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Nonuniqueness of solutions to the L_p-Minkowski problem ☆

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ABSTRACT

In this paper we show by example that there is no uniform estimate for the L_p-Minkowski problem. As a result we obtain the nonuniqueness of solutions to the problem.

1. Introduction

We consider the following L_p-Minkowski problem,

$$\det(\nabla^2 H + HI) = f H^{p-1} \quad \text{on } S^n, \tag{1.1}$$

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where H is the support function of a convex body $K = K_H$ in the Euclidean space $\mathbb{R}^{n+1}$, I is the unit matrix, f is a given positive function on the unit sphere S^n, and p is a constant. The main result of the paper is

Theorem 1.1. *For any $p \in (-n-1, 0)$, there exists a positive function $f \in C^\infty(S^n)$ such that equation (1.1) admits two different solutions.*

To prove Theorem 1.1, we will construct a smooth, positive function f, which is radially symmetric in $x_1, \cdots, x_n$, and also symmetric in x_{n+1}, such that equation (1.1) has a solution with small volume. By [7] there is another solution whose volume has a positive lower bound depending only on n, $\inf_{S^n} f$ and $\sup_{S^n} f$. Therefore we have two different solutions for equation (1.1).

The L_p-Minkowski problem was introduced by Lutwak [16] and has been studied by many authors [4,5,7,11,14,15,17–21]. One of the main questions is the uniqueness of solutions. In particular, a solution to (1.1) is also a self-similar solution to the Gauss curvature flow, of which the uniqueness has been extensively studied.

Denote $q = 1-p$. When $f \equiv 1$, the uniqueness has been obtained in the case $q = n$ for all n [9] and in the case $n = 2$ and $q = 1$ [1]. When $n = 1$, $q = 1$, and f is symmetric with respect to the origin, namely $f(\theta) = f(\theta + \pi)$, the uniqueness was obtained in [10,13]. When $q < -n$, the uniqueness can be obtained by the maximum principle [7]. One can also find some uniqueness results in the case $q < 0$ for symmetric f in [16]. In the discrete case a uniqueness was established for the L_0-Minkowski problem in [20].

On the other hand, it is well known that when $q = n+2$, all ellipsoids with the volume of the unit ball are solutions of (1.1) with $f \equiv 1$. The uniqueness is more delicate in the case $q > n+2$, at least when $n = 1$ [3]. In [7] it was shown that the solution may not be unique if $q \in (1, n+2)$ and is very close to $n+2$. In contrast to the uniqueness in [10,13] for $n = 1$, $q = 1$, and symmetric f, a surprising nonuniqueness result was discovered in [22] for nonsymmetric f, also in the case $n = 1$, $q = 1$.

The uniqueness of solutions for the case $0 < q < n+2$ attracted much attentions as it is related to the limit shape of Gauss curvature flows of convex hypersurfaces and has received considerable investigations. See [1,2,6,8,9] and the above discussion. The uniqueness of solutions has been conjectured for a number of special cases, including in particular the case $f \equiv 1$ and $q = 1$ [12]. Our theorem above shows that for general positive function f, there may be more than one solution to (1.1), for all $q \in (1, n+2)$ and all dimensions n. It implies that the limit shape of anisotropic curvature flows is usually not unique.

We will first prove Theorem 1.1 for the case $p \in (-n-1, -1)$ in Sections 2 and 3, and then for the case $p \in [-1, 0)$ in Section 4. In Section 2 we construct an f_ε such that equation (1.1) has a solution H_ε of small volume. Then in Section 3 we show that there is a solution to (1.1) of which the volume has a positive lower bound. In Section 4 we extend the example to the case $p \in [-1, 0)$.

2. A solution with small volume

Denote $\delta = n + 2 - q \in (0, n+1)$. Let

$$M_\varepsilon = \mathrm{diag}(1, \cdots, 1, \varepsilon) = \begin{pmatrix} I & 0 \\ 0 & \varepsilon \end{pmatrix} \tag{2.1}$$

be a matrix, where $0 < \varepsilon \le 1$ is a small constant and as before I is the unit $n \times n$ matrix.

Consider the equation

$$\det(\nabla^2 h + hI)(x) = |M_\varepsilon x|^{-\delta}, \quad x \in S^n. \tag{2.2}$$

Equation (2.2) is just the classical Minkowski problem. As its right hand side satisfies the necessary condition $\int_{S^n} x_k |M_\varepsilon x|^{-\delta} = 0$ for all $1 \le k \le n+1$, there is a solution, which is unique up to translation, to the equation.

Let h_ε be the unique solution to (2.2) such that the centre of the associated convex body K_{h_ε} is located at the origin. Note that h_ε is radially symmetric in $x_1, \cdots, x_n$ and symmetric in x_{n+1}. Define

$$H_\varepsilon(x) = (\det M_\varepsilon)^{\frac{2}{n+q}} \cdot \left|M_\varepsilon^{-1} x\right| \cdot h_\varepsilon\left(\frac{M_\varepsilon^{-1} x}{\left|M_\varepsilon^{-1} x\right|}\right), \tag{2.3}$$

then H_ε is the support function of a convex body K_{H_ε}, and K_{H_ε} can be obtained from K_{h_ε} by making the coordinate transform $x \to M_\varepsilon^{-1} x$ and then making a dilation $x \to (\det M_\varepsilon)^{\frac{2}{n+q}} x$.

Lemma 2.1. *The function H_ε satisfies the equation*

$$\det(\nabla^2 H_\varepsilon + H_\varepsilon I) = \frac{\hat{h}_\varepsilon^q}{H_\varepsilon^q} \quad \text{on } S^n, \tag{2.4}$$

where

$$\hat{h}_\varepsilon(x) = h_\varepsilon\left(\frac{M_\varepsilon^{-1} x}{|M_\varepsilon^{-1} x|}\right). \tag{2.5}$$

Proof. Let

$$u_\varepsilon(x) = \left|{M_\varepsilon}^{-1} x\right| \cdot h_\varepsilon\left(\frac{{M_\varepsilon}^{-1} x}{\left|{M_\varepsilon}^{-1} x\right|}\right). \tag{2.6}$$

By the invariance of the quantity $h_\varepsilon^{n+2} \det(\nabla^2 h_\varepsilon + h_\varepsilon I)$ under unimodular affine transformations, see Proposition 7.1 in [7] or formula (2.12) in [15], we have

$$\det(\nabla^2 u_\varepsilon + u_\varepsilon I)(x) = \det(\nabla^2 h_\varepsilon + h_\varepsilon I)\left(\frac{M_\varepsilon^{-1} x}{|M_\varepsilon^{-1} x|}\right) \cdot \frac{(\det M_\varepsilon^{-1})^2}{|M_\varepsilon^{-1} x|^{n+2}}.$$

Here we note that M_ε^{-1} is not a unimodular transformation. Since h_ε satisfies equation (2.2), we have

$$\det(\nabla^2 u_\varepsilon + u_\varepsilon I)(x) = |M_\varepsilon^{-1}x|^\delta \cdot \frac{(\det M_\varepsilon^{-1})^2}{|M_\varepsilon^{-1}x|^{n+2}} = \frac{1}{(\det M_\varepsilon)^2 |M_\varepsilon^{-1}x|^q}. \tag{2.7}$$

Therefore by the definition of H_ε, (2.3), one gets

$$\begin{aligned}\det(\nabla^2 H_\varepsilon + H_\varepsilon I)(x) &= (\det M_\varepsilon)^{\frac{2n}{n+q}} \det(\nabla^2 u_\varepsilon + u_\varepsilon I)(x) \\ &= \frac{1}{(\det M_\varepsilon)^{\frac{2q}{n+q}} \left|M_\varepsilon^{-1}x\right|^q} \\ &= \frac{\hat{h}_\varepsilon^q}{H_\varepsilon^q}. \qquad \square\end{aligned}$$

To estimate the volume of the convex body K_{H_ε}, one needs to study the convex body K_{h_ε}, or equivalently the support function h_ε. When $\delta \in (0, n)$, we have the following uniform estimates.

Lemma 2.2. *When $2 < q < n + 2$, there exists a positive constant C, independent of $\varepsilon \in (0, 1]$, such that*

$$C^{-1} \le h_\varepsilon \le C \quad \text{on } S^n. \tag{2.8}$$

Proof. One can easily see that the area of $\partial K_{h_\varepsilon}$ is uniformly bounded from above. In fact, since

$$|M_\varepsilon x| \ge \sqrt{1 - x_{n+1}^2} \quad \forall\, x \in S^n \text{ and } \varepsilon \in (0, 1],$$

and noting that $\delta = n + 2 - q < n$, we have

$$\begin{aligned}\text{area}(\partial K_{h_\varepsilon}) &= \int_{S^n} \det(\nabla^2 h_\varepsilon + h_\varepsilon I) \\ &= \int_{S^n} |M_\varepsilon x|^{-\delta} \\ &\le \int_{S^n} (1 - x_{n+1}^2)^{-\delta/2} \\ &\le C,\end{aligned}$$

where C is a positive constant depending on n, δ but independent of ε. By the isoperimetric inequality, we obtain

$$\mathrm{vol}(K_{h_\varepsilon}) \le C_n\, \mathrm{area}(\partial K_{h_\varepsilon})^{\frac{n+1}{n}} \le C. \tag{2.9}$$

Assume h_ε attains its maximum at point $x_\varepsilon \in S^n$, namely $h_\varepsilon(x_\varepsilon) = \max_{S^n} h_\varepsilon$. By convexity and recalling that h_ε is symmetric, we have

$$h_\varepsilon(x) \ge \max h_\varepsilon \cdot \langle x_\varepsilon, x\rangle \quad \forall\, x \in S^n.$$

Observing that

$$|M_\varepsilon x| \le 1 \quad \forall\, x \in S^n \text{ and } \varepsilon \in (0,1],$$

by equation (2.2), we have

$$\begin{aligned}
\mathrm{vol}(K_{h_\varepsilon}) &= \frac{1}{n+1}\int_{S^n} h_\varepsilon \det(\nabla^2 h_\varepsilon + h_\varepsilon I)\\
&= \frac{1}{n+1}\int_{S^n} h_\varepsilon(x)|M_\varepsilon x|^{-\delta}\\
&\ge \frac{1}{n+1}\int_{S^n} h_\varepsilon(x)\\
&\ge C_n \max h_\varepsilon \cdot \int_{S^n_\varepsilon} \langle x_\varepsilon, x\rangle\\
&= C_n \max h_\varepsilon,
\end{aligned}$$

where $S^n_\varepsilon = \{x \in S^n : \langle x_\varepsilon, x\rangle > 0\}$. Therefore we obtain from (2.9) that

$$\max h_\varepsilon \le C_n\, \mathrm{vol}(K_{h_\varepsilon}) \le C. \tag{2.10}$$

The second inequality of (2.8) is proved.

To prove the first inequality of (2.8), we make use of the concept of minimum ellipsoid of a convex body. Let E_{h_ε} be the minimum ellipsoid of K_{h_ε}. Then we have

$$\frac{1}{n+1} E_{h_\varepsilon} \subset K_{h_\varepsilon} \subset E_{h_\varepsilon}.$$

By the symmetry of K_{h_ε}, the centre of E_{h_ε} is at the origin. Let $R_{1,\varepsilon} \ge \cdots \ge R_{n+1,\varepsilon}$ be the lengths of the semi-axis of E_{h_ε}. Then

$$\begin{aligned}
R_{1,\varepsilon} &\le (n+1)\max h_\varepsilon,\\
R_{n+1,\varepsilon} &\le (n+1)\min h_\varepsilon.
\end{aligned}$$

Hence

$$\begin{aligned} \operatorname{vol}(K_{h_\varepsilon}) &\le \operatorname{vol}(E_\varepsilon) \\ &\le \omega_{n+1} R_{1,\varepsilon}^n R_{n+1,\varepsilon} \\ &\le C_n (\max h_\varepsilon)^n \cdot \min h_\varepsilon, \end{aligned}$$

where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. By (2.10) it follows that

$$1 \le C_n (\max h_\varepsilon)^{n-1} \cdot \min h_\varepsilon.$$

The first inequality of (2.8) follows. □

Now let

$$f_\varepsilon = \hat{h}_\varepsilon^q. \tag{2.11}$$

In view of (2.5) and (2.8), there exist two positive constants C_1, C_2, independent of ε, such that

$$C_1 \le f_\varepsilon \le C_2. \tag{2.12}$$

From (2.4), H_ε is a solution to

$$\det(\nabla^2 H_\varepsilon + H_\varepsilon I) = \frac{f_\varepsilon}{H_\varepsilon^q} \quad \text{on } S^n. \tag{2.13}$$

By (2.3), we have the volume estimate

$$\begin{aligned} \operatorname{vol}(K_{H_\varepsilon}) &= (\det M_\varepsilon)^{\frac{2n+2}{n+q}} \cdot \det M_\varepsilon^{-1} \cdot \operatorname{vol}(K_{h_\varepsilon}) \\ &= (\det M_\varepsilon)^{\frac{\delta}{n+q}} \operatorname{vol}(K_{h_\varepsilon}). \end{aligned} \tag{2.14}$$

Note that $\det M_\varepsilon = \varepsilon$. Hence when $0 < \delta < n$, the volume of K_{H_ε} can be as small as we want, provided ε is sufficiently small.

Remark. From the transform (2.3) and estimate (2.8), one easily sees that there is no uniform upper bound for the solution H_ε, when $\varepsilon > 0$ is small.

3. A variational solution

In the paper [7], the variational problem

$$\sup_h \left\{ \inf_{y \in K_h} J[h(x) - y \cdot x] : \operatorname{vol}(K_h) = 1 \right\} \tag{3.1}$$

is studied. Here h is the support function of a non-degenerate convex body K_h with volume $\mathrm{vol}(K_h)$. The functional $J[h]$ is given by

$$J[h] = \frac{1}{q-1} \int_{S^n} \frac{f}{h^{q-1}}.$$

For convenience we recall the following existence result in [7].

Lemma 3.1. *(See [7].) Let $1 < q < n+2$ and $f \in L^\infty(S^n)$ be a positive function satisfying (2.12). Then the variational problem (3.1) admits a maximiser H which satisfies, for some Lagrange multiplier $\lambda > 0$,*

$$\det(\nabla^2 H + HI) = \frac{\lambda f}{H^q} \quad \textit{on } S^n. \tag{3.2}$$

When $f(x) = f(-x)$, one can consider the problem (3.1) in the restricted class $h(x) = h(-x)$. One can follow the same arguments as in [7] to show that a maximiser H exists and its corresponding convex body is centrally symmetric.

By this lemma, we obtain a solution to equation (1.1). Let

$$\tilde{H} = \lambda^{-\frac{1}{n+q}} H. \tag{3.3}$$

Then $\tilde{H}$ is a solution to equation

$$\det(\nabla^2 \tilde{H} + \tilde{H} I) = \frac{f}{\tilde{H}^q} \quad \text{on } S^n. \tag{3.4}$$

The volume of the corresponding convex body $K_{\tilde{H}}$ is

$$\mathrm{vol}(K_{\tilde{H}}) = \lambda^{-\frac{n+1}{n+q}}. \tag{3.5}$$

We need to estimate the Lagrange constant λ. It is given as follows. Multiplying H to both sides of equation (3.2) and taking integration, we have

$$\begin{aligned} 1 = \mathrm{vol}(K_H) &= \frac{1}{n+1} \int_{S^n} H \det(\nabla^2 H + HI) \\ &= \frac{\lambda}{n+1} \int_{S^n} \frac{f}{H^{q-1}} \\ &= \frac{(q-1)\lambda}{n+1} J[H]. \end{aligned} \tag{3.6}$$

The Blaschke–Santalo inequality for centrally symmetric convex bodies is given by

$$V(h) \int_{S^n} \frac{1}{h^{n+1}} dS(x) \le \frac{\omega_n^2}{n+1},$$

where $h(x) = h(-x)$ is the support function of any centrally symmetric convex body and ω_n is the volume of the unit ball in $\mathbb{R}^n$. Using this inequality, we have

$$\begin{aligned} J(H) &= \frac{1}{q-1}\int_{S^n} \frac{f}{H^{q-1}} \\ &= \frac{\sup f}{q-1}\Big(\int_{S^n} \frac{1}{H^{q-1}}\Big)^{\frac{q-1}{n+1}}\Big(\int_{S^n} 1\Big)^{\frac{n+2-q}{n+1}} \\ &\le \frac{C_n \sup f}{q-1}. \end{aligned}$$

On the other hand, let h_0 be the constant function $\omega_{n+1}^{-1/(n+1)}$ which is the support function of the ball centred at the origin whose volume is equal to 1. From the variational characterisation of H, we have

$$J(H) \ge H(h_0) = \frac{C_n' \sup f}{q-1}.$$

From (3.6), we thus obtain

$$C_n^{-1} \inf_{S^n} f \le \lambda^{-1} \le C_n \sup_{S^n} f.$$

By (3.5), we see that there is a positive constant C_n depending only on n, such that

$$C_n^{-1}(\inf_{S^n} f)^{\frac{n+1}{n+q}} \le \mathrm{vol}(K_{\tilde{H}}) \le C_n(\sup_{S^n} f)^{\frac{n+1}{n+q}}. \tag{3.7}$$

Now we let $f = f_\varepsilon$ be the function in (2.13). Denote the corresponding variational solution by $\tilde{H}_\varepsilon$, which is the support function of a convex body $K_{\tilde{H}_\varepsilon}$. By virtue of (2.12) and (3.7), one has the estimates

$$C_3 \le \mathrm{vol}(K_{\tilde{H}_\varepsilon}) \le C_4,$$

where C_3, C_4 are positive constants depending only on n, q, C_1 and C_2. But recall that for the solution H_ε given in (2.3), the volume $\mathrm{vol}(K_{H_\varepsilon}) \to 0$ as $\varepsilon \to 0$. Hence $\tilde{H}_\varepsilon$ and H_ε are two different solutions. We have therefore obtained the nonuniqueness of solutions to (2.13) for small $\varepsilon > 0$, in the case $q \in (2, n+2)$.

Remark. As f is rotationally symmetric with respect to the x_{n+1}-axis, one may consider the supremum in (3.1) in the family of rotationally symmetric convex bodies and obtain a rotationally symmetric solution. Therefore for equation (2.13) the two solutions we obtained are both rotationally symmetric with respect to the x_{n+1}-axis, and so are smooth.

We also remark that the idea of our proof is based on the observation that $\mathrm{vol}(K_{H_\varepsilon})$ is small. But when $q = 1$ and $q = n + 2$, one can prove that for any solution to (1.1), the volume of the associated convex body has a positive lower bound. Therefore our construction of more than one solution does not cover the case $q = 1$. In fact, as mentioned in the introduction, when $n = 1$ and $q = 1$, the solution is unique when f is symmetric [10,13].

4. The case $1 < q \leq 2$

We have proved Theorem 1.1 for the case $2 < q < n + 2$. In this section we show that the solutions H_ε and $\tilde{H}_\varepsilon$ given in (2.3) and (3.3) are different even when $1 < q \leq 2$, provided ε are sufficiently small.

Recall that K_{h_ε} is radially symmetric in $x_1, \cdots, x_n$ and symmetric in x_{n+1}. We denote $R_1 = R_{1,\varepsilon} = h_\varepsilon(1, 0, \cdots, 0)$, $R_{n+1} = R_{n+1,\varepsilon} = h_\varepsilon(0, \cdots, 0, 1)$. Namely R_1 and R_{n+1} are respectively the values of h_ε on the equator and at the north pole of the sphere S^n.

Lemma 4.1. *There exists a constant $C > 0$, independent of $\varepsilon \in (0, 1]$, such that*

$$R_{n+1} < CR_1. \tag{4.1}$$

Proof. Denote $S_\eta = \{x \in S^n : |x_{n+1}| < \eta\}$ and $S^c_\eta = S^n \setminus S_\eta$, where $\eta \in (0, 1)$ is a constant. Let $\Gamma_\eta = \{p \in \partial K_{h_\varepsilon} : G(p) \in S_\eta\}$ and $\Gamma^c_\eta = \partial K_{h_\varepsilon} \setminus \Gamma_\eta$, where G is the Gauss map of $\partial K_{h_\varepsilon}$, namely $G(p)$ is the unit outer normal of $\partial K_{h_\varepsilon}$ at the point p.

If (4.1) is not true, we have

$$\mathrm{area}(\Gamma_{1/2}) >> \mathrm{area}(\Gamma^c_{1/2}).$$

On the other hand, by equation (2.2) we have

$$\mathrm{area}(\Gamma_{1/2}) = \int_{S_{1/2}} \det(\nabla^2 h_\varepsilon + h_\varepsilon I) = \int_{S_{1/2}} |M_\varepsilon x|^{-\delta}. \tag{4.2}$$

Note that

$$\sup_{S_\eta} |M_\varepsilon x|^{-\delta} \leq \inf_{S^c_\eta} |M_\varepsilon x|^{-\delta} \quad \forall\, \eta \in (0, 1) \text{ and } \varepsilon \in (0, 1].$$

Hence the right hand side of (4.2)

$$\leq C_n \int_{S^c_{1/2}} |M_\varepsilon x|^{-\delta} = C_n \int_{S^c_{1/2}} \det(\nabla^2 h_\varepsilon + h_\varepsilon I) = C_n\, \mathrm{area}(\Gamma^c_{1/2}). \tag{4.3}$$

We reach a contradiction. □

By equation (2.2), we have as $\varepsilon \to 0$ that

$$\operatorname{area}(\partial K_{h_\varepsilon}) = \int\limits_{S^n} |M_\varepsilon x|^{-\delta} = \begin{cases} (C + o(1))\varepsilon^{q-2} & \text{if } 1 < q < 2, \\ (C + o(1))|\log \varepsilon| & \text{if } q = 2, \end{cases} \tag{4.4}$$

where C is a positive constant independent of ε. On the other hand, by Lemma 4.1 we have

$$C^{-1} R_1^n \le \operatorname{area}(\partial K_{h_\varepsilon}) \le C {R_1}^n.$$

From (4.4) it follows that

$$\begin{gathered} C^{-1}\varepsilon^{\frac{q-2}{n}} \le R_1 \le C\varepsilon^{\frac{q-2}{n}} \quad \text{if } 1 < q < 2, \\ C^{-1}|\log \varepsilon|^{1/n} \le R_1 \le C|\log \varepsilon|^{1/n} \quad \text{if } q = 2. \end{gathered} \tag{4.5}$$

Observing that

$$\operatorname{vol}(K_{h_\varepsilon}) = C R_1^n R_{n+1},$$

by (2.14) we obtain

$$\operatorname{vol}(K_{H_\varepsilon}) = \varepsilon^{\frac{\delta}{n+q}} \operatorname{vol}(K_{h_\varepsilon}) = \begin{cases} C\varepsilon^{\frac{(q-1)(n+q-2)}{n+q}} R_{n+1} & \text{if } 1 < q < 2, \\ C\varepsilon^{\frac{\delta}{n+q}} |\log \varepsilon| R_{n+1} & \text{if } q = 2. \end{cases} \tag{4.6}$$

Now consider the solution $\tilde{H}_\varepsilon$ given in (3.3). From (3.7), we have

$$\operatorname{vol}(K_{\tilde{H}_\varepsilon}) \ge C(\inf_{S^n} f_\varepsilon)^{\frac{n+1}{n+q}}.$$

Noting that $f_\varepsilon = \hat{h}_\varepsilon^q$ and that $\inf_{S^n} \hat{h}_\varepsilon = \inf_{S^n} h_\varepsilon$, we see that

$$\operatorname{vol}(K_{\tilde{H}_\varepsilon}) \ge C(\inf_{S^n} h_\varepsilon)^{\frac{q(n+1)}{n+q}} \ge C R_{n+1}^{\frac{q(n+1)}{n+q}}. \tag{4.7}$$

In order to prove that $\tilde{H}_\varepsilon$ and H_ε are two different solutions, we need that,

$$\operatorname{vol}(K_{\tilde{H}_\varepsilon}) > \operatorname{vol}(K_{H_\varepsilon}). \tag{4.8}$$

By (4.6) and (4.7), it suffices to have

$$R_{n+1} > \begin{cases} C\varepsilon^{\frac{n+q-2}{n}} & \text{if } 1 < q < 2, \\ C\varepsilon|\log\varepsilon|^{\frac{n+2}{n}} & \text{if } q = 2. \end{cases} \tag{4.9}$$

So it suffices to give a lower bound for R_{n+1} to complete Theorem 1.1. In fact we have the following

Lemma 4.2. *For $1 < q \leq 2$, we have*

$$R_{n+1} \approx \begin{cases} \varepsilon^{\frac{(2-q)(n-1)}{n}} & \textit{if } 1 < q < 2, \\ |\log\varepsilon|^{\frac{1-n}{n}} & \textit{if } q = 2. \end{cases} \tag{4.10}$$

Here the notation "≈" means the ratio of the two sides has uniform positive upper and lower bounds.

Proof. Let u_ε be as (2.6), K_{u_ε} be the corresponding convex body. From (2.7), we have

$$\text{area}(\partial K_{u_\varepsilon}) = \int\limits_{S^n} \frac{1}{\varepsilon^2|M_\varepsilon^{-1}x|^q} = (C + o(1))\varepsilon^{-1}, \quad \forall\, q > 1. \tag{4.11}$$

Denote

$$r_1 = r_{1,\varepsilon} = u_\varepsilon(1, 0, \cdots, 0) = R_1,$$
$$r_{n+1} = r_{n+1,\varepsilon} = u_\varepsilon(0, \cdots, 0, 1) = \varepsilon^{-1}R_{n+1}.$$

Since

$$\text{area}(\partial K_{u_\varepsilon}) \approx r_1^n + r_1^{n-1}r_{n+1},$$

multiplying both sides by ε, and noting (4.11), we obtain

$$1 \approx \varepsilon R_1^n + R_1^{n-1}R_{n+1}.$$

On account of (4.5), $\varepsilon R_1^n << 1$, which leads to

$$1 \approx R_1^{n-1}R_{n+1}. \tag{4.12}$$

By (4.5) again, we have (4.10). □

Combining (4.9) and (4.10), we obtain (4.8). Therefore the case $1 < q \leq 2$ of Theorem 1.1 is also proved.

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