1} = d_i + c_{i+1} = 0$, while the relations (123) require $d_i = c_i - 1$. The simplest choice $c_1 = 1$ and $c_2 = 0$ then gives the Serre relations (124), and this choice also guarantee that the modified generators commute with the tilde generators. ■

Remark 4.4. The above construction as well as the class of positive principal series representations has been generalized to arbitrary split real quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ corresponding to any simple Lie algebra $\mathfrak{g}$ by the second author in [20, 21], and will appear in a separate publication. □

4.2 Tori realizations and the Langlands dual

We note that the representation for the modified quantum generators is still given by positive essentially self-adjoint operators, however, they are constructed from the "half" tori $\{e^{\pi b u_{ij}}, e^{2\pi b p_{ij}}\}$. It turns out that there exists a unitary transformation that realizes the action in terms of the full tori $\{e^{2\pi b u_{ij}}, e^{2\pi b p_{ij}}\}$. Let the quantum tori $\mathbb{T}_{\mathfrak{q}\tilde{\mathfrak{q}}}^{n(n-1)/2}$ be the algebra

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

of Laurent polynomials generated by positive self-adjoint $\mathbf{u}_{ij}, \mathbf{v}_{ij}, \tilde{\mathbf{u}}_{ij}, \tilde{\mathbf{v}}_{ij}$ for $1 \le i < j \le n$ such that

$$\mathbf{u}_{ij}\mathbf{v}_{ij} = \mathfrak{q}\mathbf{v}_{ij}\mathbf{u}_{ij}, \quad \tilde{\mathbf{u}}_{ij}\tilde{\mathbf{v}}_{ij} = \tilde{\mathfrak{q}}\tilde{\mathbf{v}}_{ij}\tilde{\mathbf{u}}_{ij}, \tag{128}$$

which can be realized by

$$\mathbf{u}_{ij} = e^{2\pi b u_{ij}}, \quad \mathbf{v}_{ij} = e^{2\pi b p_{ij}}, \tag{129}$$

and similarly for $\tilde{\mathbf{u}}_{ij}$ and $\tilde{\mathbf{v}}_{ij}$ with b replaced by b^{-1}. Then we obtain the following theorem.

Theorem 4.5. We have an embedding

$$\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{sl}(n,\mathbb{R})) \hookrightarrow \mathbb{T}_{\mathfrak{q}\tilde{\mathfrak{q}}}^{n(n-1)/2}. \tag{130}$$

□

Proof. By the explicit expressions for the operators of $\mathbf{E}_i$ and $\mathbf{F}_i$ constructed in the previous sections (modified by the K factors), we note that all operators are sums of terms that q^2 commute with each other. Therefore, there exists a unitary transformation such that we can diagonalize the symplectic form corresponding to these q^2 commuting terms and obtain a realization in terms of the standard tori. Since the representation is irreducible, the dimension follows. Explicitly, it can be obtained from multiplication by the unitary functions (with the corresponding change of actions):

$$\begin{aligned} e^{\pi i u_{ij} u_{kl}} &: 2p_{ij} \mapsto 2p_{ij} + u_{kl}, \quad 2p_{kl} \mapsto 2p_{kl} + u_{ij}, \quad (i,j) \neq (k,l), \\ e^{\frac{1}{2}\pi i (u_{ij})^2} &: 2p_{ij} \mapsto 2p_{ij} + u_{ij}, \end{aligned}$$

constructed inductively from the expression of $E_i^k(\mathbf{u})$ (using the notation of Section 2.4) by solving the shifts needed in each of p_{ij}, so that the representation of $\mathbf{E}_i$, $\mathbf{F}_i$ are represented by the full tori (129). Note that the expression is not unique, since only the exponents of u_{ij} modulo 2 is relevant for a full tori realization, however, they are equivalent again by applying appropriate shifts as above.

All properties of the positive representations are then preserved. ■

Finally, we calculate the commutant of this representation.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

Proposition 4.6. The commutant for the positive representation of $\mathbf{U}_q(\mathfrak{sl}(n,\mathbb{R})$ is generated by $\tilde{\mathbf{E}}_i$ and $\tilde{\mathbf{F}}_i$ and elements of the form

$$\tilde{\mathbf{K}}_1^{\frac{k}{n}}\tilde{\mathbf{K}}_2^{\frac{2k}{n}}\cdots\tilde{\mathbf{K}}_{n-k}^{\frac{(n-k)k}{n}}\cdots\tilde{\mathbf{K}}_{n-2}^{\frac{2(n-k)}{n}}\tilde{\mathbf{K}}_{n-1}^{\frac{n-k}{n}}, \tag{131}$$

for $1\leq k\leq n-1$ an integer. □

Proof. Since $\tilde{\mathbf{E}}_i$ and $\tilde{\mathbf{F}}_i$ do not commute with $\mathbf{E}_j$ and $\mathbf{F}_j$ in the strong sense, any fractional powers of $\tilde{\mathbf{E}}_i$ and $\tilde{\mathbf{F}}_i$ will not commute simultaneously with each individual components $\mathbf{E}_i^k$ and $\mathbf{F}_i^k$. For the $\tilde{\mathbf{K}}_i$ generators, using (120) and (121) amounts to solving a standard $n-1$ simultaneous set of linear equations in $n-1$ variables. ■

Remark 4.7. Note that these elements (131) correspond to the fundamental weights while $\mathbf{K}_i$ correspond to simple roots, thus $\mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$ can be viewed as the "adjoint" quantum group. One can also define its Langlands dual the "simply connected" quantum group $\mathbf{U}_{\mathfrak{q}}({}^L\mathfrak{g}_{\mathbb{R}})$ by adjoining elements of the form given in (131). Then Proposition 4.6 can be interpreted as the statement that the commutant of $\mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$ is in fact its Langlands dual quantum group $\mathbf{U}_{\tilde{\mathfrak{q}}}({}^L\mathfrak{g}_{\mathbb{R}})$. □

5 Future Perspectives

Our construction of the positive principal series representations for the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ as a certain q-deformation of the minimal principal series for $\mathfrak{g}_{\mathbb{R}}$ suggests a strong parallel between the quantum and classical theories, similar to the parallel between finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g}_c)$ and $\mathfrak{g}_c$. However, in the split real case there is also a fundamental difference between the quantum and classical theories first observed by Ponsot and Teschner [33] for $\mathfrak{sl}(2,\mathbb{R})$: the positive principal series representation of the modular double are closed under the tensor product in the sense of the direct integral decomposition. We conjecture that the closure of the positive principal series representations of the modular double is still valid for the higher rank case. One way to prove this conjecture would be to show first that the additional properties (i) and (ii) in the introduction satisfied for the positive representations characterize this class. In fact, it is easy to show that both properties are preserved under the tensor product, namely from Proposition 3.8 one has

$$(\Delta\mathbf{e}_j)^{\frac{1}{b^2}}=\Delta\tilde{\mathbf{e}}_j,\quad (\Delta\mathbf{f}_j)^{\frac{1}{b^2}}=\Delta\tilde{\mathbf{f}}_j, \tag{132}$$

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

which immediately implies the conjecture about the closure of the tensor product. Another approach to the proof would be to use a realization of positive representations in the quantum counterparts of the regular $L^2(G_{\mathbb{R}})$ or quasi regular $L^2(G_{\mathbb{R}}/K)$.

While the tensor product structure of the positive principal series representations destroys the parallel between the quantum and classical theories for the split real algebras, it creates a remarkable parallel between representation theories of the quantum group $\mathcal{U}_q(\mathfrak{g}_c)$ and the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$. In fact, thanks to the closure of tensor products of positive representations one can define a continuous version of the braided tensor category for $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ following the well-established example of $\mathcal{U}_q(\mathfrak{g}_c)$. In the case of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2,\mathbb{R})$, the structure of the braided tensor category has extensively been studied by Teschner et al. [5, 32, 33]. A generalization of their results to an arbitrary simply laced case is an interesting direction for the future research.

The existence of the braided tensor category of positive representations of the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ opens an extensive program proposed by the first author in [14]. Namely, one can try to find the analogs of various remarkable results and constructions that were discovered and studied in relation to the braided tensor category of the finite-dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{g}_c)$. This program is not entirely new since the different partial results already exist primarily in the case of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2,\mathbb{R})$, and it was behind the work of Teschner et al. However, our construction of the positive representations for higher rank algebras strongly indicates that all the results for $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ can be generalized to other types of split real quantum groups and therefore one can envision future perspectives for the positive principal series representations comparable to the past developments related to finite-dimensional representations of the quantum groups initiated by Drinfeld and Jimbo.

In particular, we would like to mention the following three directions:

(1) Topological quantum field theory (TQFT) and Chern–Simons–Witten (CSW) theory.
(2) Equivalence of categories of affine Lie algebras and quantum groups.
(3) Geometrization and categorification of quantum groups and its representations.

Below we make a few comments on the versions of the above three directions for the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ and the positive principal series representations.

(1) The first examples of TQFT introduced in [1, 38, 45] were based on a subquotient category of finite-dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$ at the root of unity q [35]. An alternative geometric approach to the same class of TQFT's

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

has been suggested by Witten [46] and is known as the CSW model for a compact group G_c. In the split real case the category of positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ studied in [5, 32, 33] was suggested by Teschner (see introduction in [41]) as an alternative approach to the construction of a new class of TQFT's that arise from the quantization of the Teichmüller spaces [7, 23]. This construction of TQFT's has been completed by Raj [34]. The geometric approach based on CSW model for a split real group $G_{\mathbb{R}}$ has been extensively studied in a recent work [8]. It is intimately related to three-dimensional hyperbolic geometry and is still in the beginning of its development.

(2) The equivalence of categories of highest weight representations of affine Lie algebras and quantum groups were extensively studied in [25]. The explicit construction of the equivalence can be simplified by considering an additional category of representations of W-algebras; see [39]. In the split real case it is still an open problem to construct a principal series of representations of the affine Lie algebra $\hat{\mathfrak{g}}_{\mathbb{R}}$ even for the case $\mathfrak{g}_{\mathbb{R}}=\mathfrak{sl}(2,\mathbb{R})$. However, one can discuss the equivalence of categories of representations of the modular double $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ and the W-algebra associated to $\mathfrak{g}_{\mathbb{R}}$. In the case when $\mathfrak{g}_{\mathbb{R}}=\mathfrak{sl}(2,\mathbb{R})$ the W-algebra is the Virasoro algebra and there is a strong evidence that the appropriate category of representations of the Virasoro algebra is associated to the Liouville model [41].

(3) The first geometric construction of the finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ (as well as their affine counterparts) based on the gauge theory has been discovered by Nakajima in [31]. By considering various categories of sheaves on the Nakajima varieties one obtains a categorification of these representations [6]. Since the work of Nakajima, other geometric and categorical constructions of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ have been found [13, 28]. In the past year, physicists have observed a remarkable relation between CSW theory for the split real group $G_{\mathbb{R}}$ and the $N=2$ super-symmetric gauge theory on a three-dimensional sphere [9, 40]. This work can be considered as a first step towards a geometrization of the category of positive representations of the modular double à la Nakajima, but a lot more work is needed to get full analogs of geometrization and categorification of the finite-dimensional representations.

The three directions of development of positive representations of the modular double of a quantum group can be complemented by various others, such as the study of these representations at roots of unity, generalizations to the affine and Kac–Moody types, counterparts of the geometric realizations of quantum groups via homology of configuration spaces, and any directions that the reader can suggest in addition. Although one cannot predict which of these directions will be particularly fruitful,

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

it is clear that we are entering a new stage in the representation theory of quantum groups.

Acknowledgements

The results of this paper were announced at the workshops in Banff and Aarhus in August and October 2011, respectively. We are grateful to the organizers for creative atmosphere that led to interesting discussions of our results.

Funding

This research was supported by the NFS grant DMS-100163 (to I.B.F.) and by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan (to I.C.H.I.).

References

[1] Atiyah, M. "Topological quantum field theories." *Publications Mathématiques. Institut de Hautes Études Scientifiques* 68, no. 1 (1989): 175–86.

[2] Awata, H., M. Noumi, and S. Odake. "Heisenberg realization for $\mathcal{U}_q(\mathfrak{sl}_n)$ on the flag manifold." *Letters in Mathematical Physics* 30, no. 1 (1994): 35–43.

[3] Berenstein, A., S. Fomin, and A. Zelevinsky. "Parametrization of canonical bases and totally positive matrices." *Advances in Mathematics* 122, no. 1 (1996): 49–149.

[4] Berenstein, A. and A. Zelevinsky. "Quantum cluster algebras." *Advances in Mathematics* 195, no. 2 (2005): 405–55.

[5] Bytsko, A. G. and K. Teschner. "R-operator, co-product and Haar-measure for the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$." *Communications in Mathematical Physics* 240, no. 1 (2003): 171–96.

[6] Cautis, S., J. Kamnitzer, and A. Licata. "Coherent sheaves on quiver varieties and categorification." (2011): preprint arXiv:1104.0352v1 [math.AG].

[7] Chekhov, L. and V. Fock. "Quantum Teichmüller space." *Theoretical and Mathematical Physics* 120, no. 3 (1999): 1245–59.

[8] Dimofte, T. "Quantum Riemann surfaces in Chern–Simons theory." (2011): arXiv:hep-th/1102.4847v3.

[9] Dimofte, T., D. Gaiotto, and S. Gukov. "Gauge theories labelled by three manifolds." (2011): preprint arXiv:hep-th/1108.4389.

[10] Drinfeld, V. G. "Hopf algebras and the quantum Yang–Baxter equation." *Doklady Akademii Nauk SSSR* 283, no. 5 (1985): 1060–4.

[11] Faddeev, L. D. "Discrete Heisenberg–Weyl group and modular group." *Letters in Mathematical Physics* 34, no. 3 (1995): 249–54.

[12] Faddeev, L. D. 'Modular double of quantum group." *Mathematical Physics Studies* 21 (2000): 149–56.

[13] Feigin, B., M. Finkelberg, I. Frenkel, and L. Rybnikov, "Gelfand–Tsetlin algebras and cohomology rings of Laumon spaces." *Selecta Mathematica (New Series)* 17, no. 2 (2011): 337–61.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

[14] Frenkel, I. B. Lecture at the workshop "Twenty-Five Years of Representation Theory of Quantum Groups", (video), BIRS, August 2011.

[15] Gerasimov, A., S. Kharchev, and D. Lebedev. "Representation theory and quantum integrability." *Progress in Mathematics* 237 (2005): 133–56.

[16] Harish-Chandra. "Spherical functions on a semisimple Lie groups II." *American Journal of Mathematics* 80 (1958): 553–613.

[17] Harish-Chandra. "Harmonic analysis on real reductive groups III. The Maass–Selberg relations and the Plancherel formula." *Annals of Mathematics* 104, no. 1 (1976): 117–201.

[18] Ip, I. "Representation of the quantum plane, its quantum double and harmonic analysis on $GL_q^+(2, R)$." *Selecta Mathematica*, to appear, (2011). doi: 10.1007/s00029-012-0112-4.

[19] Ip, I. "Gauss-Lusztig decomposition of $GL_q^+(n, \mathbb{R})$ and representations by q-tori." (2011): preprint arXiv:1111.4025.

[20] Ip, I. "Positive representations of split real simply-laced quantum groups." (2012): preprint arXiv:1203:2018.

[21] Ip, I. "Positive representations of split real quantum groups of type B_n, C_n, F_4, and G_2." (2012): preprint arXiv:1205.2940.

[22] Jimbo, M. "A q-difference analogue of $\mathcal{U}(\mathfrak{g})$ and the Yang–Baxter equation." *Letters in Mathematical Physics* 10, no. 1 (1985): 63–9.

[23] Kashaev, R. "Quantization of Teichmüller spaces and the quantum dilogarithm." *Letters in Mathematical Physics* 43, no. 2 (1998): 105–15.

[24] Kashaev, R. and Yu. Volkov. "From the tetrahedron equation to universal R-matrices." *AMS Translations, Series* (2) 201 (2000) 79–89.

[25] Kazhdan, D. and G. Lusztig. "Tensor structures arising from affine Lie algebras I, II." *Journal of the American Mathematical Society* 6, no. 4 (1993) 905–1011. "Tensor structures arising from affine Lie algebras III, IV." *Journal of the American Mathematical Society* 7, no. 2 (1994): 335–453.

[26] Klimyk, A. U. "Representations of the quantum algebra $\mathcal{U}_{q,s}(\mathfrak{su}_{1,1})$." *Journal of Physics. A. Mathematical and General* 26, no. 24 (1993): 7503–10.

[27] Knapp, A. *Representation Theory of Semisimple Groups: An Overview Based on Examples*. Princeton, NJ: Princeton University Press, 1986.

[28] Licata, A. "Framed torsion-free sheaves on $\mathbb{CP}^2$, Hilbert schemes, and representations of infinite dimensional Lie algebras." *Advances in Mathematics* 226, no. 2 (2011): 1057–95.

[29] Lusztig, G. "Total Positivity in Reductive Groups." *Lie Theory and Geometry: In Honor of B. Kostant*, 531–68. Progress in Mathematics 123. Boston, MA: Birkhauser, 1994.

[30] Masuda, T., K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi, and K. Ueno. "Unitary representations of the quantum group $SU_q(1, 1)$: I. Structure of the dual space of $\mathcal{U}_q(\mathfrak{sl}(2))$." *Letters in Mathematical Physics* 19, no. 3 (1990) 187–94.

[31] Nakajima, H. "Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras." *Duke Mathematical Journal* 76, no. 2 (1994): 365–416.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019

[32] Ponsot, B. and J. Teschner. "Liouville bootstrap via harmonic analysis on a noncompact quantum group." (1999): preprint arXiv: hep-th/9911110.

[33] Ponsot, B. and J. Teschner. "Clebsch–Gordan and Racah–Wigner coefficients for a continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$." *Communications in Mathematical Physics* 224, no. 3 (2001): 613–55.

[34] Raj, R. Private communication.

[35] Reshetikhin, N. and V. Turaev. "Invariants of 3-manifolds via link polynomials and quantum groups." *Inventiones Mathematicae* 103, no. 1 (1991): 547–97.

[36] Ruijsenaars, S. N. M. "A unitary joint eigenfunction transform for the $A\Delta O$'s $\exp(\mathrm{i}a_{\pm}\mathrm{d}/\mathrm{d}z) + \exp(2\pi z/a_{\mp})$." *Journal of Nonlinear Mathematical Physics* 12, Suppl. 2 (2005): 253–94.

[37] Schmüdgen, K. "Operator representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$." *Letters in Mathematical Physics* 37, no. 2 (1996): 211–22.

[38] Segal, G. "Two-Dimensional Conformal Field Theories and Modular Functors." *IXth International Congress on Mathematical Physics. Proceedings of the Congress held at University College of Swansea, Swansea, July 17–27, 1988*, edited by B. Simon, A. Truman and I. M. Davies, xiv+534 pp. Bristol: Adam Hilger Ltd., 1989. ISBN: 0-85274-250-9 81-06.

[39] Styrkas, K. Quantum Groups, Conformal Field Theories, and Duality of Tensor Categories. PhD Thesis, Yale University, 1998.

[40] Terashima, Y. and M. Yamazaki. "SL$(2,\mathbb{R})$ Chern–Simons, Liouville, and gauge theory on duality walls." *Journal of High Energy Physics* 2011, no. 8 (2011): 1–46.

[41] Teschner, J. "An analogue of a modular functor from quantized Teichmüller theory." *Handbook of Teichmüller Theory*, I (2007): 685–760. *IRMA Lectures in Mathematics and Theoretical Physics*.

[42] Twietmeyer, E. "Real forms of $\mathcal{U}_q(\mathfrak{g})$." *Letters in Mathematical Physics* 24, no. 1 (1992): 49–58.

[43] Vilenkin, N. *Special Functions and the Theory of Group Representations*. Monographs 22. Providence, RI: American Mathematical Society, 1968.

[44] Volkov, A. Yu. "Noncommutative hypergeometry." *Communications in Mathematical Physics* 258, no. 2 (2005): 257–73.

[45] Witten, E. "Topological quantum field theory." *Communications in Mathematical Physics* 117, no. 3 (1988): 353–86.

[46] Witten, E. "Quantum field theory and the Jones polynomial." *Communications in Mathematical Physics* 121, no. 3 (1989): 351–99.

Downloaded from https://academic.oup.com/imrn/article-abstract/2014/8/2126/690191 by Tsinghua University Library user on 23 September 2019