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Existence of solutions to the Orlicz–Minkowski problem ☆

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ABSTRACT

In this paper the Orlicz–Minkowski problem, a generalization of the classical Minkowski problem, is studied. Using the variational method, we obtain a new existence result of solutions to this problem for general measures.

1. Introduction

In recent years, the Orlicz–Brunn–Minkowski theory in convex geometry has been built up gradually and is developing rapidly. It can be viewed as the recent develop-

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ment of the classical Brunn–Minkowski theory, and has attracted great attention from many scholars, see for example [4,7–9,11,13,14,16,17,21,22,27,29,30,32–35,37–40,44] and references therein. In the Brunn–Minkowski theory, it is well known that the classical Minkowski problem is of central importance, and has many applications. In the new Orlicz–Brunn–Minkowski theory, the corresponding Minkowski problem is called the *Orlicz–Minkowski problem*.

Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a given continuous function. For a convex body $K \subset \mathbb{R}^n$ with the origin $0 \in K$, the Orlicz surface area measure is defined as $\varphi(h_K)dS_K$. Here h_K is the support function of K, and S_K is the surface area measure. The Orlicz–Minkowski problem, first proposed in [10], asks what are the necessary and sufficient conditions for a Borel measure μ on the unit sphere S^{n-1} to be a multiple of the Orlicz surface area measure of a convex body K. Namely, this problem is to find a convex body $K \subset \mathbb{R}^n$ such that

$$c\,\varphi(h_K)dS_K = d\mu \text{ on } S^{n-1} \tag{1.1}$$

for some positive constant c. Since the Orlicz surface area measure depends on φ, it is also called L_φ-surface area measure. Correspondingly, the Orlicz–Minkowski problem is sometimes called the L_φ-Minkowski problem.

When φ is a constant function, Eq. (1.1) is just the classical Minkowski problem. When $\varphi(s) = s^{1-p}$, Eq. (1.1) reduces to the L_p-Minkowski problem, which has been extensively studied, see e.g. [1–3,12,15,17–20,23–26,28,31,41,43] and Schneider's book [36], and corresponding references therein.

When the Radon–Nikodym derivative of μ with respect to the spherical measure on S^{n-1} exists, namely $d\mu = f dx$ for a non-negative integrable function f, the equation (1.1) can be written as

$$c\,\varphi(h_K)\det(\nabla^2 h_K + h_K I) = f \text{ on } S^{n-1}, \tag{1.2}$$

where $\nabla^2 h_K = (\nabla_{ij} h_K)$ is the Hessian matrix of covariant derivatives of h_K with respect to an orthonormal frame on S^{n-1}, and I is the unit matrix of order $n - 1$. This is a Monge–Ampère type equation.

Eq. (1.1) has a variational structure, which can be used to prove the existence of solutions [10,14,37]. Haberl, Lutwak, Yang and Zhang [10] considered the even Orlicz–Minkowski problem under the assumption

(A) $\varphi : (0, +\infty) \to (0, +\infty)$ *is a continuous function such that* $\phi(t) = \int_0^t 1/\varphi(s)ds$ *exists for every* $t > 0$ *and is unbounded as* $t \to +\infty$.

They proved the following

Theorem 1.1 *([10, Theorem 2]). Suppose* **(A)** *is satisfied. If* μ *is an even finite Borel measure on* S^{n-1} *which is not concentrated on any great sub-sphere of* S^{n-1}*, then there* *exists an origin symmetric convex body K in $\mathbb{R}^n$ and a number $c > 0$ satisfying (1.1). Moreover, one can require that the Orlicz-norm of h_K is equal to 1.*

We note that the Orlicz-norm of h_K in this theorem is defined with respect to $\phi(t) = \int_0^t 1/\varphi(s)ds$ and μ. Denoting it by $\|h_K\|_{\phi,\mu}$, we have that

$$\|h_K\|_{\phi,\mu} = \inf\left\{\lambda > 0 : \frac{1}{\mu(S^{n-1})}\int\limits_{S^{n-1}} \phi\Big(\frac{h_K}{\lambda}\Big)d\mu \leq \phi(1)\right\}.$$

One can consult [10, Section 4] for more properties about this norm. We will use this notation throughout this paper.

Huang and He [14] studied the general (not necessarily even) Orlicz–Minkowski problem and obtained the following result.

Theorem 1.2 *([14, Theorem 1.2]). In addition to* **(A)**, *further suppose that $\varphi(s)$ tends to $+\infty$ as $s \to 0^+$. If μ is a finite Borel measure on S^{n-1} which is not concentrated in any closed hemisphere of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ and a number $c > 0$ satisfying (1.1). Moreover, one can require that the Orlicz-norm $\|h_K\|_{\phi,\mu}$ is equal to 1.*

We note that Theorem 1.1 includes the even L_p-Minkowski problem for $p > 0$, and Theorem 1.2 includes the general L_p-Minkowski problem for $p > 1$. There is no result about the general Orlicz–Minkowski problem which can include the general L_p-Minkowski problem for $0 < p < 1$. In this paper, we will fill this gap. We obtain the following

Theorem 1.3. *In addition to* **(A)**, *further suppose that φ is non-decreasing and $\varphi(s)$ tends to 0 as $s \to 0^+$. If μ is a finite Borel measure on S^{n-1} which is not concentrated in any closed hemisphere of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ and a number $c > 0$ satisfying (1.1). Moreover, one can require that the volume of h_K is equal to 1.*

One can see that $\varphi(s) = s^{1-p}$ with $0 < p < 1$ satisfies the assumptions of Theorem 1.3. Therefore this theorem includes the general L_p-Minkowski problem for $0 < p < 1$. Actually there was even no existence result about the general L_p-Minkowski problem for $0 < p < 1$ when our paper was completed, while now there is one existence result about this L_p-Minkowski problem [5].

The method of proving Theorem 1.3 is the variational method, which was used to study the Orlicz–Minkowski problem in [10,14] and the L_p-Minkowski problem in [6,42]. However our method is not a direct generalization of these previous methods, since [10] is for the origin symmetric case, while in [6,14,42] extremum problems were considered in a class of support functions of convex bodies, which additionally need to analyze related properties of extremum convex bodies when computing variations, see Lemmas 5.5–5.6

in [6] or Lemmas 3.5–3.6 in [42]. To overcome these difficulties, in this paper we use a new technique combining the functionals given in [6] and [10], and making use of Alexandrov bodies to compute an extremum problem in the class of positive continuous functions on S^{n-1}. No additional properties of extremum convex bodies will be needed in our method.

The paper is organized as follows. In section 2, we provide some preliminaries about convex bodies. In section 3, we state Theorem 3.1 which is more general than Theorem 1.3 and prove a special discrete case of the theorem. Then we complete the proof of the theorem in section 4.

2. Preliminaries

In this section we state some notations and basic facts about convex bodies which will be used throughout this paper. For general references about convex bodies, one can consult [36].

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. For a convex body K, we use $\operatorname{int} K$ to denote the interior of K. The support function of a convex body, denoted by h_K, is given by

$$h_K(x) := \max_{\xi\in K} \xi \cdot x, \quad x \in S^{n-1},$$

where "$\cdot$" denotes the inner product in the Euclidean space $\mathbb{R}^n$. It is well known that a convex body is uniquely determined by its support function, and the convergence of a sequence of convex bodies is equivalent to the uniform convergence of the corresponding support functions on S^{n-1}. The Blaschke selection theorem says that every bounded sequence of convex bodies has a subsequence that converges to a convex body.

Denote the set of positive continuous functions on S^{n-1} by $C^+(S^{n-1})$. For $g \in C^+(S^{n-1})$ and a closed subset $\omega \subset S^{n-1}$ not lying in any closed hemisphere, define the *Alexandrov body associated with* (g, ω) as

$$K := \bigcap_{x\in\omega} \{\xi \in \mathbb{R}^n : \xi \cdot x \le g(x)\}.$$

One can see that K is a bounded convex body and $0 \in K$. Note that

$$h_K(x) \le g(x), \quad x \in \omega.$$

We write $\operatorname{vol}(g, \omega)$ for the volume of the Alexandrov body associated with (g, ω). For the concept of Alexandrov body, there is a useful variational formula due to Alexandrov, see e.g. [36, Lemma 7.5.3].

Lemma 2.1. *Let $\epsilon > 0$. Assume $G_t(x) = G(t, x) : (-\epsilon, \epsilon) \times \omega \to (0, +\infty)$ is continuous. If there is a continuous function g on ω such that*

$$\lim_{t\to 0^+}\frac{G_t - G_0}{t} = g \qquad \textit{uniformly on } \omega,$$

then

$$\lim_{t\to 0^+}\frac{\mathrm{vol}(G_t,\omega)-\mathrm{vol}(G_0,\omega)}{t} = \int_\omega g(x)dS_K(x),$$

where K is the Alexandrov body associated with (G_0,ω). The same assertion holds if the one-sided limit $\lim_{t\to 0^+}$ is replaced by $\lim_{t\to 0^-}$ or by $\lim_{t\to 0}$.

For a finite Borel measure μ on S^{n-1}, denote its support set by $\mathrm{supp}(\mu)$, and its total mass $\mu(S^{n-1})$ by $|\mu|$.

3. A special discrete case

In this paper, instead of proving Theorem 1.3 directly, we will prove the following

Theorem 3.1. *Suppose $\phi : (0,+\infty) \to (0,+\infty)$ is an increasing concave C^1 function satisfying that $\lim_{t\to+\infty}\phi(t) = +\infty$, $\phi'(t) > 0$ and $\lim_{t\to 0^+}\phi'(t) = +\infty$. If μ is a finite Borel measure on S^{n-1} which is not concentrated in any closed hemisphere of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ and a number $c > 0$ satisfying*

$$\frac{c}{\phi'(h_K)}dS_K = d\mu \ \textit{on}\ S^{n-1}. \tag{3.1}$$

Moreover, one can require the volume of h_K is equal to any given number $v > 0$.

One can easily prove Theorem 1.3 by virtue of Theorem 3.1.

Proof of Theorem 1.3. Given φ as in Theorem 1.3, we define ϕ as

$$\phi(t) = \int_0^t 1/\varphi(s)ds, \quad \forall t > 0.$$

By assumption **(A)**, ϕ is an increasing C^1 function in $(0,+\infty)$ satisfying

$$\lim_{t\to+\infty}\phi(t) = +\infty.$$

Note $\phi' = 1/\varphi > 0$ is non-increasing, then ϕ is a concave function. Also

$$\lim_{t\to 0^+}\phi'(t) = \lim_{t\to 0^+}\frac{1}{\varphi(t)} = +\infty.$$

So ϕ satisfies all assumptions of Theorem 3.1. By this theorem, there exists a convex body K in $\mathbb{R}^n$ and a number $c > 0$ satisfying

$$\frac{c}{\phi'(h_K)} dS_K = d\mu \text{ on } S^{n-1},$$

namely

$$c\,\varphi(h_K) dS_K = d\mu \text{ on } S^{n-1},$$

which is just equation (1.1). Theorem 3.1 also says that the volume of h_K can be equal to any given positive number v. We choose $v = 1$. Now Theorem 1.3 is proved to be true. □

From now on, we only focus on Theorem 3.1.

In this section, we mainly prove the following lemma, which is a special discrete case of Theorem 3.1.

Lemma 3.2. *Suppose $\phi : (0, +\infty) \to (0, +\infty)$ is an increasing concave C^2 function satisfying that $\lim_{t\to+\infty} \phi(t) = +\infty$, $\phi'(t) > 0$, $\lim_{t\to 0^+} \phi'(t) = +\infty$, and $\phi''(t) < 0$. If μ is a finite discrete measure on S^{n-1} which is not concentrated in any closed hemisphere of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ containing the origin in its interior, and a number $c > 0$ satisfying Eq. (3.1). Moreover, one can require the volume of h_K is equal to any given number $v > 0$.*

We use a variational method to prove this Lemma. For any fixed positive constant v, we consider the following minimizing problem:

$$\inf \left\{ \sup_{\xi \in K_g} J[g(x) - \xi \cdot x] : g \in C^+(S^{n-1}),\ \mathrm{vol}(K_g) = v \right\}, \tag{3.2}$$

where K_g is the Alexandrov body associated with $(g, \mathrm{supp}(\mu))$, and

$$J[g] = \int_{S^{n-1}} \phi(g(x)) d\mu(x) = \int_{\mathrm{supp}(\mu)} \phi(g(x)) d\mu(x). \tag{3.3}$$

Note when $\xi \in K_g$ and $x \in \mathrm{supp}(\mu)$, there is

$$g(x) - \xi \cdot x \geq h_{K_g}(x) - \xi \cdot x \geq 0.$$

By the assumptions of ϕ, we can define $\phi(0)$ as $\lim_{t\to 0^+} \phi(t)$ which exists and is finite. Note $\phi(0) \geq 0$. Therefore $J[g(x) - \xi \cdot x]$ in (3.2) is well-defined. The proof of Lemma 3.2 will be carried out in the following Lemmas 3.3–3.6 and finished after Lemma 3.6.

Lemma 3.3. *Assume that* $\phi : (0, +\infty) \to (0, +\infty)$ *is an increasing concave* C^1 *function satisfying* $\lim_{t\to 0^+} \phi'(t) = +\infty$, *and that* μ *is a finite discrete measure on* S^{n-1} *which is not concentrated in any closed hemisphere of* S^{n-1}. *Then for every non-negative continuous function* g *on* S^{n-1} *with* K_g *having nonempty interior, there is at least one point of* K_g, *denoted by* ξ_g, *such that*

$$J[g(x) - \xi_g \cdot x] = \sup_{\xi \in K_g} J[g(x) - \xi \cdot x]. \tag{3.4}$$

And for any such point, we have $\xi_g \in \operatorname{int} K_g$. *If* ϕ *is additionally strictly concave, then* ξ_g *is unique, and depends continuously on* g *when* $g \in C^+(S^{n-1})$.

Proof. Define $G : K_g \to \mathbb{R}$ as

$$G(\xi) := J[g(x) - \xi \cdot x] = \int\limits_{S^{n-1}} \phi(g(x) - \xi \cdot x) d\mu(x).$$

We claim that G is concave with respect to ξ. In fact, for λ_1, $\lambda_2 \in (0, 1)$ with $\lambda_1 + \lambda_2 = 1$, and ξ_1, $\xi_2 \in K_g$, we have

$$\begin{aligned} G(\lambda_1\xi_1 + \lambda_2\xi_2) &= \int\limits_{S^{n-1}} \phi[g(x) - (\lambda_1\xi_1 + \lambda_2\xi_2) \cdot x] d\mu(x) \\ &= \int\limits_{S^{n-1}} \phi[\lambda_1(g(x) - \xi_1 \cdot x) + \lambda_2(g(x) - \xi_2 \cdot x)] d\mu(x) \\ &\geq \int\limits_{S^{n-1}} [\lambda_1\phi(g(x) - \xi_1 \cdot x) + \lambda_2\phi(g(x) - \xi_2 \cdot x)] d\mu(x) \\ &= \lambda_1 G(\xi_1) + \lambda_2 G(\xi_2). \end{aligned}$$

Here we have used the concavity of ϕ. If ϕ is additionally strictly concave, when the above equality holds, there must be

$$g(x) - \xi_1 \cdot x = g(x) - \xi_2 \cdot x, \quad \forall\, x \in \operatorname{supp}(\mu),$$

namely

$$(\xi_1 - \xi_2) \cdot x = 0, \quad \forall\, x \in \operatorname{supp}(\mu).$$

Recall μ is not concentrated on any closed hemisphere, then $\operatorname{supp}(\mu)$ spans the whole space $\mathbb{R}^n$. Thus $\xi_1 = \xi_2$, which implies G is strictly concave on K_g.

Note G is continuous on the convex body K_g, there exists at least one point $\xi_g \in K_g$ such that

$$G(\xi_g) = \sup_{\xi \in K_g} G(\xi),$$

which is just (3.4). We need to prove $\xi_g \in \operatorname{int} K_g$. Otherwise, suppose $\xi_g \in \partial K_g$. We will prove that for some $e \in S^{n-1}$ and small $\lambda > 0$, $\xi_g + \lambda e \in \operatorname{int} K_g$ and $G(\xi_g + \lambda e) > G(\xi_g)$, which leads to a contradiction.

Recall the definition of K_g:

$$K_g = \bigcap_{x \in \operatorname{supp}(\mu)} \left\{ \xi \in \mathbb{R}^n : \xi \cdot x \leq g(x) \right\},$$

there must exist one $x \in \operatorname{supp}(\mu)$ such that

$$\xi_g \cdot x = g(x),$$

since otherwise $\xi_g \cdot x + \delta < g(x)$ for some $\delta > 0$ and every $x \in \operatorname{supp}(\mu)$, which would imply $\xi_g \in \operatorname{int} K_g$.

Now we write $\operatorname{supp}(\mu)$ as the union of two disjoint nonempty sets:

$$\operatorname{supp}(\mu) = A \cup B, \tag{3.5}$$

where

$$A := \left\{ x \in \operatorname{supp}(\mu) : \xi_g \cdot x = g(x) \right\},$$
$$B := \left\{ x \in \operatorname{supp}(\mu) : \xi_g \cdot x < g(x) \right\}.$$

By virtue of the assumption that K_g has nonempty interior, one can find a unit vector $e \in S^{n-1}$ such that

$$e \cdot x < 0 \text{ for every } x \in A. \tag{3.6}$$

Since $\operatorname{supp}(\mu)$ is a discrete set by the assumption, B is a closed subset of S^{n-1}. Then there exists a $\lambda_0 > 0$ satisfying

$$\xi_g \cdot x + 2\lambda_0 < g(x), \quad \forall x \in B.$$

Thus for any $0 < \lambda < 2\lambda_0$, we have

$$(\xi_g + \lambda e) \cdot x < g(x), \quad \forall x \in \operatorname{supp}(\mu).$$

By the definition of K_g,

$$\xi(\lambda) := \xi_g + \lambda e \in \operatorname{int} K_g.$$

We want to prove

$$G(\xi(\lambda)) > G(\xi_g)$$

for sufficiently small λ, which is a contradiction.

Recalling (3.5) and $\phi(0)$ is well defined, we have

$$\begin{aligned} G(\xi(\lambda)) - G(\xi_g) &= \int_{A\cup B} \phi[g(x) - \xi(\lambda)\cdot x]d\mu(x) - \int_{A\cup B} \phi[g(x) - \xi_g\cdot x]d\mu(x) \\ &= \int_A \big(\phi[g(x) - \xi(\lambda)\cdot x] - \phi(0)\big)d\mu(x) \\ &\quad + \int_B \big(\phi[g(x) - \xi(\lambda)\cdot x] - \phi[g(x) - \xi_g\cdot x]\big)d\mu(x). \end{aligned} \tag{3.7}$$

Note that

$$g(x) - \xi(\lambda)\cdot x = -\lambda e\cdot x, \quad \forall x \in A.$$

And we can strengthen (3.6) as

$$e\cdot x < -\delta_0 < 0 \quad \forall x \in A$$

for some constant $\delta_0 > 0$. Then the first integral in the end of (3.7)

$$\begin{aligned} \int_A \big(\phi[g(x) - \xi(\lambda)\cdot x] - \phi(0)\big)d\mu(x) &= \int_A \big(\phi(-\lambda e\cdot x) - \phi(0)\big)d\mu(x) \\ &\geq \int_A \big(\phi(\delta_0\lambda) - \phi(0)\big)d\mu(x) \\ &= [\phi(\delta_0\lambda) - \phi(0)]\mu(A). \end{aligned} \tag{3.8}$$

Here we have used that ϕ is increasing. To estimate the last integral in (3.7), we note that when $0 < \lambda < \lambda_0$ and $x \in B$, there is

$$\begin{aligned} g(x) - \xi(\lambda)\cdot x &= g(x) - \xi_g\cdot x - \lambda e\cdot x \\ &> 2\lambda_0 - \lambda \\ &> \lambda_0. \end{aligned}$$

Recalling ϕ is concave, we have

$$|\phi[g(x) - \xi(\lambda)\cdot x] - \phi[g(x) - \xi_g\cdot x]| \leq \phi'(\lambda_0)| - \lambda e\cdot x| \leq \lambda\phi'(\lambda_0),$$

which implies that

$$\int_B |\phi[g(x) - \xi(\lambda) \cdot x] - \phi[g(x) - \xi_g \cdot x]| d\mu(x) \leq \lambda \phi'(\lambda_0) \mu(B). \tag{3.9}$$

By (3.8) and (3.9), when $0 < \lambda < \lambda_0$ we simplify (3.7) as

$$G(\xi(\lambda)) - G(\xi_g) \geq [\phi(\delta_0 \lambda) - \phi(0)]\mu(A) - \lambda \phi'(\lambda_0)\mu(B).$$

By the assumptions of ϕ,

$$\lim_{t \to 0^+} \frac{\phi(t) - \phi(0)}{t} = \lim_{t \to 0^+} \phi'(t) = +\infty.$$

Hence, we can choose positive numbers δ and

$$M > \frac{\phi'(\lambda_0)\mu(B)}{\delta_0 \mu(A)},$$

such that

$$\phi(t) - \phi(0) > Mt, \quad \forall 0 < t < \delta.$$

Now for $0 < \lambda < \min\{\lambda_0, \delta/\delta_0\}$, we have

$$G(\xi(\lambda)) - G(\xi_g) \geq [M\delta_0 \mu(A) - \phi'(\lambda_0)\mu(B)]\lambda > 0,$$

which is impossible. So ξ_g can not be on ∂K_g, namely $\xi_g \in \operatorname{int} K_g$.

If ϕ is additionally strictly concave, then G is also strictly concave on K_g. So ξ_g must be unique. Let $g \in C^+(S^{n-1})$, and $\{g_k\} \subset C^+(S^{n-1})$ be any sequence of functions uniformly converging to g on S^{n-1}. We want to prove that ξ_{g_k} converges to ξ_g in $\mathbb{R}^n$. Note that $K_{g_k} \to K_g$ and $\xi_{g_k} \in K_{g_k}$, therefore $\{\xi_{g_k}\}$ is bounded. For any convergent subsequence $\left\{\xi_{g_{k_i}}\right\} \subset \{\xi_{g_k}\}$, we need to prove its limit, say ξ_0, equals ξ_g.

Observe that for any $\xi \in K_g$, there exists a sequence of $\xi_{k_i} \in K_{g_{k_i}}$ which converges to ξ. Then

$$\begin{aligned} G(\xi) &= J[g(x) - \xi \cdot x] \\ &= \lim_{k_i} J[g_{k_i}(x) - \xi_{k_i} \cdot x] \\ &\leq \lim_{k_i} J[g_{k_i}(x) - \xi_{g_{k_i}} \cdot x] \\ &= J[g(x) - \xi_0 \cdot x] \\ &= G(\xi_0), \end{aligned}$$

which implies that

$$G(\xi_0) = \sup_{\xi \in K_g} G(\xi).$$

By the uniqueness of ξ_g, we have

$$\xi_0 = \xi_g.$$

The proof of this lemma is completed. □

Lemma 3.4. *Under the assumptions of Lemma 3.2, the minimizing problem* (3.2) *has a solution h.*

Proof. Let m be the infimum of (3.2), namely

$$m = \inf \left\{ \sup_{\xi \in K_g} J[g(x) - \xi \cdot x] : g \in C^+(S^{n-1}),\ \mathrm{vol}(K_g) = v \right\}.$$

By $\phi \geq 0$, we see $m \geq 0$.

Let $\{g_k\} \subset C^+(S^{n-1})$, $\mathrm{vol}(K_{g_k}) = v$ be a minimizing sequence. Denote the support function of K_{g_k} by h_k. Then

$$h_k(x) \leq g_k(x), \quad \forall x \in \mathrm{supp}(\mu),$$

and $K_{h_k} = K_{g_k}$. Since $0 \in \mathrm{int}\, K_{g_k}$, h_k is positive on S^{n-1}. For any $\xi \in K_{h_k} = K_{g_k}$, by the monotonicity of ϕ,

$$\begin{aligned} J[h_k(x) - \xi \cdot x] &= \int\limits_{\mathrm{supp}(\mu)} \phi(h_k(x) - \xi \cdot x) d\mu(x) \\ &\leq \int\limits_{\mathrm{supp}(\mu)} \phi(g_k(x) - \xi \cdot x) d\mu(x) \\ &= J[g_k(x) - \xi \cdot x], \end{aligned}$$

which implies

$$\sup_{\xi \in K_{h_k}} J[h_k(x) - \xi \cdot x] \leq \sup_{\xi \in K_{g_k}} J[g_k(x) - \xi \cdot x].$$

Therefore

$$\lim_{k \to +\infty} \sup_{\xi \in K_{h_k}} J[h_k(x) - \xi \cdot x] = m.$$

Namely $\{h_k\}$ is also a minimizing sequence of (3.2). Recalling Lemma 3.3, we have $\xi_{h_k} \in \operatorname{int} K_{h_k}$ such that

$$J[h_k(x) - \xi_{h_k} \cdot x] = \sup_{\xi \in K_{h_k}} J[h_k(x) - \xi \cdot x].$$

Note h_k is also the support function of K_{h_k}, by a translation transform we can always assume $\xi_{h_k} = 0$. This fact will be used a few times.

We claim that $\{h_k\}$ is uniformly bounded on S^{n-1}. If not, we can assume

$$\lim_{k \to +\infty} \max_{x \in S^{n-1}} h_k(x) = +\infty.$$

Write $R_k = \max_{x \in S^{n-1}} h_k(x)$. For each k, there exists $x_k \in S^{n-1}$ such that $h_k(x_k) = R_k$. Since $\{x_k\} \subset S^{n-1}$, there exists a convergent subsequence. Without loss of generality, we assume

$$x_k \to x_0 \in S^{n-1} \text{ when } k \to +\infty.$$

Recall $\operatorname{supp}(\mu)$ is not concentrated on any closed hemisphere, there is some $\bar{x} \in \operatorname{supp}(\mu)$ such that $\bar{x} \cdot x_0 > 0$. Write $\delta = \frac{1}{2}\bar{x} \cdot x_0$, then $\delta > 0$ and for sufficiently large k, e.g. $k \geq k_0$, we have

$$\bar{x} \cdot x_k > \delta.$$

By the definition of support function, there is

$$h_k(\bar{x}) \geq R_k(\bar{x} \cdot x_k) > R_k\delta, \quad k \geq k_0.$$

Note ϕ is increasing, $\lim_{t \to +\infty} \phi(t) = +\infty$ and μ is a finite discrete measure, we have

$$\begin{aligned} m &= \lim_{k \to +\infty} J[h_k] \\ &= \lim_{k \to +\infty} \int_{S^{n-1}} \phi(h_k(x)) d\mu(x) \\ &\geq \lim_{k \to +\infty} \phi(h_k(\bar{x}))\mu(\bar{x}) \\ &\geq \lim_{k \to +\infty} \phi(R_k\delta)\mu(\bar{x}) \to +\infty. \end{aligned}$$

However, by Lemma 3.3, m must be finite. This is a contradiction. Thus $\{h_k\}$ is uniformly bounded.

By the Blaschke selection theorem, there is a subsequence of $\{h_k\}$ which uniformly converges to some support function h on S^{n-1}. Correspondingly K_{h_k} converges to K_h which is the convex body determined by h. Obviously $h \geq 0$ on S^{n-1}, $\operatorname{vol}(K_h) = v$, and

$$J[h] = m.$$

For any $\xi \in K_h$, there exists $\xi_k \in K_{h_k}$ such that $\xi_k \to \xi$ as $k \to +\infty$. Then

$$\begin{aligned} J[h(x) - \xi \cdot x] &= \lim_{k\to+\infty} J[h_k(x) - \xi_k \cdot x] \\ &\le \lim_{k\to+\infty} J[h_k(x) - \xi_{h_k} \cdot x] \\ &= \lim_{k\to+\infty} J[h_k(x)] \\ &= J[h(x)], \end{aligned}$$

which implies that

$$J[h] = \sup_{\xi\in K_h} J[h(x) - \xi \cdot x].$$

By Lemma 3.3, $0 \in \operatorname{int} K_h$. Therefore $h > 0$ on S^{n-1}. Hence, we see that h is a solution to the minimizing problem (3.2), and h is the support function of K_h. □

In the following we prove that the solution h obtained in Lemma 3.4 is also a solution to (3.1) for some $c > 0$.

For any given $\eta \in C(S^{n-1})$, let

$$q_t = h + t\eta \quad \text{for } t \ge 0.$$

By $h \in C^+(S^{n-1})$, $q_t \in C^+(S^{n-1})$ for sufficiently small t. By Lemma 2.1, we have

Lemma 3.5.

$$\lim_{t\to0^+} \frac{\operatorname{vol}(K_{q_t}) - \operatorname{vol}(K_h)}{t} = \int_{\operatorname{supp}(\mu)} \eta dS_{K_h}(x).$$

Let $g_t(x) = \beta(t)q_t(x)$ where

$$\beta(t) = \operatorname{vol}(K_{q_t})^{-1/n} v^{1/n}.$$

Then $g_t \in C^+(S^{n-1})$, and $\operatorname{vol}(K_{g_t}) = v$. Note $g_0(x) = h(x)$, and

$$\lim_{t\to0^+} \frac{g_t(x) - g_0(x)}{t} = \eta(x) + \beta'(0)h(x) \text{ uniformly on } S^{n-1}. \tag{3.10}$$

Also by Lemma 3.5,

$$\begin{aligned}\beta'(0) &= -\frac{v^{1/n}}{n}\operatorname{vol}(K_{q_t})^{-1/n-1}\frac{d\operatorname{vol}(K_{q_t})}{dt}\Big|_{t=0} \\ &= -\frac{1}{nv}\int_{S^{n-1}} \eta dS_{K_h}(x).\end{aligned} \tag{3.11}$$

For each g_t, $\xi(t) := \xi_{g_t} \in \mathbb{R}^n$ is well defined by Lemma 3.3.

Lemma 3.6. *$\xi(t)$ is Lipschitz continuous with respect to t.*

Proof. Since $\sup_{\xi\in K_{g_t}} J[g_t(x) - \xi \cdot x]$ is attained at $\xi = \xi(t)$, we have

$$\int_{S^{n-1}} \phi'(g_t(x) - \xi(t)\cdot x)x d\mu(x) = 0. \tag{3.12}$$

Recalling $\xi(0) = \xi_h = 0$, and taking $t = 0$ in the above equality, we have

$$\int_{S^{n-1}} \phi'(h(x))x d\mu(x) = 0. \tag{3.13}$$

Recalling that $\phi'' < 0$ in $(0, +\infty)$, and subtracting (3.13) from (3.12), we get

$$\int_{S^{n-1}} \phi''(\tau)[g_t(x) - \xi(t)\cdot x - h(x)]x d\mu(x) = 0,$$

where $\tau : S^{n-1} \times (0, +\infty) \to \mathbb{R}$ and $\tau(x, t)$ is between $g_t(x) - \xi(t) \cdot x$ and $h(x)$. Then

$$\int_{S^{n-1}} \phi''(\tau)[g_t(x) - h(x)]x d\mu(x) = \int_{S^{n-1}} \phi''(\tau)(\xi(t)\cdot x)x d\mu(x).$$

Taking the inner product with $\xi(t)$, we have

$$\int_{S^{n-1}} \phi''(\tau)[g_t(x) - h(x)](\xi(t)\cdot x) d\mu(x) = \int_{S^{n-1}} \phi''(\tau)(\xi(t)\cdot x)^2 d\mu(x). \tag{3.14}$$

Note that when t is small,

$$\begin{aligned}\sup_{x\in S^{n-1}} |g_t(x) - h(x)| &= \sup_{x\in S^{n-1}} |(\beta(t) - 1)h(x) + t\beta(t)\eta(x)| \\ &\le C[|\beta(t) - 1| + t\beta(t)] \\ &\le Ct\end{aligned}$$

for some positive constant C which is independent of x and t. Since $h > 0$, and $g_t(x) - \xi(t)\cdot x$ converges to $h(x)$ uniformly on S^{n-1} when $t \to 0^+$, we can assume $\frac{1}{2}\min_x h(x) <$

$\tau < 2\max_x h(x)$. Therefore there exist two positive constants C_1 and C_2 depending only on h and ϕ'' such that

$$-C_1 \le \phi''(\tau) \le -C_2.$$

Thus we can estimate (3.14) as

$$\begin{aligned} C_2 \int\limits_{S^{n-1}} (\xi(t)\cdot x)^2 d\mu(x) &\le \int\limits_{S^{n-1}} (-\phi''(\tau))[g_t(x) - h(x)](\xi(t)\cdot x) d\mu(x) \\ &\le C_1 C t \int\limits_{S^{n-1}} |\xi(t)\cdot x| d\mu(x) \qquad (3.15) \\ &\le C_1 C|\mu| \cdot |\xi(t)| t. \end{aligned}$$

Recall μ is not concentrated on any closed hemisphere, there exists $C_3 > 0$ depending only on μ, such that

$$\int\limits_{S^{n-1}} (y\cdot x)^2 d\mu(x) \ge C_3, \quad \forall y \in S^{n-1}.$$

Then

$$\int\limits_{S^{n-1}} (\xi(t)\cdot x)^2 d\mu(x) \ge C_3 |\xi(t)|^2.$$

Therefore, it follows from (3.15) that

$$|\xi(t)| \le \frac{C_1 C|\mu|}{C_2 C_3} t,$$

which is the desired result. □

Now, we are going to finish the proof of Lemma 3.2. Let

$$J(t) := J[g_t(x) - \xi(t)\cdot x].$$

Note $\xi(0) = 0$ and $J(0) = J[h]$. Since h is a minimizer of (3.2), we have

$$J(t) \ge J(0)$$

for all small $t \ge 0$. Thus

$$\lim_{t_k \to 0^+} \frac{J(t_k) - J(0)}{t_k} \ge 0 \qquad (3.16)$$

for any convergent subsequence $\{t_k\}$. By Lemma 3.6, we can assume without loss of generality that

$$\lim_{t_k\to 0^+}\frac{\xi(t_k)-\xi(0)}{t}=\gamma.$$

Recalling (3.10), we see that (3.16) is simplified as

$$\int_{S^{n-1}}\phi'(h)[\eta(x)+\beta'(0)h(x)-\gamma\cdot x]d\mu(x)\ge 0, \tag{3.17}$$

which, together with (3.13), implies

$$\int_{S^{n-1}}\phi'(h)[\eta(x)+\beta'(0)h(x)]d\mu(x)\ge 0.$$

By (3.11), we obtain

$$\int_{S^{n-1}}\phi'(h)\eta d\mu-c\int_{S^{n-1}}\eta dS_{K_h}\ge 0,$$

where

$$c=\frac{1}{nv}\int_{S^{n-1}}\phi'(h)hd\mu.$$

Replacing η by $-\eta$, we see that

$$\int_{S^{n-1}}\phi'(h)\eta d\mu-c\int_{S^{n-1}}\eta dS_{K_h}=0$$

for all $\eta\in C(S^{n-1})$. Thus

$$\phi'(h)d\mu-c\,dS_{K_h}=0,$$

namely

$$\frac{c}{\phi'(h)}dS_{K_h}=d\mu,$$

which means that h solves equation (3.1). Obviously $c>0$. The proof of Lemma 3.2 is completed.

4. The general case

In the previous section, we have proved Lemma 3.2. This lemma says that Theorem 3.1 holds under the additional assumptions: $\phi \in C^2(0, +\infty)$ with $\phi'' < 0$ and μ is a discrete measure on S^{n-1}. In this section, we will remove these additional assumptions to complete the proof of Theorem 3.1. We will use approximations to achieve this aim. First we remove the assumptions: $\phi \in C^2(0, +\infty)$ with $\phi'' < 0$. Namely we prove the following:

Lemma 4.1. *Suppose $\phi : (0, +\infty) \to (0, +\infty)$ is an increasing concave C^1 function satisfying that $\lim_{t\to+\infty} \phi(t) = +\infty$, $\phi'(t) > 0$ and $\lim_{t\to 0^+} \phi'(t) = +\infty$. If μ is a finite discrete measure on S^{n-1} which is not concentrated in any closed hemisphere of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ and a $c > 0$ satisfying*

$$\frac{c}{\phi'(h_K)} dS_K = d\mu \text{ on } S^{n-1}.$$

Moreover, one can require the volume of h_K is equal to any given number $v > 0$. And $h_K \in C^+(S^{n-1})$ is a minimizer of

$$\inf \left\{ \sup_{\xi \in K_g} J[g(x) - \xi \cdot x] : g \in C^+(S^{n-1}),\ \mathrm{vol}(K_g) = v \right\}. \tag{4.1}$$

Here K_g is the Alexandrov body associated with $(g, \mathrm{supp}(\mu))$, and J is given by (3.3).

Proof. Assume $\rho \in C^\infty(\mathbb{R})$ is a non-negative smooth function compactly supported in $[-1, 0]$, and

$$\int_{\mathbb{R}} \rho(t)dt = 1.$$

Let $\rho_\epsilon(t) = \epsilon^{-1}\rho(t/\epsilon)$ for $\epsilon > 0$. Then ρ_ϵ is an approximation to the identity. Let $\tilde{\phi}$ be the extension of ϕ from $(0, +\infty)$ to $\mathbb{R}$, given by

$$\tilde{\phi}(t) := \begin{cases} \phi(t), & \text{if } t > 0, \\ \lim\limits_{t\to 0^+} \phi(t), & \text{if } t = 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Then $\tilde{\phi}$ is non-decreasing and non-negative in $\mathbb{R}$. Let $\tilde{\phi}_\epsilon$ be the convolution product of $\tilde{\phi}$ and ρ_ϵ, namely for any $t \in \mathbb{R}$,

$$\tilde{\phi}_\epsilon(t) := (\tilde{\phi} * \rho_\epsilon)(t)$$

$$= \int_{\mathbb{R}} \tilde{\phi}(t-\tau)\rho_\epsilon(\tau)d\tau$$

$$= \int_{-\epsilon}^{0} \tilde{\phi}(t-\tau)\rho_\epsilon(\tau)d\tau.$$

Then $\tilde{\phi}_\epsilon$ is a non-negative C^∞ function in $\mathbb{R}$. For any $t_2 > t_1$, we have

$$\tilde{\phi}_\epsilon(t_2) - \tilde{\phi}_\epsilon(t_1) = \int_{\mathbb{R}} [\tilde{\phi}(t_2-\tau) - \tilde{\phi}(t_1-\tau)]\rho_\epsilon(\tau)d\tau \geq 0,$$

where the inequality is due to the monotonicity of $\tilde{\phi}$ in $\mathbb{R}$. Therefore $\tilde{\phi}_\epsilon$ is non-decreasing in $\mathbb{R}$, and then $\tilde{\phi}'_\epsilon \geq 0$. For any $t > 0$, we also have

$$\begin{aligned}\tilde{\phi}_\epsilon(t) &= \int_{-\epsilon}^{0} \tilde{\phi}(t-\tau)\rho_\epsilon(\tau)d\tau \\ &\geq \int_{-\epsilon}^{0} \tilde{\phi}(t)\rho_\epsilon(\tau)d\tau \\ &= \tilde{\phi}(t) = \phi(t),\end{aligned}$$

which implies that

$$\lim_{t\to+\infty} \tilde{\phi}_\epsilon(t) \geq \lim_{t\to+\infty} \phi(t) = +\infty.$$

Next we show $\tilde{\phi}_\epsilon$ is also concave in $(0, +\infty)$. In fact, for any $t_2 > t_1 > 0$, there is

$$\begin{aligned}\tilde{\phi}_\epsilon\Big(\frac{t_1+t_2}{2}\Big) &= \int_{-\epsilon}^{0} \tilde{\phi}\Big(\frac{t_1+t_2}{2} - \tau\Big)\rho_\epsilon(\tau)d\tau \\ &= \int_{-\epsilon}^{0} \tilde{\phi}\Big(\frac{t_1-\tau+t_2-\tau}{2}\Big)\rho_\epsilon(\tau)d\tau \\ &\geq \int_{-\epsilon}^{0} \frac{1}{2}[\tilde{\phi}(t_1-\tau) + \tilde{\phi}(t_2-\tau)]\rho_\epsilon(\tau)d\tau \\ &= \frac{1}{2}[\tilde{\phi}_\epsilon(t_1) + \tilde{\phi}_\epsilon(t_2)],\end{aligned}$$

where the inequality is true since $\tilde{\phi} = \phi$ is concave in $(0, +\infty)$. So $\tilde{\phi}_\epsilon$ is a non-negative, non-decreasing concave C^∞ function in $(0, +\infty)$ satisfying that $\lim_{t\to+\infty} \tilde{\phi}_\epsilon(t) = +\infty$, $\tilde{\phi}'_\epsilon(t) \geq 0$, and $\tilde{\phi}''_\epsilon(t) \leq 0$.

Now define ϕ_ϵ as

$$\phi_\epsilon(t) := \tilde{\phi}_\epsilon(t) + \epsilon\alpha(t), \quad \forall t > 0,$$

where

$$\alpha(t) = \frac{\sqrt{t}}{1+\sqrt{t}}.$$

Direct computations show that

$$\alpha'(t) = \frac{1}{2(1+\sqrt{t})^2\sqrt{t}} > 0,$$
$$\alpha''(t) = -\frac{1+3\sqrt{t}}{4(1+\sqrt{t})^3 t^{3/2}} < 0.$$

Recalling the above properties about $\tilde{\phi}_\epsilon$, we see that ϕ_ϵ is a positive, increasing and concave C^∞ function in $(0, +\infty)$ with $\lim_{t\to+\infty} \phi_\epsilon(t) = +\infty$, $\phi'_\epsilon(t) > 0$ and $\phi''_\epsilon(t) < 0$. Observing $\tilde{\phi}_\epsilon$ is smooth in $\mathbb{R}$, and $\lim_{t\to 0^+} \alpha'(t) = +\infty$, we obtain

$$\begin{aligned}\lim_{t\to 0^+} \phi'_\epsilon(t) &= \lim_{t\to 0^+} \tilde{\phi}'_\epsilon(t) + \epsilon \lim_{t\to 0^+} \alpha'(t) \\ &= \tilde{\phi}'_\epsilon(0) + \epsilon \lim_{t\to 0^+} \alpha'(t) \\ &= +\infty.\end{aligned}$$

Hence ϕ_ϵ satisfies all the assumptions on ϕ in Lemma 3.2.

Now applying Lemma 3.2 on ϕ_ϵ, there exists $h_\epsilon \in C^+(S^{n-1})$ which is a minimizer of

$$\inf\left\{\sup_{\xi\in K_g} J_\epsilon[g(x) - \xi\cdot x] : g \in C^+(S^{n-1}),\ \mathrm{vol}(K_g) = v\right\},$$

where

$$J_\epsilon[g] = \int_{S^{n-1}} \phi_\epsilon(g(x))d\mu(x).$$

Moreover h_ϵ satisfies the following

$$\frac{c_\epsilon}{\phi'_\epsilon(h_\epsilon)} dS_{K_{h_\epsilon}} = d\mu, \tag{4.2}$$

where

$$c_\epsilon = \frac{1}{nv} \int\limits_{S^{n-1}} \phi'_\epsilon(h_\epsilon) h_\epsilon d\mu. \tag{4.3}$$

And h_ϵ is the support function of K_{h_ϵ}.

For $0 < \epsilon < 1$, let

$$m_\epsilon = J_\epsilon[h_\epsilon].$$

We claim m_ϵ is uniformly bounded from above. In fact, denote K_μ the Alexandrov body associated with $(1, \operatorname{supp}(\mu))$. Here 1 means the constant function on S^{n-1}. Let

$$\bar{g} \equiv \left(\frac{v}{\operatorname{vol}(K_\mu)}\right)^{1/n},$$

we have $K_{\bar{g}} = \bar{g} K_\mu$, and then $\operatorname{vol}(K_{\bar{g}}) = g^n \operatorname{vol}(K_\mu) = v$. By definition, there is

$$\begin{aligned} m_\epsilon &\le \sup_{\xi \in K_{\bar{g}}} J_\epsilon[\bar{g} - \xi \cdot x] \\ &= \sup_{\xi \in K_{\bar{g}}} \int\limits_{\operatorname{supp}(\mu)} \phi_\epsilon(\bar{g} - \xi \cdot x) d\mu(x) \\ &\le \int\limits_{\operatorname{supp}(\mu)} \phi_\epsilon(\operatorname{diam}(K_{\bar{g}})) d\mu(x) \\ &= \phi_\epsilon(\operatorname{diam}(K_{\bar{g}})) |\mu|. \end{aligned}$$

Note that when $0 < \epsilon < 1$

$$\phi_\epsilon(t) < \phi(t + \epsilon) + \epsilon < \phi(t + 1) + 1,$$

we have

$$m_\epsilon < [\phi(\operatorname{diam}(K_{\bar{g}}) + 1) + 1] \cdot |\mu|. \tag{4.4}$$

Next, we prove $\{h_\epsilon\}$ is uniformly bounded on S^{n-1}. If not, there exists a sequence $\{\epsilon_k\}$ such that

$$\lim_{k \to +\infty} \max_{x \in S^{n-1}} h_{\epsilon_k}(x) = +\infty.$$

Write $R_{\epsilon_k} = \max_{x \in S^{n-1}} h_{\epsilon_k}(x)$. For each ϵ_k, there exists $x_{\epsilon_k} \in S^{n-1}$ such that $h_{\epsilon_k}(x_{\epsilon_k}) = R_{\epsilon_k}$. Since $\{x_{\epsilon_k}\} \subset S^{n-1}$, there exists a convergent subsequence. Without loss of generality, we assume

$$x_{\epsilon_k} \to x_0 \in S^{n-1} \text{ when } k \to +\infty.$$

Recall $\operatorname{supp}(\mu)$ is not concentrated on any closed hemisphere, there is some $\bar{x} \in \operatorname{supp}(\mu)$ such that $\bar{x} \cdot x_0 > 0$. Write $\delta = \frac{1}{2}\bar{x} \cdot x_0 > 0$, then for sufficiently large k, e.g. $k \geq k_0$, we have

$$\bar{x} \cdot x_{\epsilon_k} > \delta. \tag{4.5}$$

By the definition of support function, there is

$$h_{\epsilon_k}(\bar{x}) \geq R_{\epsilon_k}(\bar{x} \cdot x_{\epsilon_k}) > R_{\epsilon_k}\delta, \quad k \geq k_0. \tag{4.6}$$

Note ϕ_ϵ is increasing, $\phi_\epsilon(t) > \phi(t)$ for $t > 0$, and $\lim_{t\to+\infty} \phi(t) = +\infty$, we have

$$\begin{aligned} m_{\epsilon_k} &= \int_{S^{n-1}} \phi_{\epsilon_k}(h_{\epsilon_k}(x))d\mu(x) \\ &\geq \phi_{\epsilon_k}(h_{\epsilon_k}(\bar{x}))\mu(\bar{x}) \\ &\geq \phi_{\epsilon_k}(R_{\epsilon_k}\delta)\mu(\bar{x}) \\ &\geq \phi(R_{\epsilon_k}\delta)\mu(\bar{x}) \to +\infty. \end{aligned} \tag{4.7}$$

However, by (4.4), m_{ϵ_k} is uniformly bounded. This is a contradiction. Thus $\{h_\epsilon\}$ is uniformly bounded, namely there exists some positive constant C_1 such that

$$\max_{x\in S^{n-1}} h_\epsilon(x) \leq C_1, \quad \forall\ 0 < \epsilon < 1. \tag{4.8}$$

By the Blaschke selection theorem, we can assume h_ϵ converges to some support function h uniformly on S^{n-1} when $\epsilon \to 0^+$. Correspondingly K_{h_ϵ} converges to K_h. Note that $\operatorname{vol}(h) = v$ and $h \geq 0$ on S^{n-1}. We claim that if non-negative g_ϵ converges to some g uniformly on S^{n-1}, then

$$\lim_{\epsilon\to 0^+} \sup_{\xi\in K_{g_\epsilon}} J_\epsilon[g_\epsilon(x) - \xi \cdot x] = \sup_{\xi\in K_g} J[g(x) - \xi \cdot x]. \tag{4.9}$$

In fact, let $\hat{\xi}_\epsilon$ be a point in K_{g_ϵ} such that

$$J_\epsilon[g_\epsilon(x) - \hat{\xi}_\epsilon \cdot x] = \sup_{\xi\in K_{g_\epsilon}} J_\epsilon[g_\epsilon(x) - \xi \cdot x].$$

Since K_{g_ϵ} converges to K_g, one can assume $\hat{\xi}_\epsilon$ converges to some $\hat{\xi} \in K_g$. By our construction, ϕ_ϵ converges to ϕ uniformly on any closed interval of $[0, +\infty)$, then

$$
\begin{aligned}
\lim_{\epsilon\to 0^+} J_\epsilon[g_\epsilon(x) - \hat{\xi}_\epsilon \cdot x] &= \lim_{\epsilon\to 0^+} \int\limits_{S^{n-1}} \phi_\epsilon(g_\epsilon(x) - \hat{\xi}_\epsilon \cdot x) d\mu(x) \\
&= \int\limits_{S^{n-1}} \phi(g(x) - \hat{\xi} \cdot x) d\mu(x) \\
&= J[g(x) - \hat{\xi} \cdot x] \\
&\leq \sup_{\xi \in K_g} J[g(x) - \xi \cdot x].
\end{aligned} \tag{4.10}
$$

On the other hand, for any $\xi \in K_g$, there exists $\xi_\epsilon \in K_{g_\epsilon}$ such that ξ_ϵ converges to ξ as $\epsilon \to 0^+$. Then

$$
\begin{aligned}
J[g(x) - \xi \cdot x] &= \int\limits_{S^{n-1}} \phi(g(x) - \xi \cdot x) d\mu(x) \\
&= \lim_{\epsilon\to 0^+} \int\limits_{S^{n-1}} \phi_\epsilon(g_\epsilon(x) - \xi_\epsilon \cdot x) d\mu(x) \\
&= \lim_{\epsilon\to 0^+} J_\epsilon[g_\epsilon(x) - \xi_\epsilon \cdot x] \\
&\leq \lim_{\epsilon\to 0^+} J_\epsilon[g_\epsilon(x) - \hat{\xi}_\epsilon \cdot x],
\end{aligned}
$$

which implies that

$$
\sup_{\xi \in K_g} J[g(x) - \xi \cdot x] \leq \lim_{\epsilon\to 0^+} J_\epsilon[g_\epsilon(x) - \hat{\xi}_\epsilon \cdot x]. \tag{4.11}
$$

Combining (4.10) and (4.11), we have obtained (4.9).

From (4.9), we have

$$
\lim_{\epsilon\to 0^+} J_\epsilon[h_\epsilon] = \lim_{\epsilon\to 0^+} \sup_{\xi \in K_{h_\epsilon}} J_\epsilon[h_\epsilon(x) - \xi \cdot x] = \sup_{\xi \in K_h} J[h(x) - \xi \cdot x],
$$

namely

$$
J[h] = \sup_{\xi \in K_h} J[h(x) - \xi \cdot x]. \tag{4.12}
$$

Recalling Lemma 3.3, we see ξ_h can be chosen as 0. Then $0 \in \operatorname{int} K_h$, namely $h \in C^+(S^{n-1})$. Now for any $g \in C^+(S^{n-1})$ with $\mathrm{vol}(K_g) = v$, by (4.9), there is

$$
\begin{aligned}
J[h] &= \lim_{\epsilon\to 0^+} J_\epsilon[h_\epsilon] \\
&\leq \lim_{\epsilon\to 0^+} \sup_{\xi \in K_g} J_\epsilon[g(x) - \xi \cdot x] \\
&= \sup_{\xi \in K_g} J[g(x) - \xi \cdot x],
\end{aligned}
$$

which together with (4.12) implies that h is a minimizer of (4.1).

Since $h > 0$ on S^{n-1} and $h_\epsilon \to h$ uniformly, there exists some positive constant C_2 such that

$$\min_{x\in S^{n-1}} h_\epsilon(x) \geq C_2$$

for sufficiently small ϵ, say $0 < \epsilon < \epsilon_0$. Recalling (4.8), we have

$$C_2 \leq h_\epsilon(x) \leq C_1, \quad \forall x \in S^{n-1} \text{ and } \epsilon \in (0, \epsilon_0).$$

By the definition of ϕ_ϵ, ϕ'_ϵ converges to ϕ' uniformly on $[C_2, C_1]$ when $\epsilon \to 0^+$. Then

$$\phi'_\epsilon(h_\epsilon) \rightrightarrows \phi'(h) \text{ uniformly on } S^{n-1}.$$

Now passing to the limit in (4.2) and (4.3), we obtain

$$\frac{c}{\phi'(h)} dS_{K_h} = d\mu,$$

where

$$c = \frac{1}{nv} \int\limits_{S^{n-1}} \phi'(h) h d\mu.$$

Obviously c is positive. In this way, we have completed the proof of this lemma. □

Based on Lemma 4.1 and using approximation, we can remove the restriction that μ is discrete, and thus prove Theorem 3.1.

Proof of Theorem 3.1. As was shown in [36, Theorem 8.2.2], for a given finite Borel measure μ on S^{n-1} which is not concentrated in any closed hemisphere, one can find a sequence of finite discrete measures $\{\mu_j\}$ on S^{n-1} weakly converging to μ, and each of them is not concentrated in any closed hemisphere. Also we can require that $|\mu_j| = |\mu|$. For each μ_j, applying Lemma 4.1, there exists a support function $h_j \in C^+(S^{n-1})$ and a $c_j > 0$ satisfying

$$\frac{c_j}{\phi'(h_j)} dS_{K_j} = d\mu_j \text{ on } S^{n-1}, \tag{4.13}$$

where K_j is the convex body determined by h_j. Moreover, one can require the volume of h_j is equal to any given number $v > 0$, and h_j is a minimizer of

$$\inf \left\{ \sup_{\xi \in K_{g,\mu_j}} J_j[g(x) - \xi \cdot x] : g \in C^+(S^{n-1}), \ \mathrm{vol}(K_{g,\mu_j}) = v \right\}. \tag{4.14}$$

Here K_{g,μ_j} is the Alexandrov body associated with $(g, \mathrm{supp}(\mu_j))$, and J_j is given by

$$J_j[g] = \int_{S^{n-1}} \phi(g(x))d\mu_j(x) = \int_{\text{supp}(\mu_j)} \phi(g(x))d\mu_j(x).$$

Denote the minimum of (4.14) by m_j, namely $m_j = J_j[h_j]$. We claim that m_j is uniformly bounded from above. In fact, denote K_{μ_j} the Alexandrov body associated with $(1, \text{supp}(\mu_j))$, and set

$$\bar{g}_j \equiv \left(\frac{v}{\text{vol}(K_{\mu_j})}\right)^{1/n}.$$

Then we have $K_{\bar{g}_j,\mu_j} = \bar{g}_j K_{\mu_j}$, and $\text{vol}(K_{\bar{g}_j,\mu_j}) = \bar{g}_j^n \,\text{vol}(K_{\mu_j}) = v$. By definition, there is

$$\begin{aligned} m_j &\leq \sup_{\xi \in K_{\bar{g}_j,\mu_j}} J_j[\bar{g}_j - \xi \cdot x] \\ &= \sup_{\xi \in K_{\bar{g}_j,\mu_j}} \int_{\text{supp}(\mu_j)} \phi(\bar{g}_j - \xi \cdot x)d\mu_j(x) \\ &\leq \int_{\text{supp}(\mu_j)} \phi(\text{diam}(K_{\bar{g}_j,\mu_j}))d\mu_j(x) \\ &= \phi(\bar{g}_j \,\text{diam}(K_{\mu_j}))|\mu_j|. \end{aligned} \tag{4.15}$$

We now prove $\text{diam}(K_{\mu_j})$ is uniformly bounded from above. Otherwise, without loss of generality, one can find a sequence of $\{\xi_j\}$ such that $\xi_j \in K_{\mu_j}$ and

$$\lim_{j \to +\infty} |\xi_j| = +\infty.$$

Let $\tilde{\xi}_j = \xi_j / |\xi_j|$, then $\tilde{\xi}_j \in S^{n-1}$. We can assume

$$\lim_{j \to +\infty} \tilde{\xi}_j = \tilde{\xi} \in S^{n-1}.$$

Recall $\text{supp}(\mu)$ is not concentrated in any closed hemisphere, there is one $\tilde{x} \in \text{supp}(\mu)$ such that

$$\tilde{\xi} \cdot \tilde{x} > 0. \tag{4.16}$$

Note that for any neighborhood of $\tilde{x}$, say $O(\tilde{x})$, we have

$$\liminf_{j \to +\infty} \mu_j(O(\tilde{x})) \geq \mu(O(\tilde{x})) > 0,$$

which implies

$$O(\tilde{x}) \cap \text{supp}(\mu_j) \neq \emptyset \text{ for infinitely many } j.$$

Thus there exists a subsequence $\{j_i\} \subset \{j\}$ and $\tilde{x}_{j_i} \in \text{supp}(\mu_{j_i})$ such that

$$\lim_{j_i} \tilde{x}_{j_i} = \tilde{x}.$$

For each j_i, since $\xi_{j_i} \in K_{\mu_{j_i}}$, by definition,

$$\xi_{j_i} \cdot \tilde{x}_{j_i} \leq 1,$$

namely

$$\tilde{\xi}_{j_i} \cdot \tilde{x}_{j_i} \leq \frac{1}{|\xi_{j_i}|}.$$

Passing to the limit, we obtain

$$\tilde{\xi} \cdot \tilde{x} \leq 0,$$

which is a contradiction with (4.16). Thus there exists a positive constant C such that

$$\text{diam}(K_{\mu_j}) \leq C \text{ for all } j. \tag{4.17}$$

Note the unit ball B^n in $\mathbb{R}^n$ is contained in K_{μ_j} for each j, there is

$$\bar{g}_j \leq \left(\frac{v}{\text{vol}(B^n)}\right)^{1/n} \quad \forall j. \tag{4.18}$$

Combining (4.15), (4.17) and (4.18), there is a positive constant C_1 such that

$$m_j \leq \phi(C_1)|\mu_j| = \phi(C_1)|\mu| \text{ for all } j. \tag{4.19}$$

Now with (4.19) instead of (4.4), we can prove that h_j is uniformly bounded from above, just by the arguments from (4.4) to (4.8), but changing $\bar{x}$ into a small neighborhood of $\bar{x}$ in estimates (4.5)–(4.7). Therefore there exists a positive constant C_2 such that

$$\max_{x \in S^{n-1}} h_j(x) \leq C_2 \text{ for all } j. \tag{4.20}$$

By the Blaschke selection theorem, we can assume h_j converges to some support function h uniformly on S^{n-1} when $j \to +\infty$. Correspondingly, K_j converges to the convex body K determined by h. Note that $\text{vol}(h) = v$ and $h \geq 0$ on S^{n-1}. By the assumptions on ϕ, we see $1/\phi'$ is continuous on $[0, C_2]$. Thus when $j \to +\infty$,

$$\frac{1}{\phi'(h_j)} \rightrightarrows \frac{1}{\phi'(h)} \text{ uniformly on } S^{n-1}. \tag{4.21}$$

Integrating (4.13), we have

$$
\begin{aligned}
c_j &= \frac{1}{nv} \int\limits_{S^{n-1}} \phi'(h_j) h_j d\mu_j \\
&\leq \frac{1}{nv} \int\limits_{S^{n-1}} [\phi(h_j) - \phi(0)] d\mu_j \\
&\leq \frac{1}{nv} \int\limits_{S^{n-1}} [\phi(C_2) - \phi(0)] d\mu_j \\
&= \frac{1}{nv} [\phi(C_2) - \phi(0)] \cdot |\mu|.
\end{aligned}
$$

Then we assume without loss of generality that

$$\lim_{j \to +\infty} c_j = c \geq 0. \tag{4.22}$$

With (4.21) and (4.22), we pass to the limit in (4.13) and then obtain

$$\frac{c}{\phi'(h)} dS_K = d\mu \text{ on } S^{n-1}.$$

Obviously c can not be zero, namely $c > 0$. The proof of Theorem 3.1 is completed. □

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