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Nonexistence of maximizers for the functional of the centroaffine Minkowski problem

Jian Lu

Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023*, China*
*Email: lj-tshu*04*@*163*.com*

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Abstract The centroaffine Minkowski problem is studied, which is the critical case of the L_p-Minkowski problem. It admits a variational structure that plays an important role in studying the existence of solutions. In this paper, we find that there is generally no maximizer of the corresponding functional for the centroaffine Minkowski problem.

Keywords centroaffine Minkowski problem, Monge-Ampère equation, variational structure, Blaschke-Santaló inequality

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1 Introduction

The centroaffine Minkowski problem [8] is to find necessary and sufficient conditions about a given positive function $\tilde{f}$, such that $\tilde{f}$ is the centroaffine curvature of a convex body containing the origin in $\mathbb{R}^{n+1}$. In the smooth case, the centroaffine Minkowski problem is equivalent to solving the following Monge-Ampère type equation:

$$\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on} \quad S^n, \tag{1.1}$$

where f is the given positive function (in fact $f = 1/\tilde{f}$), H is the support function of a bounded convex body X in $\mathbb{R}^{n+1}$, I is the unit matrix, and $\nabla^2 H = (\nabla_{ij} H)$ is the Hessian matrix of covariant derivatives of H with respect to an orthonormal frame on S^n.

Equation (1.1) is a special case of the L_p-Minkowski problem introduced by Lutwak [21], which has attracted great attention (see for example [4, 10–12, 20, 22, 25, 30] and the references therein). Equation (1.1) has applications in anisotropic Gauss curvature flows (see [9, 27]), and image processing (see [2]). It can be reduced to a singular Monge-Ampère equation in the half Euclidean space $\mathbb{R}^{n+1}_+$, and its regularity was strongly studied in [14, 15].

Equation (1.1) has a natural variational structure, which involves a type of Mahler volume for convex bodies. Consider the following maximizing problem:

$$\sup_X \inf_{\xi \in X} \mathrm{vol}(X) \cdot \frac{1}{n+1} \int_{S^n} \frac{f(x) dS(x)}{(H(x) - \xi \cdot x)^{n+1}}, \tag{1.2}$$

where X is any bounded convex body in $\mathbb{R}^{n+1}$, ξ is any point in X, $\mathrm{vol}(X)$ is the volume of the convex body X, and H is the support function of X. Note that for any X, there exists a unique $\xi \in X$ such that the infimum is attained. By the arguments of [8], any maximizer of this problem provides a solution to (1.1) after rescaling by a constant. When $f \equiv 1$, (1.2) is exactly the Mahler volume for X. For the upper bound of Mahler volume, it is given by the famous Blaschke-Santaló inequality in convex geometry (see [23]):

$$\sup_X \inf_{\xi \in X} \mathrm{vol}(X) \cdot \frac{1}{n+1} \int_{S^n} \frac{dS(x)}{(H(x) - \xi \cdot x)^{n+1}} = \omega_{n+1}^2, \tag{1.3}$$

where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$, and the supremum is attained at any ellipsoid. So the maximizing problem (1.2) is obviously bounded from above, and its maximizers may be not uniformly bounded. This means that there is no a priori estimates for maximizers to (1.2). Actually, this feature originates from (1.1) itself, which remains invariant under projective transforms on S^n (see [8, 20]). When f is a constant function, all ellipsoids centered at the origin are the solutions to (1.1) (see [5]). Equation (1.1) is similar, in some aspects, to the prescribed scalar curvature problem on S^n (see [6, 24]), but more complicated due to the lack of a Liouville type theorem after blow-up.

For $n = 1$, (1.1) is reduced to a semi-linear ordinary differential equation. The existence of solutions was investigated in [1, 3, 7, 9, 10, 16, 17, 26]. For the higher n-dimension, there are few existence results about (1.1) except several special cases; see [19, 20] for the rotationally symmetric case, [13] for the mirror symmetric case, and [30] for the discrete case.

The variational method is the major way to obtain the existence of solutions to the L_p-Minkowski problem. For subcritical cases $p > -n-1$, see [4, 8, 28, 29]. For the critical case $p = -n-1$, namely (1.1), almost all existence results mentioned above for general dimension n were obtained by the variational method corresponding to (1.2). In this paper, we find an interesting fact: the variational method cannot be used to obtain a solution to (1.1) in the general case.

We say f is even, if $f(-x) = f(x)$ for all $x \in S^n$.

Theorem 1.1. *If f is a continuous and even function on S^n, then*

$$\sup_X \inf_{\xi \in X} \mathrm{vol}(X) \cdot \frac{1}{n+1} \int_{S^n} \frac{f(x)dS(x)}{(H(x) - \xi \cdot x)^{n+1}} = f_{\max}\omega_{n+1}^2, \tag{1.4}$$

where $f_{\max} = \sup_{x \in S^n} f(x)$. When f is non-constant, the supremum cannot be attained.

Recall that when $f \equiv 1$, (1.4) is just the Blaschke-Santaló inequality (1.3), and the supremum is attained at any ellipsoid. Although the variational method was used to obtain a solution to (1.1) for some special symmetric cases and the discrete case, it unfortunately cannot be applied to (1.1) when f is a general even function without additional restrictions by Theorem 1.1.

This paper consists of two sections. We prove Theorem 1.1 in the next section.

2 Nonexistence of a maximizer

In this section, we prove Theorem 1.1. First, note that the Mahler type volume

$$\inf_{\xi \in X} \mathrm{vol}(X) \cdot \frac{1}{n+1} \int_{S^n} \frac{f(x)dS(x)}{(H(x) - \xi \cdot x)^{n+1}}$$

is invariant under any dilation of the convex body X. So (1.4) is equivalent to

$$\sup_{|X| = \omega_{n+1}} \inf_{\xi \in X} \frac{1}{n+1} \int_{S^n} \frac{f(x)dS(x)}{(H(x) - \xi \cdot x)^{n+1}} = f_{\max}\omega_{n+1}. \tag{2.1}$$

Here, $|X|$ denotes the volume of X, namely $\mathrm{vol}(X)$. For any support function H, let

$$J[H] = \frac{1}{n+1} \int_{S^n} \frac{f}{H^{n+1}}. \tag{2.2}$$

Now, Theorem 1.1 is equivalent to the following.

Theorem 2.1. *If f is a continuous and even function on S^n, then*

$$\sup_{|X|=\omega_{n+1}} \inf_{\xi\in X} J[H(x)-\xi\cdot x] = f_{\max}\omega_{n+1}. \tag{2.3}$$

When f is non-constant, the supremum cannot be attained.

The proof of Theorem 2.1 is based on the invariance of the functional J under unimodular linear transforms. For any convex body X in $\mathbb{R}^{n+1}$, after performing a unimodular linear transform $A^{\mathrm{T}} \in SL(n+1)$, it becomes into another convex body X_A, namely $X_A = A^{\mathrm{T}}(X)$. In the following, we use H_A to denote the support function of X_A. Then

$$H_A(x) = |Ax|\cdot H\left(\frac{Ax}{|Ax|}\right), \quad x\in S^n, \tag{2.4}$$

where H is the support function of X (see [20] for more details about this type of transforms). The invariance of J is a direct corollary of [18, Lemma 5.1].

Lemma 2.2 (See [18, Lemma 5.1]). *For any integral function g on S^n, and any matrix $A\in GL(n+1)$, we have the following variable substitution for integration*:

$$\int_{S^n} g(y)\,dS(y) = \int_{S^n} g\left(\frac{Ax}{|Ax|}\right)\cdot\frac{|\det A|}{|Ax|^{n+1}}\,dS(x). \tag{2.5}$$

Proof. For completeness, here we provide the proof given in [18].

We first claim: for any homogeneous function φ in $\mathbb{R}^{n+1}$ of degree zero, there is

$$\int_{B^{n+1}} \varphi(y)\,dy = \frac{1}{n+1}\int_{S^n}\varphi(y)\,dS(y), \tag{2.6}$$

where B^{n+1} is the unit ball in $\mathbb{R}^{n+1}$. In fact, recall that

$$\int_{B^{n+1}} \varphi(y)\,dy = \int_0^1 dr\int_{S^n(r)}\varphi(y)\,dS(y),$$

where $S^n(r)$ is the n-sphere centered at the origin with radius r. Since φ is homogeneous of degree zero, the above equality becomes into

$$\begin{aligned}\int_{B^{n+1}} \varphi(y)\,dy &= \int_0^1 r^n dr\int_{S^n}\varphi(y)\,dS(y)\\ &= \frac{1}{n+1}\int_{S^n}\varphi(y)\,dS(y),\end{aligned}$$

which implies the claim.

Now we can apply the variable substitution for integration in $\mathbb{R}^{n+1}$ to prove (2.5). Extending g on S^n as a homogeneous function of degree zero in $\mathbb{R}^{n+1}$, noting (2.6) and using the variable substitution

$$y = \frac{|x|}{|Ax|}Ax, \quad x\in\mathbb{R}^{n+1},$$

we obtain

$$\begin{aligned}\int_{S^n} g(y)\,dS(y) &= (n+1)\int_{B^{n+1}} g(y)\,dy\\ &= (n+1)\int_{B^{n+1}} g\left(\frac{|x|}{|Ax|}Ax\right)\cdot\left|\det y'_x\right|\,dx.\end{aligned}$$

By the direct computations, one can see

$$|\det y'_x| = \frac{|\det A|\,|x|^{n+1}}{|Ax|^{n+1}}.$$

Therefore,

$$\int_{S^n} g(y)\,dS(y) = (n+1)\int_{B^{n+1}} g\left(\frac{|x|}{|Ax|}Ax\right)\cdot\frac{|\det A|\,|x|^{n+1}}{|Ax|^{n+1}}\,dx. \tag{2.7}$$

Noting the integrand on the right-hand side of (2.7) is homogeneous of degree zero, and again applying (2.6), we obtain

$$\int_{S^n} g(y)\,dS(y) = \int_{S^n} g\left(\frac{Ax}{|Ax|}\right)\cdot\frac{|\det A|}{|Ax|^{n+1}}\,dS(x).$$

This completes the proof of the lemma. □

By (2.5) of this lemma, one immediately obtains the invariance about J. Namely, for any integral function f on S^n, any support function H, and any matrix $A\in SL(n+1)$, we have

$$\int_{S^n}\frac{f}{H^{n+1}} = \int_{S^n}\frac{f_A}{H_A^{n+1}},\quad f_A(x) = f\left(\frac{Ax}{|Ax|}\right). \tag{2.8}$$

Let $\sigma_n := (n+1)\omega_{n+1}$ be the area of the unit n-sphere, and $J_{\sup}$ be the supremum on the left-hand side of (2.3).

Lemma 2.3. *If f is a continuous and even function on S^n, then the corresponding $J_{\sup}$ is equal to $f_{\max}\omega_{n+1}$.*

Proof. Let X be any convex body with volume ω_{n+1}, H be its support function, and $\xi\in X$. Since

$$J[H(x)-\xi\cdot x] \leqslant f_{\max}\cdot\frac{1}{n+1}\int_{S^n}\frac{dS(x)}{(H(x)-\xi\cdot x)^{n+1}},$$

we have

$$\begin{aligned}\inf_{\xi\in X} J[H(x)-\xi\cdot x] &\leqslant f_{\max}\cdot\inf_{\xi\in X}\frac{1}{n+1}\int_{S^n}\frac{dS(x)}{(H(x)-\xi\cdot x)^{n+1}}\\ &\leqslant f_{\max}\cdot\omega_{n+1},\end{aligned}$$

where the second inequality holds due to the Blaschke-Santaló inequality (1.3). Thus,

$$J_{\sup}\leqslant f_{\max}\omega_{n+1}. \tag{2.9}$$

On the other hand, for any matrix $A\in SL(n+1)$, let $H_A(x) = |Ax|$, $x\in S^n$ be the support function of the ellipsoid $E_A := A^{\mathrm{T}}(B^{n+1})$, where B^{n+1} is the unit ball in $\mathbb{R}^{n+1}$. Since f is even, we easily see that

$$\inf_{\xi\in E_A} J[H_A(x)-\xi\cdot x] = J[H_A].$$

Noting the volume of E_A is ω_{n+1}, by the definition of $J_{\sup}$, we have

$$\begin{aligned}J_{\sup} &\geqslant J[H_A]\\ &= \frac{1}{n+1}\int_{S^n}\frac{f}{H_A^{n+1}}\\ &= \frac{1}{n+1}\int_{S^n} f_{A^{-1}},\end{aligned}$$

where the last equality comes from (2.8), and $f_{A^{-1}}(x) = f(\frac{A^{-1}x}{|A^{-1}x|})$. Noting that $A\in SL(n+1)$ is arbitrary, we obtain

$$J_{\sup}\geqslant\sup_{A\in SL(n+1)}\frac{1}{n+1}\int_{S^n} f_{A^{-1}}. \tag{2.10}$$

We claim that the right-hand side of (2.10) equals $f_{\max}\omega_{n+1}$. In fact, we can assume $f_{\max}$ is attained at the north and south poles without loss of generality, namely $f(\pm e_{n+1}) = f_{\max}$, where $e_{n+1} = (0,\ldots,0,1)$. Let

$$A_k^{-1} = \mathrm{diag}(k^{-\frac{1}{n+1}},\ldots,k^{-\frac{1}{n+1}},k^{\frac{n}{n+1}})\in SL(n+1),\quad k>0.$$

Then for any $x=(x_1,\dots,x_{n+1})\in S^n$ with $x_{n+1}\neq 0$,

$$\begin{aligned}f_{A_k^{-1}}(x)&=f\bigg(\frac{k^{-1}x_1,\dots,k^{-1}x_n,x_{n+1}}{\sqrt{k^{-2}x_1^2+\dots+k^{-2}x_n^2+x_{n+1}^2}}\bigg)\\&\to f(\pm e_{n+1}),\quad \text{as}\quad k\to+\infty.\end{aligned}$$

Therefore, by the bounded convergence theorem we have

$$\begin{aligned}\sup_{A\in SL(n+1)}\frac{1}{n+1}\int_{S^n}f_{A^{-1}}&\geqslant\lim_{k\to+\infty}\frac{1}{n+1}\int_{S^n}f_{A_k^{-1}}\\&=\frac{1}{n+1}\int_{S^n}f(\pm e_{n+1})\\&=\frac{1}{n+1}f_{\max}\sigma_n\\&=f_{\max}\omega_{n+1}.\end{aligned}$$

It is obvious that

$$\sup_{A\in SL(n+1)}\frac{1}{n+1}\int_{S^n}f_{A^{-1}}\leqslant f_{\max}\omega_{n+1}.$$

Hence,

$$\sup_{A\in SL(n+1)}\frac{1}{n+1}\int_{S^n}f_{A^{-1}}=f_{\max}\omega_{n+1}.$$

Now, (2.10) becomes into

$$J_{\sup}\geqslant f_{\max}\omega_{n+1},$$

which together with (2.9) leads to the conclusion of this lemma. □

Now, we prove Theorem 2.1.

Proof of Theorem 2.1. Because of Lemma 2.3, it only needs to prove that the supremum cannot be attained when f is non-constant. We prove it by contradiction. Assume $\hat{H}$ is a maximizer to the left-hand side of (2.3), i.e.,

$$J[\hat{H}]=\inf_{\xi}J[\hat{H}(x)-\xi\cdot x]=J_{\sup}.$$

By virtue of the Blaschke-Santaló inequality (1.3), let $\tilde{\xi}$ be the point such that

$$\frac{1}{n+1}\int_{S^n}\frac{dS(x)}{(\hat{H}(x)-\tilde{\xi}\cdot x)^{n+1}}\leqslant\omega_{n+1}.$$

Noting f is continuous and non-constant, we have

$$\begin{aligned}J[\hat{H}]&\leqslant J[\hat{H}(x)-\tilde{\xi}\cdot x]\\&=\frac{1}{n+1}\int_{S^n}\frac{f(x)dS(x)}{(\hat{H}(x)-\tilde{\xi}\cdot x)^{n+1}}\\&<f_{\max}\cdot\frac{1}{n+1}\int_{S^n}\frac{dS(x)}{(\hat{H}(x)-\tilde{\xi}\cdot x)^{n+1}}\\&\leqslant f_{\max}\omega_{n+1},\end{aligned}$$

i.e.,

$$J_{\sup}<f_{\max}\omega_{n+1},$$

which contradicts Lemma 2.3. This completes the proof of Theorem 2.1. □

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