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Curvature estimates for stable free boundary minimal hypersurfaces

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Abstract. In this paper, we prove uniform curvature estimates for immersed stable free boundary minimal hypersurfaces satisfying a uniform area bound, which generalize the celebrated Schoen–Simon–Yau interior curvature estimates up to the free boundary. Our curvature estimates imply a smooth compactness theorem which is an essential ingredient in the min-max theory of free boundary minimal hypersurfaces developed by the last two authors. We also prove a monotonicity formula for free boundary minimal submanifolds in Riemannian manifolds for any dimension and codimension. For 3-manifolds with boundary, we prove a stronger curvature estimate for properly embedded stable free boundary minimal surfaces without a-priori area bound. This generalizes Schoen's interior curvature estimates to the free boundary setting. Our proof uses the theory of minimal laminations developed by Colding and Minicozzi.

1. Introduction

Let (M^m, g) be an m-dimensional Riemannian manifold and $N^n \subset M^m$ an embedded n-dimensional submanifold called the *constraint submanifold*. If we consider the k-dimensional area functional on the space of immersed k-submanifolds $\Sigma^k \subset M^m$ with boundary $\partial\Sigma$ lying on the constraint submanifold N, the critical points are called *free boundary minimal submanifolds*. These are minimal submanifolds $\Sigma \subset M$ meeting N orthogonally along $\partial\Sigma$ (cf. Definition 2.2). Such a critical point is said to be *stable* (cf. Definition 2.4) if it minimizes area up to second order. The purpose of this paper is three-fold. First, we prove uniform curvature estimates (Theorem 1.1) for *immersed* stable free boundary minimal hypersurfaces satisfying a uniform area bound. Second, we prove a monotonicity formula (Theorem 3.4) near the

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boundary for free boundary minimal submanifolds in any dimension and codimension. Finally, we use Colding–Minicozzi's theory of minimal laminations (adapted to the free boundary setting) to establish a stronger curvature estimate (Theorem 1.2) for *properly embedded* stable free boundary minimal surfaces in compact Riemannian 3-manifolds with boundary, *without* assuming a uniform area bound on the minimal surfaces.

Curvature estimates for immersed stable minimal hypersurfaces in Riemannian manifolds were first proved in the celebrated work of Schoen, Simon and Yau in [18]. Such curvature estimates have profound applications in the theory of minimal hypersurfaces. For example, Pitts [14] made use of Schoen–Simon–Yau's estimates in an essential way to establish the regularity of minimal hypersurfaces Σ constructed by min-max methods, for $2 \leq \dim \Sigma \leq 5$ due to the dimension restriction in [18]. Shortly after, Schoen and Simon [17] generalized these curvature estimates to any dimension (but still for codimension one, i.e. *hypersurfaces*) for *embedded* stable minimal hypersurfaces, which enabled them to complete Pitts' program for $\dim \Sigma > 5$.

In this paper, we establish uniform curvature estimates in the free boundary setting. The theorem below follows from our curvature estimates near the free boundary (Theorem 4.1) and the interior curvature estimates [18]. We refer the readers to Section 2 for the precise definitions.

Theorem 1.1. *Assume $2 \leq n \leq 6$. Let M^{n+1} be a Riemannian manifold and $N^n \subset M$ an embedded hypersurface, both without boundary. Suppose that $U \subset M$ is an open subset with compact closure. If $(\Sigma, \partial\Sigma) \looparrowright (U, N \cap U)$ is an immersed (embedded when $n = 6$) stable free boundary minimal hypersurface with* $\mathrm{Area}(\Sigma) \leq C_0$*, then*

$$|A^\Sigma|^2(x) \leq \frac{C_1}{\operatorname{dist}^2_M(x, \partial U)} \quad \text{for all } x \in \Sigma,$$

where $C_1 > 0$ is a constant depending only on C_0, U and $N \cap U$.

An important consequence of Theorem 1.1 is a smooth compactness theorem for stable free boundary minimal hypersurfaces which are *almost properly embedded* (cf. [13]). As in the work [14], this is a key ingredient in the regularity part of the min-max theory for free boundary minimal hypersurfaces in compact Riemannian manifolds with boundary, which is developed in [13] by the last two authors. We remark that any compact Riemannian manifold Ω with boundary $\partial\Omega = N$ can be extended to a closed Riemannian manifold M with Ω as a compact domain. Hence, our curvature estimates above can be applied in this situation as well.

Our proof of the curvature estimates uses a contradiction argument. If the curvature estimates do not hold, we can apply a blow-up argument to a sequence of counterexamples together with a reflection principle to obtain a nonflat complete stable immersed minimal hypersurface Σ_∞ in $\mathbb{R}^{n+1}$ *without boundary*. We then apply the Bernstein theorem in [18, Theorem 2] (which only holds for $2 \leq n \leq 5$) or [17, Theorem 3] (when $n = 6$ for embedded hypersurface) to conclude that Σ_∞ is flat, hence resulting in a contradiction. Using Ros' estimates [15, Theorem 9 and Corollary 11] for one-sided stable minimal surfaces, our result also holds true when $n = 2$ if one removes the two-sided condition. When $n \geq 7$, the stable free boundary minimal hypersurface may contain a singular set with Hausdorff codimension at least seven. This follows from similar arguments as in [17]. To keep this paper less technical, the details will appear in a forthcoming paper.

The classical monotonicity formula plays an important role in the regularity theory for minimal submanifolds, even without the stability assumption. Unfortunately, it ceases to hold once the ball hits the boundary of the minimal submanifold. Therefore, to study the boundary regularity of free boundary minimal submanifolds, we need a monotonicity formula which holds for balls centered at points lying on the constraint submanifold N. By an isometric embedding of M into some Euclidean space $\mathbb{R}^L$, we establish a monotonicity formula (Theorem 3.4) for free boundary minimal submanifolds relative to *Euclidean* balls of $\mathbb{R}^L$ centered at points on the constraint submanifold N.

We remark that Grüter and Jost proved in [10, Theorem 3.1] a version of monotonicity formula and used it to establish an important Allard-type regularity theorem for varifolds with free boundary. However, the monotonicity formula they obtained contains an extra term involving the mass of the varifold in a reflected ball (whose center may not lie on the constraint submanifold N), which makes it difficult to apply in some situations (in [13] for example). In contrast, our monotonicity formula (Theorem 3.4) does not require any reflection which makes it more readily applicable. Moreover, the formula holds in the Riemannian manifold setting for stationary varifolds with free boundary in any dimension and codimension. We expect that our monotonicity formula might be useful in the regularity theory for other natural free boundary problem in calibrated geometries (see for example [4] and [11]). We would like to mention that other monotonicity formulas have been proved for free boundary minimal submanifolds in a Euclidean unit ball (see [3, 21]).

Consider now the case of a compact Riemannian 3-manifold M with boundary ∂M; by the remark in the paragraph after Theorem 1.1, we can assume that M is a compact subdomain of a larger Riemannian manifold $\widetilde{M}$ without boundary and $N = \partial M$ is the constraint submanifold. Furthermore, if we assume that the free boundary minimal surface Σ is *properly embedded* in M (i.e. $\Sigma \subset M$ and $\Sigma \cap \partial M = \partial\Sigma$), then we prove a stronger uniform curvature estimate similar to the one in Theorem 1.1, but *independent of the area of* Σ.

Theorem 1.2. *Let (M^3, g) be a compact Riemannian 3-manifold with nonempty boundary ∂M. Then there exists a constant $C_2 > 0$ depending only on the geometry of M and ∂M such that if $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ is a compact, properly embedded stable minimal surface with free boundary, then*

$$\sup_{x\in\Sigma} |A|^2(x) \le C_2.$$

Remark 1.3. For simplicity, we assume that Σ is compact in Theorem 1.2. This ensures that Σ has no boundary points lying in the interior of M. Without the compactness assumption, similar uniform estimates still hold as long as we stay away from the points in $\overline{\Sigma} \setminus \Sigma$ inside the interior of M as in Theorem 1.1. Note that Σ is always locally two-sided under the embeddedness assumption.

Our proof of Theorem 1.2 involves the theory of minimal laminations which require the minimal surface to be *embedded*. In view of the celebrated interior curvature estimates for stable *immersed* minimal surfaces in 3-manifolds by Schoen [16] (see also [6] and [15]), we conjecture that the embeddedness of Σ is unnecessary.

Conjecture 1.4. Theorem 1.2 holds even when Σ is immersed.

The organization of the paper is as follows. In Section 2, we give the basic definitions for free boundary minimal submanifolds in any dimension and codimension and discuss the notion of stability in the hypersurface case. In Section 3, we prove the monotonicity formula (Theorem 3.4) for stationary varifolds with free boundary near the free boundary in any dimension and codimension. In Section 4, we prove our main curvature estimates (Theorem 4.1) for stable free boundary minimal hypersurfaces near the free boundary. In Section 5, we prove the stronger curvature estimate (Theorem 1.2) in the case of *properly embedded* stable free boundary minimal surfaces in a Riemannian 3-manifold with boundary. In Section 6, we prove a general convergence result for free boundary minimal submanifolds (in any dimension and codimension) satisfying uniform bounds on area and the second fundamental form. Finally, in Section 7, we prove a lamination convergence result for free boundary minimal surfaces in a three-manifold with uniform bound depending only on the second fundamental form of the minimal surfaces.

2. Free boundary minimal submanifolds

In this section, we give the definition of *free boundary minimal submanifolds* (Definition 2.2) and the notion of *stability* (Definition 2.4) in the hypersurface case. We also prove a reflection principle (Lemma 2.6) which will be useful in subsequent sections.

Let (M, g) be an m-dimensional Riemannian manifold, and let $N \subset M$ be an embedded n-dimensional constraint submanifold. We will always assume that M, N are smooth without boundary unless otherwise stated. Suppose that Σ is a k-dimensional smooth manifold with boundary $\partial\Sigma$ (possibly empty).

Definition 2.1. We use $(\Sigma, \partial\Sigma) \looparrowright (M, N)$ to denote an immersion $\varphi : \Sigma \to M$ such that $\varphi(\partial\Sigma) \subset N$. If, furthermore, φ is an embedding, we denote it as $(\Sigma, \partial\Sigma) \subset (M, N)$. An embedded submanifold $(\Sigma, \partial\Sigma) \subset (M, N)$ is said to be *proper* if $\varphi(\Sigma) \cap N = \varphi(\partial\Sigma)$.

Definition 2.2. We say that $(\Sigma, \partial\Sigma) \subset (M, N)$ is an immersed (resp. embedded) *free boundary minimal submanifold* if

(i) $\varphi : \Sigma \to M$ is a minimal immersion (resp. embedding),

(ii) Σ meets N orthogonally along $\partial\Sigma$.

Remark 2.3. Condition (ii), is often called the *free boundary condition*. Note that both conditions (i) and (ii) are local properties.

Free boundary minimal submanifolds can be characterized variationally as critical points to the k-dimensional area functional of (M, g) among the class of all immersed k-submanifolds $(\Sigma, \partial\Sigma) \looparrowright (M, N)$. Given a smooth one-parameter family of immersions

$$\varphi_t : (\Sigma, \partial\Sigma) \looparrowright (M, N), \quad t \in (-\epsilon, \epsilon),$$

whose variation vector field

$$X(x) = \frac{d}{dt}\Big|_{t=0} \varphi_t(x)$$

is compactly supported in Σ, the *first variational formula* (cf. [6, Section 1.3]) says that

$$\frac{d}{dt}\bigg|_{t=0} \operatorname{Area}(\varphi_t(\Sigma)) = \int_\Sigma \operatorname{div}_\Sigma X \, da = -\int_\Sigma X \cdot H \, da + \int_{\partial\Sigma} X \cdot \eta \, ds, \tag{2.1}$$

where H is the mean curvature vector of the immersion $\varphi_0 : \Sigma \to M$ with outward unit conormal η, da and ds are the induced measures on Σ and $\partial\Sigma$, respectively. Since $\varphi_t(\partial\Sigma) \subset N$ for all t, it follows that the variation vector field X must be tangent to N along $\partial\Sigma$. Therefore, $\varphi : (\Sigma, \partial\Sigma) \looparrowright (M, N)$ is a free boundary minimal submanifold if and only if (2.1) vanishes for all compactly supported variational vector field X with $X(p) \in T_p N$ for all $p \in \partial\Sigma$, which is equivalent to conditions (i) and (ii) in Definition 2.2.

Since free boundary minimal submanifolds are critical points to the area functional, we can look at the second variation and study their stability. Roughly speaking, a free boundary minimal submanifold is said to be *stable* if the second variation is nonnegative. For simplicity and our purpose, we will only consider the *hypersurface* case, i.e. $\dim \Sigma = \dim N = m - 1$, where $m = \dim M$. Recall that an immersion $\varphi : \Sigma \to M$ is said to be *two-sided* if there exists a globally defined continuous unit normal vector field ν on Σ.

Definition 2.4. Let $\varphi : (\Sigma, \partial\Sigma) \looparrowright (M, N)$ be an immersed free boundary minimal hypersurface. Then φ is said to be *stable* if it is two-sided and satisfies the stability inequality, i.e.

$$\begin{aligned} 0 &\le \frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{Area}(\varphi_t(\Sigma)) \\ &= \int_\Sigma |\nabla_\Sigma f|^2 - (|A^\Sigma|^2 + \operatorname{Ric}(\nu, \nu)) f^2 \, da - \int_{\partial\Sigma} A^N(\nu, \nu) f^2 \, ds, \end{aligned}$$

where $\varphi_t : (\Sigma, \partial\Sigma) \looparrowright (M, N)$ is any compactly supported variation of $\varphi_0 = \varphi$ with variation field $X = f\nu$, A^Σ and A^N are the second fundamental forms of Σ and N in M, respectively, and Ric is the Ricci curvature of M.

Remark 2.5. The sign convention of A^N in Definition 2.4 is taken such that $A^N \ge 0$ if $N = \partial\Omega$ is the boundary of a convex domain in M.

One particularly important example is as follows: $M = \mathbb{R}^{n+1}$ and $N = \mathbb{R}^n = \{x_1 = 0\}$. Let $\mathbb{R}^{n+1}_+ = \{x_1 \ge 0\}$ and let $\theta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection map across $\mathbb{R}^n$. We have the following reflection principle that relates free boundary minimal hypersurfaces with minimal hypersurfaces without boundary.

Lemma 2.6 (Reflection principle). *If $(\Sigma, \partial\Sigma) \looparrowright (\mathbb{R}^{n+1}, \mathbb{R}^n)$ is an immersed stable free boundary minimal hypersurface, then $\Sigma \cup \theta(\Sigma)$ is an immersed stable minimal hypersurface (without boundary) in $\mathbb{R}^{n+1}$.*

Proof. Since minimality is preserved under the isometry θ of $\mathbb{R}^{n+1}$ and that Σ is orthogonal to $\mathbb{R}^n$ along $\partial\Sigma$, it follows that $\Sigma \cup \theta(\Sigma)$ is a C^1 minimal hypersurface in $\mathbb{R}^{n+1}$ without boundary. Higher regularity for minimal hypersurfaces implies that it is indeed smooth across $\partial\Sigma$. Stability follows directly from the definition since the boundary term in the stability inequality of Definition 2.4 vanishes for $N = \mathbb{R}^n$. □

3. Monotonicity formula

In this section, we prove a monotonicity formula (Theorem 3.4) for stationary varifolds with free boundary (cf. Definition 3.1) in Riemannian manifolds for any dimension and codimension. The monotonicity formula for free boundary minimal submanifolds is then a direct corollary.

Throughout this section, we will always assume that M is isometrically embedded into the Euclidean space $\mathbb{R}^L$ (such an embedding exists by Nash's isometric embedding theorem) and $N \subset M$ is a closed n-dimensional constraint submanifold. We will denote $\widetilde{B}(p,r)$ to be the open Euclidean ball in $\mathbb{R}^L$ with center p and radius $r > 0$.[1)] The second fundamental form of M in $\mathbb{R}^L$ is denoted by A^M.

We begin with a discussion on the notion of *stationary varifolds with free boundary*. Let $\mathcal{V}_k(M)$ denote the closure (with respect to the weak topology) of rectifiable *k-varifolds* in $\mathbb{R}^L$ which is supported in M (cf. [14, 2.1(18)(g)]). As usual, the *weight* of a varifold $V \in \mathcal{V}_k(M)$ is denoted by $\|V\|$. We refer the readers to the standard reference [19] on varifolds.

We use $\mathfrak{X}(M,N)$ to denote the space of smooth vector fields X compactly supported on $\mathbb{R}^L$ such that $X(x) \in T_xM$ for all $x \in M$ and $X(p) \in T_pN$ for all $p \in N$. Any such vector field $X \in \mathfrak{X}(M,N)$ generates a one-parameter family of diffeomorphisms $\phi_t : M \to M$ with $\phi_t(N) = N$ and the first variation of a varifold $V \in \mathcal{V}_k(M)$ along X is defined by

$$\delta V(X) := \frac{d}{dt}\bigg|_{t=0} \|(\phi_t)_\sharp V\|(M),$$

where $(\phi_t)_\sharp V \in \mathcal{V}_k(M)$ denotes the pushforward of the varifold V by the diffeomorphism ϕ_t (cf. [14, 2.1(18)(h)]).

Definition 3.1. A k-varifold $V \in \mathcal{V}_k(M)$ is said to be *stationary with free boundary on N* if $\delta V(X) = 0$ for all $X \in \mathfrak{X}(M,N)$.

This generalizes the notion of free boundary minimal submanifolds to allow singularities. By the first variation formula for varifolds [19, Section 39.2], a k-varifold $V \in \mathcal{V}_k(M)$ is stationary with free boundary on N if and only if

$$\int_{G_k(\mathbb{R}^L)} \operatorname{div}_S X(x)\, dV(x,S) = 0 \tag{3.1}$$

for all $X \in \mathfrak{X}(M,N)$. If X is not tangent to M but $X(p) \in T_pN$ for all $p \in N$, then (3.1) implies that

$$\int_{G_k(\mathbb{R}^L)} \operatorname{div}_S X(x)\, dV(x,S) = \int_{G_k(\mathbb{R}^L)} X(x) \cdot \operatorname{tr}_S A^M \, dV(x,S), \tag{3.2}$$

where $S \subset T_xM$ is an arbitrary k-plane, and

$$\operatorname{tr}_S A^M = \sum_{i=1}^{k} A^M(e_i, e_i)$$

for an orthonormal basis $\{e_1, \dots, e_k\}$ of S.

[1)] Note that our notation is different from that in [10] where $\widetilde{B}$ is used to denote some kind of reflected ball.

The key idea to derive our monotonicity formula near a base point $p \in N$ is to find a special test vector field X which is asymptotic (near p) to the radial vector field centered at p and, at the same time, tangential along the constraint submanifold N. Our choice of X is largely motivated by [2, 10]. For convenience of the readers, we provide details on some of the preliminary results below.

Let us review some local geometry of the k-dimensional closed constraint submanifold N in $\mathbb{R}^L$ essentially following the discussions in [2, Section 2]. We always identify a linear subspace $P \subset \mathbb{R}^L$ with its orthogonal projection $P \in \mathrm{Hom}(\mathbb{R}^L, \mathbb{R}^L)$ onto this subspace. Using this notion, we define the maps $\tau, \nu : N \to \mathrm{Hom}(\mathbb{R}^L, \mathbb{R}^L)$ to be

$$\tau(p) := T_p N \quad \text{and} \quad \nu(p) := (T_p N)^{\perp},$$

where $T_p N$ is the tangent space of N in $\mathbb{R}^L$, and $(T_p N)^{\perp}$ is the orthogonal complement of $T_p N$ in $\mathbb{R}^L$.

To bound the turning of N inside $\mathbb{R}^L$, we define as in [2] a global geometric quantity

$$\kappa := \inf\left\{ t \geq 0 : |\nu(x)(y - x)| \leq \frac{t}{2}|y - x|^2 \text{ for all } x, y \in N \right\}.$$

By the compactness and smoothness of N, $\kappa \in [0, \infty)$ and thus one can define the *radius of curvature for* N to be

$$R_0 := \kappa^{-1} \in (0, \infty]. \tag{3.3}$$

Let ξ be the nearest point projection map onto N and let $\rho(\,\cdot\,) := \mathrm{dist}_{\mathbb{R}^L}(\,\cdot\,, N)$ be the distance function to N *in* $\mathbb{R}^L$, both defined on a tubular neighborhood of N. More precisely, if we define the open set

$$A := \bigcup_{p \in N} \widetilde{B}(p, R_0)$$

which is an open neighborhood of N inside $\mathbb{R}^L$, we have the following from [2, Lemma 2.2].

Lemma 3.2. *With the definitions as above, ξ, ρ, τ, ν are well-defined and smooth on A. Moreover, we have the following estimates:*

$$\|D\nu_p(v)\| \leq \kappa |v| \qquad \textit{for all } p \in N,\ v \in T_p N, \tag{3.4}$$

$$\|D\xi_a\| \leq \frac{1}{1 - \kappa\rho(a)} \qquad \textit{for all } a \in A, \tag{3.5}$$

$$|\xi(a) - p| \leq \frac{|a - p|}{1 - \kappa|a - p|} \qquad \textit{for all } p \in N,\ a \in \widetilde{B}(p, R_0). \tag{3.6}$$

Proof. See [2, Lemma 2.2]. □

From now on, we fix a point $p \in N$. Without loss of generality, we can assume that $p = 0$ after a translation in $\mathbb{R}^L$. By Lemma 3.2, we can define a smooth map $\zeta : \widetilde{B}(0, R_0) \to \mathbb{R}^L$ by

$$\zeta(x) := -\nu(\xi(x))\xi(x).$$

Note that $-\zeta(x)$ is the normal component (with respect to $T_{\xi(x)} N$) of the vector $\xi(x) - p$ (which is equal to $\xi(x)$ when $p = 0$). See Figure 1.

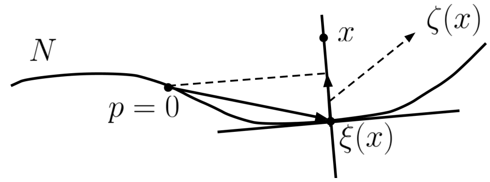

Figure 1. Definition of ζ.

Lemma 3.3. *Fix any $s \in (0, R_0)$. If we let $\gamma = \frac{R_0}{2(R_0-s)^2}$, then*

$$\|D\zeta_x\| \leq 2\gamma|x| \quad \text{and} \quad |\zeta(x)| \leq \gamma|x|^2 \quad \text{for all } x \in \widetilde{B}(0,s). \tag{3.7}$$

Proof. Fix $s \in (0, R_0)$ and any $x \in \widetilde{B}(0,s)$. As $D\xi_x(v) \in T_{\xi(x)}N$ for any $v \in \mathbb{R}^L$, we have $\nu(\xi(x))D\xi_x(v) = 0$ for any v, thus

$$\begin{aligned} D\zeta_x(v) &= -[D\nu_{\xi(x)} \circ D\xi_x(v)](\xi(x)) - \nu(\xi(x))D\xi_x(v) \\ &= -[D\nu_{\xi(x)} \circ D\xi_x(v)](\xi(x)). \end{aligned}$$

Therefore, we have by (3.4), (3.5), (3.6), $\rho(x) \leq |x|$ and $D\xi_x(v) \in T_{\xi(x)}N$,

$$\|D\zeta_x\| \leq \kappa \cdot \frac{1}{1-\kappa\rho(x)} \cdot \frac{|x|}{1-\kappa|x|} \leq \frac{R_0}{(R_0-|x|)^2}|x| \leq 2\gamma|x|.$$

The estimate for $|\zeta(x)|$ follows from a line integration from $x = 0$ using that $\zeta(0) = 0$. □

We can now state our monotonicity formula.

Theorem 3.4 (Monotonicity formula). *Assume that M is an embedded m-dimensional submanifold in $\mathbb{R}^L$ with second fundamental form A^M bounded by some constant $\Lambda > 0$, i.e. $|A^M| \leq \Lambda$. Suppose that $N \subset M$ is a closed embedded n-dimensional submanifold, and $V \in \mathcal{V}_k(M)$ is a stationary k-varifold with free boundary on N. For any point $p \in N$ and $0 < \sigma < \rho < \frac{1}{2}R_0$ as defined in* (3.3)*, we have*

$$e^{\Lambda_1\sigma}\frac{\|V\|(\widetilde{B}(p,\sigma))}{\sigma^k} \leq e^{\Lambda_1\rho}\frac{\|V\|(\widetilde{B}(p,\rho))}{\rho^k} - \int_{G_k(\widetilde{A}(p,\sigma,\rho))} \frac{e^{\Lambda_1 r}|\nabla_S^\perp r|^2}{(1+\gamma r)r^k}\, dV(x,S).$$

Here $\gamma = \frac{2}{R_0}$ is defined in Lemma 3.3 *(with $s = \frac{1}{2}R_0$), $\Lambda_1 := k(\Lambda + 3\gamma)$, $r(x) := |x-p|$, $\nabla_S^\perp r$ is the projection of ∇r to the orthogonal complement $S^\perp$ of the k-plane $S \subset \mathbb{R}^L$, and $G_k(\widetilde{A}(p,\sigma,\rho)) := \widetilde{A}(p,\sigma,\rho) \times G(L,k)$ is the restriction of the k-dimensional Grassmannian on $\mathbb{R}^L$ restricted to $\widetilde{A}(p,\sigma,\rho) := \widetilde{B}(p,\rho) \setminus \widetilde{B}(p,\sigma)$.*

Proof. As before, we can assume that $p = 0$ by a translation in $\mathbb{R}^L$. The monotonicity formula will be obtained by choosing a suitable test vector field X in (3.2). Define

$$X(x) := \varphi(r)(x + \zeta(x)),$$

where $r = |x|$ and $\varphi \geq 0$ is a smooth cutoff function with $\varphi' \leq 0$, and $\varphi(r) = 0$ for $r \geq \frac{1}{2}R_0$. When $x \in N$, we have $\xi(x) = x$ and thus

$$x + \zeta(x) = x - \nu(x)x = \tau(x)x \in T_xN.$$

Hence $X(x) \in T_xN$ for all $x \in N$, and (3.2) holds true for such X.

For any k-dimensional subspace $S \subset \mathbb{R}^L$, by the definition of X,

$$\begin{aligned}\operatorname{div}_S X(x) &= \varphi(r)(\operatorname{div}_S x + \operatorname{div}_S \zeta(x)) + \varphi'(r)\nabla^S r \cdot (x + \zeta(x)) \\ &= \varphi(r)(k + \operatorname{div}_S \zeta(x)) + \varphi'(r)[r(1 - |\nabla_S^{\perp} r|^2) + \nabla^S r \cdot \zeta(x)].\end{aligned}$$

By (3.7), we have the estimates

$$|\operatorname{div}_S \zeta(x)| \le k\|D\zeta_x\| \le 2k\gamma r \quad \text{and} \quad |\nabla^S r \cdot \zeta(x)| \le |\zeta(x)| \le \gamma r^2.$$

Using the fact that $\varphi \ge 0$ and $\varphi' \le 0$, we have the following estimates:

$$\operatorname{div}_S X(x) \ge \varphi(r)(k - 2k\gamma r) + \varphi'(r)[r(1 - |\nabla_S^{\perp} r|^2) + \gamma r^2]$$

and

$$|X(x)| \le \varphi(r)(|x| + |\zeta(x)|) \le \varphi(r)(r + \gamma r^2).$$

Plugging these estimates into (3.2) and using the bound $|A^M| \le \Lambda$,

$$\begin{aligned}&\int \varphi'(r) r(1+\gamma r)\, d\|V\| + k \int \varphi(r)\, d\|V\| \\ &\quad \le \int \varphi'(r) r |\nabla_S^{\perp} r|^2\, dV(x,S) + k\Lambda \int_\Sigma \varphi(r) r(1+\gamma r)\, d\|V\| + 2k\gamma \int_\Sigma \varphi(r) r\, d\|V\|.\end{aligned}$$

Fix a smooth cutoff function $\phi : [0,\infty) \to [0,1]$ such that $\phi' \le 0$ and $\phi(s) = 0$ for $s \ge 1$. For any $\rho \in (0, \frac{1}{2}R_0)$, if we define $\varphi(r) = \phi(\frac{r}{\rho})$, then it is a cutoff function satisfying all the assumptions above. Moreover, $r\varphi'(r) = -\rho \frac{d}{d\rho}\varphi(\frac{r}{\rho})$. Plugging into the inequality above, using the fact that $\phi(\frac{r}{\rho}) = 0$ for $r \ge \rho$,

$$\begin{aligned}&-\rho(1+\gamma\rho)\frac{d}{d\rho}\int \phi\left(\frac{r}{\rho}\right) + k\int \phi\left(\frac{r}{\rho}\right) \\ &\quad \le -\rho\frac{d}{d\rho}\int \phi\left(\frac{r}{\rho}\right)|\nabla_S^{\perp} r|^2 + k\Lambda\rho(1+\gamma\rho)\int \phi\left(\frac{r}{\rho}\right) + 2k\gamma\rho\int \phi\left(\frac{r}{\rho}\right).\end{aligned}$$

Adding $k\gamma\rho \int \phi(\frac{r}{\rho})$ to both sides of the inequality, we obtain

$$\begin{aligned}&-\rho(1+\gamma\rho)\frac{d}{d\rho}\int \phi\left(\frac{r}{\rho}\right) + k(1+\gamma\rho)\int \phi\left(\frac{r}{\rho}\right) \\ &\quad \le -\rho\frac{d}{d\rho}\int \phi\left(\frac{r}{\rho}\right)|\nabla_S^{\perp} r|^2 + k\rho[\Lambda(1+\gamma\rho) + 3\gamma]\int \phi\left(\frac{r}{\rho}\right).\end{aligned}$$

Denote $I(\rho) = \int \phi(\frac{r}{\rho}) d\|V\|$ and $J(\rho) = \int \phi(\frac{r}{\rho})|\nabla_S^{\perp} r|^2\, dV(x,S)$. Then we have

$$(1+\gamma\rho)\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^k}\right) \ge \frac{J'(\rho)}{\rho^k} - k[\Lambda(1+\gamma\rho) + 3\gamma]\frac{I(\rho)}{\rho^k},$$

which clearly implies

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^k}\right) + k(\Lambda + 3\gamma)\frac{I(\rho)}{\rho^k} \ge \frac{J'(\rho)}{(1+\gamma\rho)\rho^k}.$$

Therefore, we can rewrite it into the form

$$\frac{d}{d\rho}\left(e^{k(\Lambda+3\gamma)\rho}\frac{I(\rho)}{\rho^k}\right) \ge \frac{e^{k(\Lambda+3\gamma)\rho}}{(1+\gamma\rho)\rho^k} J'(\rho).$$

The monotonicity formula follows by letting ϕ approach the characteristic function of the interval $[0,1]$. □

4. Curvature estimates

In this section, we prove our main curvature estimates (Theorem 4.1) which imply Theorem 1.1. The estimates hold for immersed stable free boundary minimal hypersurfaces in any closed Riemannian manifold (M, g) with constraint hypersurface $N \subset M$. Moreover, the estimates are local and *uniform* in the sense that the constants only depend on the geometry of M and N, and the area of the minimal hypersurface. As in the previous section, we will continue to assume that the $(n+1)$-dimensional closed Riemannian manifold (M^{n+1}, g) is isometrically embedded into $\mathbb{R}^L$ and $N \subset M$ is a compact embedded hypersurface in M with $\partial N = \emptyset$.

Denote $B(p, r) \subset M$ as the open geodesic ball of M centered at p with radius $r > 0$. Since the intrinsic distance on M and the extrinsic distance on $\mathbb{R}^L$ are equivalent near a given point $p \in M$, we can without loss of generality assume that the monotonicity formula (Theorem 3.4) holds true for geodesic balls when the radius is less than some $R_0 > 0$ (depending only on (M, N) and the embedding to $\mathbb{R}^L$). Now we can state our main curvature estimates near the boundary.

Theorem 4.1. *Let $2 \le n \le 6$. Suppose that $M^{n+1} \subset \mathbb{R}^L$, N and R_0 are given as above. Let $p \in N$ and $0 < R < R_0$. If $(\Sigma, \partial\Sigma) \looparrowright (B(p, R), N \cap B(p, R))$ is an immersed (embedded when $n = 6$) stable free boundary minimal hypersurface satisfying the area bound* $\mathrm{Area}(\Sigma \cap B(p, R)) \le C_0$, *then*

$$\sup_{x \in \Sigma \cap B(p, \frac{R}{2})} |A^\Sigma|(x) \le C_1,$$

where $C_1 > 0$ is a constant depending on C_0, M and N.

Proof. The proof is by a contradiction argument which will be divided into three steps. First, if the assertion is false, then we can carry out a blow-up argument to obtain a limit after a suitable rescaling. Second, we show that if the limit satisfies certain area growth condition, it has to be a flat hyperplane which would give a contradiction to the choice of the blow-up sequence. Finally, we check that the limit indeed satisfies the area growth condition using the monotonicity formula (Theorem 3.4).

Step 1: The blow-up argument. Suppose that the assertion is false; then there exists a sequence

$$(\Sigma_i, \partial\Sigma_i) \looparrowright (B(p, R), N \cap B(p, R))$$

of immersed (embedded when $n = 6$) stable free boundary minimal hypersurfaces such that

$$\mathrm{Area}(\Sigma_i \cap B(p, R)) \le C_0, \tag{4.1}$$

but as $i \to \infty$, we have

$$\sup_{x \in \Sigma_i \cap B(p, \frac{R}{2})} |A^{\Sigma_i}|(x) \to \infty.$$

Therefore, we can pick a sequence of points $x_i \in \Sigma_i \cap B(p, \frac{R}{2})$ such that $|A^{\Sigma_i}|(x_i) \to \infty$. By compactness we can assume that $x_i \to x \in B(p, \frac{2R}{3})$. By the Schoen–Simon–Yau interior curvature estimates [18] (or Schoen–Simon's curvature estimates [17] when $n = 6$), we must have $x \in N$, and moreover, the connected component of $\Sigma_i \cap B(p, R)$ that passes through x_i must have a nonempty free boundary component lying on $N \cap B(p, R)$. Define a sequence of

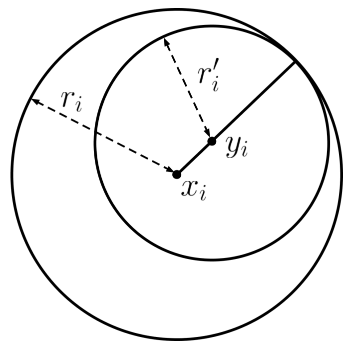

Figure 2. The balls $B(x_i, r_i)$ and $B(y_i, r_i')$.

positive numbers

$$r_i := (|A^{\Sigma_i}|(x_i))^{-\frac{1}{2}}.$$

Then we have $r_i \to 0$ and $r_i |A^{\Sigma_i}|(x_i) \to \infty$ as $i \to \infty$. Now, choose $y_i \in \Sigma_i \cap B(x_i, r_i)$ so that it achieves the maximum of

$$\sup_{y \in \Sigma_i \cap B(x_i, r_i)} |A^{\Sigma_i}|(y) \operatorname{dist}_M(y, \partial B(x_i, r_i)). \tag{4.2}$$

Let $r_i' := r_i - \operatorname{dist}_M(y_i, x_i)$ (see Figure 2). Note that $r_i' \to 0$ as $r_i' \le r_i \to 0$. Moreover, the same point $y_i \in \Sigma_i \cap B(x_i, r_i)$ also achieves the maximum of

$$\sup_{y \in \Sigma_i \cap B(y_i, r_i')} |A^{\Sigma_i}|(y) \operatorname{dist}_M(y, \partial B(y_i, r_i')). \tag{4.3}$$

Define $\lambda_i := |A^{\Sigma_i}|(y_i)$. Then we have $\lambda_i \to \infty$ since $r_i' \to 0$ and

$$\begin{aligned}
\lambda_i r_i' &= |A^{\Sigma_i}|(y_i) \operatorname{dist}_M(y_i, \partial B(y_i, r_i')) \\
&= |A^{\Sigma_i}|(y_i) \operatorname{dist}_M(y_i, \partial B(x_i, r_i)) \\
&\ge |A^{\Sigma_i}|(x_i) \operatorname{dist}_M(x_i, \partial B(x_i, r_i)) \\
&= r_i |A^{\Sigma_i}|(x_i) \to +\infty,
\end{aligned}$$

where the inequality above follows from (4.2).

Let $\eta_i : \mathbb{R}^L \to \mathbb{R}^L$ be the blow-up maps $\eta_i(z) := \lambda_i(z - y_i)$ centered at y_i. Denote $(M_i', N_i') := (\eta_i(M), \eta_i(N))$ and let $B'(0, r)$ be the open geodesic ball in M_i' of radius $r > 0$ centered at $0 \in M_i'$. We get a blow-up sequence of immersed stable free boundary minimal hypersurfaces

$$(\Sigma_i', \partial\Sigma_i') := (\eta_i(\Sigma_i), \eta_i(\partial\Sigma_i)) \looparrowright (B'(0, \lambda_i R), N' \cap B'(0, \lambda_i R)).$$

Note that we have $|A^{\Sigma_i'}|(0) = \lambda_i^{-1} |A^{\Sigma_i}|(y_i) = 1$ for every i, and the connected component of Σ_i' passing through 0 must have nonempty free boundary lying on $N_i' \cap B'(0, \lambda_i R)$. For each fixed $r > 0$, we have $\lambda_i^{-1} r < r_i'$ for all i sufficiently large since $\lambda_i r_i' \to +\infty$. Hence, if $x \in \Sigma_i' \cap B'(0, r)$, then $\eta_i^{-1}(x) \in \Sigma_i \cap B(y_i, \lambda_i^{-1} r) \subset \Sigma_i \cap B(y_i, r_i')$. Using (4.3), we have

$$|A^{\Sigma_i'}|(x) \le \frac{\lambda_i r_i'}{\lambda_i r_i' - r} \tag{4.4}$$

since $\operatorname{dist}_M(\eta_i^{-1}(x), \partial B(y_i, r_i')) \ge r_i' - \lambda_i^{-1} r$ for all i sufficiently large (depending on the fixed $r > 0$). Note that the right hand side of (4.4) approaches 1 as $i \to \infty$.

Step 2: The contradiction argument. By the smoothness of M and that $y_i \to x \in M$, we clearly have $B'(0, \lambda_i r_i')$ converging to $T_x M$ smoothly and locally uniformly in $\mathbb{R}^L$. However, as y_i does not necessarily lie on N, we have to consider two cases of convergence scenario:

- Case I: $\liminf_{i\to\infty} \lambda_i \operatorname{dist}_{\mathbb{R}^L}(y_i, N) = \infty$.
- Case II: $\liminf_{i\to\infty} \lambda_i \operatorname{dist}_{\mathbb{R}^L}(y_i, N) < \infty$.

For Case I, the rescaled constraint surface $N' \cap B'(0, \lambda_i R)$ will escape to infinity as $i \to \infty$ and therefore disappear in the limit. For Case II, after passing to a subsequence, $N' \cap B'(0, \lambda_i R)$ smoothly and locally uniformly converge to some n-dimensional affine subspace $P \subset \mathbb{R}^L$.

Assume for now that the blow-ups Σ_i' satisfy a *uniform Euclidean area growth* with respect to the geodesic balls in M_i, i.e. there exists a uniform constant $C_2 > 0$ such that for each fixed $r > 0$, when i is sufficiently large (depending possibly on r), we have

$$\operatorname{Area}(\Sigma_i' \cap B'(0, r)) \le C_2 r^n. \tag{4.5}$$

By using either the classical convergence theorem for minimal submanifolds with bounded curvature (for Case I) or Theorem 6.1 (for Case II), there exists a subsequence of the connected component of Σ_i' passing through 0 converging smoothly and locally uniformly to either

- a complete, immersed stable minimal hypersurface Σ_∞^1 in $T_x M$, or
- an immersed stable free boundary minimal hypersurface $(\Sigma_\infty^2, \partial\Sigma_\infty^2) \looparrowright (T_x M, P)$ such that $\partial\Sigma_\infty^2 \neq \emptyset$,

satisfying the same Euclidean area growth as in (4.5) *for all* $r > 0$ with Σ_i' replaced by Σ_∞^1 or Σ_∞^2. When $n = 6$, $\Sigma_\infty^1, \Sigma_\infty^2$ are both embedded by our assumption. In the first case, the classical Bernstein theorem [18, Theorem 2] (when $2 \le n \le 5$) or [17, Theorem 3] (when $n = 6$) implies that Σ_∞^1 is a flat hyperplane in $T_x M$, which is a contradiction as $|A^{\Sigma_\infty^1}|(0) = 1$. In the second case, as the constraint hypersurface P is a hyperplane in $T_x M$, we can double Σ_∞^2 as in Lemma 2.6 by reflecting across P to obtain a complete, immersed (embedded when $n = 6$) stable minimal hypersurface in $T_x M$ with Euclidean area growth. This gives the same contradiction as in the first case.

Step 3: The area growth condition. It remains now to establish the uniform Euclidean area growth for Σ_i' in (4.5). This is essentially a consequence of the monotonicity formula (Theorem 3.4). In the following, $C_3, C_4, \dots$ will be used to denote constants depending only on (M, N) and the embedding $M \subset \mathbb{R}^L$.

Let $d_i := \operatorname{dist}_M(y_i, N)$ and let $z_i \in N$ be the nearest point projection (in M) of y_i to N. Hence $d_i \to 0$ by the choice of y_i. We have to consider two cases:

- Case 1: $\liminf_{i\to\infty} \lambda_i d_i = \infty$.
- Case 2: $\liminf_{i\to\infty} \lambda_i d_i < \infty$.

Let us first consider Case 1. Fix $r > 0$. Since $\lambda_i d_i \to \infty$, we have for all i sufficiently large (depending on r),

$$B(y_i, \lambda_i^{-1} r) \subset B(y_i, d_i) \subset B(z_i, 2d_i) \subset B(z_i, \tfrac{R}{2}) \subset B(p, R). \tag{4.6}$$

Note that $B(y_i, d_i) \cap N = \emptyset$, by the interior monotonicity formula [19, Theorem 17.6] and the inclusions in (4.6), we have for i sufficiently large,

$$\operatorname{Area}(\Sigma_i \cap B(y_i, \lambda_i^{-1} r)) \leq C_3 \frac{\operatorname{Area}(\Sigma_i \cap B(y_i, d_i))}{d_i^n} (\lambda_i^{-1} r)^n.$$

Using $d_i \to 0$, (4.6) and the boundary monotonicity formula (Theorem 3.4), we have for i sufficiently large,

$$\operatorname{Area}(\Sigma_i \cap B(y_i, \lambda_i^{-1} r)) \leq 2^n C_4 \frac{\operatorname{Area}(\Sigma_i \cap B(z_i, \frac{R}{2}))}{(\frac{R}{2})^n} (\lambda_i^{-1} r)^n.$$

Finally, using (4.6) and (4.1), for i sufficiently large we have

$$\operatorname{Area}(\Sigma_i \cap B(y_i, \lambda_i^{-1} r)) \leq (2^{2n} C_4 C_0 R^{-n}) \cdot (\lambda_i^{-1} r)^n,$$

which implies (4.5). This finishes the proof for Case 1.

Now we consider Case 2, i.e. $\lambda_i d_i$ is uniformly bounded for all i. By a similar argument as above, we have

$$B(y_i, \lambda_i^{-1} r) \subset B(z_i, d_i + \lambda_i^{-1} r) \subset B(z_i, \tfrac{R}{2}) \subset B(p, R)$$

for all i sufficiently large (for any fixed $r > 0$). By exactly the same arguments as in Case 1, we have

$$\operatorname{Area}(\Sigma_i \cap B(y_i, \lambda_i^{-1} r)) \leq C_0 2^n C_5 R^{-n} \left(1 + \frac{\lambda_i d_i}{r}\right)^n \cdot (\lambda_i^{-1} r)^n.$$

Since $\lambda_i d_i$ is uniformly bounded, for r sufficiently large independent of i, estimate (4.5) is satisfied. This proves Case 2 and thus completes the proof of Theorem 4.1. □

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 using the same blow-up arguments as in the proof of Theorem 4.1. However, since we do not assume a uniform area bound of the minimal surfaces, we may not get a single stable minimal surface in the blow-up limit. Nonetheless, with the extra *embeddedness* assumption, the blow-up sequence would still subsequentially converge to a *minimal lamination*. Roughly speaking, a minimal lamination in a 3-manifold M^3 is a disjoint collection $\mathscr{L}$ of embedded minimal surfaces Λ (called the *leaves* of the lamination) such that $\bigcup_{\Lambda \in \mathscr{L}} \Lambda$ is a closed subset of M. In [5], Colding and Minicozzi proved that a sequence of minimal laminations with uniformly bounded curvature subsequentially converges to a limit minimal lamination. For our purpose, we will generalize the notion of minimal laminations to include the case with free boundary.

Throughout this section, we will denote M^3 to be a compact 3-manifold with boundary ∂M, and without loss of generality, suppose that M is a compact subdomain of another closed Riemannian 3-manifold $\widetilde{M}$. We denote the half-space

$$\mathbb{R}^3_+ := \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^1 \geq 0\},$$

whose boundary is given by the plane $\mathbb{R}^2_1 = \partial \mathbb{R}^3_+ = \{x^1 = 0\}$. First, let us recall the definition of minimal lamination from [5].

Definition 5.1 ([5, Appendix B]). Let $\Omega \subset \widetilde{M}$ be an open subset. A *minimal lamination of* Ω is a collection $\mathcal{L}$ of disjoint, embedded, connected minimal surfaces, denoted by Λ (called the *leaves* of the lamination) such that $\cup_{\Lambda \in \mathcal{L}} \Lambda$ is a closed subset of Ω. Moreover, for each $x \in \Omega$, there exists a neighborhood U of x in Ω and a local chart (U, Φ) with $\Phi(U) \subset \mathbb{R}^3$ so that in these coordinates the leaves in $\mathcal{L}$ pass through $\Phi(U)$ in slices of the form $(\mathbb{R}^2 \times \{t\}) \cap \Phi(U)$.

Now we can define minimal laminations with free boundary.

Definition 5.2. A *minimal lamination of M^3 with free boundary on ∂M* is a collection $\mathcal{L}$ of disjoint, embedded, connected minimal surfaces with (possibly empty) free boundary on ∂M, denoted by Λ, such that $\bigcup_{\Lambda \in \mathcal{L}} \Lambda$ is a closed subset of M. Moreover, for each $x \in M$, one of the following holds:

(i) $x \in M \setminus \partial M$ and there exists an open neighborhood U of x in $M \setminus \partial M$ such that the set $\{\Lambda \cap U : \Lambda \in \mathcal{L}\}$ is a minimal lamination of U.

(ii) $x \in \partial M$ and there exist a relatively open neighborhood $\widetilde{U}$ of x in M and a local coordinate chart $(\widetilde{U}, \widetilde{\Phi})$ such that $\widetilde{\Phi}(\widetilde{U}) \subset \mathbb{R}^3_+$ and $\widetilde{\Phi}(\partial M \cap \widetilde{U}) \subset \partial \mathbb{R}^3_+$ so that in these coordinates the leaves in $\mathcal{L}$ pass through the chart in slices of the form $(\mathbb{R}^2 \times \{t\}) \cap \widetilde{\Phi}(\widetilde{U})$.

(iii) $x \in \partial M$ and there exists an open neighborhood U of x in $\widetilde{M}$ such that $\{\Lambda \cap U : \Lambda \in \mathcal{L}\}$ is a minimal lamination of U.

Remark 5.3. Note that the leaves Λ of the lamination $\mathcal{L}$ in Definition 5.2 may not be properly embedded in M. For example, Λ may touch ∂M in the interior of Λ in case (iii).

In the special case that $M^3 = \mathbb{R}^3_+$, by the maximum principle [6, Corollary 1.28] we know that all leaves of the lamination $\mathcal{L}$ are properly embedded (except when $\Lambda = \partial \mathbb{R}^3_+$). Therefore Lemma 2.6 implies the following reflection principle for minimal lamination with free boundary.

Lemma 5.4 (Lamination reflection principle). *If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3_+$ with free boundary on $\partial \mathbb{R}^3_+$, then $\{\Lambda \cup \theta(\Lambda) : \Lambda \in \mathcal{L}\}$ is a minimal lamination of $\mathbb{R}^3$ (in the sense of Definition* 5.1*).*

We need the following convergence result, whose proof is postponed until Section 7.

Theorem 5.5. *Let (M^3, g) be a compact Riemannian* 3*-manifold with nonempty boundary ∂M. If $\mathcal{L}_i$ is a sequence of minimal laminations of M with free boundary on ∂M with uniformly bounded curvature, i.e. there exists a constant $C > 0$ such that*

$$\sup\{|A^\Lambda|^2(x) : x \in \Lambda \in \mathcal{L}_i\} \leq C,$$

then a subsequence of $\mathcal{L}_i$ converges in the C^α topology for any $\alpha < 1$ to a Lipschitz lamination $\mathcal{L}$ with minimal leaves in M and free boundary on ∂M.

Proof of Theorem 1.2. We follow the same contradiction argument as in the proof of Theorem 4.1 and adopt the same notions therein. After a blow-up process, we again face two types of convergence scenario. By Colding–Minicozzi's convergence theorem for minimal laminations with bounded curvature [5, Proposition B.1] (for Case I) and Theorem 5.5 (for Case II), a subsequence of blow-ups converges to

- a minimal lamination $\tilde{\mathcal{L}}$ in $T_x M \simeq \mathbb{R}^3$, or
- a minimal lamination $\mathcal{L}$ in a half-space $H \simeq \mathbb{R}^3_+$ with free boundary on ∂H.

In the second case, we can apply the lamination reflection principle (Lemma 5.4) to obtain a minimal lamination $\tilde{\mathcal{L}}$ in $T_x M \simeq \mathbb{R}^3$. By the blow-up assumption, we know that the origin $0 \in \mathbb{R}^3$ is in the support of $\tilde{\mathcal{L}}$, and the curvature of the leaf Λ_0 passing through 0 is exactly 1 at 0, i.e. $|A^{\Lambda_0}|(0) = 1$.

Now we analyze the structure of the minimal lamination $\tilde{\mathcal{L}} \subset \mathbb{R}^3$ for both cases. We refer to [12] for well-known terminologies for minimal laminations. If $\Lambda \in \tilde{\mathcal{L}}$ is an accumulating leaf, then either Λ or its double cover $\tilde{\Lambda}$ is a complete, stable minimal surface in $\mathbb{R}^3$, which must be an affine plane by the Bernstein theorem in $\mathbb{R}^3$ (see [7, 8]). Therefore, the leaf Λ_0 passing through 0 must be an isolated leaf. Since all the surfaces in the sequence Σ_i' are stable with free boundary, the smooth convergence of Σ_i' to $\tilde{\mathcal{L}}$ or $\mathcal{L}$ and the reflection principle (Lemma 2.6) imply that Λ_0 is a complete, stable, minimal surface in $\mathbb{R}^3$. This again violates the Bernstein theorem as $|A^{\Lambda_0}|(0) = 1$ by our construction. Therefore, we arrive at a contradiction and finish the proof of Theorem 1.2. □

6. Convergence of free boundary minimal submanifolds

In this section, we prove a general convergence result (Theorem 6.1) for free boundary minimal submanifolds with uniformly bounded second fundamental form. Note that this convergence result does not require stability and holds in any dimension and codimension.

To facilitate our discussion, let us first review some basic properties of *Fermi coordinates*. Let $N^n \subset M^{n+1}$ be an embedded hypersurface (without boundary) in the Riemannian manifold (M, g). We can assume that both N and M are complete. Fix a point $p \in N$. We let $(x_1, \dots, x_n)$ be the geodesic normal coordinates of N centered at p, and let $t = \operatorname{dist}_M(\cdot, N)$ be the signed distance function from N which is well-defined and smooth in a neighborhood of p inside M. Therefore, for $r_0 > 0$ sufficiently small, there exists a diffeomorphism, called a *Fermi coordinate chart*,

$$\begin{aligned}\phi : B^{n+1}_{r_0}(0) \subset T_p M &\to U \subset M,\\ (t, x_1, \dots, x_n) &\mapsto \phi(t, x_1, \dots, x_n),\end{aligned}$$

such that $U \cap N = \phi(\{t = 0\})$. Here, $B^{n+1}_{r_0}(0)$ is the open Euclidean ball of $T_p M \cong \mathbb{R}^{n+1}$ of radius $r_0 > 0$ centered at 0. We refer the readers to [13] for a more detailed discussion on Fermi coordinates. The components of the metric g in Fermi coordinates satisfy $g_{tt} = 1$ and $g_{x_i t} = 0$ for $i = 1, \dots, n$.

Let $(\Sigma, \partial\Sigma) \subset (M, N)$ be smooth embedded free boundary minimal k-dimensional submanifold, with $1 \le k \le n$. Fix any $p \in \partial\Sigma \subset N$, and let $\phi : B^{n+1}_{r_0}(0) \to U$ be a Fermi coordinate chart as above centered at p. After a rotation we can assume that

$$T_p(\partial\Sigma) = \{x_k = \dots = x_n = 0 = t\} \cong \mathbb{R}^{k-1}.$$

Since Σ meets N orthogonally along $\partial\Sigma$, after picking a choice on the sign of t, the tangent half-space $T_p\Sigma$ is given by

$$T_p\Sigma = \{x_k = \cdots = x_n = 0,\ t \geq 0\} \cong \mathbb{R}^k_+.$$

Hence, under the Fermi coordinates in a neighborhood of p, Σ can be written as a graph of $u = (u_1, \ldots, u_{n+1-k})$ which is an $\mathbb{R}^{n+1-k}$-valued function of $(t, x') = (t, x_1, \ldots, x_{k-1})$ in a domain of $\mathbb{R}^k_+$, i.e. $\phi^{-1}(\Sigma) = \{(t, x', u(t, x'))\} \subset \mathbb{R}^{n+1}_+$. Moreover, $\phi^{-1}(\partial\Sigma)$ is given by the same graph with $t = 0$. Since $\frac{\partial}{\partial t}$ is a unit normal vector field along $N \cap U$, it is clear that the free boundary condition along $\partial\Sigma$ is equivalent to

$$\frac{\partial u_\ell}{\partial t}(0, x') = 0 \quad \text{for } \ell = 1, \ldots, n+1-k. \tag{6.1}$$

We now state the convergence result for free boundary minimal submanifolds with uniformly bounded area and the second fundamental form.

Theorem 6.1. *Suppose we have a sequence $(\Sigma_j, \partial\Sigma_j) \looparrowright (M^{n+1}, N^n)$ of immersed free boundary minimal k-dimensional submanifolds, where $1 \leq k \leq n$, with uniformly bounded area and second fundamental form, i.e. there exist positive constants $C_0, C_1 > 0$ such that*

$$\operatorname{Area}(\Sigma_j) \leq C_0 \quad \textit{and} \quad \sup_{\Sigma_j} |A^{\Sigma_j}| \leq C_1$$

for all j, then after passing to a subsequence, $(\Sigma_j, \partial\Sigma_j)$ converges smoothly and locally uniformly to $(\Sigma_\infty, \partial\Sigma_\infty) \looparrowright (M, N)$ which is a smooth immersed free boundary minimal k-dimensional submanifold.

Proof. The convergence away from N follows from the classical convergence results. By the second fundamental form bound, we can cover N by balls (of a uniform size) under Fermi coordinates centered at $p \in N$ such that each Σ_j can be written as graphs over some domain of $T_p\Sigma_j$ with uniformly bounded gradient (see [6, Section 2.2]). By using the uniform area bound together with the monotonicity formula (Theorem 3.4), there is a uniform upper bound on the number of sheets of the graphs. After passing to a subsequence, the number of sheets remains constant for all j and each sheet is a graph over a k-dimensional subspace of T_pM or a k-dimensional half-space orthogonal to T_pN. The first case again follows from the classical interior convergence result. The second case follows from standard elliptic PDE theory with Neumann boundary conditions (6.1) (see [1] for example). □

7. Convergence of free boundary minimal lamination

Finally, we give the proof of Theorem 5.5 which was used in Section 5.

Proof of Theorem 5.5. For simplicity we will assume that each lamination $\mathcal{L}_i$ has finitely many leaves where the number of leaves may depend on i; this will suffice for our application. For any interior point $x \in M \setminus \partial M$, the argument used in the proof of [5, Proposition B.1] implies the convergence in a small neighborhood of x in $M \setminus \partial M$. Hence, we only need to deal with the convergence near a boundary point $x \in \partial M$.

Fix $p \in \partial M$ and let $N = \partial M$. The theorem will follow once we construct uniform coordinate charts in a small neighborhood of p in the Fermi coordinate system as in Section 6. Let φ be a Fermi coordinate chart in a relatively open neighborhood U of p in M, i.e.

$$\varphi : U \subset M \to \widetilde{U} \subseteq \mathbb{R}^3_+,$$

such that $\varphi(p) = 0$ and $\varphi(N \cap U) = \{x_1 = 0\} \cap \widetilde{U}$. Here, (x_1, x_2, x_3) are the local Fermi coordinate system centered at p (i.e. $t = x_1$). Suppose that $B^+_{4r_0} \subset \widetilde{U}$ for some small r_0 to be chosen later, where $B^+_{4r_0} = B_{4r_0} \cap \{x_1 \geq 0\}$ denotes the half ball in $\mathbb{R}^3_+$ with radius $4r_0$ centered at the origin.

Next, we will construct uniform coordinate charts on $\varphi^{-1}(B^+_{r_0})$. Note that for each i and every $\Lambda \in \mathcal{L}_i$, we have $\sup_\Lambda |A^\Lambda|^2 \leq C$. We may choose r_0 sufficiently small so that $C r_0$ is as small as we wish. Then for each fixed i,

$$\bigcup_{\Lambda \in \mathcal{L}_i} \varphi(\Lambda \cap U) \cap B^+_{4r_0}$$

gives a finite number of disconnected surfaces with bounded curvature in the Fermi coordinate system.

Since the lamination has uniformly bounded curvature, by the tilt estimates [6, Lemma 2.4], there exists a constant $\delta > 0$ such that for each lamination $\mathcal{L}_i$, we have the following two cases:

(i) None of the leaves of $\mathcal{L}_i$ meets $\partial\mathbb{R}^3_+$ in $B^+_{\delta r_0}$ (except possibly for one leaf touching $\partial\mathbb{R}^3_+$ tangentially at some points).

(ii) There exists a leaf of $\mathcal{L}_i$ meeting $\partial\mathbb{R}^3_+$ along some nonempty free boundary.

For case (i), we can construct uniform coordinate charts as in the proof of [5, Proposition B.1] in a neighborhood of the larger manifold $\widetilde{M}$. For case (ii), we claim that in $B^+_{2\delta r_0}$, all leaves of $\mathcal{L}_i$ which intersect $B^+_{\delta r_0}$ must meet $\partial\mathbb{R}^3_+$ along some nonempty free boundary; otherwise, the tilt estimates will imply that two leaves intersect somewhere in $B^+_{r_0}$ which contradicts the assumption that all leaves are disjoint. Note that the tilt estimates in [6, Lemma 2.4] only use the uniform curvature bound of leaves in $\mathcal{L}_i$, but not the minimal surface equations.

Now, we focus on case (ii). For simplicity, we use r_0 to denote δr_0. The free boundary condition and the choice of Fermi coordinates imply that these surfaces meet $\partial\mathbb{R}^3_+$ orthogonally in the Euclidean metric. Going to a further subsequence (possibly with r_0 even smaller), for fixed i, every sheet of

$$\bigcup_{\Lambda \in \mathcal{L}_i} \varphi(\Lambda \cap U) \cap B^+_{2r_0}$$

which intersects $B^+_{r_0}$ is a graph with small gradient over a subset of certain fixed plane perpendicular to $\partial\mathbb{R}^3_+$ (which can be chosen as $\mathbb{R}^2 \times \{0\} := \{x^3 = 0\}$ after a rotation keeping $\partial\mathbb{R}^3_+$ fixed as a set) containing a half ball of radius r_0 (see [6, Lemma 2.4]).

We will show that in a concentric half ball of smaller radius in $B^+_{2r_0}$, the sequence of laminations converges in the C^α topology to a lamination for any $\alpha < 1$. The coordinate chart Φ required by the definition of a lamination will be given by the Arzela–Ascoli theorem as a limit of a sequence of bi-Lipschitz maps

$$\Phi_i : B^+_{2r_0} \to \mathbb{R}^3_+$$

with bounded bi-Lipschitz constants, and Φ will be defined on a slightly smaller concentric half ball $B^+_{sr_0}$ for some $s > 0$ to be determined. Furthermore, we will show that for each i fixed

$$\Phi_i\bigg(B^+_{sr_0} \cap \varphi\bigg(\bigcup_{\Lambda\in\mathcal{L}_i} \Lambda \cap U\bigg)\bigg)$$

is the union of subsets of planes which are each parallel to $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3_+$.

Set the map Φ_i by letting

$$\Phi_i^{-1}(y_1, y_2, y_3) = (y_1, y_2, \phi_i(y_1, y_2, y_3)),$$

where ϕ_i is defined as follows: order the sheets of $B^+_{2r_0} \cap \varphi(\bigcup_{\Lambda\in\mathcal{L}_i} \Lambda \cap U)$ as $\Lambda_{i,k}$ for $k \geq 1$ by increasing values of x_3 and let $\Lambda_{i,k}$ be the graph of the function $f_{i,k}$ over (part of) the $\mathbb{R}^2 \times \{0\}$ plane. In the following we only need to consider those sheets $\Lambda_{i,k}$ where $\Lambda_{i,k} \cap B^+_{r_0} \neq \emptyset$, since we eventually will work on a much smaller concentric half ball. The domain of such $f_{i,k}$ contains the half ball of radius r_0 centered at the origin of the $\mathbb{R}^2 \times \{0\}$ plane. Again as Cr_0 can be chosen small enough, we can assume that $|\nabla f_{i,k}|$ are as small as we want. Moreover, the free boundary condition satisfied by $\Lambda_{i,k}$ is equivalent to the Neumann boundary condition

$$\frac{\partial f_{i,k}(0,\cdot)}{\partial x_1} = 0.$$

Set $w_{i,k} = f_{i,k+1} - f_{i,k}$. In the following, Δ, ∇, and div will be with respect to the Euclidean metric on $\mathbb{R}^2 \times \{0\}$. By a standard computation (cf. [6, Chapter 7] or [20, equation (7)]), we have

$$\operatorname{div}((a + \mathrm{Id})\nabla w_{i,k}) + b\nabla w_{i,k} + c w_{i,k} = 0 \quad \text{and} \quad \frac{\partial w_{i,k}(0,\cdot)}{\partial x_1} = 0, \tag{7.1}$$

where a is a matrix-valued function, b is a vector-valued function, and c is simply a real-valued function.

Note that a, b, and c depend on i, but the norms of a, b, c can be made uniformly small if Cr_0 is small enough and if we rescale our ambient manifold by a large factor. By (7.1), and the Harnack inequality (see [9, Section 8.20] and [1, Section 6]) applied to the positive function $w_{i,k}$ gives

$$\sup_{\mathbb{B}^+_{2sr_0}} w_{i,k} \leq C_1 \inf_{\mathbb{B}^+_{2sr_0}} w_{i,k}, \tag{7.2}$$

where C_1 depends only on the norms of a, b and c. Here, $\mathbb{B}^+_t$ is the half ball in $\mathbb{R}^2 \times \{0\}$ with radius t and center 0. Set $\mathbf{M}_{i,k} = f_{i,k}(0,0)$. In the region $\{(y_1, y_2, y_3) \in \mathbb{B}^+_{r_0} \times [\mathbf{M}_{i,k}, \mathbf{M}_{i,k+1}]\}$, define the function ϕ_i by

$$\phi_i(y_1, y_2, y_3) = f_{i,k}(y_1, y_2) + \frac{y_3 - \mathbf{M}_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} w_{i,k}(y_1, y_2).$$

Hence,

$$\Phi_i^{-1}(y_1, y_2, f_{i,k}(0,0)) = (y_1, y_2, f_{i,k}(y_1, y_2));$$

that is, Φ_i maps $\Lambda_{i,k}$ to a subset of the plane $\mathbb{R}^2 \times \{f_{i,k}(0,0)\}$.

Note that $\phi_i(0,0,0) = 0$. Moreover, we have

$$\nabla\phi_i = \nabla f_{i,k} + \frac{y_3 - \mathbf{M}_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} \nabla w_{i,k} + \frac{w_{i,k}}{\mathbf{M}_{i,k+1} - \mathbf{M}_{i,k}} \frac{\partial}{\partial y_3}. \tag{7.3}$$

By (7.2) and (7.3), we know that for each i the map Φ_i restricted to $B^+_{sr_0} \subseteq \mathbb{R}^3_+$ is bi-Lipschitz with uniformly bounded bi-Lipschitz constant.

By the Arzela–Ascoli theorem, a subsequence of Φ_i converges in the C^α topology for any $\alpha < 1$ to a Lipschitz coordinate chart Φ with the properties that are required. By standard elliptic regularity theory, the leaves are either minimal surfaces (for the first case) or minimal surfaces with free boundary on N (for the second case). □

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