J. Differential Equations 254 (2013) 983–1005

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Rotationally symmetric solutions to the L_p-Minkowski problem ☆

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ARTICLE INFO

Article history:
Received 12 September 2012
Available online 22 October 2012

Keywords:
Monge–Ampère equation
Minkowski problem
A priori estimates
Existence of solutions

ABSTRACT

In this paper we study the L_p-Minkowski problem for $p = -n - 1$, which corresponds to the critical exponent in the Blaschke–Santalo inequality. We first obtain volume estimates for general solutions, then establish a priori estimates for rotationally symmetric solutions by using a Kazdan–Warner type obstruction. Finally we give sufficient conditions for the existence of rotationally symmetric solutions by a blow-up analysis. We also include an existence result for the L_p-Minkowski problem which corresponds to the super-critical case of the Blaschke–Santalo inequality.

1. Introduction

Let f be a positive function on the unit sphere S^n. In this paper we are concerned with the solvability of the equation

$$\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on } S^n, \tag{1.1}$$

where H is the support function of a bounded convex body K in the Euclidean space $\mathbb{R}^{n+1}$, I is the unit matrix, $\nabla^2 H = (\nabla_{ij} H)$ is the covariant derivatives of H with respect to an orthonormal frame on S^n.

☆ The first author was supported by the Natural Science Foundation of China (Grant No. 11131005) and the Doctoral Programme Foundation of Institution of Higher Education of China. The second author was supported by the Australian Research Council.

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.10.008

Eq. (1.1) is the L_p-Minkowski problem of Lutwak [16] with $p = -n-1$. It is called the centroaffine Minkowski problem in [9], and is of particular interest due to its invariance under projective transformations on S^n. This equation also arises in a number of applications. It describes self-similar solutions to the anisotropic curve shortening flow [3,10]. The associated parabolic equation also received considerable interest in image processing [2]. Eq. (1.1) corresponds to the critical case of the Blaschke–Santalo inequality [18], and its existence of solution is a rather complicated problem. The situation is similar, in some aspects, to the Nirenberg problem and the prescribing scalar curvature problem on the sphere, which involve critical exponents of the Sobolev inequalities and have been extensively studied [6,8,12,14,15]. For Eq. (1.1), it is known that when f is constant, all ellipsoids centered at the origin are solutions to (1.1) [5]. So one cannot obtain a priori estimates for solutions without additional assumptions on f. Similarly to the prescribing scalar curvature problem, there exist obstructions for the existence of solutions, such as a Kazdan–Warner type one in [9].

Eq. (1.1) has been studied in a number of papers, see [1,7,11,13,20] for the case $n = 1$, and [9,16,17] for $n > 1$. When $n = 1$, (1.1) is a nonlinear ordinary differential equation, which arises in the investigation of self-similar solutions to the anisotropic curve shortening flow [3,10]. Sufficient conditions for the existence of solutions have been found in [1,7,11,13,20] by different methods. In this paper we study the n-dimensional case of Eq. (1.1) for $n \geqslant 1$, especially when f is a rotationally symmetric function.

First we have the following volume estimates.

Theorem 1.1. *There exist positive constants $C_n, \tilde{C}_n$, depending only on n, such that for any solution H to Eq. (1.1), we have*

$$C_n \sqrt{f_{\min}} \leqslant |K| \leqslant \tilde{C}_n \sqrt{f_{\max}}, \tag{1.2}$$

where $f_{\min} = \inf_{S^n} f$, $f_{\max} = \sup_{S^n} f$, and

$$|K| = \frac{1}{n+1} \int\limits_{S^n} H \det\bigl(\nabla^2 H + HI\bigr)$$

is the volume of the corresponding convex body K.

Next we consider a priori estimates and existence of rotationally symmetric solutions, that is, solutions which are rotationally symmetric with respect to the x_{n+1}-axis in $\mathbb{R}^{n+1}$. In the spherical coordinates, a rotationally symmetric function f on S^n can be regarded as a function on $[0, \pi]$, such that $f(\theta) = f(x_1, \dots, x_{n+1})$ with $x_{n+1} = \cos\theta$. In particular $f(0)$ is the value of f at the north pole and $f(\pi)$ is the value of f at the south pole. Using the superscript $'$ to denote $\frac{d}{d\theta}$, we introduce the following two quantities associated with f,

$$ni(f) = \begin{cases} -f''(\frac{\pi}{2}), & n \geqslant 2, \\ \int_0^\pi (f'(\theta) - f'(\frac{\pi}{2})) \tan\theta \, d\theta, & n = 1, \end{cases}$$

and

$$pi(f) = \int\limits_0^\pi f'(\theta) \cot\theta \, d\theta.$$

Note that by the rotational symmetry, we have $f'(0) = f'(\pi) = 0$.

Theorem 1.2. *Assume that $f \in C^2(S^n)$ when $n \neq 2$ and $f \in C^6(S^n)$ when $n = 2$, that f is positive, rotationally symmetric, and that $f'(\frac{\pi}{2}) = 0$, $ni(f) \neq 0$ and $pi(f) \neq 0$. Then there exist positive constants $C, \tilde{C}$ depending only on n and f, such that for any rotationally symmetric solution H to Eq.* (1.1)*, we have*

$$C \leqslant H \leqslant \tilde{C}. \tag{1.3}$$

By the above a priori estimate, we then have the following existence result.

Theorem 1.3. *Under assumptions of Theorem* 1.2*, if $ni(f) < 0$ and $pi(f) > 0$, then Eq.* (1.1) *admits a rotationally symmetric solution.*

The proof of the a priori estimates (1.3) is inspired by [1,13], which treats the one dimensional case of the above problem, and by [6], which treats prescribing scalar curvature problem on the sphere. For this approach, we need the rotational symmetry to conclude the uniqueness of solutions in a limiting procedure. For the prescribing scalar curvature problem, the corresponding uniqueness is a consequence of the Liouville theorem.

With the a priori estimates, one can study the existence of solutions by the topological degree theory, as was in [1,13] for the one dimensional case. In this paper we choose a different approach to the existence, namely by a blow-up analysis. However, additional conditions are needed in this approach, just as in the approach by the degree method [1,13]. The blow-up analysis is of some interest itself, as it may apply to the non-rotationally symmetric case as well. We plan to explore this approach further in a subsequent work. In this paper we use the Kazdan–Warner type obstruction to establish the a priori estimates (1.3) and will restrict ourselves to the rotationally symmetric case only. Note also that even in the case $n = 1$, our conditions are different from those in [1,13,20].

The paper is organized as follows. In Section 2, we recall an obstruction for the existence of solutions in [9] and prove Theorem 1.1. Then we prove the a priori estimates, Theorem 1.2, in Section 3, and the existence Theorem 1.3 in Section 4. In Section 5, we prove an existence result for the rotationally symmetric solutions to L_p-Minkowski problem, in the super-critical case of the Blaschke–Santalo inequality.

The first author would like to thank his supervisor, Professor Huaiyu Jian, for many discussions.

2. A necessary condition and volume estimates

In this section we recall a necessary condition introduced in [9] and give an upper and lower bounds for volume estimates.

Let B be an arbitrarily given $(n + 1) \times (n + 1)$ matrix. The matrix generates a projective vector field ξ, given by

$$\xi(x) = Bx - \big(x^T Bx\big)x, \quad x \in S^n. \tag{2.1}$$

It was proved that a solution to Eq. (1.1) must satisfy the following necessary condition.

Proposition 2.1. *Let H be a C^3-solution to Eq.* (1.1)*. Then for the projective vector field ξ given by* (2.1)*, we have*

$$\int\limits_{S^n} \frac{\nabla_\xi f}{H^{n+1}} = 0. \tag{2.2}$$

This proposition was proved in [9] using the gnomonic projection. Here we prove it by the moving frame method. The idea of the proof is essentially the same. First we prove the following integral identities on S^n.

Lemma 2.2. *For any C^3-function u on S^n, and any tangent vector field ξ of form (2.1), we have*

$$\int_{S^n} u\det\big(\nabla^2 u+uI\big)\operatorname{div}\xi=(n+1)\int_{S^n}\det\big(\nabla^2 u+uI\big)\nabla_\xi u, \tag{2.3}$$

$$\int_{S^n} u\nabla_\xi\det\big(\nabla^2 u+uI\big)=-(n+2)\int_{S^n}\det\big(\nabla^2 u+uI\big)\nabla_\xi u. \tag{2.4}$$

Proposition 2.1 follows readily from this lemma. Indeed, if H is a solution to Eq. (1.1), then

$$\begin{aligned}\int_{S^n}\frac{\nabla_\xi f}{H^{n+1}}&=\int_{S^n}\frac{\nabla_\xi(H^{n+2}\det(\nabla^2 H+HI))}{H^{n+1}}\\&=\int_{S^n}(n+2)\det\big(\nabla^2 H+HI\big)\nabla_\xi H+H\nabla_\xi\det\big(\nabla^2 H+HI\big)\\&=0.\end{aligned}$$

In the following we will denote by $u_{,i}=\nabla_i u$ and $u_{,ij}=\nabla_{ji}u$ the covariant derivatives of u, in an orthonormal frame on S^n. We also denote as usual that $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Proof of Lemma 2.2. For simplicity we denote $(u_{,ij}+u\delta_{ij})$ by (A_{ij}), and by A^{ij} the cofactor of A_{ij}. One easily sees that $A_{ij,k}$ is a symmetric tensor and $A^{ij}_{,j}=0$. Hence the following integration by parts holds for any smooth functions φ,ψ on S^n,

$$\int A^{ij}\varphi_{,ij}\psi=-\int A^{ij}\varphi_{,i}\psi_{,j}=\int A^{ij}\varphi\psi_{,ij}. \tag{2.5}$$

All integrals in the proof of Lemma 2.2 is over the unit sphere S^n.

Write $\xi=\xi^k e_k$, then $\xi^k=x^T_{,k}Bx$. Calculating its covariant derivatives, we get

$$\xi^k_{,i}=x^T_{,k}Bx_{,i}-x^TBx\delta_{ki}=\big(x^TBx_{,i}\big)_{,k}, \tag{2.6}$$

$$\xi^k_{,ij}=-\xi^k\delta_{ij}-\xi^j\delta_{ki}-x^TBx_{,i}\delta_{jk}-x^TBx_{,j}\delta_{ki}. \tag{2.7}$$

Hence

$$\begin{aligned}n\int\det(A_{ij})\nabla_\xi u&=\int A^{ij}\big(\nabla^2 u+uI\big)\xi^k u_{,k}\\&=\int A^{ij}u\big(\xi^k u_{,k}\big)_{,ij}+A^{ij}u\delta_{ij}\xi^k u_{,k}\\&=\int uA^{ij}\big(\xi^k_{,ij}u_{,k}+\xi^k_{,i}u_{,kj}+\xi^k_{,j}u_{,ki}+\xi^k u_{,kij}\big)+uA^{ij}\delta_{ij}\xi^k u_{,k}\\&=\int uA^{ij}u_{,k}\big(\xi^k_{,ij}+\xi^k\delta_{ij}\big)+2uA^{ij}\xi^k_{,i}u_{,kj}+uA^{ij}\xi^k(A_{ki,j}-u_{,j}\delta_{ki}).\end{aligned}$$

Taking into account of (2.7) and the symmetry of $A_{ij,k}$, the above equality reads

$$\begin{aligned} n\int \det(A_{ij})\nabla_\xi u &= \int -uA^{ij}u_{,k}\big(\xi^j\delta_{ki} + x^T Bx_{,i}\delta_{jk} + x^T Bx_{,j}\delta_{ki}\big) \\ &\quad + \int 2uA^{ij}\xi^k_{,i}u_{,kj} + uA^{ij}\xi^k A_{ij,k} - uA^{ij}u_{,j}\xi^i \\ &= \int -2uA^{ij}u_{,j}\xi^i - 2uA^{ij}u_{,j}x^T Bx_{,i} + 2uA^{ij}\xi^k_{,i}u_{,kj} + u\xi^k(\det A_{ij})_{,k} \\ &= \int -A^{ij}\big(u^2\big)_{,j}\big(\xi^i + x^T Bx_{,i}\big) + 2uA^{ij}\xi^k_{,i}u_{,kj} + \int u\nabla_\xi \det(A_{ij}). \end{aligned} \tag{2.8}$$

By virtue of (2.5) and (2.6), the first integral can be simplified as follows,

$$\begin{aligned} \int u^2A^{ij}\big(\xi^i + x^T Bx_{,i}\big)_{,j} + 2uA^{ij}\xi^k_{,i}u_{,kj} &= \int u^2A^{ij}\big(\xi^i_{,j} + \xi^j_{,i}\big) + 2uA^{ij}\xi^k_{,i}u_{,kj} \\ &= \int 2uA^{ij}\big(u\xi^j_{,i} + \xi^k_{,i}u_{,kj}\big) \\ &= \int 2uA^{ij}\xi^k_{,i}A_{kj} \\ &= \int 2u\det(A_{ij})\delta^i_k\xi^k_{,i} \\ &= 2\int u\det(A_{ij})\operatorname{div}\xi. \end{aligned}$$

Hence (2.8) becomes

$$n\int \det(A_{ij})\nabla_\xi u = 2\int u\det(A_{ij})\operatorname{div}\xi + \int u\nabla_\xi \det(A_{ij}). \tag{2.9}$$

On the other hand, the divergence theorem gives

$$\int \det(A_{ij})\nabla_\xi u + \int u\nabla_\xi \det(A_{ij}) + \int u\det(A_{ij})\operatorname{div}\xi = 0. \tag{2.10}$$

Now the lemma follows immediately from (2.9) and (2.10). □

An important property of Eq. (1.1) is its invariance under the projective transformation group $SL(n+1)$. More precisely, let H be a solution to this equation and K the associated convex body in $\mathbb{R}^{n+1}$. Then after making a unimodular linear transformation $A^T \in SL(n+1)$, the convex body K is changed to K_A with support function H_A. We have

$$H_A(x) = |Ax|\cdot H\left(\frac{Ax}{|Ax|}\right), \quad x\in S^n. \tag{2.11}$$

Indeed, by the definition of support function,

$$\begin{aligned} H_A(x) &= \sup_{p_A \in K_A} p_A^T x \\ &= \sup_{p \in K} \left(A^T p\right)^T x \\ &= \sup_{p \in K} p^T A x \\ &= |Ax| \cdot H\left(\frac{Ax}{|Ax|}\right). \end{aligned}$$

One can also verify, see [9], that H_A solves the equation

$$\det\left(\nabla^2 H_A + H_A I\right) = \frac{f_A}{H_A^{n+2}}, \qquad f_A(x) = f\left(\frac{Ax}{|Ax|}\right). \tag{2.12}$$

To understand formula (2.11), it is helpful to consider the corresponding convex body. The support function is the distance from the origin to the tangent plane, and (2.11) is the formula which tells how the distance changes under linear transformation.

It is known that for any non-degenerate convex body K, there is a unique ellipsoid E which attains the minimum volume among all ellipsoids containing K [19]. This ellipsoid E is called the *minimum ellipsoid* of K [19], which satisfies

$$\frac{1}{n+1} E \subset K \subset E, \tag{2.13}$$

where $\alpha E = \{\alpha(x - x_0) + x_0 \mid x \in E\}$ and x_0 is the center of E. We say K is *normalized* if E is a ball.

Next we consider the volume estimate for the solution H. Let K be the convex body with support function H. Recall that the volume of K is given by

$$\begin{aligned} |K| &= \frac{1}{n+1} \int_{S^n} H \det\left(\nabla^2 H + HI\right) \\ &= \frac{1}{n+1} \int_{S^n} \frac{f}{H^{n+1}}. \end{aligned}$$

Lemma 2.3. *There exists a positive constant C_n depending only on n, such that for any solution H to Eq. (1.1) we have*

$$H_{\min} \cdot H_{\max}^n \cdot |K| \geqslant C_n f_{\min}. \tag{2.14}$$

Proof. By extending H to $\mathbb{R}^{n+1}$ such that it is homogeneous of degree one and by the convexity of H, one sees that $|\nabla H| \leqslant H_{\max} := \sup_{S^n} H$. Hence for any fixed point $x_0 \in S^n$, we have

$$H(x) \leqslant H(x_0) + H_{\max}|x - x_0| \quad \forall x \in S^n, \tag{2.15}$$

where $|\cdot|$ means the standard metric in $\mathbb{R}^{n+1}$.

Direct computation shows

$$\begin{aligned}\int_{S^n}\frac{1}{H^{n+1}} &\geqslant \int_{S^n}\frac{1}{(H(x_0)+H_{\max}|x-x_0|)^{n+1}}\\ &=\int_0^{\pi}\frac{\sigma_n\sin^{n-1}\theta}{(H(x_0)+H_{\max}2\sin\frac{\theta}{2})^{n+1}}\,d\theta\\ &\geqslant\int_0^{\frac{\pi}{2}}\frac{C_n\theta^{n-1}}{(H(x_0)+H_{\max}\theta)^{n+1}}\,d\theta\\ &\geqslant\frac{C_n}{H(x_0)H_{\max}^n}\int_0^{\frac{\pi}{2}}\frac{t^{n-1}}{(1+t)^{n+1}}\,dt,\end{aligned}\tag{2.16}$$

where the spherical coordinate system with respect to x_0 is used and σ_n is the area of unit sphere in $\mathbb{R}^n$. Thus we have

$$H(x_0)H_{\max}^n\int_{S^n}\frac{1}{H^{n+1}}\geqslant C_n$$

for a different constant C_n. Since x_0 is any given point, we obtain

$$H_{\min}\cdot H_{\max}^n\int_{S^n}\frac{1}{H^{n+1}}\geqslant C_n.\tag{2.17}$$

Therefore

$$H_{\min}\cdot H_{\max}^n\cdot|K|=H_{\min}\cdot H_{\max}^n\cdot\frac{1}{n+1}\int_{S^n}\frac{f}{H^{n+1}}\geqslant C_n f_{\min}. \qquad\Box$$

Proof of Theorem 1.1. As the estimates (1.2) are invariant under unimodular linear transformation, we only need to prove it for normalized H. Let R be the radius of the minimum ellipsoid of H (actually a ball), then

$$\omega_{n+1}\left(\frac{R}{n+1}\right)^{n+1}\leqslant|K|\leqslant\omega_{n+1}R^{n+1},\tag{2.18}$$

where ω_{n+1} is the volume of unit ball in $\mathbb{R}^{n+1}$.

Noting that

$$H_{\min}\cdot H_{\max}^n\leqslant H_{\max}^{n+1}\leqslant(2R)^{n+1}\leqslant\frac{(2n+2)^{n+1}}{\omega_{n+1}}|K|,$$

by virtue of Lemma 2.3, one immediately gets the first inequality.

On the other hand,

$$\begin{aligned}
|K| &= \frac{1}{n+1}\int_{S^n} H\det\left(\nabla^2 H + HI\right) \\
&\leqslant \frac{1}{n+1}\left(\int_{S^n} H^{n+2}\det\left(\nabla^2 H + HI\right)\right)^{\frac{1}{n+2}}\left(\int_{S^n}\det\left(\nabla^2 H + HI\right)\right)^{\frac{n+1}{n+2}} \\
&= \frac{1}{n+1}\left(\int_{S^n} f\right)^{\frac{1}{n+2}}\left(\int_{S^n}\det\left(\nabla^2 H + HI\right)\right)^{\frac{n+1}{n+2}}.
\end{aligned}$$

The last integral is equal to the area of the convex hypersurface ∂K with support function H, namely

$$\left(\int_{S^n}\det\left(\nabla^2 H + HI\right)\right)^{\frac{n+1}{n+2}} = area(H)^{\frac{n+1}{n+2}} \leqslant \left(\sigma_{n+1}R^n\right)^{\frac{n+1}{n+2}}.$$

Hence we obtain

$$|K| \leqslant C_n (f_{\max})^{\frac{1}{n+2}}\left(R^n\right)^{\frac{n+1}{n+2}}.$$

Namely

$$R \leqslant C_n f_{\max}^{\frac{1}{2n+2}},$$

which together with (2.18) leads to the second inequality of (1.2). □

We note that when $n = 1$, similar volume estimates were obtained in [1] for centro-symmetric solutions. For normalized solution we then have

Corollary 2.4. *There exist positive constants $C_n, \tilde{C}_n$ depending only on n, such that for any normalized solution H to Eq. (1.1),*

$$C_n f_{\min} f_{\max}^{-\frac{2n+1}{2n+2}} \leqslant H \leqslant \tilde{C}_n f_{\max}^{\frac{1}{2n+2}}.$$

3. A priori estimates

From now on we only consider rotationally symmetric solutions to Eq. (1.1). In this case, f must also be rotationally symmetric, and the obstruction (2.2) can be written as

Proposition 3.1. *Let H be a rotationally symmetric C^3-solution to Eq. (1.1). Then*

$$\int_0^\pi \frac{f'(\theta)\sin^n\theta\cos\theta}{H^{n+1}(\theta)}\,d\theta = 0. \tag{3.1}$$

Proof. When $n=1$, (3.1) can be proved directly by integration by parts [1,13].

Let ξ be the vector field given by (2.1) with $B=(b_{\alpha\beta})$. Then

$$\nabla_\xi f = f'(\theta)x_{,\theta}^T Bx = f'(\theta)\big(\cot\theta x^T - v(\theta)\big)Bx,$$

where $v(\theta)=(0,\dots,0,\csc\theta)$. Therefore

$$\int_{S^n} \frac{\nabla_\xi f}{H^{n+1}} = \int_0^\pi \frac{f'(\theta)\,d\theta}{H^{n+1}(\theta)} \int_{S_\theta} x_{,\theta}^T Bx,$$

where $S_\theta=\{x\in S^n\colon\ x_{n+1}=\cos\theta\}$. Direct computation shows

$$\begin{aligned}
\int_{S_\theta} x_{,\theta}^T Bx &= \cot\theta \int_{S_\theta} x^T Bx - v(\theta)B\int_{S_\theta} x \\
&= \cot\theta \sum_\alpha b_{\alpha\alpha}\int_{S_\theta} x^\alpha x^\alpha\,d\sigma - b_{n+1,n+1}\cot\theta\int_{S_\theta} d\sigma \\
&= \left(\frac{\sum_{\alpha=1}^n b_{\alpha\alpha}}{n} - b_{n+1,n+1}\right)\cot\theta\sin^2\theta\int_{S_\theta} d\sigma \\
&= \frac{\operatorname{tr} B-(n+1)b_{n+1,n+1}}{n}\cdot\sigma_n\sin^n\theta\cos\theta.
\end{aligned}$$

Hence

$$\int_{S^n}\frac{\nabla_\xi f}{H^{n+1}} = \frac{\operatorname{tr} B-(n+1)b_{n+1,n+1}}{n}\cdot\sigma_n\int_0^\pi \frac{f'(\theta)\sin^n\theta\cos\theta}{H^{n+1}(\theta)}\,d\theta.$$

Thus (3.1) holds. □

For a rotationally symmetric solution H to (1.1), one can choose matrix

$$A=\operatorname{diag}\big(a^{\frac{1}{n+1}},\dots,a^{\frac{1}{n+1}},a^{-\frac{n}{n+1}}\big) \tag{3.2}$$

such that H_A is normalized, where $a>0$, and H_A, f_A are defined in (2.11) and (2.12). To prove Theorem 1.2, we have two cases to consider, that is either $a\to\infty$ or $a\to 0^+$.

From (2.12), we can write f_A as

$$f_A(\theta)=f\big(\gamma_a(\theta)\big), \tag{3.3}$$

where

$$\begin{aligned}
\gamma_a(\theta) &= \arccos\left(\frac{\cos\theta}{i_a(\theta)}\right), \\
i_a(\theta) &= \sqrt{a^2\sin^2\theta+\cos^2\theta}.
\end{aligned} \tag{3.4}$$

In fact,

$$Ax = \left(a^{\frac{1}{n+1}}x_1, \ldots, a^{\frac{1}{n+1}}x_n, a^{-\frac{n}{n+1}}x_{n+1}\right)^T,$$
$$|Ax| = a^{-\frac{n}{n+1}}\sqrt{a^2\left((x_1)^2 + \cdots + (x_n)^2\right) + (x_{n+1})^2}. \tag{3.5}$$

Therefore in the spherical coordinates we have

$$\frac{Ax}{|Ax|} = \left(\cdot, \frac{\cos\theta}{i_a(\theta)}\right)^T,$$

which implies

$$f_A(\theta) = f\left(\cdot, \frac{\cos\theta}{i_a(\theta)}\right) = f\left(\gamma_a(\theta)\right).$$

First we prove two auxiliary lemmas.

Lemma 3.2. *Let $\varphi_a \in C[0,\pi]$ be a sequence of uniformly bounded functions. If φ_a converges to a constant $\varphi_\infty > 0$ locally uniformly in $(0,\pi)$ as $a \to +\infty$, then*

$$\int_0^\pi \varphi_a(\theta)\left(f'\left(\gamma_a(\theta)\right) - f'(\pi/2)\right)\frac{a^3 \sin^n\theta\cos\theta}{i_a^2(\theta)}\,d\theta$$
$$= \begin{cases} C_n\varphi_\infty(ni(f) + o(1)), & n \geqslant 3, \\ \varphi_\infty \log a^2(ni(f) + o(1)), & n = 2, \\ \varphi_\infty a(ni(f) + o(1)), & n = 1, \end{cases} \tag{3.6}$$

where $C_n = \int_0^\pi \sin^{n-3}\theta\cos^2\theta\,d\theta$.

Proof. Let Λ_a denote the integral on the left hand side of (3.6).

If $n \geqslant 3$, we write Λ_a as

$$\Lambda_a = \int_0^\pi \varphi_a(\theta)\cdot\left(f'\left(\gamma_a(\theta)\right) - f'(\pi/2)\right)a\tan\theta\cdot\frac{a^2\sin^{n-1}\theta\cos^2\theta}{i_a^2(\theta)}\,d\theta.$$

When $a \to +\infty$, one easily verifies that

$$\left|\left(f'\left(\gamma_a(\theta)\right) - f'(\pi/2)\right)a\tan\theta\right| \leqslant \sup_{[0,\pi]}\left|f''\right|,$$
$$\left(f'\left(\gamma_a(\theta)\right) - f'(\pi/2)\right)a\tan\theta \to -f''(\pi/2),$$

where the convergence is uniform on any closed interval of $(0,\pi)$. By the bounded convergence theorem, we obtain

$$\lim_{a\to+\infty}\Lambda_a = \int_0^\pi \varphi_\infty\cdot\left(-f''(\pi/2)\right)\cdot\sin^{n-3}\theta\cos^2\theta\,d\theta = -C_n\varphi_\infty f''(\pi/2).$$

Namely

$$\Lambda_a = C_n\varphi_\infty \cdot \big(-f''(\pi/2)+o(1)\big).$$

If $n=2$, we shall use Taylor expansion to evaluate Λ_a. Denote $\tilde f(t) = f(\arccos t)$. Then $\tilde f \in C^3[-1,1]$ if $f \in C^6(S^2)$[1] and

$$\begin{aligned} f'\big(\gamma_a(\theta)\big) &= -\tilde f'\left(\frac{\cos\theta}{i_a(\theta)}\right)\cdot\frac{a\sin\theta}{i_a(\theta)} \\ &= -\left(\tilde f'(0) + \tilde f''(0)\frac{\cos\theta}{i_a(\theta)} + O(1)\frac{\cos^2\theta}{i_a^2(\theta)}\right)\cdot\frac{a\sin\theta}{i_a(\theta)}. \end{aligned}$$

Hence

$$\begin{aligned} \Lambda_a = &-\int_0^\pi \varphi_a(\theta)\tilde f''(0)\frac{a^4\sin^3\theta\cos^2\theta}{i_a^4(\theta)}\,d\theta - \int_0^\pi O(1)\frac{a^4\sin^3\theta\cos^3\theta}{i_a^5(\theta)}\,d\theta \\ &-\int_0^\pi \varphi_a(\theta)\left(\tilde f'(0)\frac{a\sin\theta}{i_a(\theta)} + f'(\pi/2)\right)\frac{a^3\sin^2\theta\cos\theta}{i_a^2(\theta)}\,d\theta. \end{aligned}$$

Noting $\tilde f'(0) = -f'(\frac{\pi}{2})$ and $\tilde f''(0) = f''(\frac{\pi}{2})$, one sees that, as $a\to+\infty$,

$$\begin{aligned} \Lambda_a &= -\big(\varphi_\infty + o(1)\big)f''(\pi/2)\big(1+o(1)\big)\log a^2 + O(1) \\ &\quad -\int_0^\pi \varphi_a(\theta)f'(\pi/2)\left(1-\frac{a\sin\theta}{i_a(\theta)}\right)\frac{a^3\sin^2\theta\cos\theta}{i_a^2(\theta)}\,d\theta \\ &= \varphi_\infty\cdot\big(-f''(\pi/2)+o(1)\big)\log a^2 - a\int_0^\pi O(1)\left(1-\frac{a\sin\theta}{i_a(\theta)}\right)\cos\theta\,d\theta \\ &= \varphi_\infty\cdot\big(-f''(\pi/2)+o(1)\big)\log a^2 + O(1) \\ &= \varphi_\infty\cdot\big(-f''(\pi/2)+o(1)\big)\log a^2. \end{aligned}$$

[1] We note that $\tilde f \in C^2$ is not sufficient. For example, let $\tilde f'(t) = 1 + t - (1-t^2)^{3/2}$. Then $\tilde f$ is C^2 but not C^3. Observing that $\tilde f'(0) = 0$ and $\tilde f''(0) = 1$, one can compute

$$\int_0^{\frac{\pi}{2}} \tilde f'\left(\frac{\cos\theta}{i_a(\theta)}\right)\frac{a\cos\theta}{i_a(\theta)} = \frac{\pi+1+2\log 2}{2} + o(1),$$

but

$$\int_0^{\frac{\pi}{2}} \left(\tilde f'(0) + \tilde f''(0)\frac{\cos\theta}{i_a(\theta)} + o(1)\frac{\cos\theta}{i_a(\theta)}\right)\frac{a\cos\theta}{i_a(\theta)} = \frac{\pi}{2} + o(1).$$

They are not equal. The reason is that the $o(1)$ above is not really small near $\theta = 0$.

If $n=1$, applying the variable substitution $\theta=\gamma_{a^{-1}}(t)$ (more details of this substitution is given below), we find that

$$\Lambda_a=\int_0^{\pi}\varphi_a\big(\gamma_{a^{-1}}(t)\big)\big(f'(t)-f'(\pi/2)\big)\frac{a^3\sin t\cos t}{\sin^2 t+a^2\cos^2 t}\,dt$$
$$=a\int_0^{\pi}\varphi_a\big(\gamma_{a^{-1}}(t)\big)\big(f'(t)-f'(\pi/2)\big)\tan t\cdot\frac{a^2\cos^2 t}{\sin^2 t+a^2\cos^2 t}\,dt.$$

Noting that

$$\big|\big(f'(t)-f'(\pi/2)\big)\tan t\big|\leqslant\sup_{[0,\pi]}\big|f''\big|,$$

we have

$$\lim_{a\to+\infty}a^{-1}\Lambda_a=\int_0^{\pi}\varphi_\infty\cdot\big(f'(t)-f'(\pi/2)\big)\tan t\,dt.$$

Hence

$$\Lambda_a=\varphi_\infty\,a\cdot\left(\int_0^{\pi}\big(f'(t)-f'(\pi/2)\big)\tan t\,dt+o(1)\right).$$

This lemma is proved. □

Lemma 3.3. *Let φ_a be a sequence of continuous, uniformly bounded functions on $[0,\pi]$. Assume that φ_a converges a.e. to a function $\varphi_0>0$ as $a\to0^+$. Then*

$$\int_0^{\pi}\varphi_a(\theta)f'\big(\gamma_a(\theta)\big)\frac{\sin^n\theta\cos\theta}{i_a^2(\theta)}\,d\theta=\varphi_0(\pi/2)\cdot\big(pi(f)+o(1)\big). \tag{3.7}$$

Proof. Let Λ_a denote the integral on the left hand side of (3.7). Consider the variable substitution

$$\theta=\gamma_{a^{-1}}(t)=\arccos\left(\frac{a\cos t}{j_a(t)}\right),$$

where

$$j_a(t)=\sqrt{\sin^2 t+a^2\cos^2 t}. \tag{3.8}$$

Direct computation shows

$$\cos\theta=\frac{a\cos t}{j_a(t)},$$

$$\sin\theta = \frac{\sin t}{j_a(t)},$$
$$i_a(\theta) = \frac{a}{j_a(t)},$$
$$d\theta = \frac{a}{j_a^2(t)}\,dt.$$

Then we find that

$$\begin{aligned}
\Lambda_a &= \int_0^\pi \varphi_a\big(\gamma_{a^{-1}}(t)\big) f'(t)\left(\frac{\sin t}{j_a(t)}\right)^n \cdot \frac{a\cos t}{j_a(t)} \cdot \left(\frac{j_a(t)}{a}\right)^2 \cdot \frac{a}{j_a^2(t)}\,dt \\
&= \int_0^\pi \varphi_a\big(\gamma_{a^{-1}}(t)\big) f'(t)\frac{\sin^n t\cos t}{j_a^{n+1}(t)}\,dt \\
&= \int_0^\pi \varphi_a\big(\gamma_{a^{-1}}(t)\big)\cdot f'(t)\cot t\cdot\frac{\sin^{n+1} t}{j_a^{n+1}(t)}\,dt.
\end{aligned}$$

Observing that

$$\big|f'(t)\cot t\big| \leqslant \sup_{[0,\pi]}\big|f''\big|,$$
$$\varphi_a\big(\gamma_{a^{-1}}(t)\big) \to \varphi_0(\pi/2) \quad \text{a.e.},$$

we obtain by the bounded convergence theorem that

$$\lim_{a\to 0^+}\Lambda_a = \int_0^\pi \varphi_0(\pi/2)\cdot f'(t)\cot t\,dt.$$

Hence

$$\Lambda_a = \varphi_0(\pi/2)\cdot\left(\int_0^\pi f'(t)\cot t\,dt + o(1)\right). \qquad \square$$

Now we use Lemmas 3.2 and 3.3 to obtain the a priori estimates (1.3).

Proof of Theorem 1.2. By Theorem 1.1, we only need to obtain a uniform positive lower bound for rotationally symmetric solutions. Suppose to the contrary that there exists a sequence of rotationally symmetric solutions H_k to Eq. (1.1) such that $\min_{S^n} H_k \to 0^+$ as $k\to\infty$. For each k, there exists a matrix

$$A_k = \operatorname{diag}\big(a_k^{\frac{1}{n+1}},\dots,a_k^{\frac{1}{n+1}},a_k^{-\frac{n}{n+1}}\big), \tag{3.9}$$

such that H_{A_k}, given by (2.11), is a normalized rotationally symmetric solution to (2.12). We have either $a_k\to\infty$ or $a_k\to 0^+$.

By virtue of (3.1), we have the following equalities

$$
\begin{aligned}
0 &= \int_0^\pi \frac{f'_{A_k}(\theta)\sin^n\theta\cos\theta}{H_{A_k}^{n+1}(\theta)}\,d\theta \\
&= \int_0^\pi \frac{f'(\gamma_{a_k}(\theta))}{H_{A_k}^{n+1}(\theta)}\cdot\frac{a_k\sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,d\theta.
\end{aligned}
\tag{3.10}
$$

By Blaschke's selection theorem, we may assume that H_{A_k} converges uniformly to some support function H_{A_∞} on S^n, which is also normalized and rotationally symmetric. By the weak convergence of the Monge–Ampère equation, H_{A_∞} is a generalized solution to

$$\det\left(\nabla^2 H + HI\right) = \frac{f_{A_\infty}}{H^{n+2}} \quad \text{on } S^n, \tag{3.11}$$

where

$$f_{A_\infty} = \begin{cases} f(\pi/2) & \text{if } a_k \to \infty, \\ f(0)\chi_{\{x_{n+1}>0\}} + f(\pi)\chi_{\{x_{n+1}<0\}} & \text{if } a_k \to 0^+, \end{cases} \tag{3.12}$$

where χ is the characteristic function.

In the case of $a_k \to +\infty$, $f_{A_\infty} \equiv f(\pi/2)$ is a constant. In this case, a solution to (3.11) is an elliptic affine sphere. Hence it must be an ellipsoid [5]. But the solution is normalized, so it must be a sphere. Hence $H_{A_\infty} \equiv f(\pi/2)^{\frac{1}{2n+2}}$. Applying Lemma 3.2 to (3.10) and recalling our assumption that $f'(\frac{\pi}{2}) = 0$, we have $ni(f) = 0$.

In the case $a_k \to 0^+$, f_{A_∞} is equal to two different constants on the north and south hemispheres. In this case, the solution H_{A_∞} to (3.11) is strictly convex and C^1 smooth[4]. Applying Lemma 3.3 to (3.10), we see $pi(f) = 0$. In both cases we reach a contradiction with our assumptions on f. Thus the theorem is proved. □

Remark. From the above proof, one sees that estimates (1.3) holds uniformly for $\varepsilon \in (0, 1]$ for rotationally symmetric solutions to the following equation

$$\det\left(\nabla^2 H + HI\right) = \frac{1+\varepsilon f}{H^{n+2}} \quad \text{on } S^n, \tag{3.13}$$

provided f satisfies the conditions in Theorem 1.2.

4. Existence of solutions

In this section we prove the existence of solutions to Eq. (1.1). First we recall the existence of solutions to the equation

$$\det\left(\nabla^2 H + HI\right) = \frac{\lambda f}{H^p} \quad \text{on } S^n, \tag{4.1}$$

where $p \in (0, n+2)$ is a constant, f is a bounded, measurable function satisfying $0 < f_{\min} \leqslant f \leqslant f_{\max} < \infty$, and λ is the Lagrange multiplier. This is the p-Minkowski problem introduced by Lutwak [16]. When $p < n+2$, Eq. (4.1) corresponds to the sub-critical case of the Blaschke–Santalo inequality, and the existence of solutions to (4.1) for $p \in (0, n+2)$ was established in [9]. It was proved that for any given $\delta := n+2-p \in (0, n+2)$, there exists a solution H_δ to (4.1) with volume

$$|K_\delta| = \frac{1}{n+1} \int_{S^n} H_\delta \det\left(\nabla^2 H_\delta + H_\delta I\right) = 1, \tag{4.2}$$

and

$$\lambda = \lambda_\delta = (n+1)\left[\int_{S^n} \frac{f}{(H_\delta)^{p-1}}\right]^{-1}. \tag{4.3}$$

where K_δ is the convex body associated with H_δ. The solution H_δ is a maximizer of

$$\sup_{|K|=1} \inf_{\xi \in K} J(H - \xi \cdot x), \tag{4.4}$$

where the supremum is taken among all convex bodies K with volume 1, the infimum is taken among all points $\xi \in K$, and H is the support function of K. The functional J is given by

$$J(H) = \frac{1}{p-1} \int \frac{f}{H^{p-1}} \quad \text{if } p \neq 1;$$

and

$$J(H) = -\int f \log H \quad \text{if } p = 1.$$

The above existence was proved in [9] for general function f. If f is rotationally symmetric, then one may restrict to rotationally symmetric convex bodies such that the solution obtained in [9] is also rotationally symmetric. In the following we assume that f is rotationally symmetric and consider rotationally symmetric solutions only.

We want to prove that as $\delta \to 0^+$, H_δ converges to a solution H_0 of (1.1). Making a unimodular linear transform A_δ^T such that $\mathbb{K}_\delta := A_\delta^T(K_\delta)$ is normalized, let h_δ denote the support function of $\mathbb{K}_\delta$. Then by (2.12), h_δ satisfies,

$$\det\left(\nabla^2 h + hI\right) = \frac{\lambda_\delta f_\delta(\hat{H}_\delta)^\delta}{h^{n+2}} \quad \text{on } S^n, \tag{4.5}$$

where $f_\delta(x) = f(\frac{A_\delta x}{|A_\delta x|})$ and $\hat{H}_\delta(x) = H_\delta(\frac{A_\delta x}{|A_\delta x|})$.

Lemma 4.1. *There exists a constant $c_0 > 0$, depending only on n, $f_{\min}$, and $f_{\max}$, such that*

$$\lambda_\delta \leqslant c_0. \tag{4.6}$$

Proof. The upper bound for λ_δ follows from its definition (4.3) and the fact that the solution H_δ is a maximizer of (4.4). □

Lemma 4.2. *There exists a constant $c_1 > 0$, depending only on n, $f_{\min}$, and $f_{\max}$, such that as $\delta \to 0^+$,*

$$\lambda_\delta \geqslant c_1. \tag{4.7}$$

Proof. One can prove (4.7) easily if H_δ is uniformly bounded. Indeed, if $\lambda_\delta \to 0$ as $\delta \to 0$, then the right hand side of (4.1) vanishes on the part $\{x \in S^n \mid H_\delta(x) > 0\}$. It implies $\int_{H_\delta > 0} \det(\nabla^2 H_\delta + H_\delta I)$, that is the area measure of $\partial K_\delta \cap \{H_\delta > 0\}$, vanishes. But this is impossible by the volume restriction (4.2).

In the following we consider the case when H_δ is not uniformly bounded. Since the solution is rotationally symmetric, as before we express H_δ as a function of $\theta \in [0, \pi]$, such that $H_\delta(0)$ is the value of H_δ at the north pole and $H_\delta(\pi)$ the value at the south pole. Then there are two possibilities: $H_\delta(\frac{\pi}{2}) \to 0$ and $H_\delta(\frac{\pi}{2}) \to \infty$.

Denote

$$\beta^+ = H_\delta(0), \qquad \beta^- = -H_\delta(\pi), \qquad \beta = \beta^+ - \beta^-, \quad \text{and} \quad r = H_\delta\left(\frac{\pi}{2}\right);$$

and

$$\alpha^+ = h_\delta(0), \qquad \alpha^- = -h_\delta(\pi), \qquad \alpha = \alpha^+ - \alpha^-, \quad \text{and} \quad R = h_\delta\left(\frac{\pi}{2}\right).$$

Then the convex body K_δ is contained in the cylinder

$$C_\delta = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \beta^- \leqslant x_{n+1} \leqslant \beta^+,\ \sum_{i=1}^n x_i^2 < r^2 \right\};$$

and the normalized convex body $\mathbb{K}_\delta$ is contained in the cylinder

$$\mathbb{C}_\delta = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \alpha^- \leqslant x_{n+1} \leqslant \alpha^+,\ \sum_{i=1}^n x_i^2 < R^2 \right\}.$$

Since $\mathbb{K}_\delta$ is normalized, we have $C_1 \leqslant \alpha, R \leqslant C_2$ for some positive constants C_1, C_2 depending only on n.

Case I. $H_\delta(\frac{\pi}{2}) \to 0$. In this case, for any given $t > 0$, and any point $z \in \Lambda_{1,t} := \partial K \cap \{\beta^- + t\beta \leqslant x_{n+1} \leqslant \beta^+ - t\beta\}$, by the rotational symmetry and the convexity of K_δ, one easily verifies that

$$t H_\delta(\gamma_z) \leqslant H_\delta\left(\frac{\pi}{2}\right) \leqslant t^{-1} H_\delta(\gamma_z), \tag{4.8}$$

where $\gamma_z \in S^n$ is the unit outer normal of K_δ at z. Denote

$$\Gamma_{1,t} := \partial \mathbb{K}_\delta \cap \left\{ \alpha^- + t\alpha \leqslant x_{n+1} \leqslant \alpha^+ - t\alpha \right\},$$

which corresponds to $\Lambda_{1,t}$ before the normalization. Then $h_\delta(\gamma_z) \geqslant C > 0$ on $\Gamma_{1,t}$, here we also use γ_z to denote the unit outer normal of $\partial \mathbb{K}_\delta$ at z. Hence if $\lambda_\delta \to 0$, the right hand side of (4.5) converges uniformly to zero on $\{\gamma_z \in S^n \mid z \in \Gamma_{1,t}\} =: \Gamma_{1,t}^*$. It means by Eq. (4.5) that the area measure

$$|\Gamma_{1,t}| = \int_{\Gamma_{1,t}^*} \det\left(\nabla^2 h + hI\right) \to 0.$$

This is impossible as $\mathbb{K}_\delta$ is normalized.

Case II. $H_\delta(\frac{\pi}{2}) \to \infty$. In this case, both $H_\delta(0)$ and $H_\delta(\pi)$ converge to 0. Without loss of generality we assume that $H_\delta(0) \geqslant H_\delta(\pi)$. Denote by $S^{n,+}$ and $S^{n,-}$ the north and south hemispheres, respectively. For any given $t > 0$, and any point $z \in \Lambda^+_{2,t} := \{x \in \partial K_\delta \mid \sum_{i=1}^n x_i^2 \leqslant (1-t)^2 r^2,\ \gamma_z \in S^{n,+}\}$, similarly to (4.8) we have

$$tH_\delta(\gamma_z) \leqslant H_\delta(0) \leqslant t^{-1} H_\delta(\gamma_z). \tag{4.9}$$

Hence $(H_\delta(\gamma_z))^\delta$ is uniformly bounded. Denote

$$\Gamma^\pm_{2,t} := \left\{ z \in \partial \mathbb{K}_\delta \;\middle|\; \sum_{i=1}^n z_i^2 \leqslant (1-t)^2 R^2,\ \gamma_z \in S^{n,\pm} \right\}.$$

By the volume constraint (4.2) and recall that $H_\delta(0) \geqslant H_\delta(\pi)$, we have

$$h_\delta(\gamma_z) \geqslant C > 0 \quad \forall z \in \Gamma^+_{2,t}. \tag{4.10}$$

Hence if $\lambda_\delta \to 0$, the right hand side of (4.5) converges to zero uniformly on $\{\gamma_z \in S^n \mid z \in \Gamma^+_{2,t}\}$. It means by Eq. (4.5) that the area measure of $\Gamma^+_{2,t}$ converges to zero, which is a contradiction as $\mathbb{K}_\delta$ is normalized. $\square$

Lemma 4.3. *Suppose $p \in [n, n+2)$. If $H_\delta \leqslant C$ on S^n for some positive constant $C > 0$, then $H_\delta \geqslant C' > 0$, where C' depends only on n, C, c_0, c_1, $f_{\min}$ and $f_{\max}$.*

Proof. By (2.15), one sees that if $p \in [n, n+2)$ and if $\inf_{S^n} H_\delta$ is small, then $\int_{S^n} \frac{\lambda_\delta f}{H_\delta^p}$ is very large. But on the other hand

$$\int_{S^n} \frac{\lambda_\delta f}{H_\delta^p} = \int_{S^n} \det\bigl(\nabla^2 H_\delta + H_\delta I\bigr)$$

is equal to the area of ∂K_δ, which is uniformly bounded. $\square$

Lemma 4.4. *There exist two positive constants c_2, c_3 such that for $\delta \in (0, 2]$,*

$$c_2 \leqslant (\hat{H}_\delta)^\delta \leqslant c_3 \quad \text{on } S^n. \tag{4.11}$$

Proof. If the solution H_δ is uniformly bounded, by Lemma 4.3, we have $(H_\delta)^\delta \to 1$ uniformly on S^n and (4.11) holds. Therefore it suffices to consider the case when H_δ is not uniformly bounded. As in the proof of Lemma 4.2, we express H_δ as a function on $[0, \pi]$, and consider the two separate cases, namely $H_\delta(\frac{\pi}{2}) \to 0$ and $H_\delta(\frac{\pi}{2}) \to \infty$.

By the volume constraint (4.2), we have

$$\sup_{S^n} H_\delta \leqslant C_n \Bigl[\inf_{S^n} H_\delta\Bigr]^{-n}.$$

Hence the second inequality of (4.11) follows from the first one.

We prove the first inequality of (4.11) by contradiction. In the first case, namely when $\hat{H}_\delta(\frac{\pi}{2}) \to 0$,[2] note that by (4.8), $(\hat{H}_\delta(\gamma_z))^\delta$ converges to the same limit uniformly for all $z \in \Gamma_{1,t}$. If the limit is zero, the right hand side of (4.5) converges to zero uniformly on $\Gamma_{1,t}$. Hence the area measure of $\Gamma_{1,t}$ converges to zero, which is a contradiction.

In the second case, namely when $\hat{H}_\delta(\frac{\pi}{2}) \to \infty$, we see that by (4.9), $(\hat{H}_\delta(\gamma_z))^\delta$ converges to the same limit uniformly for all $z \in \Gamma^+_{2,t}$. If the limit is zero, by (4.10), the right hand side of (4.5) converges to zero uniformly on $\Gamma^+_{2,t}$. Hence the area measure of $\Gamma^+_{2,t}$ converges to zero, also a contradiction. Therefore $(\hat{H}_\delta(\gamma_z))^\delta$ converges to a positive constant on $\Gamma^+_{2,t}$.

The above proof also applies to $\Gamma^-_{2,t}$ provided (4.10) holds on $\Gamma^-_{2,t}$. In the following we prove (4.10) on $\Gamma^-_{2,t}$. By (2.11), $h_\delta(x_0) = |A_\delta(x_0)|\hat{H}_\delta(x_0)$, where x_0 is the north pole of S^n. By (4.10) and since $\mathbb{K}_\delta$ is normalized, there is a positive upper and lower bound for $h_\delta(x_0) = h_\delta(\theta)_{|\theta=0}$. Hence $\lim_{\delta\to 0}|A_\delta(x_0)|^\delta = \lim_{\delta\to 0}|\hat{H}_\delta(x_0)|^\delta$, which is positive by the last paragraph. In the rotationally symmetric case, the matrix A_δ has the form

$$A_\delta = \operatorname{diag}\left(a_\delta^{\frac{1}{n+1}}, \ldots, a_\delta^{\frac{1}{n+1}}, a_\delta^{-\frac{n}{n+1}}\right).$$

Hence if $|A_\delta(x_0)|^\delta$ converges to a positive constant, then $|A_\delta(x)|^\delta$ converges to positive constants for all $x \in S^n$.

By (2.11) we can write Eq. (4.5) in the form

$$\det\left(\nabla^2 h + hI\right) = \frac{\lambda_\delta f_\delta |A_\delta x|^\delta}{h^p}, \quad x \in S^n. \tag{4.12}$$

We have shown that $|A_\delta x|^\delta$ converges uniformly to a positive constant. As $\mathbb{K}_\delta$ is normalized, h_δ is uniformly bounded. Hence by the argument of Lemma 4.3, $h_\delta > C$ on S^n, namely (4.10) holds on $\Gamma^-_{2,t}$. □

We can strengthen Lemma 4.4 to the following

Lemma 4.5. *There exists a positive constant $c_4 > 0$ such that for any $x \in S^n$, not on the equator,*

$$\left(\hat{H}_\delta(x)\right)^\delta \to c_4 \quad as\ \delta \to 0. \tag{4.13}$$

Proof. In the proof of Lemma 4.4, we have shown that h_δ is uniformly bounded and strictly positive, and $|Ax|^\delta$ converges to a positive constant. Hence $(\hat{H}_\delta(\gamma_z))^\delta$ converges to the same positive constant on $\Gamma^+_{2,t}$ and $\Gamma^-_{2,t}$. Since the right hand side of (4.5) is uniformly bounded and strictly positive, the hypersurface $\partial\mathbb{K}_\delta$ is strictly convex and C^1 smooth [4]. Hence (4.13) holds as the constant $t > 0$ is arbitrarily chosen. □

Remark. Lemma 4.5 can be strengthened to

$$\left(\hat{H}_\delta(x)\right)^\delta \to 1 \quad \text{as } \delta \to 0, \tag{4.13a}$$

uniformly on the whole sphere S^n. Indeed, one can prove that the sup of (4.4) is continuous for $p \in (0, n+2]$, up to $p = n+2$. Therefore λ_δ is continuous as $\delta \to 0$. From the proof of Lemma 4.4, we have $c_4 \leqslant 1$. If $c_4 < 1$, namely if (4.13a) is not true, from Eq. (4.5) one can show that sup of (4.4) is not continuous at $p = n+2$.

---

[2] Recall the relation $\hat{H}_\delta(x) = H_\delta(\frac{A_\delta x}{|A_\delta x|})$. For points on $\Lambda_{1,t}$ and $\Lambda_{2,t}$, the corresponding function is H_δ, and for points on $\Gamma_{1,t}$ and $\Gamma_{2,t}$, the corresponding function is $\hat{H}_\delta$, where $\Lambda_{1,t}$, $\Gamma_{1,t}$ etc. are the notation introduced in the proof of Lemma 4.2.

We are in position to prove the existence of solutions to (1.1) (Theorem 1.3). It suffices to prove the following

Lemma 4.6. *Under assumptions of Theorem 1.3, the sequence of solutions H_δ is uniformly bounded as $\delta \to 0$.*

Proof. Write Eq. (4.1) as

$$\det\big(\nabla^2 H + HI\big) = \frac{\lambda f H^\delta}{H^{n+2}} \quad \text{on } S^n. \tag{4.14}$$

Let H_δ be the solution of (4.14) and regard it as a function of $\theta \in [0, \pi]$. Suppose there is a sequence $\delta \to 0$ such that $\sup_{S^n} H_\delta \to \infty$. Denote $a_\delta \approx [H_\delta(\pi/2)]^{-n-1}$ and make the linear transform

$$A_\delta = \operatorname{diag}\big(a_\delta^{\frac{1}{n+1}}, \dots, a_\delta^{\frac{1}{n+1}}, a_\delta^{-\frac{n}{n+1}}\big)$$

such that $\mathbb{K}_\delta = A_\delta^T(K_\delta)$ is normalized. Then h_δ, the support function of $\mathbb{K}_\delta$, satisfies the equation

$$\det\big(\nabla^2 h + hI\big) = \frac{\lambda f_\delta(\hat{H}_\delta)^\delta}{h^{n+2}} \quad \text{on } S^n, \tag{4.15}$$

where by (2.11) and (2.12),

$$f_\delta(x) = f\left(\frac{A_\delta x}{|A_\delta x|}\right), \qquad h_\delta(x) = |A_\delta x| \hat{H}_\delta(x), \qquad \hat{H}_\delta(x) = H_\delta\left(\frac{A_\delta x}{|A_\delta x|}\right).$$

For simplicity we will drop the subscript δ if no confusion arises. In the spherical coordinates, by (3.3) and (3.5), we see

$$f_\delta(\theta) = f\big(\gamma_a(\theta)\big), \qquad h = E_a \hat{H}_\delta, \quad E_a = a^{-\frac{n}{n+1}} i_a.$$

Denote

$$\hat{f} = f_\delta E_a^{-\delta},$$

then

$$f_\delta(\hat{H}_\delta)^\delta = \hat{f} h^\delta.$$

Applying the necessary condition (3.1) to Eq. (4.15), we get

$$\begin{aligned} 0 &= \int_0^\pi \frac{(h^\delta \hat{f})' \sin^n\theta \cos\theta}{h^{n+1}(\theta)}\, d\theta \\ &= \int_0^\pi \frac{\hat{f}' \sin^n\theta\cos\theta}{h^{p-1}}\, d\theta + \int_0^\pi \frac{(h^\delta)' \hat{f} \sin^n\theta\cos\theta}{h^{n+1}}\, d\theta. \end{aligned} \tag{4.16}$$

Note that

$$\frac{(h^\delta)'}{h^{n+1}} = \frac{\delta}{1-p}\big(h^{1-p}\big)',$$

and use integration by part, we see that the second integral of (4.16) becomes

$$\begin{aligned}
&\frac{\delta}{1-p}\int_0^\pi \big(h^{1-p}\big)'\hat f \sin^n\theta\cos\theta\, d\theta \\
&\quad = \frac{\delta}{p-1}\left(\int_0^\pi \frac{\hat f' \sin^n\theta\cos\theta}{h^{p-1}}\, d\theta + \int_0^\pi \frac{\hat f(\sin^n\theta\cos\theta)'}{h^{p-1}}\, d\theta\right).
\end{aligned}$$

Substituting it into (4.16), and multiplying both sides by $p-1$, we have

$$0 = (n+1)\int_0^\pi \frac{\hat f' \sin^n\theta\cos\theta}{h^{p-1}}\, d\theta + \delta\int_0^\pi \frac{\hat f(\sin^n\theta\cos\theta)'}{h^{p-1}}\, d\theta. \tag{4.17}$$

Using that

$$\begin{aligned}
\hat f' &= E_a^{-\delta} f_\delta' - \delta f_\delta E_a^{-\delta-1} E_a' \\
&= E_a^{-\delta} f'\big(\gamma_a(\theta)\big) a\, i_a^{-2} - \delta f_\delta E_a^{-\delta-1} E_a',
\end{aligned}$$

we can write (4.17) as

$$\begin{aligned}
&(n+1)\int_0^\pi \frac{E_a^{-\delta}}{h^{p-1}}\cdot f'\big(\gamma_a(\theta)\big)\frac{a\sin^n\theta\cos\theta}{i_a^2(\theta)}\, d\theta \\
&\quad = \delta\int_0^\pi \frac{E_a^{-\delta} f_\delta}{h^{p-1}}\big[(n+1)E_a^{-1}E_a'\sin^n\theta\cos\theta - \big(\sin^n\theta\cos\theta\big)'\big]\, d\theta.
\end{aligned} \tag{4.18}$$

Let I_δ denote the integral on the right hand side of (4.18), then we see

$$I_\delta = \int_0^\pi \frac{E_a^{-\delta} f_\delta}{h^{p-1}}\cdot\frac{a^2\sin^2\theta - n\cos^2\theta}{a^2\sin^2\theta + \cos^2\theta}\sin^{n-1}\theta\, d\theta. \tag{4.19}$$

On the other hand, by Blaschke's selection theorem, we may assume that h_δ converges uniformly to some support function h_0 on S^n, which is also normalized and rotationally symmetric. By the weak convergence of the Monge–Ampère equation, h_0 is a generalized solution to

$$\det\big(\nabla^2 h + hI\big) = \frac{c_4\lambda_0 f_0}{h^{n+2}} \quad \text{on } S^n, \tag{4.20}$$

where

$$f_0 = \begin{cases} f(\pi/2) & \text{if } a_\delta \to \infty, \\ f(0)\chi_{\{x_{n+1}>0\}} + f(\pi)\chi_{\{x_{n+1}<0\}} & \text{if } a_\delta \to 0^+. \end{cases}$$

In the case of $a_\delta \to +\infty$, f_0 is a constant. In this case, a solution to (4.20) is an elliptic affine sphere. Hence it must be an ellipsoid [5]. But the solution is normalized, so it must be a sphere. Therefore $h_0 \equiv (c_4\lambda_0 f(\pi/2))^{\frac{1}{2n+2}}$. Recalling that $E_a^{-\delta} \to c_4$, by the bounded convergence theorem we obtain from (4.19) that

$$\lim_{\delta\to 0} I_\delta = \sqrt{c_4\lambda_0^{-1} f(\pi/2)} \int_0^\pi \sin^{n-1}\theta\, d\theta =: C_0.$$

By our assumption that $f'(\pi/2) = 0$ and applying Lemma 3.2 to the left hand side of (4.18), we see that (4.18) becomes into

$$\big(C_1 + o(1)\big)\delta = \begin{cases} (ni(f) + o(1))\frac{1}{a^2}, & n \geqslant 3, \\ (ni(f) + o(1))\frac{\log a^2}{a^2}, & n = 2, \\ (ni(f) + o(1))\frac{1}{a}, & n = 1, \end{cases} \tag{4.21}$$

where C_1 is a positive constant depending only on n, c_4, λ_0 and $f(\pi/2)$.

In the case $a_\delta \to 0^+$, f_0 is equal to two different constants on the north and south hemispheres. In this case, the solution h_0 to (4.20) is strictly convex and C^1 smooth [4]. By the bounded convergence theorem, we obtain from (4.19) that

$$\lim_{\delta\to 0} I_\delta = -n \int_0^\pi \frac{c_4 f_0(\theta)}{h_0^{n+1}(\theta)} \sin^{n-1}\theta\, d\theta =: -C_0.$$

Applying Lemma 3.3 to the left hand side of (4.18), we see that (4.18) becomes into

$$\big(-C_1 + o(1)\big)\delta = \big(pi(f) + o(1)\big)a, \tag{4.22}$$

where C_1 is a positive constant depending only on n, c_4, λ_0, $f(0)$ and $f(\pi)$.

By our assumption, $ni(f) < 0$ and $pi(f) > 0$. Hence neither (4.21) nor (4.22) holds. In both cases we reach a contradiction. Thus the lemma is proved. □

Remark. Using the topological degree argument [1,13], one may also prove the existence when $ni(f) > 0$ and $pi(f) < 0$. In the high dimensions the degree argument is more complicated than that in [13] as one needs to work out the kernel of the linearized operator of (4.1). Here we choose the above blow-up argument and we plan to explore this approach further in a subsequent work for the case when f is not rotationally symmetric, using the fact that H_δ is a maximizer of (4.4).

5. Rotationally symmetric solutions in the super-critical case

In this section we consider the existence of rotationally symmetric maximizers of

$$\sup_{|K|=1} \inf_{\xi\in K} J(H - \xi\cdot x), \tag{5.1}$$

where as in Section 4, the supremum is taken among all convex bodies K with volume 1, the infimum is taken among all points $\xi \in K$, H is the support function of K, and the functional J is given by

$$J(H)=\frac{1}{p-1}\int\frac{f}{H^{p-1}},\quad p>n+2,$$

where $p>n+2$ corresponds to the supercritical case of the Blaschke–Santalo inequality.

When $p>n+2$, the supremum is usually equal to infinity. But in the special case when convex bodies K are rotationally symmetric and f vanishes at $\theta=0$, $\frac{\pi}{2}$, and π, we show that the supremum can be attained by a convex body. From the argument in [9], the associated support function satisfies the Euler equation (4.1) with the Lagrange multiple λ given by (4.3), if f vanishes only at finitely many points. If f vanishes in an open set, the solution must be understood in a generalized sense.

Theorem 5.1. *Let $f\in C[0,\pi]$ be a bounded, nonnegative function satisfying $f(0)=f(\frac{\pi}{2})=f(\pi)=0$ and $f\not\equiv 0$ elsewhere. Suppose*

$$\begin{aligned}
&f(\theta)\leqslant C\theta^{\alpha}\quad \textit{near } \theta=0,\\
&f(\theta)\leqslant C|\pi-\theta|^{\alpha}\quad \textit{near } \theta=\pi,\\
&f(\theta)\leqslant C|\theta-\pi/2|^{\alpha}\quad \textit{near } \theta=\pi/2,
\end{aligned}\tag{5.2}$$

where $\alpha>\frac{n}{n+1}(p-n-2)$. Then there is a maximizer of (5.1)*.*

Proof. We denote $\inf_{\xi\in K}J(H-\xi\cdot x)$ by M_K. Let K_j be a maximizing sequence of (5.1), and E_j be an ellipsoid such that $E_j\subset K_j\subset(n+1)E_j$, see (2.13). One easily sees that $M_{K_j}\leqslant M_{E_j}$. To show that K_j is uniformly bounded, it suffices to show that E_j is uniformly bounded. Since K_j is rotationally symmetric, E_j is also rotationally symmetric and so it can be given by

$$E_j=\left\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}\;\middle|\;\frac{x_1^2+\cdots+x_n^2}{a_j^2}+\frac{x_{n+1}^2}{b_j^2}<1\right\}.$$

Since the supremum is invariant by a translation of the convex body, we assume that the origin is the center of E_j. Let

$$\mathcal{C}_j=\left\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}\;\middle|\;\left(x_1^2+\cdots+x_n^2\right)^{1/2}<a_j,\ |x_{n+1}|<b_j\right\}$$

be a cylinder. We show that if $\max(a_j,b_j)\to\infty$, then $M_{\mathcal{C}_j}\to 0$. It implies that $M_{E_j}\leqslant M_{\mathcal{C}_j/2}\to 0$ (by the homogeneity of the functional J). But since $f\geqslant 0$ and $f\not\equiv 0$, the supremum of (5.1) is positive. This contradiction implies that K_j is uniformly bounded.

First we consider the case $a_j\to\infty$. Denote H_j the support function of $\mathcal{C}_j$. Then

$$\begin{aligned}
&H_j(\theta)=b_j\cos\theta+a_j\sin\theta,\quad \theta\in[0,\pi/4],\\
&H_j(\theta)\geqslant a_j/2,\quad \theta\in[\pi/4,\pi/2].
\end{aligned}$$

Note that by the homogeneity of the functional J, we may assume that $a_j^n b_j=1$. Note that for $\theta\in[\pi/2,\pi]$, the computation is the same. Hence we have

$$\begin{aligned}
M_{\mathcal{C}_j}&\leqslant J(H_j)\\
&\leqslant Ca_j^{1-p}+C\int_0^{\pi/4}\frac{\theta^{\alpha+n-1}}{(b_j+a_j\theta)^{p-1}}
\end{aligned}$$

$$= Ca_j^{1-p} + C \int_0^{\pi/4} \frac{\theta^{\alpha+n-1}}{(a_j^{-n} + a_j\theta)^{p-1}}$$

$$\leqslant Ca_j^{1-p} + C \int_0^{\pi/4} \frac{\theta^{\alpha+n-1}}{a_j^{-n(p-1)}(1 + a_j^{n+1}\theta)^{p-1}}.$$

By direct computation, we then obtain

$$M_{\mathcal{C}_j} \leqslant \begin{cases} Ca_j^{1-p} & \text{if } \alpha > p-n-1, \\ Ca_j^{\alpha(n+1)-n(p-n-2)} \log a_j & \text{if } \alpha = p-n-1, \\ Ca_j^{\alpha(n+1)-n(p-n-2)} & \text{if } \alpha < p-n-1 \end{cases}$$
$$\to 0 \quad \text{as } a_j \to \infty.$$

Next we consider the case $b_j \to \infty$. In this case, we have

$$H_j(\theta) = a_j \sin\theta + b_j \cos\theta, \quad \theta \in [\pi/4, \pi/2],$$
$$H_j(\theta) \geqslant b_j/2, \quad \theta \in [0, \pi/4].$$

Making the change $\phi = \pi/2 - \theta$, we obtain by the above computation that $M_{\mathcal{C}_j} \to 0$ as $b_j \to \infty$. This completes the proof. □

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