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A priori estimates and existence of solutions to the prescribed centroaffine curvature problem ☆

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ABSTRACT

In this paper we study the prescribed centroaffine curvature problem in the Euclidean space $\mathbb{R}^{n+1}$. This problem is equivalent to solving a Monge–Ampère equation on the unit sphere. It corresponds to the critical case of the Blaschke–Santaló inequality. By approximation from the subcritical case, and using an obstruction condition and a blow-up analysis, we obtain sufficient conditions for the a priori estimates, and the existence of solutions up to a Lagrange multiplier.

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1. Introduction

Given a hypersurface M in the Euclidean space $\mathbb{R}^{n+1}$, the centroaffine curvature κ of M at point p is by definition equal to K/d^{n+2}, where K is the Gauss curvature and d is

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the distance from the origin to the tangent plane of M at p. The centroaffine curvature κ was first discovered by Tzitzéica [31] in 1908. It is invariant under transformations in $\mathrm{SL}(n+1)$ and is an elementary quantity in the affine differential geometry and in the theory of convex bodies [8,13,14,21–23,25,27]. It appears naturally in geometric objects such as affine normal and affine spheres, and plays a fundamental role in the study of many geometric problems [1–3,9,15,16,19].

In this paper, we consider the following prescribed centroaffine curvature problem. Given a positive function f in $\mathbb{R}^{n+1}$, find proper conditions on f such that there exists a closed convex hypersurface M in $\mathbb{R}^{n+1}$ surrounding the origin, of which the centroaffine curvature κ at a point $p \in M$ is equal to $f(p)$.

The corresponding prescribed mean curvature and Gauss curvature problems, namely the problems with the centroaffine curvature replaced by the mean curvature or the Gauss curvature, were raised by Yau [34]. In the case of Gauss curvature, the problem was studied in [10,26,30,33].

Let H be the support function of the polar body of M. We show in Section 2 that the problem is equivalent to solving the following Monge–Ampère equation,

$$\det(\nabla^2 H + HI)(x) = \frac{f(x/H)}{H^{n+2}} \quad x \in S^n, \tag{1.1}$$

where $\nabla^2 H = (\nabla_{ij} H)$ is the covariant derivatives of H with respect to an orthonormal frame on the unit sphere S^n, and I is the unit matrix.

Problem (1.1) is closely related to the L_p-Minkowski problem [22],

$$\det(\nabla^2 H + HI)(x) = f(x)H^{p-1} \quad x \in S^n, \tag{1.2}$$

where f is a positive function on S^n. The L_p-Minkowski problem is an extension of the famous Minkowski problem (the case when $p = 1$), and has been extensively studied recently [5,9,11,18,24,32]. In particular a solution to the L_p-Minkowski problem is also a self-similar solution to an anisotrophic Gauss curvature flow, of which the asymptotic behaviour of solutions has attracted much attentions [4,12].

Equation (1.1), or equation (1.2) in the case $p = -n - 1$, is referred to as the centroaffine Minkowski problem [9]. The centroaffine Minkowski problem is also of interest in the image processing. In image processing, one hopes that the deformation of image is invariant when one looks at the picture from different angles. In other words, the image processing should be invariant under projective transformations. For this purpose an evolution equation was introduced in [2], which becomes a centroaffine Minkowski problem if one deals with self-similar solutions.

In this paper we establish the a priori estimates and existence of solutions to equation (1.1). As the centroaffine curvature is invariant under projective transformations on S^n, all ellipsoids of volume $|B_1(0)|$ have constant centroaffine curvature 1, namely they are all solutions to (1.1) with $f \equiv 1$. Therefore conditions are needed for the uniform estimate of H. In fact an obstruction was found in [9] for the existence of solutions, which means that for some f there is no solution to equation (1.1).

On the other hand, equation (1.1) corresponds to the critical exponent case of the Blaschke–Santaló inequality [9]. Therefore one may employ the variational approach to study the problem. As is well known, the critical exponent case is usually very complicated. So we will first prove an existence result in the subcritical case, and then establish the a priori estimates and prove the existence of solutions by approximation.

For a support function H, denote the volume of the associated convex body by $\operatorname{vol}(H)$, namely

$$\operatorname{vol}(H) = \frac{1}{n+1} \int\limits_{S^n} H \det(\nabla^2 H + HI). \tag{1.3}$$

Then we have

Theorem 1.1. *Let f be a bounded and positive function in $\mathbb{R}^{n+1}$. For any positive constants $v > 0$ and $q \in [n+1, n+2)$, there exist a number $\lambda > 0$ and a positive support function $H \in C^{1,\gamma}(S^n)$ for some $\gamma \in (0,1)$ with volume $\operatorname{vol}(H) = v$, which solve the equation*

$$\det(\nabla^2 H + HI)(x) = \frac{\lambda f(x/H)}{H^q} \quad x \in S^n. \tag{1.4}$$

The above theorem deals with the subcritical case $q < n+2$. By approximation to the critical exponent case $q = n+2$, and using a blow-up analysis and the necessary condition in [9], we find sufficient conditions for the a priori estimates of solutions.

Theorem 1.2. *Let $f \in C^1(\mathbb{R}^{n+1})$ be a positive function. Assume that*

$$f(x) = f(\infty) + \frac{\beta + o(1)}{|x|^\alpha} \text{ as } |x| \to \infty, \tag{1.5}$$

for constants $\alpha > 0$, $f(\infty) > 0$, and $\beta \neq 0$, and

$$\text{either } f(x) > f(\infty) \text{ or } f(x) < f(\infty) \ \ \forall x \in \mathbb{R}^{n+1}. \tag{1.6}$$

Let H be a solution to (1.1). Then we have the a priori estimates

$$C^{-1} \le H \le C, \tag{1.7}$$

where C is a positive constant depending only on n and f.

By Theorems 1.1 and 1.2, and using approximation, we obtain the following existence result for equation (1.1).

Theorem 1.3. *Let $f \in C^1(\mathbb{R}^{n+1})$ be a positive function. Assume that f satisfies (1.5) with $\beta > 0$ and*

$$f(x) > f(\infty) \;\forall\, x \in \mathbb{R}^{n+1}. \tag{1.8}$$

Then for any positive constant $v > 0$, there exists a number λ and a support function $H \in C^{2,\gamma}$ $(\forall\, \gamma \in (0,1))$ with $\mathrm{vol}(H) = v$ which solve the equation

$$\det(\nabla^2 H + HI)(x) = \frac{\lambda f(x/H)}{H^{n+2}} \quad x \in S^n. \tag{1.9}$$

Remark. In Theorem 1.1, it suffices to assume that f is a positive and bounded function. In this case, the solution is $C^{1,\gamma}$ for some $\gamma \in (0,1)$ [7]. In Theorems 1.2 and 1.3, we use condition (2.7) and so $f \in C^1(\mathbb{R}^{n+1})$ is needed. The assumption $f \in C^1(\mathbb{R}^{n+1})$ implies that $H \in C^{2,\gamma}(S^n)$ for all $\gamma \in (0,1)$.

The prescribed centroaffine curvature problem is related to the Blaschke–Santaló inequality

$$\sup_H \inf_{\xi \in K} \mathrm{vol}(H) \int_{S^n} \frac{1}{(H - \xi \cdot x)^{n+1}} d\sigma_{S^n} \le \frac{\omega_n^2}{n+1}, \tag{1.10}$$

just as the prescribed scalar curvature problem related to the Sobolev inequality. Here $d\sigma_{S^n}$ denotes the volume element of S^n, and $\omega_n = \int_{S^n} d\sigma_{S^n}$. The prescribed scalar curvature equation on the sphere

$$-\Delta_{S^n} u + \frac{1}{2} n(n-2)u = R(x)u^p \quad \text{on } S^n \tag{1.11}$$

has been studied by numerous authors (see, e.g. [28]), where $p = \frac{n+2}{n-2}$. To prove the a priori estimates and the existence of solutions to (1.1), we use the blow-up approach. The first step is to prove the existence of a solution H_q to (1.4) in the sub-critical case $q \in [n+1, n+2)$. Even in the sub-critical case, the Monge–Ampère equation (1.2) is more complicated than (1.11). It is well known that the solutions to (1.11) are uniformly bounded in the sub-critical case $1 < p < \frac{n+2}{n-2}$. But this is not true for equation (1.2). There may exist infinitely many solutions to (1.2) which are not uniformly bounded in the subcritical case [17].

The second step is to find conditions on f such that H_q is uniformly bounded for all $q \in [n+1, n+2)$. Suppose that $\sup_{x \in S^n} H_q(x) \to \infty$ as $q \to n+2$. We normalize the associated convex body to get a new support function $\tilde{H}_q$, of which the limit $\tilde{H}_\infty$ satisfies equation (1.1) for a different function f_∞. In order that this blow-up argument works, it is crucial to have a classification of the limit shape $\tilde{H}_\infty$.

However, unlike the prescribed scalar curvature equation (1.11), where the limit of the blow-up sequence is unique (by a Liouville type theorem), the problem (1.1) is more difficult, because the limit f_∞ is a function, not a constant in general. To overcome this difficulty, we assume that

$$\lim_{|p|\to+\infty} f(p) = const, \tag{1.12}$$

so that f_∞ is a constant and hence $\tilde{H}_\infty$ must be a sphere.

Condition (1.12) alone is not sufficient, as there is no uniform estimate even if $f \equiv 1$. To establish the a priori estimates (1.7), we use the necessary condition (2.7) and a blow-up analysis, by which we arrive at the condition (1.5), which is a strengthening of (1.12). The blow-up argument was also used in [20] for the rotationally symmetric case of the L_p-Minkowski problem. In this paper we consider the non-symmetric case and the analysis is more delicate, as there are many different cases to deal with. See Lemma 4.1.

This paper is organized as follows. In Section 2, we show that the prescribed centroaffine curvature problem is equivalent to equation (1.1). In Section 3, we prove the existence of solutions in the subcritical case, namely Theorem 1.1. In Section 4, we establish the a priori estimates in Theorem 1.2 by a delicate blow-up analysis. Finally we prove Theorem 1.3 in Section 5.

2. The Monge–Ampère equation

Given a bounded, positive function f in $\mathbb{R}^{n+1}$, the prescribed centroaffine curvature problem is to find a convex hypersurface M such that

$$\kappa(p) = f(p) \quad \forall\, p \in M, \tag{2.1}$$

where $\kappa(p)$ is the centroaffine curvature of M at p. When M is smooth and strictly convex, $\kappa(p)$ can be expressed as in [9]

$$\kappa(p) = \frac{1}{H^{n+2}(x)\det(\nabla^2 H + HI)(x)},$$

where $x \in S^n$ is the unit outer normal of M at p and H is the support function of M, given by

$$H(x) = \sup\{\langle x, p\rangle \mid p \in M\} \qquad x \in S^n.$$

Extend H to $\mathbb{R}^{n+1}$ such that it is homogeneous of degree 1. Denote the gradient of H in $\mathbb{R}^{n+1}$ by $\overline{\nabla}$. It is well known that

$$p = \overline{\nabla} H(x) = \nabla H(x) + H(x)x.$$

Thus (2.1) can be written as

$$H^{n+2}(x)\det(\nabla^2 H + HI)(x) = \left[f\left(\overline{\nabla} H(x)\right)\right]^{-1} \quad x \in S^n. \tag{2.2}$$

Let M^* be the boundary of the polar set of the convex body enclosed by M [27]. Let ρ^* be the radial function of M^*, such that

$$M^* = \{x\rho^*(x) \mid x \in S^n\}.$$

Denote by H^* the support function of M^*. Then by definition,

$$\rho^*(x) = 1/H(x),$$

which implies that

$$\overline{\nabla} H = -\frac{\overline{\nabla}\rho^*}{(\rho^*)^2}.$$

Hence in terms of ρ^*, equation (2.2) can be rewritten as

$$\frac{\det\left(-\rho^*\nabla^2\rho^* + 2(\nabla\rho^*)^T\nabla\rho^* + (\rho^*)^2 I\right)}{(\rho^*)^{4n+2}} = \left[f\left(\frac{-\overline{\nabla}\rho^*}{(\rho^*)^2}\right)\right]^{-1}. \tag{2.3}$$

On the other hand, let x^* be the unit outer normal of M^* at point $\rho^*(x)x$. We have

$$x^* = -\frac{\overline{\nabla}\rho^*}{|\overline{\nabla}\rho^*|}(x),$$

which implies that

$$\frac{x^*}{H^*(x^*)} = -\frac{\overline{\nabla}\rho^*}{(\rho^*)^2}(x). \tag{2.4}$$

It is known that the Gauss curvature of M^* at this point is given by

$$K^* = \frac{1}{\det\left(\nabla^2 H^* + H^* I\right)(x^*)} = \frac{\det\left(-\rho^*\nabla^2\rho^* + 2(\nabla\rho^*)^T\nabla\rho^* + (\rho^*)^2 I\right)}{(\rho^*)^{2n-2}|\overline{\nabla}\rho^*|^{n+2}}(x).$$

Hence equation (2.3) can also be written as

$$\begin{aligned}\det\left(\nabla^2 H^* + H^* I\right)(x^*) &= f\left(-\overline{\nabla}\rho^*/(\rho^*)^2\right)\left(\frac{|\overline{\nabla}\rho^*|}{(\rho^*)^2}\right)^{n+2}(x)\\ &= \frac{f\left(x^*/H^*\right)}{(H^*)^{n+2}}(x^*),\end{aligned}$$

where the second equality is due to (2.4). Thus if H is a solution to equation (2.2), then the support function of its polar body, H^*, satisfies the following equation

$$\det\left(\nabla^2 H^* + H^* I\right)(x^*) = \frac{f\left(x^*/H^*\right)}{(H^*)^{n+2}} \quad x^* \in S^n,$$

which is exactly the equation (1.1).

An important property of equation (1.1) is its invariance under projective transformations on S^n [9]. That is if H is a solution to (1.1), then H_A with $A \in SL(n+1)$ given by

$$H_A(x) = |Ax| \cdot H\left(\frac{Ax}{|Ax|}\right) \quad x \in S^n, \tag{2.5}$$

satisfies the following equation

$$\det(\nabla^2 H_A + H_A I) = \frac{f(Ax/H_A)}{H_A^{n+2}} \quad x \in S^n. \tag{2.6}$$

In the paper [9] the authors also found a necessary condition for the existence of solutions to (1.1). That is, if H is a solution to (1.1), then

$$\int_{S^n} \frac{\nabla_\xi [f(x/H)](x)}{H^{n+1}} d\sigma_{S^n} = 0 \tag{2.7}$$

for any projective vector field ξ generated by any square matrix B of order $n+1$, namely

$$\xi(x) = Bx - (x^T Bx)x, \quad x \in S^n. \tag{2.8}$$

In the following we may drop $d\sigma_{S^n}$, when the integral over S^n is under the standard metric.

3. Existence of solutions in the subcritical case

In this section we sketch the proof of Theorem 1.1. The proof is similar to that in [9, Section 5], where the existence of solutions for the case $f = f(x)$ was obtained. We refer the reader to [9] for more details.

For $q \in [n+1, n+2)$, we denote

$$F(x,t) = \int_t^{+\infty} \frac{f(x/s)}{s^q} ds, \tag{3.1}$$

where $x \in S^n$ and $t > 0$, and

$$J[H] = \int_{S^n} F(x, H(x)). \tag{3.2}$$

Consider the maximizing problem

$$M_v =: \sup_{H \in \mathcal{S}_v} \inf_{y \in K_H} J[H(x) - y \cdot x], \tag{3.3}$$

where v is a positive constant, and $\mathcal{S}_v$ is the set of support functions such that the volume of the associated convex body is equal to v.

Observe that

$$\frac{f_{\inf}}{q-1}\cdot\frac{1}{t^{q-1}} \le F(x,t) \le \frac{f_{\sup}}{q-1}\cdot\frac{1}{t^{q-1}} \quad \forall\, x\in S^n. \tag{3.4}$$

Given $H \in \mathcal{S}_v$, since $q \in [n+1, n+2)$, one easily sees that $J[H(x) - y\cdot x] \to \infty$ whenever $y \in K_H$ and y converges to a boundary point of ∂K_H. Hence there exists a point $y = y(H) \in K_H$ such that the infimum $\inf_{y\in K_H} J[H(x) - y\cdot x]$ is attained at y.

We claim that there exist two positive constants C_1, C_2, depending only on n and f, such that

$$\frac{C_1}{q-1}v^{-\frac{q-1}{n+1}} \le M_v \le \frac{C_2}{q-1}v^{-\frac{q-1}{n+1}}. \tag{3.5}$$

In fact, by the Hölder inequality, we have

$$\begin{aligned} J[H(x)-y\cdot x] &\le \frac{f_{\sup}}{q-1}\int_{S^n}\frac{1}{(H(x)-y\cdot x)^{q-1}} \\ &\le \frac{C_n f_{\sup}}{q-1}\left(\int_{S^n}\frac{1}{(H(x)-y\cdot x)^{n+1}}\right)^{\frac{q-1}{n+1}}. \end{aligned}$$

By the Blaschke–Santaló inequality (1.10), we obtain

$$\inf_{y\in K_H} J[H(x)-y\cdot x] \le \frac{C_n f_{\sup}}{q-1}\cdot \mathrm{vol}(K_H)^{-\frac{q-1}{n+1}} = \frac{C_2}{q-1}v^{-\frac{q-1}{n+1}},$$

which implies the second inequality of (3.5). Letting $H \equiv [(n+1)v/\omega_n]^{1/(n+1)}$, we have

$$\begin{aligned} M_v &\ge \inf_{y\in K_H} J[H(x)-y\cdot x] \\ &\ge \frac{f_{\inf}}{q-1}\int_{S^n}\frac{1}{H^{q-1}} \\ &\ge \frac{C_n f_{\inf}}{q-1}v^{-\frac{q-1}{n+1}}, \end{aligned}$$

which is the first inequality of (3.5).

Lemma 3.1. *For a given positive constant $v > 0$, there exists a support function $H \in \mathcal{S}_v$ such that $M_v = \inf_{y\in K_H} J[H(x) - y\cdot x]$.*

Proof. Let $\{H_j\} \subset \mathcal{S}_v$ be a maximizing sequence. We show that $\{H_j\}$ is uniformly bounded. Suppose to the contrary that $\max_{x\in S^n} H_j(x) \to +\infty$ as $j \to +\infty$. Denote the minimum ellipsoid of K_{H_j} by E_j. Let z_j be the centre of E_j. We have

$$\inf_{y\in K_{H_j}} J[H_j(x) - y\cdot x] \le J[H_j - z_j \cdot x] \le \frac{f_{\sup}}{q-1}\int_{S^n} \frac{1}{(H_j(x) - z_j\cdot x)^{q-1}}.$$

By a translation of coordinates, we assume that z_j is the origin. Then the support function of E_j can be expressed as $|B_j x|$ for some positive definite matrix B_j of order $n+1$. As E_j is the minimum ellipsoid of K_{H_j}, we have

$$\frac{1}{n+1}|B_j x| \le H_j(x) \le |B_j x| \quad \forall\, x \in S^n. \tag{3.6}$$

It follows that

$$\inf_{y\in K_{H_j}} J[H_j(x) - y\cdot x] \le C_n f_{\sup} \int_{S^n} \frac{1}{|B_j x|^{q-1}}. \tag{3.7}$$

From (3.6), we see that $C_n^{-1} v \le \det B_j \le C_n v$. By assumption, $\max_{x\in S^n} |B_j x| \to +\infty$ as $j \to +\infty$. Hence when $q < n+2$, one infers that [9]

$$\lim_{j\to+\infty} \int_{S^n} \frac{1}{|B_j x|^{q-1}} = 0,$$

which together with (3.7) implies that $M_v = 0$, contradicting with (3.5). Therefore $\{H_j\}$ is uniformly bounded.

By Blaschke's selection theorem, there is a subsequence of $\{H_j\}$ which converges uniformly to a maximizer H of the problem (3.3) and H is uniformly bounded. □

Proof of Theorem 1.1. For any given positive constant $v > 0$, by Lemma 3.1, there exists a support function $H \in \mathcal{S}_v$ such that $M_v = \inf_{y\in K_H} J[H(x) - y\cdot x]$. From the proofs of Corollary 5.4 and Lemmas 5.5–5.6 in [9], one sees that the maximizer H is a generalized solution to

$$\det(\nabla^2 H + HI)(x) = \frac{\lambda f(x/H)}{H^q} \quad \forall x \in S^n,$$

where λ is the Lagrange multiplier.

When $q \in [n+1, n+2)$, one easily verifies that H is positive. For if $\inf H = 0$, then $J[H] = \infty$, which is in contradiction with (3.5). Hence by [7], H is strictly convex and $C^{1,\gamma}$ smooth, for some $\gamma \in (0,1)$. This completes the proof of Theorem 1.1. □

We remark that if furthermore $f \in C^1(\mathbb{R}^{n+1})$, then by [6,29], we have $H \in C^{2,\gamma}(S^n)$ for any $\gamma \in (0,1)$.

4. A priori estimates in the critical case

In this section we use the necessary condition (2.7) and a blow-up analysis to prove the a priori estimates (1.7) in Theorem 1.2.

Let $A_k \in SL(n+1)$ be a sequence of diagonal matrices,

$$A_k = \operatorname{diag}(s_{1,k}, \cdots, s_{n+1,k}), \tag{4.1}$$

where $s_{1,k} \geq \cdots \geq s_{n+1,k} > 0$ and $s_{1,k} \to +\infty$ and $s_{n+1,k} \to 0$ as $k \to +\infty$. We successively define integers $l_1, l_2, \cdots$ as follows:

$$\begin{aligned} l_1 &:= \max_j \left\{ j : \lim_k s_{j,k} = +\infty \right\}, \\ l_i &:= \max_j \left\{ j : \lim_k \frac{s_{j,k}}{s_{l_{i-1},k}} = +\infty \right\} \quad \text{for } i = 2, 3, \cdots . \end{aligned} \tag{4.2}$$

By choosing a subsequence we may assume all the limits exist or equal infinity. The procedure in (4.2) must end in finite steps, say, at step m. Then we have

$$1 \leq l_m < \cdots < \cdots < l_1 \leq n < n+1.$$

Lemma 4.1. *Assume that φ_k, ψ_k are two sequences of uniformly bounded functions on S^n, converging uniformly to functions φ, ψ as $k \to +\infty$. Assume that $\varphi \in C^1(S^n)$ and ψ is a positive constant. Consider the following integral*

$$\Lambda_k := \int_{S^n} \varphi_k(x) \zeta(\psi_k(x) A_k x) \frac{1}{|A_k x|^\alpha}, \tag{4.3}$$

where $\alpha > 0$ is a constant, and $\zeta \in C(\mathbb{R}^{n+1})$ is a bounded function satisfying

(a) $\zeta(y) = O(|y|^\alpha)$ *as* $y \to 0$,
(b) $\zeta(y) = \zeta_\infty + o(1)$ *as* $y \to \infty$.

Then as $k \to +\infty$, we have the following estimates.

(i) *When $\alpha > l_1$,*

$$\Lambda_k = \frac{1}{s_{1,k} \cdots s_{l_1,k}} \left(\psi^{\alpha - l_1} \int_{S^{n-l_1}} \varphi(0, v) d\sigma_{S^{n-l_1}} \int_{u \in \mathbb{R}^{l_1}} \frac{\zeta(u, \psi N v)}{|(u, \psi N v)|^\alpha} du + o(1) \right), \tag{4.4}$$

where $N = \lim_{k \to \infty} \operatorname{diag}(s_{l_1+1,k}, \cdots, s_{n,k}, 0)$ (we allow all the limits to be zero).

(ii) *When* $\alpha = l_1$,

$$\Lambda_k = \frac{C \log s_{l_1,k}}{s_{1,k} \cdots s_{l_1,k}} \left(\zeta_\infty \int\limits_{S^{n-l_1}} \varphi(0,v) d\sigma_{S^{n-l_1}} + o(1) \right). \tag{4.5}$$

(iii) *When* $l_i < \alpha < l_{i-1}$ *for* $i = 2, \cdots, m$,

$$\Lambda_k = \frac{C}{s_{1,k} \cdots s_{l_i,k} \cdot s_{l_{i-1},k}^{\alpha - l_i}} \left(\zeta_\infty \int\limits_{S^{n-l_i}} \frac{\varphi(0,v)}{|Nv|^{\alpha - l_i}} d\sigma_{S^{n-l_i}} + o(1) \right), \tag{4.6}$$

where $N = \lim_{k\to\infty} \operatorname{diag}\left(\frac{s_{l_i+1,k}}{s_{l_{i-1},k}}, \frac{s_{l_i+2,k}}{s_{l_{i-1},k}}, \cdots, \frac{s_{l_{i-1}-1,k}}{s_{l_{i-1},k}}, \frac{s_{l_{i-1},k}}{s_{l_{i-1},k}}, 0, \cdots, 0 \right)$ *is a matrix of order* $n + 1 - l_i$.

(iv) *When* $\alpha = l_i$ *for* $i = 2, \cdots, m$,

$$\Lambda_k = \frac{C \log(s_{l_i,k}/s_{l_{i-1},k})}{s_{1,k} \cdots s_{l_i,k}} \left(\zeta_\infty \int\limits_{S^{n-l_i}} \varphi(0,v) d\sigma_{S^{n-l_i}} + o(1) \right). \tag{4.7}$$

(v) *When* $\alpha < l_m$,

$$\Lambda_k = \frac{1}{s_{l_m,k}^{\alpha}} \left(\zeta_\infty \int\limits_{S^n} \frac{\varphi(x)}{|\tilde{A}x|^\alpha} d\sigma_{S^n} + o(1) \right), \tag{4.8}$$

where $\tilde{A} = \lim_{k\to\infty} \operatorname{diag}\left(\frac{s_{1,k}}{s_{l_m,k}}, \frac{s_{2,k}}{s_{l_m,k}}, \cdots, \frac{s_{l_m-1,k}}{s_{l_m,k}}, \frac{s_{l_m,k}}{s_{l_m,k}}, 0, \cdots, 0 \right)$ *is a matrix of order* $n + 1$.

In the above, C *is a positive constant.*

Proof. We prove this lemma case by case.

Case (i): $\alpha > l_1$. For convenience, we denote $l := l_1$, $x = (u, v)$ and

$$A_k = \begin{pmatrix} M_k & 0 \\ 0 & N_k \end{pmatrix}, \tag{4.9}$$

where

$$u = (x_1, \cdots, x_l), \quad v = (x_{l+1}, \cdots, x_{n+1}), \tag{4.10}$$

and M_k and N_k are diagonal matrices of order l and $n + 1 - l$, respectively. Denote

$$S^n_* = \{x = (u, v) \in S^n \mid 1/2 \le |u| \le 1\} . \tag{4.11}$$

By the coarea formula,

$$\begin{aligned}\int_{S^n_*} \frac{1}{|M_k u|^\alpha} &= \int_{1/2\le|u|\le1} \frac{du}{\rho(u)} \int_{|v|=\rho(u)} \frac{1}{|M_k u|^\alpha} d\sigma \\ &= \omega_{n-l} \int_{1/2\le|u|\le1} \frac{1}{|M_k u|^\alpha} \rho(u)^{n-l-1} du,\end{aligned} \tag{4.12}$$

where $\rho(u) = \sqrt{1-|u|^2}$. We have

$$\begin{aligned}\int_{1/2\le|u|\le1} \frac{1}{|M_k u|^\alpha} \rho(u)^{n-l-1} du &= \int_{\frac{1}{2}}^{1} \rho(r)^{n-l-1} dr \int_{|u|=r} \frac{1}{|M_k u|^\alpha} d\sigma \\ &= \int_{\frac{1}{2}}^{1} r^{l-\alpha-1} \rho(r)^{n-l-1} dr \int_{|u|=1} \frac{1}{|M_k u|^\alpha} d\sigma \\ &= C_{l,\alpha} \int_{S^{l-1}} \frac{1}{|M_k u|^\alpha} d\sigma,\end{aligned}$$

where $C_{l,\alpha}$ is a positive constant depending only on n, l and α. By [18, Lemma 3.1],

$$\int_{S^{l-1}} \frac{1}{|M_k u|^\alpha} d\sigma = \frac{1}{\det M_k} \int_{S^{l-1}} \left|M_k^{-1} u\right|^{\alpha-l} d\sigma.$$

Hence from (4.12) we obtain

$$\int_{S^n_*} \frac{1}{|M_k u|^\alpha} = \frac{\omega_{n-l} C_{l,\alpha}}{\det M_k} \int_{S^{l-1}} \left|M_k^{-1} u\right|^{\alpha-l} d\sigma. \tag{4.13}$$

When $\alpha - l > 0$, there exist positive constant $\tilde{C}$ depending only on l and α, such that

$$\tilde{C}^{-1} \left|M_k^{-1} u\right|^{\alpha-l} \le \left|\frac{u_1}{s_{1,k}}\right|^{\alpha-l} + \cdots + \left|\frac{u_l}{s_{l,k}}\right|^{\alpha-l} \le \tilde{C} \left|M_k^{-1} u\right|^{\alpha-l}.$$

Hence by (4.13),

$$\int_{S^n_*} \frac{1}{|M_k u|^\alpha} = \frac{C_k}{\det M_k}\left(\frac{1}{s_{1,k}^{\alpha-l}} + \cdots + \frac{1}{s_{l,k}^{\alpha-l}}\right) = \frac{C_k}{\det M_k}\left(\mathrm{tr} M_k^{-1}\right)^{\alpha-l}, \tag{4.14}$$

where C_k is a positive constant independent of M_k, and $\tilde{C}^{-1} \le \frac{C_k}{\omega_{n-l}C_{l,\alpha}} \le \tilde{C}$.

Now we compute Λ_k. By the coarea formula, we have

$$\begin{aligned} I_k &:= \int_{S^n \setminus S^n_*} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha} \\ &= \int_{|u|<1/2} \frac{du}{\rho(u)} \int_{|v|=\rho(u)} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha} d\sigma. \end{aligned}$$

Let $v = \rho(u)\tilde{v}$. We have

$$\begin{aligned} I_k &= \int_{|u|<1/2} \rho^{n-l-1}(u)du \int_{|\tilde{v}|=1} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha} d\sigma \\ &= \int_{|\tilde{v}|=1} d\sigma \int_{|u|<1/2} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha}\rho^{n-l-1}(u)du \\ &=: \int_{|\tilde{v}|=1} \Phi_k(\tilde{v})d\sigma. \end{aligned} \tag{4.15}$$

Let $u = M_k^{-1}\tilde{u}$. Then

$$\Phi_k(\tilde{v}) = \frac{1}{\det M_k}\int_{|M_k^{-1}\tilde{u}|<1/2} \varphi_k(x)\zeta\left(\psi_k(x)(\tilde{u}, N_k v)\right)\frac{1}{|\tilde{u}, N_k v|^\alpha}\rho^{n-l-1}(u)d\tilde{u}, \tag{4.16}$$

where $|\tilde{u}, N_k v|$ is an abbreviation of $|(\tilde{u}, N_k v)|$. Therefore

$$\begin{aligned} |\det M_k \cdot \Phi_k(\tilde{v})| &\le C \int_{|M_k^{-1}\tilde{u}|<1/2} \left|\zeta\left(\psi_k(x)(\tilde{u}, N_k v)\right)\right| \frac{1}{|\tilde{u}, N_k v|^\alpha} d\tilde{u} \\ &\le C \int_{\tilde{u}\in\mathbb{R}^l} \left|\zeta\left(\psi_k(x)(\tilde{u}, N_k v)\right)\right| \frac{1}{|\tilde{u}, N_k v|^\alpha} d\tilde{u}, \end{aligned}$$

which is integrable by our assumptions (a), (b) and $\alpha > l$. Applying the dominated convergence theorem to (4.16), we obtain, as $k \to +\infty$,

$$\Phi_k(\tilde{v}) = \frac{1}{\det M_k}\left(\int\limits_{\tilde{u}\in\mathbb{R}^l} \varphi(0,\tilde{v})\zeta\left(\psi\cdot(\tilde{u},N\tilde{v})\right)\frac{1}{|\tilde{u},N\tilde{v}|^{\alpha}}d\tilde{u} + o(1)\right),$$

where $N := \lim_k N_k$. Inserting the above formula into (4.15), we obtain

$$I_k = \frac{1}{\det M_k}\left(\int\limits_{|\tilde{v}|=1} \varphi(0,\tilde{v})d\sigma(\tilde{v}) \int\limits_{\tilde{u}\in\mathbb{R}^l} \zeta\left(\psi\cdot(\tilde{u},N\tilde{v})\right)\frac{1}{|\tilde{u},N\tilde{v}|^{\alpha}}d\tilde{u} + o(1)\right). \tag{4.17}$$

Note that

$$\begin{aligned}|\Lambda_k - I_k| &\le C\int\limits_{S^n_*}\frac{1}{|M_k u|^{\alpha}}\\ &\le \frac{C}{\det M_k}\left(\mathrm{tr}M_k^{-1}\right)^{\alpha-l} = \frac{o(1)}{\det M_k}.\end{aligned}$$

Hence from (4.17),

$$\begin{aligned}\Lambda_k &= \frac{1}{\det M_k}\left(\int\limits_{|\tilde{v}|=1} \varphi(0,\tilde{v})d\sigma(\tilde{v}) \int\limits_{\tilde{u}\in\mathbb{R}^l} \frac{\zeta\left(\psi\cdot(\tilde{u},N\tilde{v})\right)}{|\tilde{u},N\tilde{v}|^{\alpha}}d\tilde{u} + o(1)\right)\\ &= \frac{1}{\det M_k}\left(\psi^{\alpha-l}\int\limits_{|\tilde{v}|=1} \varphi(0,\tilde{v})d\sigma(\tilde{v}) \int\limits_{\tilde{u}\in\mathbb{R}^l} \frac{\zeta\left(\tilde{u},\psi N\tilde{v}\right)}{|\tilde{u},\psi N\tilde{v}|^{\alpha}}d\tilde{u} + o(1)\right).\end{aligned}$$

We obtain (4.4).

Case (ii): $\alpha = l_1$. For convenience, we denote again $l := l_1$, and use the same notations in (4.10) and (4.9).

Before computing Λ_k, we estimate an integral first. Denote

$$T_k := \left\{x = (u,v) \in S^n \mid |u| < 1/2,\ |M_k u| \ge s_{l,k}/2\right\}. \tag{4.18}$$

By the coarea formula, similarly to (4.12), we have

$$\begin{aligned}\int\limits_{T_k}\frac{1}{|M_k u|^{\alpha}} &= \omega_{n-l}\int\limits_{|u|<1/2,\ |M_k u|\ge s_{l,k}/2}\frac{1}{|M_k u|^{\alpha}}\rho(u)^{n-l-1}du\\ &= C\omega_{n-l}\int\limits_{|u|<1/2,\ |M_k u|\ge s_{l,k}/2}\frac{1}{|M_k u|^{\alpha}}du\end{aligned} \tag{4.19}$$

for a constant $C \in (1/2^n, 2)$. Let

$$u := s_{l,k}\, M_k^{-1}\tilde{u}. \tag{4.20}$$

Then

$$\int_{T_k} \frac{1}{|M_k u|^\alpha} = \frac{C\omega_{n-l}}{\det M_k \cdot s_{l,k}^{\alpha-l}} \int_{|s_{l,k}M_k^{-1}\tilde{u}|<1/2,\, |\tilde{u}|\geq 1/2} \frac{1}{|\tilde{u}|^\alpha} d\tilde{u}. \tag{4.21}$$

Observing that $s_{l,k}M_k^{-1}$ is a diagonal matrix whose diagonal entries are in ascending order and the last one is equal to 1, we have

$$\int_{|s_{l,k}M_k^{-1}\tilde{u}|<1/2,\, |\tilde{u}|\geq 1/2} \frac{1}{|\tilde{u}|^\alpha} d\tilde{u} \leq \int_{\tilde{u}\in\mathbb{R}^{l-1}\times(-1/2,1/2),\, |\tilde{u}|\geq 1/2} \frac{1}{|\tilde{u}|^\alpha} d\tilde{u}$$
$$< +\infty,$$

where the last inequality holds when $\alpha \geq l$. (Although in this case $\alpha = l$, we deduce the following (4.22) and (4.27) for $\alpha \geq l$, which will be used in case (iii).) Thus there exists a positive constant C depending only on n and l, such that

$$\int_{T_k} \frac{1}{|M_k u|^\alpha} \leq \frac{C}{\det M_k \cdot s_{l,k}^{\alpha-l}}. \tag{4.22}$$

To estimate Λ_k, without loss of generality, we assume that

$$s_{l,k} > 2, \quad s_{l+1,k} \leq 1, \quad \forall\, k \geq 1.$$

Denote

$$F_k := \{x = (u,v) \in S^n \mid |M_k u| \geq 1\}, \tag{4.23}$$
$$G_k := \{x = (u,v) \in S^n \mid |M_k u| < s_{l,k}/2\}. \tag{4.24}$$

Then $F_k \cup G_k = S^n$, and

$$|u| < 1/2 \quad \forall\, x = (u,v) \in G_k, \tag{4.25}$$
$$S^n \backslash G_k = S^n_* \cup T_k. \tag{4.26}$$

By (4.14) and (4.22) and since $\alpha \geq l$, there exists a positive constant C depending only on n, l and α, such that

$$\int_{S^n\backslash G_k} \frac{1}{|M_k u|^\alpha} \leq \frac{C}{\det M_k \cdot s_{l,k}^{\alpha-l}}. \tag{4.27}$$

Observe that

$$|A_k x| \leq \sqrt{2}|M_k u|, \quad \forall\, x = (u,v) \in F_k. \tag{4.28}$$

Similarly to (4.19), we have

$$|S^n \backslash F_k| = \omega_{n-l} \int\limits_{|M_k u|<1} \rho^{n-l-1}(u)du$$
$$= C\omega_{n-l} \int\limits_{|M_k u|<1} du \tag{4.29}$$
$$= \frac{C}{\det M_k}.$$

Since $F_k \cup G_k = S^n$, we can write Λ_k as

$$\Lambda_k = \left(\int\limits_{S^n\backslash F_k} + \int\limits_{F_k \cap G_k} + \int\limits_{S^n\backslash G_k} \right) \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right) \frac{1}{|A_k x|^\alpha} \tag{4.30}$$
$$=: I_k + II_k + III_k.$$

Noting that the integrand is bounded by our assumptions (a) and (b), we see from (4.29) that

$$|I_k| \le C \int\limits_{S^n\backslash F_k} d\sigma_{S^n} \le \frac{C}{\det M_k}.$$

By (4.27) we also have

$$|III_k| \le C \int\limits_{S^n\backslash G_k} \frac{1}{|M_k u|^\alpha} \le \frac{C}{\det M_k}.$$

Therefore (4.30) can be written as

$$\Lambda_k = II_k + \frac{O(1)}{\det M_k} \quad \text{as } k \to +\infty. \tag{4.31}$$

To estimate II_k, first computing as in (4.19), (4.20) and (4.21), we have

$$\int\limits_{F_k \cap G_k} \frac{1}{|M_k u|^\alpha} \le \int\limits_{F_k \cap G_k} \frac{1}{|M_k u|^l}$$
$$= \frac{C\omega_{n-l}}{\det M_k} \int\limits_{1/s_{l,k} \le |\tilde{u}| < 1/2} \frac{1}{|\tilde{u}|^l} d\tilde{u}$$

$$= \frac{C\omega_{n-l}\omega_{l-1}}{\det M_k} \int_{1/s_{l,k}}^{1/2} \frac{1}{r} dr \tag{4.32}$$

$$= \frac{C\omega_{n-l}\omega_{l-1}}{\det M_k} \log\left(\frac{s_{l,k}}{2}\right)$$

$$= (C + o(1)) \frac{\log s_{l,k}}{\det M_k},$$

as $k \to +\infty$, where C is a positive constant independent of k.

We claim that for any bounded function $\eta \in C(\mathbb{R}^{n+1})$ satisfying

$$\lim_{y\to\infty} \eta(y) = 0, \tag{4.33}$$

and any positive constant $\lambda_k \geq 1$,

$$\int_{F_k\cap G_k} \eta\left(\psi_k(x)\lambda_k A_k x\right) \frac{1}{|A_k x|^\alpha} = o(1) \frac{\log s_{l,k}}{\det M_k}, \quad \text{as } k \to +\infty. \tag{4.34}$$

In fact, denote

$$q(r) := \sup_{|y|\geq r} |\eta(y)| \quad r \in [0, +\infty).$$

Then q is bounded and monotonically decreasing. By (4.33) it satisfies $\lim_{r\to+\infty} q(r) = 0$. Observing that

$$\begin{aligned}
\int_{F_k\cap G_k} |\eta\left(\psi_k(x)\lambda_k A_k x\right)| \frac{1}{|A_k x|^\alpha} &\leq \int_{F_k\cap G_k} q\left(|\psi_k(x)\lambda_k A_k x|\right) \frac{1}{|A_k x|^\alpha} \\
&\leq \int_{F_k\cap G_k} q\left(\frac{\psi}{2}|A_k x|\right) \frac{1}{|A_k x|^\alpha} \\
&\leq \int_{F_k\cap G_k} q\left(|M_k u|\right) \frac{1}{|M_k u|^\alpha},
\end{aligned} \tag{4.35}$$

where without loss of generality, we have assumed that $\psi \geq 2$. Again computing as in (4.19), (4.20) and (4.21), we have

$$\begin{aligned}
\int_{F_k\cap G_k} q\left(|M_k u|\right) \frac{1}{|M_k u|^\alpha} &\leq \int_{F_k\cap G_k} q\left(|M_k u|\right) \frac{1}{|M_k u|^l} \\
&= \frac{C\omega_{n-l}}{\det M_k} \int_{1/s_{l,k}\leq|\tilde{u}|<1/2} \frac{q\left(|s_{l,k}\tilde{u}|\right)}{|\tilde{u}|^l} d\tilde{u}
\end{aligned} \tag{4.36}$$

$$= \frac{C\omega_{n-l}\omega_{l-1}}{\det M_k} \int\limits_{1/s_{l,k}}^{1/2} \frac{q\left(s_{l,k}t\right)}{t} dt$$

$$= \frac{C\omega_{n-l}\omega_{l-1}}{\det M_k} \int\limits_{1}^{s_{l,k}/2} \frac{q(r)}{r} dr.$$

Since $q(t) \to 0$ as $t \to \infty$, it is easy to see that

$$\int\limits_{1}^{s_{l,k}/2} \frac{q(r)}{r} dr = o(1) \log s_{l,k}$$

as $k \to \infty$. Hence (4.36) becomes

$$\int\limits_{F_k \cap G_k} q\left(|M_k u|\right) \frac{1}{|M_k u|^{\alpha}} = o(1) \frac{\log s_{l,k}}{\det M_k},$$

which together with (4.35) implies (4.34).

We can now compute II_k. Write

$$\zeta(y) = \zeta_\infty + \eta(y).$$

Then η satisfies (4.33). By our assumptions,

$$\begin{aligned} II_k &= \int\limits_{F_k \cap G_k} \left(\varphi(x) + o(1)\right)\left(\zeta_\infty + \eta\left(\psi_k(x) A_k x\right)\right) \frac{1}{|A_k x|^{\alpha}} \\ &= \zeta_\infty \int\limits_{F_k \cap G_k} \frac{\varphi(x)}{|A_k x|^{\alpha}} + \int\limits_{F_k \cap G_k} \frac{\varphi(x)\eta\left(\psi_k(x) A_k x\right)}{|A_k x|^{\alpha}} + o(1) \int\limits_{F_k \cap G_k} \frac{1}{|A_k x|^{\alpha}} \\ &= \zeta_\infty \int\limits_{F_k \cap G_k} \frac{\varphi(x)}{|A_k x|^{\alpha}} + o(1) \frac{\log s_{l,k}}{\det M_k}, \end{aligned} \tag{4.37}$$

where (4.32) and (4.34) are used in the last equality. Furthermore, by the coarea formula we have

$$\begin{aligned} \int\limits_{F_k \cap G_k} \frac{\varphi(x)}{|A_k x|^{\alpha}} &= \int\limits_{1 \le |M_k u| < s_{l,k}/2} \frac{du}{\rho(u)} \int\limits_{|v| = \rho(u)} \frac{\varphi(x)}{|A_k x|^{\alpha}} d\sigma \\ &= \int\limits_{1 \le |M_k u| < s_{l,k}/2} \rho(u)^{n-l-1} du \int\limits_{|\tilde{v}| = 1} \frac{\varphi(u, \rho(u)\tilde{v})}{|A_k x|^{\alpha}} d\sigma. \end{aligned} \tag{4.38}$$

Denote

$$\begin{aligned}\Phi_k &:= \int\limits_{1\le|M_k u|<s_{l,k}/2} \rho(u)^{n-l-1}du \int\limits_{|\tilde v|=1} \frac{\varphi(0,\tilde v)}{|A_k x|^\alpha}d\sigma \\ &= \int\limits_{|\tilde v|=1} \varphi(0,\tilde v)d\sigma \int\limits_{1\le|M_k u|<s_{l,k}/2} \frac{1}{|A_k x|^\alpha}\rho(u)^{n-l-1}du.\end{aligned} \tag{4.39}$$

Since $\varphi \in C^1(S^n)$, we have

$$|\varphi(u,\rho(u)\tilde v) - \varphi(0,\tilde v)| \le 2\,\|\varphi\|_{C^1(S^n)}\,|u|, \quad \forall\,|u|\le 1,\ |\tilde v| = 1.$$

Thus

$$\begin{aligned}\left|\int\limits_{F_k\cap G_k} \frac{\varphi(x)}{|A_k x|^\alpha} - \Phi_k\right| &\le C_\varphi \int\limits_{1\le|M_k u|<s_{l,k}/2} \rho(u)^{n-l-1}du \int\limits_{|\tilde v|=1} \frac{|u|}{|A_k x|^\alpha}d\sigma \\ &\le 2C_\varphi \int\limits_{1\le|M_k u|<s_{l,k}/2} du \int\limits_{|\tilde v|=1} \frac{|u|}{|M_k u|^\alpha}d\sigma \\ &= 2C_\varphi\omega_{n-l} \int\limits_{1\le|M_k u|<s_{l,k}/2} \frac{|u|}{|M_k u|^\alpha}du,\end{aligned} \tag{4.40}$$

where (4.25) is used in the second inequality. By the change (4.20), we obtain

$$\begin{aligned}\int\limits_{1\le|M_k u|<s_{l,k}/2} \frac{|u|}{|M_k u|^\alpha}du &\le \int\limits_{1\le|M_k u|<s_{l,k}/2} \frac{|u|}{|M_k u|^l}du \\ &= \frac{1}{\det M_k} \int\limits_{1/s_{l,k}\le|\tilde u|<1/2} \frac{|s_{l,k}\,M_k^{-1}\tilde u|}{|\tilde u|^l}d\tilde u \\ &\le \frac{1}{\det M_k} \int\limits_{|\tilde u|<1/2} \frac{1}{|\tilde u|^{l-1}}d\tilde u \\ &\le \frac{C_l}{\det M_k}.\end{aligned}$$

Hence by (4.40),

$$\left|\int\limits_{F_k\cap G_k} \frac{\varphi(x)}{|A_k x|^\alpha} - \Phi_k\right| \le \frac{C_\varphi C_l}{\det M_k}. \tag{4.41}$$

To estimate Φ_k, by (4.28), there exists a positive constant $C_{l,k} > 0$ such that

$$\begin{aligned}\int\limits_{1\le|M_k u|<s_{l,k}/2} \frac{1}{|A_k x|^\alpha}\rho(u)^{n-l-1}du &= C_{l,k}\int\limits_{F_k\cap G_k}\frac{1}{|M_k u|^\alpha} \\ &= (C+o(1))\frac{\log s_{l,k}}{\det M_k},\end{aligned} \tag{4.42}$$

where the second equality is due to (4.32). Substituting (4.42) into (4.39), we get

$$\Phi_k = \left(C\int\limits_{|\tilde v|=1}\varphi(0,\tilde v)d\sigma + o(1)\right)\frac{\log s_{l,k}}{\det M_k},$$

which, together with (4.41), implies that

$$\int\limits_{F_k\cap G_k}\frac{\varphi(x)}{|A_k x|^\alpha} = \left(C\int\limits_{|\tilde v|=1}\varphi(0,\tilde v)d\sigma + o(1)\right)\frac{\log s_{l,k}}{\det M_k}. \tag{4.43}$$

From (4.37) it then follows that

$$II_k = \left(C\zeta_\infty\int\limits_{|\tilde v|=1}\varphi(0,\tilde v)d\sigma + o(1)\right)\frac{\log s_{l,k}}{\det M_k}. \tag{4.44}$$

By (4.31), we finally obtain

$$\Lambda_k = \left(C\zeta_\infty\int\limits_{|\tilde v|=1}\varphi(0,\tilde v)d\sigma + o(1)\right)\frac{\log s_{l,k}}{\det M_k}, \tag{4.45}$$

as $k \to +\infty$, which is just (4.5).

Case (iii): $l_i < \alpha < l_{i-1}$ for some $i = 2, \cdots, m$. For convenience, we write $l := l_i$, $\tilde l := l_{i-1}$, and use the notation in (4.10) and (4.9) for x and A_k.

As before, we have

$$\begin{aligned} I_k &:= \int\limits_{G_k}\varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha} \\ &= \int\limits_{|M_k u|<s_{l,k}/2}\frac{du}{\rho(u)}\int\limits_{|v|=\rho(u)}\varphi_k(x)\zeta\left(\psi_k(x)A_k x\right)\frac{1}{|A_k x|^\alpha}d\sigma \end{aligned} \tag{4.46}$$

$$= \int_{|M_k u|<s_{l,k}/2} \rho(u)^{n-l-1} du \int_{|\tilde{v}|=1} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right) \frac{1}{|A_k x|^{\alpha}} d\sigma$$

$$= \int_{|\tilde{v}|=1} d\sigma(\tilde{v}) \int_{|M_k u|<s_{l,k}/2} du\, \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right) \frac{\rho(u)^{n-l-1}}{|A_k x|^{\alpha}},$$

where $v = \rho(u)\tilde{v}$. Making the change

$$u := s_{\tilde{l},k}\, M_k^{-1}\tilde{u}, \tag{4.47}$$

we have

$$\begin{aligned} I_k =& \frac{1}{\det M_k \cdot s_{\tilde{l},k}^{-l}} \int_{|\tilde{v}|=1} d\sigma \int_{2|\tilde{u}|<s_{l,k}/s_{\tilde{l},k}} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right) \frac{\rho(u)^{n-l-1}}{|A_k x|^{\alpha}} d\tilde{u} \\ =& \frac{1}{\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}} \int_{|\tilde{v}|=1} d\sigma \int_{2|\tilde{u}|<s_{l,k}/s_{\tilde{l},k}} \varphi_k(x)\zeta\left(\psi_k(x)A_k x\right) \frac{\rho(u)^{n-l-1}}{\left|s_{\tilde{l},k}^{-1}A_k x\right|^{\alpha}} d\tilde{u}. \end{aligned} \tag{4.48}$$

Note that $s_{\tilde{l},k}^{-1}A_k x = (\tilde{u}, s_{\tilde{l},k}^{-1}N_k\rho(u)\tilde{v})$. From (4.25) we have $\rho(u) > 1/2$. By our definitions of l and $\tilde{l}$,

$$s_{\tilde{l},k}^{-1}N_k \geq \begin{pmatrix} I_{\tilde{l}-l} & \\ & 0 \end{pmatrix},$$

where $I_{\tilde{l}-l}$ is unit matrix of order $\tilde{l}-l$. We obtain

$$\left|s_{\tilde{l},k}^{-1}A_k x\right| \geq \frac{1}{2}\sqrt{|\tilde{u}|^2 + |\tilde{v}_*|^2},$$

where $\tilde{v}_*$ denotes the first $\tilde{l}-l$ entries of $\tilde{v}$. Hence, from (4.48) we have

$$\begin{aligned} \left|\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}\, I_k\right| &\leq C \int_{|\tilde{v}|=1} d\sigma(\tilde{v}) \int_{2|\tilde{u}|<s_{l,k}/s_{\tilde{l},k}} \frac{d\tilde{u}}{(|\tilde{u}|^2 + |\tilde{v}_*|^2)^{\alpha/2}} \\ &\leq C \int_{|\tilde{v}|=1} d\sigma(\tilde{v}) \int_{\tilde{u}\in\mathbb{R}^l} \frac{d\tilde{u}}{(|\tilde{u}|^2 + |\tilde{v}_*|^2)^{\alpha/2}} \\ &= C\omega_{n-\tilde{l}} \int_{|\tilde{v}_*|\leq 1} \rho(\tilde{v}_*)^{n-\tilde{l}-1} d\tilde{v}_* \int_{\tilde{u}\in\mathbb{R}^l} \frac{d\tilde{u}}{(|\tilde{u}|^2 + |\tilde{v}_*|^2)^{\alpha/2}} \\ &< +\infty, \end{aligned} \tag{4.49}$$

where the last inequality is due to the assumption $l < \alpha < \tilde{l}$. Applying the dominated convergence theorem to (4.48), we obtain as $k \to +\infty$ that

$$
\begin{aligned}
I_k &= \frac{1}{\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}} \left(\zeta_\infty \int\limits_{|\tilde{v}|=1} \varphi(0,\tilde{v}) d\sigma \int\limits_{\tilde{u}\in\mathbb{R}^l} \frac{d\tilde{u}}{|\tilde{u}, N\tilde{v}|^\alpha} + o(1) \right) \\
&= \frac{1}{\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}} \left(C_{l,\alpha}\zeta_\infty \int\limits_{|\tilde{v}|=1} \frac{\varphi(0,\tilde{v})}{|N\tilde{v}|^{\alpha-l}} d\sigma + o(1) \right),
\end{aligned}
\tag{4.50}
$$

where $C_{l,\alpha}$ is a positive constant depending only on l and α, and

$$
N := \lim_k s_{\tilde{l},k}^{-1} N_k,
$$

which is well defined, and its first $\tilde{l} - l$ diagonal entries are finite but greater than or equal to 1, and the other $n + 1 - \tilde{l}$ diagonal entries are 0. By the definition of I_k and (4.27), we have

$$
\begin{aligned}
|\Lambda_k - I_k| &\le C \int\limits_{S^n \setminus G_k} \frac{1}{|M_k u|^\alpha} \\
&\le \frac{C}{\det M_k \cdot s_{l,k}^{\alpha-l}} = \frac{o(1)}{\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}}.
\end{aligned}
$$

Hence by (4.50) we obtain

$$
\Lambda_k = \frac{1}{\det M_k \cdot s_{\tilde{l},k}^{\alpha-l}} \left(C_{l,\alpha}\zeta_\infty \int\limits_{|\tilde{v}|=1} \frac{\varphi(0,\tilde{v})}{|N\tilde{v}|^{\alpha-l}} d\sigma(\tilde{v}) + o(1) \right),
$$

which is just (4.6).

Case (iv): $\alpha = l_i$ for some $i = 2, \cdots, m$. As before, we denote $l := l_i$, $\tilde{l} := l_{i-1}$, and use the notations in (4.10) and (4.9) for x and A_k.

For convenience we denote $A'_k := s_{\tilde{l},k}^{-1} A_k = \operatorname{diag}\left(s'_{1,k}, \cdots, s'_{n+1,k}\right)$, and as (4.9), we write $A'_k = \begin{pmatrix} M'_k & 0 \\ 0 & N'_k \end{pmatrix}$, where M'_k and N'_k are diagonal matrices of order l and $n+1-l$ respectively. Then

$$
\Lambda_k = \frac{1}{s_{\tilde{l},k}^\alpha} \int\limits_{S^n} \varphi_k(x) \zeta\left(\psi_k(x) s_{\tilde{l},k} A'_k x\right) \frac{1}{|A'_k x|^\alpha} =: \frac{1}{s_{\tilde{l},k}^\alpha} \Lambda'_k. \tag{4.51}
$$

Noting $\alpha = l$, one sees that Λ'_k is in the same form as in **Case (ii)**. Following the argument there, we have

$$\Lambda'_k = \left(\int\limits_{S^n \setminus F'_k} + \int\limits_{F'_k \cap G'_k} + \int\limits_{S^n \setminus G'_k} \right) \varphi_k(x)\zeta\left(\psi_k(x) s_{\tilde{l},k} A'_k x\right) \frac{1}{|A'_k x|^\alpha} \tag{4.52}$$
$$=: I'_k + II'_k + III'_k,$$

where

$$G'_k := \left\{x = (u, v) \in S^n \mid |M'_k u| < s'_{l,k}/2\right\},$$
$$F'_k := \{x = (u, v) \in S^n \mid |M'_k u| \geq 1\}.$$

For I'_k, since the integrand here may fail to be bounded, we need to modify the computations in **Case (ii)**. But for II'_k and III'_k, one easily sees that the computations in **Case (ii)** still work, and one has

$$II'_k + III'_k = \left(C\zeta_\infty \int\limits_{|\tilde{v}|=1} \varphi(0, \tilde{v}) d\sigma(\tilde{v}) + o(1) \right) \frac{\log s'_{l,k}}{\det M'_k}. \tag{4.53}$$

Noting that $M'_k = s^{-1}_{\tilde{l},k} M_k$, $s'_{l,k} = s^{-1}_{\tilde{l},k} s_{l,k}$, we thus have

$$\Lambda_k = \frac{1}{s^\alpha_{\tilde{l},k}} I'_k + \left(C\zeta_\infty \int\limits_{|\tilde{v}|=1} \varphi(0, \tilde{v}) d\sigma(\tilde{v}) + o(1) \right) \frac{\log(s_{l,k}/s_{\tilde{l},k})}{\det M_k}. \tag{4.54}$$

Denote

$$I_k := \frac{1}{s^\alpha_{\tilde{l},k}} I'_k = \int\limits_{S^n \setminus F'_k} \varphi_k(x)\zeta\left(\psi_k(x) A_k x\right) \frac{1}{|A_k x|^\alpha}, \tag{4.55}$$

where

$$S^n \setminus F'_k = \left\{ x = (u, v) \in S^n \mid |M_k u| < s_{\tilde{l},k} \right\}.$$

From the computations in (4.46), (4.47) and (4.48), we have

$$I_k = \frac{1}{\det M_k} \int\limits_{|\tilde{v}|=1} d\sigma(\tilde{v}) \int\limits_{|\tilde{u}|<1} d\tilde{u}\, \varphi_k(x)\zeta\left(\psi_k(x) A_k x\right) \frac{\rho(u)^{n-l-1}}{\left|s^{-1}_{\tilde{l},k} A_k x\right|^\alpha}.$$

As in (4.49) we also have

$$
\begin{aligned}
|\det M_k \cdot I_k| &\le C \int\limits_{|\tilde{v}|=1} d\sigma(\tilde{v}) \int\limits_{|\tilde{u}|<1} \frac{d\tilde{u}}{(|\tilde{u}|^2 + |\tilde{v}_*|^2)^{\alpha/2}} \\
&= C\omega_{n-\tilde{l}} \int\limits_{|\tilde{v}_*|\le 1} \rho(\tilde{v}_*)^{n-\tilde{l}-1} d\tilde{v}_* \int\limits_{|\tilde{u}|<1} \frac{d\tilde{u}}{(|\tilde{u}|^2 + |\tilde{v}_*|^2)^{\alpha/2}} \\
&< +\infty,
\end{aligned}
$$

where the last inequality holds because $\alpha = l < \tilde{l}$. Thus we obtain

$$
I_k = \frac{O(1)}{\det M_k}. \tag{4.56}
$$

Combining (4.54), (4.55) and (4.56), we have, as $k \to +\infty$,

$$
\Lambda_k = \left(C\zeta_\infty \int\limits_{|\tilde{v}|=1} \varphi(0, \tilde{v}) d\sigma(\tilde{v}) + o(1) \right) \frac{\log(s_{l,k}/s_{\tilde{l},k})}{\det M_k},
$$

which is just (4.7).

Case (v): $\alpha < l_m$. For convenience, we denote $l := l_m$, and use the notations in (4.10) and (4.9). We have

$$
s_{l,k}^{\alpha} \Lambda_k = \int\limits_{S^n} \varphi_k(x) \zeta\left(\psi_k(x) A_k x\right) \frac{1}{\left| s_{l,k}^{-1} A_k x \right|^{\alpha}}. \tag{4.57}
$$

By our assumptions, we can estimate

$$
\begin{aligned}
\left| s_{l,k}^{\alpha} \Lambda_k \right| &\le C \int\limits_{S^n} \frac{1}{\left| s_{l,k}^{-1} M_k u \right|^{\alpha}} \\
&\le C \int\limits_{S^n} \frac{1}{|u|^{\alpha}} \\
&= C\omega_{n-l} \int\limits_{|u|\le 1} \frac{1}{|u|^{\alpha}} \rho(u)^{n-l-1} du \\
&< +\infty,
\end{aligned}
$$

where the last inequality holds because $\alpha < l$. Applying the dominated convergence theorem to (4.57), we obtain

$$
\lim_k s_{l,k}^{\alpha} \Lambda_k = \zeta_\infty \int\limits_{S^n} \varphi(x) \frac{1}{|\tilde{A}x|^{\alpha}},
$$

where $\tilde{A} := \lim_k s_{l,k}^{-1} A_k$. Hence (4.8) holds. We have completed the proof. □

In the rest of the paper, we will use ζ exclusively to denote

$$\zeta(y) := [f(y) - f(\infty)] \cdot |y|^\alpha \quad \forall\, y \in \mathbb{R}^{n+1}. \tag{4.58}$$

Under the assumptions on f in Theorem 1.2, ζ satisfies the conditions in Lemma 4.1, and $\zeta_\infty = \beta$. For a matrix B of order $n+1$, denote

$$\varphi_B(x) := \mathrm{tr}B - (n+1)x^T Bx \quad \forall\, x \in S^n. \tag{4.59}$$

Lemma 4.2. (1) *Let f be as in Theorem 1.2. Let $l \in \{1, 2, \cdots, n\}$ be an integer smaller than α, and N be a diagonal matrix of order $n+1-l$. If $B = \mathrm{diag}(1, 0, \cdots, 0)$, the integral*

$$\int\limits_{S^{n-l}} \varphi_B(0, v) d\sigma \int\limits_{u\in\mathbb{R}^l} \frac{\zeta\,(u, Nv)}{|(u, Nv)|^\alpha} du$$

is positive when $f > f(\infty)$, and negative when $f < f(\infty)$.

(2) *For $l \in \{1, 2, \cdots, n\}$ and $B = \mathrm{diag}(1, 0, \cdots, 0)$, we have*

$$\int\limits_{S^{n-l}} \varphi_B(0, v) d\sigma(v) > 0.$$

(3) *Let l and $\tilde{l}$ be integers satisfying $1 \le l < \alpha < \tilde{l} \le n$, N be a diagonal matrix of order $n+1-l$ whose first $\tilde{l} - l$ diagonal entries are positive and the others are equal to 0. If $B = \mathrm{diag}(0, \cdots, 0, 1)$, then*

$$\int\limits_{S^{n-l}} \frac{\varphi_B(0, v)}{|Nv|^{\alpha - l}} d\sigma(v) < 0.$$

(4) *Let l be an integer such that $\alpha < l \le n$, $\tilde{A}$ be a diagonal matrix of order $n+1$ whose first l diagonal entries are positive and the others are 0. If $B = \mathrm{diag}(0, \cdots, 0, 1)$, then*

$$\int\limits_{S^n} \frac{\varphi_B(x)}{|\tilde{A}x|^\alpha} < 0.$$

Proof. (1) In this case, we have $\varphi_B(x) = 1 - (n+1)x_1^2$. Hence $\varphi_B(0, v) = 1$ for $v \in S^{n-l}$. So we have

$$\int\limits_{S^{n-l}} \varphi_B(0, v) d\sigma(v) \int\limits_{u\in\mathbb{R}^l} \frac{\zeta\,(u, Nv)}{|(u, Nv)|^\alpha} du = \int\limits_{S^{n-l}} d\sigma(v) \int\limits_{u\in\mathbb{R}^l} [f(u, Nv) - f(\infty)] du,$$

which is positive when $f > f(\infty)$, and negative when $f < f(\infty)$.

(2) As in (1), we have $\varphi_B(0,v)=1$ for $v\in S^{n-l}$. Hence $\int_{S^{n-l}}\varphi_B(0,v)d\sigma>0$.

(3) Denote $\gamma=\alpha-l$, $v=(\mu,\tau)$ where $\mu=\left(\mu_1,\cdots,\mu_{\tilde{l}-l}\right)$, $\tau=\left(\tau_1,\cdots,\tau_{n+1-\tilde{l}}\right)$. Correspondingly we write the matrix N in the form $N=\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}$. Then

$$\varphi_B(0,v)=1-(n+1)\tau^2_{n+1-\tilde{l}}.$$

By the coarea formula, we have

$$\begin{aligned}\int\limits_{S^{n-l}}\frac{\varphi_B(0,v)}{|Nv|^{\alpha-l}}d\sigma&=\int\limits_{S^{n-l}}\frac{1-(n+1)\tau^2_{n+1-\tilde{l}}}{|E\mu|^{\gamma}}d\sigma\\&=\int\limits_{|\mu|\le 1}\frac{d\mu}{\rho(\mu)}\int\limits_{|\tau|=\rho(\mu)}\frac{1-(n+1)\tau^2_{n+1-\tilde{l}}}{|E\mu|^{\gamma}}d\sigma(\tau)\\&=\int\limits_{|\mu|\le 1}d\mu\cdot\frac{\omega_{n-\tilde{l}}\,\rho(\mu)^{n-\tilde{l}-1}}{|E\mu|^{\gamma}}\left(1-\frac{n+1}{n+1-\tilde{l}}\rho(\mu)^2\right)\\&=\int\limits_0^1\omega_{n-\tilde{l}}\,\rho(r)^{n-\tilde{l}-1}\left(1-\frac{n+1}{n+1-\tilde{l}}\rho(r)^2\right)dr\int\limits_{|\mu|=r}\frac{d\sigma(\mu)}{|E\mu|^{\gamma}}.\end{aligned}$$

But

$$\int\limits_{|\mu|=r}\frac{d\sigma(\mu)}{|E\mu|^{\gamma}}=r^{\tilde{l}-\alpha-1}\int\limits_{|\mu|=1}\frac{d\sigma(\mu)}{|E\mu|^{\gamma}}=:r^{\tilde{l}-\alpha-1}C_{E,\gamma}.$$

Hence

$$\begin{aligned}\int\limits_{S^{n-l}}\frac{\varphi_B(0,v)}{|Nv|^{\alpha-l}}d\sigma(v)&=C_{E,\gamma}\omega_{n-\tilde{l}}\int\limits_0^1 r^{\tilde{l}-\alpha-1}\rho(r)^{n-\tilde{l}-1}\left(1-\frac{n+1}{n+1-\tilde{l}}\rho(r)^2\right)dr\\&=-C_{E,\gamma}\omega_{n-\tilde{l}}\cdot\frac{\alpha}{4}\cdot\frac{\Gamma\left(\frac{\tilde{l}-\alpha}{2}\right)\Gamma\left(\frac{n-\tilde{l}+1}{2}\right)}{\Gamma\left(\frac{n-\alpha+3}{2}\right)}\\&<0.\end{aligned}$$

(4) Denote $x=(\mu,\tau)$, where $\mu=(\mu_1,\cdots,\mu_l)$, $\tau=(\tau_1,\cdots,\tau_{n+1-l})$, and correspondingly write $\tilde{A}=\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}$. Then

$$\varphi_B(x)=1-(n+1)\tau^2_{n+1-l}.$$

As in (3), we have

$$\int_{S^n} \frac{\varphi_B(x)}{|\tilde{A}x|^\alpha} = \int_{S^n} \frac{1-(n+1)\tau_{n+1-l}^2}{|E\mu|^\alpha} d\sigma(x)$$
$$= -C_{E,\alpha}\omega_{n-l} \cdot \frac{\alpha}{4} \cdot \frac{\Gamma\left(\frac{l-\alpha}{2}\right)\Gamma\left(\frac{n-l+1}{2}\right)}{\Gamma\left(\frac{n-\alpha+3}{2}\right)}$$
$$< 0. \quad \square$$

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. Since f is positive and bounded, by the volume estimate [20, Theorem 1.1], we have

$$C_1 \inf_{\mathbb{R}^{n+1}} f^{1/2} \le \mathrm{vol}(H) \le C_2 \sup_{\mathbb{R}^{n+1}} f^{1/2}.$$

Hence it suffices to prove that there is a uniform positive lower bound for solutions to (1.1).

Suppose on the contrary that there is a sequence of solutions H_k to (1.1) with $\min_{x\in S^n} H_k(x) \to 0^+$ as $k \to +\infty$. Let M_k be the associated convex body. Then there exists a unique matrix $A_k \in SL(n+1)$ such that $A_k^T(M_k)$ is normalized [29]. Let H_{A_k} be the support function of $A_k^T(M_k)$. Then H_{A_k} is given by

$$H_{A_k}(x) = |A_k x| \cdot H_k\left(\frac{A_k x}{|A_k x|}\right) \quad x \in S^n, \tag{4.60}$$

and H_{A_k} satisfies equation (2.6). It is known that $H_{A_k} \ge c_0$ for some positive constant c_0 (Corollary 2.4 in [20]). Hence we have $\max_{x\in S^n} |A_k x| \to +\infty$, which implies that

$$|A_k x| \to +\infty \quad \text{for a.e. } x \in S^n.$$

To see this, by a rotation of the coordinates we assume that $A_k = \mathrm{diag}(a_{1,k}, \cdots, a_{n+1,k})$ with $a_{1,k} \le \cdots \le a_{n+1,k}$. Then $a_{n+1,k} \to \infty$. Hence $|A_k x| \ge |a_{n+1,k}x_{n+1}| \to \infty$ for any $x \notin \{x_{n+1} = 0\} \cap S^n$.

By Blaschke's selection theorem, we may assume that H_{A_k} converges uniformly to some support function H_∞ on S^n, which is also normalized. Note that the right hand side of equation (2.6) with A replaced by A_k converges (weakly as measure) to $f(\infty)/H_\infty^{n+2}$. By the weak convergence of the Monge–Ampère equation, H_∞ is a generalized solution to [29]

$$\det(\nabla^2 H + HI) = \frac{f(\infty)}{H^{n+2}} \quad x \in S^n.$$

Hence H_∞ is an elliptic affine sphere [8]. But recall that H_∞ is normalized, hence H_∞ is a constant. Namely,

$$H_\infty \equiv f(\infty)^{\frac{1}{2n+2}}.$$

On the other hand, applying the necessary condition (2.7) to equation (2.6), we have that

$$\int_{S^n} \frac{\nabla_\xi [f(A_k x/H_{A_k})](x)}{H_{A_k}^{n+1}} = 0,$$

which implies by integration by parts that

$$\int_{S^n} f(A_k x/H_{A_k}) \operatorname{div}\left(\frac{1}{H_{A_k}^{n+1}}\xi\right) = 0.$$

By the definition of ζ in (4.58), the above equation can be written as

$$\Lambda_k := \int_{S^n} \zeta\left(\frac{A_k x}{H_{A_k}}\right) \frac{H_{A_k}^\alpha}{|A_k x|^\alpha} \operatorname{div}\left(\frac{1}{H_{A_k}^{n+1}}\xi\right) = 0. \tag{4.61}$$

We now use Lemma 4.1 to estimate the quantity Λ_k. Denote

$$\begin{aligned} \varphi_k &= H_{A_k}^\alpha \operatorname{div}\left(\frac{1}{H_{A_k}^{n+1}}\xi\right), \\ \psi_k &= \frac{1}{H_{A_k}}. \end{aligned} \tag{4.62}$$

Recall H_{A_k} converges uniformly to H_∞ on S^n, we have

$$\psi_k \to \psi := \frac{1}{H_\infty} \tag{4.63}$$

uniformly as $k \to \infty$. Note that

$$\varphi_k = H_{A_k}^{\alpha-n-1} \operatorname{div}\xi - (n+1) H_{A_k}^{\alpha-n-2} \xi \cdot \nabla H_{A_k}.$$

Recall that for any sequence of bounded convex functions h_k, if it converges to a constant, then $Dh_k \to 0$ locally uniformly. Hence $\nabla H_{A_k} \to 0$ uniformly on the sphere S^n. It follows that

$$\varphi_k \to \varphi := H_\infty^{\alpha-n-1} \operatorname{div}\xi = H_\infty^{\alpha-n-1} \varphi_B \tag{4.64}$$

uniformly, where φ_B is defined in (4.59), and the last equality is due to the definition of ξ in (2.8). Note that (4.61) holds for any matrix B of order $n+1$.

By a rotation of the coordinates, we assume that A_k is diagonal and is given by (4.1). Define the integers $l_1, \cdots, l_m$ as in (4.2) accordingly. So the quantity Λ_k defined in (4.61) coincides with that in Lemma 4.1. From the above we see that all assumptions in Lemma 4.1 are satisfied. There are five cases in Lemma 4.1. We consider case by case in the following.

Case (i): $\alpha > l_1$. By (4.4) in Lemma 4.1,

$$\Lambda_k = \frac{C}{s_{1,k} \cdot \cdots \cdot s_{l_1,k}} \left(\int_{S^{n-l_1}} \varphi_B(0,v) d\sigma(v) \int_{u \in \mathbb{R}^{l_1}} \frac{\zeta(u, Nv)}{|(u, Nv)|^\alpha} du + o(1) \right), \tag{4.65}$$

where N is as in (4.4). By Lemma 4.2 (1), $\Lambda_k \neq 0$ (for sufficiently large k) if we choose $B = \mathrm{diag}(1, 0, \cdots, 0)$, which is in contradiction with (4.61).

Case (ii): $\alpha = l_1$. By (4.5) in Lemma 4.1,

$$\Lambda_k = \frac{\log s_{l_1,k}}{s_{1,k} \cdot \cdots \cdot s_{l_1,k}} \left(C\zeta_\infty \int_{S^{n-l_1}} \varphi_B(0,v) d\sigma(v) + o(1) \right). \tag{4.66}$$

Hence by Lemma 4.2 (2), $\Lambda_k \neq 0$ when k is sufficiently large, provided we choose $B = \mathrm{diag}(1, 0, \cdots, 0)$, again in contradiction with (4.61).

Case (iii): $l_i < \alpha < l_{i-1}$ for some $i = 2, \cdots, m$. By (4.6) we have

$$\Lambda_k = \frac{1}{s_{1,k} \cdot \cdots \cdot s_{l_i,k} \cdot s_{l_{i-1},k}^{\alpha - l_i}} \left(C\zeta_\infty \int_{S^{n-l_i}} \frac{\varphi_B(0,v)}{|Nv|^{\alpha - l_i}} d\sigma(v) + o(1) \right), \tag{4.67}$$

where N is as in (4.6). By Lemma 4.2 (3), $\Lambda_k \neq 0$ for large k if we choose $B = \mathrm{diag}(0, \cdots, 0, 1)$, again in contradiction with (4.61).

Case (iv): $\alpha = l_i$ for some $i = 2, \cdots, m$. By (4.7) we have

$$\Lambda_k = \frac{\log(s_{l_i,k}/s_{l_{i-1},k})}{s_{1,k} \cdot \cdots \cdot s_{l_i,k}} \left(C\zeta_\infty \int_{S^{n-l_i}} \varphi_B(0,v) d\sigma(v) + o(1) \right). \tag{4.68}$$

By Lemma 4.2 (2), $\Lambda_k \neq 0$ for large k if we choose $B = \mathrm{diag}(1, 0, \cdots, 0)$, but $\Lambda_k = 0$ in (4.61), a contradiction.

Case (v): $\alpha < l_m$. By (4.8) in Lemma 4.1,

$$\Lambda_k = \frac{1}{s_{l_m,k}^\alpha} \left(C\zeta_\infty \int_{S^n} \frac{\varphi_B(x)}{|\tilde{A}x|^\alpha} + o(1) \right). \tag{4.69}$$

By Lemma 4.2 (4), $\Lambda_k \neq 0$ for large k if we choose $B = \mathrm{diag}(0, \cdots, 0, 1)$, but $\Lambda_k = 0$ in (4.61), a contradiction.

We have reached a contradiction in all possible cases. This completes the proof. □

5. Existence of solutions in the critical case

In this section we prove Theorem 1.3. Denote $\delta = n + 2 - q$. For any given constant $v > 0$, by Theorem 1.1, there exists a constant λ_δ (also depending on v) and a support function H_δ with $vol(H_\delta) = v$, such that

$$\det(\nabla^2 H_\delta + H_\delta I)(x) = \frac{\lambda_\delta f(x/H_\delta)}{H_\delta^{n+2-\delta}} \quad x \in S^n. \tag{5.1}$$

We want to prove that as $\delta \to 0^+$, H_δ converges to a solution to (1.9). Note that we always use the same notation to denote a sequence and its subsequences.

Lemma 5.1. *There exists a positive constant C depending only on n, v and f, independent of δ, such that*

$$C^{-1} \le \lambda_\delta \le C. \tag{5.2}$$

Proof. Multiplying equation (5.1) by H_δ and taking integration, we obtain, by the volume formula (1.3),

$$v = \frac{\lambda_\delta}{n+1} \int_{S^n} \frac{f(x/H_\delta)}{H_\delta^{q-1}}.$$

Hence there is a positive constant C independent of δ, such that

$$C^{-1} \int_{S^n} \frac{1}{H_\delta^{q-1}} \le \lambda_\delta^{-1} \le C \int_{S^n} \frac{1}{H_\delta^{q-1}}. \tag{5.3}$$

Noting that H_δ is a maximizer of (3.3), by (3.5) we have

$$C^{-1} \le J[H_\delta] \le C, \tag{5.4}$$

where C is a positive constant independent of δ. By virtue of (3.4), there is a positive constant C depending only on n and f, such that

$$C^{-1} \int_{S^n} \frac{1}{H_\delta^{q-1}} \le J[H_\delta] \le C \int_{S^n} \frac{1}{H_\delta^{q-1}}. \tag{5.5}$$

Now combining (5.3), (5.4) with (5.5), we obtain (5.2). □

Let $A_\delta \in SL(n+1)$ be the matrix such that

$$H_{A_\delta}(x) := |A_\delta x| \cdot H_\delta\left(\frac{A_\delta x}{|A_\delta x|}\right) \quad x \in S^n \tag{5.6}$$

is normalized, see (4.60). Let s_δ be the largest eigenvalue of A_δ. If s_δ is uniformly bounded, so is H_δ. From (5.1), we see that the limit support function $H_0 := \lim_{\delta\to 0} H_\delta$ is a solution to equation (1.9) and so Theorem 1.3 is proved. Therefore it suffices to prove that s_δ is uniformly bounded.

By (2.6), H_{A_δ} satisfies the equation

$$\det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) = \frac{\lambda_\delta f(A_\delta x/H_{A_\delta})}{H^q_{A_\delta}} \cdot \frac{1}{|A_\delta x|^\delta} \quad x \in S^n, \tag{5.7}$$

or equivalently

$$\det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) = \frac{\lambda_\delta f(A_\delta x/H_{A_\delta})(\hat{H}_\delta)^\delta}{H^{n+2}_{A_\delta}} \quad x \in S^n, \tag{5.8}$$

where $\hat{H}_\delta(x) = H_\delta\left(\frac{A_\delta x}{|A_\delta x|}\right)$. To prove that s_δ is uniformly bounded, first we prove

Lemma 5.2. *There exists a positive constant C depending only on n, v and f, independent of $\delta \in (0, \frac{1}{2})$, such that*

$$s_\delta^\delta \le C. \tag{5.9}$$

Proof. By equation (5.7) and estimate (5.2), we have

$$\int_{S^n} H^q_{A_\delta} \det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) \le C \int_{S^n} \frac{1}{|A_\delta x|^\delta}.$$

By Hölder's inequality,

$$\begin{aligned} v &= \frac{1}{n+1} \int_{S^n} H_{A_\delta} \det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) \\ &\le \frac{1}{n+1} \left(\int_{S^n} H^q_{A_\delta} \det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) \right)^{\frac{1}{q}} \left(\int_{S^n} \det(\nabla^2 H_{A_\delta} + H_{A_\delta} I) \right)^{\frac{q-1}{q}} \\ &\le C \left(\int_{S^n} \frac{1}{|A_\delta x|^\delta} \right)^{\frac{1}{q}} \operatorname{area}(H_{A_\delta})^{\frac{q-1}{q}} \\ &\le C \left(\int_{S^n} \frac{1}{|A_\delta x|^\delta} \right)^{\frac{1}{q}} v^{\frac{n}{n+1} \cdot \frac{q-1}{q}}, \end{aligned}$$

where area(H_{A_δ}) is the surface area of the convex body determined by H_{A_δ}, and the last inequality is because that H_{A_δ} is normalized. Thus there is a positive constant C, independent of δ, such that

$$C \leq \int_{S^n} \frac{1}{|A_\delta x|^\delta}. \tag{5.10}$$

But by direct computation one easily verifies that

$$\int_{S^n} \frac{1}{|A_\delta x|^\delta} \leq \frac{C_n}{s_\delta^\delta} \quad \forall\, \delta \in (0, 1/2].$$

In this way, we have proved (5.9). □

Estimate (5.9) implies that $\sup_{x \in S^n} |A_\delta x|^\delta$ is uniformly bounded. Hence

$$|A_\delta x|^\delta \to C_1 \quad \text{a.e. } x \in S^n. \tag{5.11}$$

By Blaschke's selection theorem, we can assume that H_{A_δ} converges uniformly on S^n to some support function H_{A_0}.

Lemma 5.3. *Under the assumptions in Theorem 1.3, s_δ is uniformly bounded as $\delta \to 0^+$.*

Proof. Assume to the contrary that $s_\delta \to \infty$. Then

$$|A_\delta x| \to +\infty \quad \text{a.e. } x \in S^n.$$

Sending $\delta \to 0^+$, by Lemma 5.1 and the weak convergence of the Monge–Ampère measure, we obtain from (5.7) the following equation

$$\det(\nabla^2 H_{A_0} + H_{A_0} I) = \frac{\lambda_0 f(\infty)/C_1}{H_{A_0}^{n+2}} \quad \text{on } S^n.$$

Namely H_{A_0} is a generalised solution to the above equation. By the regularity theory of the Monge–Ampère equation [29], H_{A_0} is smooth. As H_{A_0} is normalized, we see that H_{A_0} is a constant [8], namely

$$H_{A_0} \equiv (\lambda_0 f(\infty)/C_1)^{\frac{1}{2n+2}}. \tag{5.12}$$

Next we apply the necessary condition (2.7) to equation (5.8), to get

$$\int_{S^n} \frac{\nabla_\xi \big[f(Ax/H_A)(\hat{H}_\delta)^\delta \big](x)}{H_A^{n+1}} = 0.$$

Here and below we omit the subscript δ in A_δ for brevity. Integration by parts gives

$$
\begin{aligned}
0 &= \int_{S^n} \frac{\nabla_\xi \big[(f(Ax/H_A)-f(\infty))(\hat{H}_\delta)^\delta\big](x)}{H_A^{n+1}} + \int_{S^n} \frac{\nabla_\xi \big[f(\infty)(\hat{H}_\delta)^\delta\big](x)}{H_A^{n+1}} \\
&= -\int_{S^n} (f(Ax/H_A)-f(\infty))(\hat{H}_\delta)^\delta \operatorname{div}\left(\frac{\xi}{H_A^{n+1}}\right) + f(\infty)\int_{S^n} \frac{\nabla_\xi (\hat{H}_\delta)^\delta(x)}{H_A^{n+1}} \qquad (5.13) \\
&=: -\Lambda_\delta + f(\infty) I_\delta.
\end{aligned}
$$

To prove the lemma, we will show that (5.13) does not hold for sufficiently small δ.

There is no loss of generality in assuming that A_δ is diagonal, namely

$$
A_\delta = \operatorname{diag}(s_{1,\delta}, \cdots, s_{n+1,\delta})
$$

with $s_{1,\delta} \geq \cdots \geq s_{n+1,\delta} > 0$. Then $s_\delta = s_{1,\delta}$. Define the integers $l_1, \cdots, l_m$ as in (4.2).

We first compute the quantity I_δ in (5.13). Observing that

$$
\begin{aligned}
\nabla_\xi (\hat{H}_\delta)^\delta &= \delta(\hat{H}_\delta)^{\delta-1}\nabla_\xi \hat{H}_\delta \\
&= \delta(\hat{H}_\delta)^{\delta-1}\nabla_\xi\left(\frac{H_A}{|Ax|}\right) \\
&= \delta(\hat{H}_\delta)^{\delta}\left(\frac{\nabla_\xi H_A}{H_A} - \frac{1}{|Ax|^2}x^T A^T A\xi\right),
\end{aligned}
$$

we get

$$
I_\delta = \delta \int_{S^n} \frac{(\hat{H}_\delta)^\delta}{H_A^{n+1}}\left(\frac{\nabla_\xi H_A}{H_A} - \frac{1}{|Ax|^2}x^T A^T A\xi\right).
$$

Recall that

$$
\xi(x) = Bx - (x^T Bx)x \quad x \in S^n.
$$

The above equation can be expressed as

$$
I_\delta = \delta \int_{S^n} \frac{(\hat{H}_\delta)^\delta}{H_A^{n+1}}\left(\frac{\nabla_\xi H_A}{H_A} - \frac{1}{|Ax|^2}x^T A^T ABx + x^T Bx\right).
$$

Hence as $\delta \to 0^+$,

$$
I_\delta = \delta\left(C\frac{\operatorname{tr} B}{n+1}\omega_n - \int_{S^n} \frac{(\hat{H}_\delta)^\delta}{H_A^{n+1}} \cdot \frac{x^T A^T ABx}{|Ax|^2} + o(1)\right), \qquad (5.14)
$$

where $C = 1/\sqrt{C_1\lambda_0 f(\infty)}$ by (5.12).

Next we compute Λ_δ. Recall ζ is given by

$$\zeta(y) = [f(y) - f(\infty)] \cdot |y|^\alpha.$$

Inserting it into Λ_δ in (5.13), we have

$$\Lambda_\delta = \int_{S^n} \zeta\left(\frac{Ax}{H_A}\right) \frac{H_A^\alpha \cdot (\hat{H}_\delta)^\delta}{|Ax|^\alpha} \operatorname{div}\left(\frac{\xi}{H_A^{n+1}}\right). \tag{5.15}$$

By (5.9), $(\hat{H}_\delta)^\delta$ is uniformly bounded. Hence in view of (5.11) and (4.62)–(4.64), we can still apply Lemma 4.1 to (5.15). According to Lemma 4.1, there are five possible cases. We consider case by case in the following.

Case (i): $\alpha > l_1$. By (4.4) in Lemma 4.1, we have

$$\Lambda_\delta = \frac{C}{s_{1,\delta} \cdot \cdots \cdot s_{l_1,\delta}} \left(\int_{S^{n-l_1}} \varphi_B(0,v) d\sigma(v) \int_{u \in \mathbb{R}^{l_1}} \frac{\zeta\,(u, Nv)}{|(u, Nv)|^\alpha} du + o(1) \right).$$

Let $B = \operatorname{diag}(1, 0, \cdots, 0)$. By Lemma 4.2 (1) and recalling the assumption (1.8), we see that $\Lambda_\delta > 0$ for small $\delta > 0$. On the other hand, we can simplify (5.14) as follows.

$$\begin{aligned}
I_\delta &= \delta \left(C \frac{1}{n+1} \omega_n - \int_{S^n} \frac{(\hat{H}_\delta)^\delta}{H_A^{n+1}} \cdot \frac{s_{1,\delta}^2 x_1^2}{s_{1,\delta}^2 x_1^2 + \cdots + s_{n+1,\delta}^2 x_{n+1}^2} + o(1) \right) \\
&= C\delta \left(\frac{1}{n+1} \omega_n - \int_{S^n} \frac{x_1^2}{s_{1,\delta}^2 x_1^2 / s_{1,\delta}^2 + \cdots + s_{l_1,\delta}^2 x_{l_1}^2 / s_{1,\delta}^2} + o(1) \right) \\
&\le C\delta \left(\frac{1}{n+1} \omega_n - \int_{S^n} \frac{{x_1}^2}{{x_1}^2 + \cdots + {x_n}^2} + o(1) \right) \\
&= C\delta \left(\frac{1}{n+1} \omega_n - \frac{1}{n} \omega_n + o(1) \right) \\
&< 0,
\end{aligned} \tag{5.16}$$

for sufficiently small $\delta > 0$. Therefore equality (5.13) can not hold for small δ.

Case (ii): $\alpha = l_1$. By (4.5) in Lemma 4.1,

$$\Lambda_\delta = \frac{\log s_{l_1,\delta}}{s_{1,\delta} \cdot \cdots \cdot s_{l_1,\delta}} \left(C\zeta_\infty \int_{S^{n-l_1}} \varphi_B(0,v) d\sigma(v) + o(1) \right).$$

Let $B = \operatorname{diag}(1, 0, \cdots, 0)$. By Lemma 4.2 (2), the integral in Λ_δ is positive. By the assumption $\beta > 0$ in Theorem 1.3, and since $\zeta_\infty = \beta$, we see that $\Lambda_\delta > 0$ for small

$\delta > 0$. In the current case, I_δ is the same as in (5.16). Therefore (5.13) can not hold for sufficiently small δ.

Case (iii): $l_i < \alpha < l_{i-1}$ for some $i = 2, \cdots, m$. By (4.6) in Lemma 4.1 we have

$$\Lambda_\delta = \frac{1}{s_{1,\delta} \cdot \cdots \cdot s_{l_i,\delta} \cdot s_{l_{i-1},\delta}^{\alpha - l_i}} \left(C\zeta_\infty \int_{S^{n-l_i}} \frac{\varphi_B(0,v)}{|Nv|^{\alpha - l_i}} d\sigma(v) + o(1) \right).$$

Now we choose $B = \mathrm{diag}\,(0, \cdots, 0, 1)$. By Lemma 4.2 (3), we see that $\Lambda_\delta < 0$ for sufficiently small δ. On the other hand, with $B = \mathrm{diag}\,(0, \cdots, 0, 1)$, from (5.14) it is easy to see that

$$\begin{aligned} I_\delta &= \delta \left(C\frac{1}{n+1}\omega_n - \int_{S^n} \frac{(\hat{H}_\delta)^\delta}{H_A^{n+1}} \cdot \frac{{s_{n+1,\delta}}^2 {x_{n+1}}^2}{{s_{1,\delta}}^2 {x_1}^2 + \cdots + {s_{n+1,\delta}}^2 {x_{n+1}}^2} + o(1) \right) \\ &= \delta \left(C\frac{1}{n+1}\omega_n + o(1) \right) > 0, \end{aligned} \tag{5.17}$$

for small δ. Therefore equality (5.13) can not hold for small $\delta > 0$.

Case (iv): $\alpha = l_i$ for some $i = 2, \cdots, m$. Applying Lemma 4.1 to (5.15), we have

$$\Lambda_\delta = \frac{\log(s_{l_i,\delta}/s_{l_{i-1},\delta})}{s_{1,\delta} \cdot \cdots \cdot s_{l_i,\delta}} \left(C\zeta_\infty \int_{S^{n-l_i}} \varphi_B(0,v) d\sigma(v) + o(1) \right).$$

Similarly as *Case (ii)*, one sees that (5.13) can not hold for sufficiently small δ.

Case (v): $\alpha < l_m$. Applying Lemma 4.1 to (5.15), we have

$$\Lambda_\delta = \frac{1}{s_{l_m,\delta}^\alpha} \left(C\zeta_\infty \int_{S^n} \varphi_B(x) \frac{1}{|\tilde{A}x|^\alpha} + o(1) \right),$$

Choose $B = \mathrm{diag}(0, \cdots, 0, 1)$. By Lemma 4.2 (4), the integral in Λ_δ is negative. Recall $\zeta_\infty = \beta > 0$, we see $\Lambda_\delta < 0$ for sufficiently small δ. But $I_\delta > 0$ by (5.17), we see equality (5.13) can not hold for sufficiently small δ.

We have now proved that, under the assumptions in Theorem 1.3, the necessary condition (5.13) does not hold in all the possible cases. This contradiction implies that s_δ is uniformly bounded when $\delta \to 0^+$. $\square$

Once H_δ is uniformly bounded, by convexity it sub-converges to a convex function H_0. By the weak convergence of the Monge–Ampère equation, H_0 is a generalized solution to (1.9). The regularity theory for the Monge–Ampère equation implies that $H_0 \in C^{2,\gamma}(S^n)$ for any $\gamma \in (0,1)$ [29].

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