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# A Sharp Comparison Theorem for Compact Manifolds with Mean Convex Boundary

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**Abstract** Let $M$ be a compact $n$-dimensional Riemannian manifold with nonnegative Ricci curvature and mean convex boundary $\partial M$. Assume that the mean curvature $H$ of the boundary $\partial M$ satisfies $H \geq (n-1)k > 0$ for some positive constant $k$. In this paper, we prove that the distance function $d$ to the boundary $\partial M$ is bounded from above by $\frac{1}{k}$ and the upper bound is achieved if and only if $M$ is isometric to an $n$-dimensional Euclidean ball of radius $\frac{1}{k}$.

**Keywords** Comparison theorem · Ricci curvature · Mean convex boundary · Rigidity

## 1 Introduction

By a classical theorem of Bonnet and Myers, if a complete $n$-dimensional Riemannian manifold $M$ has Ricci curvature at least $(n-1)k$, where $k > 0$ is a constant, then the diameter of $M$ is at most $\frac{\pi}{\sqrt{k}}$. Applying this result to the universal cover $\tilde{M}$, we see that such manifolds must be compact and have finite fundamental group. In [2], Cheng proved the rigidity theorem that if the diameter is equal to $\frac{\pi}{\sqrt{k}}$, then $M$ is isometric to the $n$-sphere with constant sectional curvature $k$.

In this paper, we prove a similar result for compact manifolds with nonnegative Ricci curvature and mean convex boundary. Our main result is the following.

**Theorem 1.1** *Let $M^n$ be a complete $n$-dimensional $(n \geq 2)$ Riemannian manifold with nonnegative Ricci curvature and mean convex boundary $\partial M$. Assume the mean*

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*curvature $H$ of $\partial M$ with respect to the inner unit normal satisfies $H \geq (n-1)k > 0$ for some constant $k > 0$. Let $d$ denote the distance function on $M$. Then,*

$$\sup_{x \in M} d(x, \partial M) \leq \frac{1}{k}. \tag{1.1}$$

*Furthermore, if we assume that $\partial M$ is compact, then $M$ is also compact and equality holds in (1.1) if and only if $M^n$ is isometric to an $n$-dimensional Euclidean ball of radius $\frac{1}{k}$.*

*Remark 1.2* For any isometric embedding of a Riemannian $m$-manifold $N$ into a metric space $X$, Gromov [5] defined the *filling radius*, Fill Rad $(N \subset X)$, to be the infimum of those numbers $\epsilon > 0$ for which $N$ bounds in the $\epsilon$-neighborhood $U_\epsilon(N) \subset X$, that is, the inclusion homomorphism of the $m$-th homology (over $\mathbb{Z}$ or $\mathbb{Z}_2$) $H_m(N) \to H_m(U_\epsilon(N))$ vanishes. Therefore, we can restate the conclusion of Theorem 1.1 as Fill Rad $(\partial M \subset M) \leq \frac{1}{k}$, and equality holds if and only if $M$ is the Euclidean ball of radius $\frac{1}{k}$.

Note that under the curvature assumptions in Theorem 1.1, the complete manifold $M$ may be *non-compact*. However, if we put a stronger convexity assumption on $\partial M$, then the boundary convexity could force $\partial M$ to be compact and hence $M$ would also be compact. In [6], Hamilton proved that any convex hypersurface in $\mathbb{R}^n$ with pinched second fundamental form is compact. We conjecture that the result can be generalized to manifolds with nonnegative Ricci curvature.

**Conjecture 1.3** *Let $M^n$ be a complete Riemannian $n$-manifold with nonempty boundary $\partial M$. Assume $M$ has nonnegative Ricci curvature and $\partial M$ is uniformly convex with respect to the inner unit normal, i.e., the second fundamental form $h$ of $\partial M$ satisfies $h \geq k > 0$ for some constant $k$. Then, $M$ is compact and $\pi_1(M)$ is finite.*

Manifolds satisfying the assumptions in Conjecture 1.3 have been studied by several authors. Some rigidity results were obtained in [9] and [10]. In [4], J. Escobar gave upper and lower estimates for the first nonzero Steklov eigenvalue for these manifolds with boundary. However, all these results are proved under the assumption that $M$ is compact. Conjecture 1.3 above would imply that this assumption is void and these manifolds have finite fundamental group.

## 2 Preliminaries

In this section, we collect some known facts which will be used in the proof of Theorem 1.1. Let $M$ be a complete $n$-dimensional Riemannian manifold with nonempty boundary $\partial M$. We denote by $\langle \, , \rangle$ the metric on $M$ as well as that induced on $\partial M$. Suppose $\gamma : [0, \ell] \to M$ is a geodesic in $M$ parameterized by arc length such that $\gamma(0)$ and $\gamma(\ell)$ lie on $\partial M$ and $\gamma(s)$ lies in the interior of $M$ for all $s \in (0, \ell)$. Assume that $\gamma$ meets $\partial M$ orthogonally, that is, $\gamma'(0) \perp T_{\gamma(0)}\partial M$ and $\gamma'(\ell) \perp T_{\gamma(\ell)}\partial M$. Hence, $\gamma$ is a critical point of the length functional as a free boundary problem. We call such $\gamma$ a *free boundary geodesic*. For any normal vector field $V$ along $\gamma$, the orthogonality condition implies that $V$ is tangent to $\partial M$ at $\gamma(0)$ and $\gamma(\ell)$, hence is an admissible variation to the free boundary problem. A direct calculation gives the second variation formula of arc length:

$$\begin{aligned}\delta^s\gamma(V,V) &= \int_0^\ell \left(\left|V'(s)\right|^2 - \left|V(s)\right|^2 K\left(\gamma'(s), V(s)\right)\right) ds \\ &\quad + \left\langle \overline{\nabla}_{V(\ell)} V(\ell), \gamma'(\ell)\right\rangle - \left\langle \overline{\nabla}_{V(0)} V(0), \gamma'(0)\right\rangle, \end{aligned} \tag{2.1}$$

where $\overline{\nabla}$ is the Riemannian connection on $M$, and $K(u,v)$ is the sectional curvature of the plane spanned by $u$ and $v$ in $M$.

Let $N$ be the inner unit normal of $\partial M$ with respect to $M$. The second fundamental form $h$ of $\partial M$ with respect to $N$ is defined by $h(u,v) = \langle \overline{\nabla}_u v, N\rangle$ for $u, v$ tangent to $\partial M$. The mean curvature $H$ of $\partial M$ with respect to $N$ is defined as the trace of $h$, that is, $H = \sum_{i=1}^{n-1} h(e_i, e_i)$ for any orthonormal basis $e_1, \dots, e_{n-1}$ of the tangent bundle $T\partial M$. The principal curvatures of $\partial M$ are defined to be the eigenvalues of $h$. Using a Frankel-type argument as in [7], we have the following lemma.

**Lemma 2.1** *Let $M$ be a compact, connected $n$-dimensional Riemannian manifold with nonempty boundary $\partial M$. Suppose $M$ has nonnegative Ricci curvature and the mean curvature $H$ of $\partial M$ with respect to the inner unit normal satisfies $H \geq (n-1)k > 0$ for some positive constant $k$. Then, $\partial M$ is connected and the map*

$$\pi_1(\partial M) \xrightarrow{i_*} \pi_1(M)$$

*induced by inclusion is surjective, i.e., $\pi_1(M, \partial M) = 0$.*

*Proof* We follow the argument given in [7]. We show under the curvature assumptions, any free boundary geodesic must be unstable as a free boundary solution. To see this, let $\gamma : [0,\ell] \to M$ be a free boundary geodesic. Fix an orthonormal basis $e_1, \dots, e_{n-1}$ of $T_{\gamma(0)}\partial M$, let $V_i(s)$ be the normal vector field along $\gamma$ obtained from $e_i$ by parallel translation, and using the second variation formula (2.1), we have

$$\sum_{i=1}^{n-1} \delta^2\gamma(V_i, V_i) = -\int_0^\ell \operatorname{Ric}(\gamma'(s), \gamma'(s))\, ds - H_{\gamma(\ell)} - H_{\gamma(0)} < 0,$$

where Ric is the Ricci curvature of $M$. Therefore, $\delta^2\gamma(V_i, V_i) < 0$ for some $i$, and therefore $\gamma$ is unstable.

Suppose $\partial M$ is not connected or $\pi_1(M, \partial M) \neq 0$. In either case, there exists a free boundary geodesic $\gamma$ which minimizes length in its homotopy class in $\pi_1(M, \partial M)$, hence is stable. This contradicts the fact that there are no stable free boundary geodesics in $M$. □

We will use the following lemma, which is a special case of Theorem 1 in [8].

**Lemma 2.2** *Let $M$ be a compact $n$-dimensional Riemannian manifold with nonempty boundary $\partial M$ and nonnegative Ricci curvature. If the mean curvature $H$ of $\partial M$ with respect to the unit inner normal satisfies*

$$H \geq \frac{n-1}{n}\frac{|\partial M|}{|M|},$$

*where $|\partial M|$ and $|M|$ denote the $(n-1)$- and $n$-dimensional volume of $\partial M$ and $M$, respectively, then $M^n$ is isometric to a Euclidean ball.*

## 3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We first prove the upper bound in (1.1). Fix any point $x$ in the interior of $M$; there exists a geodesic $\gamma : [0, \ell] \to M$ parameterized by arc length such that $\ell = d(x, \partial M)$ (the existence of such geodesic follows from the completeness of $M$). Note that $\gamma$ lies in the interior of $M$ except at $\gamma(\ell)$. We want to prove that $\ell \leq \frac{1}{k}$. The first variation formula tells us that $\gamma'(\ell)$ is orthogonal to $\partial M$ at $\gamma(\ell)$. Moreover, the second variation of $\gamma$ for any normal vector field $V$ along $\gamma$ where $V(0) = 0$ is nonnegative:

$$\delta^2\gamma(V, V) = \int_0^\ell \left(\left|V'(s)\right|^2 - \left|V(s)\right|^2 K\left(\gamma'(s), V(s)\right)\right) ds + \left\langle \overline{\nabla}_{V(\ell)} V(\ell), \gamma'(\ell)\right\rangle \geq 0. \tag{3.1}$$

Fix an orthonormal basis $e_1, \ldots, e_{n-1}$ for $T_{\gamma(\ell)}\partial M$, and let $E_i(s)$ be the parallel translate of $e_i$ along $\gamma$. Define $V_i(s) = \frac{s}{\ell} E_i(s)$. Substitute into (3.1) and sum over $i$ from 1 to $n-1$,

$$\sum_{i=1}^{n-1} \delta^2\gamma(V_i, V_i) = \int_0^\ell \left(\frac{n-1}{\ell^2} - \left(\frac{s}{l}\right)^2 \mathrm{Ric}\big(\gamma'(s), \gamma'(s)\big)\right) ds - H_{\gamma(\ell)} \geq 0. \tag{3.2}$$

Since Ric$\geq 0$ and $H \geq (n-1)k > 0$, (3.2) implies that $\frac{n-1}{l} \geq (n-1)k$. Therefore, $\ell \leq \frac{1}{k}$. Since the point $x$ is arbitrary, we have proved inequality (1.1).

Assume now that $\partial M$ is compact; then (1.1) implies that $M$ is compact. Suppose equality holds in (1.1). By rescaling the metric of $M$, we can assume that $k = 1$. Then we want to prove that $M^n$ is isometric to the $n$-dimensional Euclidean unit ball. Since $M$ is compact, there exists some $x_0$ in the interior of $M$ such that

$$d(x_0, \partial M) = 1. \tag{3.3}$$

The key step is to show that $M$ is equal to the geodesic ball of radius 1 centered at $x_0$, denoted by $B_1(x_0)$. From (3.3), it is clear that $B_1(x_0)$ is contained in $M$. Let $\rho = d(x_0, \cdot)$ denote the distance function from $x_0$. Since $M$ has nonnegative Ricci curvature, the Laplacian comparison theorem gives

$$\overline{\Delta} d \leq \frac{n-1}{d}, \tag{3.4}$$

where $\overline{\Delta}$ is the Laplacian operator on $M$, and $d = d(x, \cdot)$ is the distance function in $M$ from any point $x$.

Let $S = \{q \in \partial M : \rho(q) = 1\}$. We claim that $S = \partial M$. To prove the claim, it suffices to show that $S$ is an open and closed subset of $\partial M$, since $\partial M$ is connected by Lemma 2.1. Note that $S$ is closed by the continuity of $\rho$. It remains to prove that $S$ is open in $\partial M$. Pick any point $q \in S$; we will show that $\rho \equiv 1$ in a neighborhood of $q$ in $\partial M$. If $q$ is not a conjugate point to $x_0$ in $M$, then the geodesic sphere $\partial B_1(x_0)$ is a smooth hypersurface near $q$ in $M$, whose mean curvature with respect to the inner unit normal is at most $n-1$ by the Laplacian comparison theorem (3.4). On the other hand, $\partial M$ has mean curvature at least $n-1$ with respect to the inner unit normal by assumption. The maximum principle for hypersurfaces in manifolds [3] implies that $\partial M$ and $\partial B_1(x_0)$ coincide in a neighborhood of $q$. Hence, $\rho \equiv 1$ in a neighborhood of $q$. Therefore, $S$ is open near any $q$ which is not a conjugate point to $x_0$ in $M$. If $q$ is a conjugate point of $x_0$, we want to show that $\Delta\rho \leq 0$ in the barrier sense [1] in a neighborhood $q$, where $\Delta$ is the Laplacian operator on $\partial M$. Since $q$ is a minimum of $\rho$, we can then apply the strong maximum principle in [1] for a superharmonic function in the barrier sense to conclude that $\rho \equiv 1$ near $q$ in $\partial M$. To see why $\rho$ is superharmonic in $\partial M$, let $\epsilon > 0$ be any small constant and $p$ be any point on $\partial M$ near $q$. We have to find an upper barrier $\rho_\epsilon$ which is $C^2$ in a neighborhood of $p$ in $\partial M$, i.e., $\rho_\epsilon(p) = \rho(p)$ and $\rho_\epsilon \geq \rho$ in a neighborhood of $p$ in $\partial M$. Let $\gamma : [0, 1] \to M$ be a minimizing geodesic from $x_0$ to $p$ parameterized by arc length. Let $\delta > 0$ be a small constant to be fixed later, and define

$$\rho_\delta(\cdot) = \delta + d\big(\gamma(\delta), \cdot\big),$$

which is smooth in a neighborhood of $p$. Notice that $\rho_\delta(p) = \rho(p)$ and $\rho_\delta \geq \rho$ in a neighborhood of $p$ by the triangle inequality. By the Laplacian comparison theorem (3.4), we have

$$\overline{\Delta}\rho_\delta \leq \frac{n-1}{d(\gamma(\delta), \cdot)} = \frac{n-1}{\rho_\delta - \delta}. \tag{3.5}$$

On a neighborhood of $p$ in $\partial M$, we have

$$\Delta\rho_\delta = \overline{\Delta}\rho_\delta + H\frac{\partial\rho_\delta}{\partial N} - \operatorname{Hess}\rho_\delta(N, N), \tag{3.6}$$

where $N$ is the inner unit normal of $\partial M$ with respect to $M$, $H$ is the mean curvature of $\partial M$ with respect to $N$, and Hess $\rho_\delta$ is the Hessian of $\rho_\delta$ in $M$. Observe that

$$\rho_\delta(p) = \rho(p), \qquad \frac{\partial\rho_\delta}{\partial N}(p) = -1 \quad \text{and} \quad \operatorname{Hess}\rho_\delta(N, N)(p) = 0.$$

Choose a neighborhood $U \subset \partial M$ of $q$ such that for any $p \in U$ and $\delta > 0$ sufficiently small, we have

$$\rho_\delta \geq \rho \geq 1, \qquad \frac{\partial\rho_\delta}{\partial N} \geq -1 + \delta \quad \text{and} \quad \operatorname{Hess}\rho_\delta(N, N) \geq -\delta \tag{3.7}$$

on the neighborhood $U$. By assumption, $H \geq n-1$. We see from (3.5), (3.6), and (3.7) that in the neighborhood $U$ around $p$,

$$\Delta \rho_\delta \leq \frac{n-1}{1-\delta} - (1-\delta)(n-1) + \delta \leq \epsilon$$

if $\delta$ is sufficiently small. Since $\epsilon$ is arbitrary, this shows that $\rho$ is superharmonic near $q$ in the barrier sense and attains a local minimum at $q$. Therefore, $\rho$ is constant near $q$ by the maximum principle of [1]. This proves the claim that $S = \partial M$.

Now, we have shown that $M = B_1(x_0)$, the geodesic ball of radius 1 centered at $x_0$ in $M$. We first note that $\rho$ is smooth up to the boundary $\partial M$. This is true since any $q \in \partial M$ can be joined by a minimizing geodesic $\gamma$ of unit length from $x_0$ to $q$. As $\partial M = \partial B_1(x_0)$, $\gamma$ is orthogonal to $\partial M$ at $q$, hence is uniquely determined by $q$. Therefore, $q$ is not in the cut locus of $x_0$. Since $M$ has nonnegative Ricci curvature, the Laplacian comparison (3.4) for $\rho = d(x_0, \cdot)$ holds in the classical sense, that is,

$$\rho \overline{\Delta} \rho \leq n-1. \tag{3.8}$$

Since $|\overline{\nabla} \rho| = 1$ on $M$, $\rho \equiv 1$ and $\frac{\partial \rho}{\partial \nu} = 1$ on $\partial M$, where $\nu = -N$ is the outer unit normal of $\partial M$, integrating (3.8) over the whole manifold $M$ and applying Stokes' theorem, we get

$$|\partial M| - |M| = \int_{\partial M} \rho \frac{\partial \rho}{\partial \nu} - \int_M |\overline{\nabla} \rho|^2 = \int_M \rho \overline{\Delta} \rho \leq \int_M (n-1) = (n-1)|M|.$$

This implies that

$$\frac{1}{n} \frac{|\partial M|}{|M|} \leq 1.$$

Since the mean curvature of $\partial M$ satisfies $H \geq n-1$, by Lemma 2.2, $M$ is isometric to a Euclidean ball of radius $r$. It is clear that $r = 1$ as $M = B_1(x_0)$. This completes the proof of Theorem 1.1.

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