IMPROVED ESTIMATES FOR POLYNOMIAL ROTH TYPE THEOREMS IN FINITE FIELDS

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ABSTRACT. We prove that, under certain conditions on the function pair φ_1 and φ_2 , bilinear average $p^{-1} \sum_{y \in \mathbb{F}_p} f_1(x + \varphi_1(y)) f_2(x + \varphi_2(y))$ along curve (φ_1, φ_2) satisfies certain decay estimate. As a consequence, Roth type theorems hold in the setting of finite fields. In particular, if $\varphi_1, \varphi_2 \in \mathbb{F}_p[X]$ with $\varphi_1(0) = \varphi_2(0) = 0$ are linearly independent polynomials, then for any $A \subset \mathbb{F}_p, |A| = \delta p$ with $\delta > cp^{-\frac{1}{12}}$, there are $\gtrsim \delta^3 p^2$ triplets $x, x + \varphi_1(y), x + \varphi_2(y) \in A$. This extends a recent result of Bourgain and Chang who initiated this type of problems, and strengthens the bound in a result of Peluse, who generalized Bourgain and Chang's work. The proof uses discrete Fourier analysis and algebraic geometry.

1. INTRODUCTION

Fix a large prime p and denote $e_p(x) := e^{2\pi i \frac{x}{p}}$. For any $\varphi_1, \varphi_2 : \mathbb{F}_p \to \mathbb{F}_p$, we are interested in the bilinear average along the "curve" $\Gamma = (\varphi_1, \varphi_2)$: for any $x \in \mathbb{F}_p$,

(1.1)
$$\mathcal{A}_{\Gamma}(f_1, f_2)(x) := \frac{1}{p} \sum_{y \in \mathbb{F}_p} f_1(x + \varphi_1(y)) f_2(x + \varphi_2(y)) \, .$$

The behavior of the bilinear average relies closely to the following exponential sum associated to Γ ,

(1.2)
$$K_{\Gamma}(x,y) := \begin{cases} \frac{1}{p} \sum_{z \in \mathbb{F}_p} e_p(x\varphi_1(z) + y\varphi_2(z)) & y \neq 0; \\ 0 & y = 0. \end{cases}$$

To state our main result, we first set up some notations. For $f: \mathbb{F}_p \to \mathbb{C}$, define

$$\mathbb{E}[f] = \mathbb{E}_x[f] = \frac{1}{p} \sum_{x=0}^{p-1} f(x)$$
$$\|f\|_r = \left(\frac{1}{p} \sum_x |f(x)|^r\right)^{\frac{1}{r}}$$
$$\|f\|_{l^r} = \left(\sum_x |f(x)|^r\right)^{\frac{1}{r}}$$
$$\hat{f}(z) = \frac{1}{p} \sum_x f(x) e_p(-xz)$$

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With these notations, it is easy to verify that

$$\|f\|_{r} \leq \|f\|_{s} \text{ if } s > r \quad \text{(a special case of Hölder inequality);}$$
$$\|f\|_{2} = \|\hat{f}\|_{l^{2}} \quad \text{(Parseval)}$$
$$f(x) = \sum_{z} \hat{f}(z)e_{p}(xz) \quad \text{(Fourier inversion)}$$

We also need a notion of generalized diagonal sets.

Definition. A set $D \subset \mathbb{F}_p \times \mathbb{F}_p$ is called **generalized diagonal** if for any $x \in \mathbb{F}_p$, there are O(1) y's such that $(x, y) \in D$ and for any $y \in \mathbb{F}_p$ there are O(1) x's such that $(x, y) \in D$. The implied constant must be independent of p.

Our main theorem below provides a framework to obtain decay estimate for the bilinear operator \mathcal{A}_{Γ} associated with various function pairs (φ_1, φ_2) . Throughout the paper, $A \leq B$ denotes the statement that $|A| \leq C|B|$ for some positive constant C independent of the prime p and the coefficients of polynomials where relevant.

Theorem 1.1. Let the kernel K_{Γ} be defined as in (1.2). We define for any $h, y, y' \in \mathbb{F}_p$,

(1.3)
$$I_{\Gamma} := \sum_{x \in \mathbb{F}_p} K_{\Gamma}(x, y) \overline{K_{\Gamma}(x - h, y + h)} K_{\Gamma}(x, y') K_{\Gamma}(x - h, y' + h)$$

Suppose that the following three conditions hold:

- (1) There exists $\theta \in (0,1]$ such that $\frac{1}{p} \sum_{y \in \mathbb{F}_p} e_p(s\varphi_1(y)) \lesssim p^{-\theta}$ for any $s \neq 0$;
- (2) There exists $\alpha \in (\frac{1}{4}, 1)$ such that $K_{\Gamma}(x, y) \lesssim p^{-\alpha}$ for any $x, y \in \mathbb{F}_p$;
- (3) There exists $\beta > 1$ such that for any $h \in \mathbb{F}_p^* := \mathbb{F}_p \setminus \{0\}$ we can find a generalized diagonal set $D_{\Gamma,h}$ so that $I_{\Gamma} \leq p^{-\beta}$ for any $(y, y') \notin D_{\Gamma,h}$.

Then the bilinear average defined by (1.1) obeys

(1.4)
$$\|\mathcal{A}_{\Gamma}(f_1, f_2) - \mathbb{E}[f_1]\mathbb{E}[f_2]\|_2 \lesssim p^{-\gamma} \|f_1\|_2 \|f_2\|_2$$

with $\gamma = \min\{\theta, \alpha - \frac{1}{4}, \frac{\beta}{4} - \frac{1}{4}\}.$

Motivated by the non-conventional ergodic averages considered by Bergelson [1] and Frantzikinakis and Kra [6], Bourgain and Chang [3] are the first to consider quantitative estimate of the form (1.4). They established (1.4) with $\gamma = \frac{1}{10}$ for the quadratic monomial curve $\Gamma = (y, y^2)$, via an elegant way combining discrete Fourier analysis, explicit evaluation of quadratic Gauss sum and Bombieri's estimate for Weil sum of rational functions [2].

Peluse [13, Theorem 2.2] generalized Bourgain-Chang's result to the polynomial curve $(\varphi_1(y), \varphi_2(y))$ for any linearly independent polynomials φ_1, φ_2 . Her result also applies over arbitrary finite fields of large characteristic (and not just \mathbb{F}_p). However, she must take $\gamma = 1/16$. Her method is based on careful analysis of the dimension of varieties created by multiple applications of Cauchy-Schwartz, and an exponential sum bound due to Kowalski.

Our result improves the decay rate from $\frac{1}{16}$ to $\frac{1}{8}$ in Peluse's bound. This also improves the decay rate from $\frac{1}{10}$ to $\frac{1}{8}$ in the cases handled by Bourgain and Chang. Moreover, in the special case $\Gamma = (y, y^2)$, our approach does not rely on Bombieri's estimate: When $\Gamma = (y, y^2)$, $K_{\Gamma}(x, y)$ is a quadratic Gauss sum which can be evaluated explicitly. Condition (3) can therefore be verified by

$$|I_{\Gamma}| \le \frac{1}{p^2} \left| \sum_{x} e_p\left(-\frac{x^2}{4y}\right) e_p\left(\frac{(x-h)^2}{4(y+h)}\right) e_p\left(\frac{x^2}{4y'}\right) e_p\left(-\frac{(x-h)^2}{4(y'+h)}\right) \right| \le p^{-\frac{3}{2}} \text{ for } y \ne y',$$

using only quadratic Gauss sum estimate. Hence $\beta = \frac{3}{2}$. It is easy to check that $\theta = \alpha = \frac{1}{2}$, and thus $\gamma = \frac{1}{8}$.

To extend to the polynomial curve $\Gamma = (y, P(y))$, the condition (3) can be verified by Deligne's fundamental work on exponential sums over finite fields [5]. When extending to the bi-polynomial case, we need to use Katz's generalisation [10] of Deligne's theorem on exponential sums over smooth affine varieties.

Theorem 1.1 immediately implies a quantitative Roth type theorem:

Corollary 1.2. Let $\varphi_1, \varphi_2 : \mathbb{F}_p \to \mathbb{F}_p$ be functions satisfying conditions (1), (2) and (3) (with parameters θ , α and β , resp.) in Theorem 1.1. Then for any $A \subset \mathbb{F}_p, |A| = \delta p$ with $\delta > cp^{-\frac{2}{3}\gamma}, \gamma = \min\{\theta, \alpha - \frac{1}{4}, \frac{\beta}{4} - \frac{1}{4}\}$, there are $\gtrsim \delta^3 p^2$ triplets $x, x + \varphi_1(y), x + \varphi_2(y) \in A$.

We include its short proof (which is the same as that of Corollary 1.2 in [3]) here for the reader's convenience. Indeed, set both f_1 and f_2 to be the indicator function of the set A. By Cauchy-Schwarz inequality and (1.4),

(1.5)
$$\sum_{x,y} f(x)f(x+\varphi_1(y))f(x+\varphi_2(y)) \ge p^2 \left(\mathbb{E}[f]^3 - \|f\|_2 \|\mathcal{A}_{\Gamma}(f,f) - \mathbb{E}[f]^2\|_2\right) \gtrsim p^2 \delta^3,$$

from which the corollary follows.

One interesting case of Theorem 1.1 is the following theorem:

Theorem 1.3. Let $\Gamma = (\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_2 \in \mathbb{F}_p[X]$, $\varphi_1(0) = \varphi_2(0) = 0$. Suppose that φ_1, φ_2 are linearly independent. Then the average function \mathcal{A}_{Γ} satisfies

(1.6)
$$\|\mathcal{A}_{\Gamma}(f_1, f_2) - \mathbb{E}[f_1]\mathbb{E}[f_2]\|_2 \lesssim p^{-1/8} \|f_1\|_2 \|f_2\|_2,$$

with the implied constant depending only on the degrees of φ_1 and φ_2 .

As before, we can obtain the corresponding Roth type theorem in which the lower bound $p^{-\frac{1}{12}}$ of δ is slightly better than the bound $p^{-\frac{1}{15}}$ obtained in Bourgain-Chang's paper [3].

Corollary 1.4. Let $\varphi_1, \varphi_2 \in \mathbb{F}_p[X]$, $\varphi_1(0) = \varphi_2(0) = 0$, be linearly independent. Then for any $A \subset \mathbb{F}_p$, $|A| = \delta p$ with $\delta > cp^{-\frac{1}{12}}$, there are $\gtrsim \delta^3 p^2$ triplets $x, x + \varphi_1(y), x + \varphi_2(y) \in A$.

Remark 1.5. The results of this paper can be generalized to an arbitrary finite field \mathbb{F}_q with $q = p^m$. In this general setting, one should be careful that the degree of the polynomial should be coprime to p in order to get the Weil's estimate [4] (and using Deligne-Katz theory). However, as we are usually only interested in the case that p is very large compared with the degrees of the relative polynomials, the coprime condition is automatically satisfied.

Remark 1.6. Some rational functions could be included in our results. For instance, when $\varphi_1(y) = y, \varphi_2(y) = \frac{1}{y}$ (this case is also considered in [3]), we can get the same conclusion as in Theorem 1.3, using Kloosterman sum estimates (Corollary 3.3. in [7]).

Remark 1.7. Theorem 1.3, in the case $\varphi_1(x) = x$, implies that the polynomial $x + \varphi_2(y - x)$ is an almost strong asymmetric expander in the sense of Tao's paper [14]. It is possible that this result could also be established using [14, Theorem 3], but we do not pursue this.

We will prove Theorem 1.1 in the Section 2. In Section 3 we will verify the three conditions (1), (2), and (3) for certain polynomial pairs and henceforth prove Theorem 1.3.

2. Proof of Theorem 1.1

We prove the main theorem in this section. We follow the spirit in the second author's work on the bilinear Hilbert transform along curves in [12]. First, by using Fourier inversion for f_1 and f_2 , it is clear that

$$\mathcal{A}_{\Gamma}(f_1, f_2)(x) = \sum_{n_1, n_2} \hat{f}_1(n_1) \hat{f}_2(n_2) e_p((n_1 + n_2)x) \mathbb{E}_y[e_p(n_1\varphi_1(y) + n_2\varphi_2(y))].$$

Changing variables $n_2 = n, n_1 = s - n$, we then split the bilinear average $\mathcal{A}_{\Gamma}(f_1, f_2)(x)$ into three terms:

$$\mathcal{A}_{\Gamma}(f_1, f_2)(x) = J_1 + J_2 + J_3,$$

where

$$J_{1} = \hat{f}_{1}(0)\hat{f}_{2}(0) = \mathbb{E}[f_{1}]\mathbb{E}[f_{2}],$$

$$J_{2} = \hat{f}_{2}(0)\sum_{s\neq 0} \left(\hat{f}_{1}(s)\mathbb{E}_{y}[e_{p}(s\varphi_{1}(y))]\right)e_{p}(sx),$$

$$J_{3} = \sum_{s} \left(\sum_{n\neq 0}\hat{f}_{1}(s-n)\hat{f}_{2}(n)\mathbb{E}_{y}[e_{p}((s-n)\varphi_{1}(y) + n\varphi_{2}(y))]\right)e_{p}(sx).$$

By the assumption (1), when $s \neq 0$, we get

(2.1)
$$\mathbb{E}_{y}[e_{p}(s\varphi_{1}(y))] = \frac{1}{p}\sum_{y}e_{p}(s\varphi_{1}(y)) \lesssim \frac{1}{p^{\theta}}$$

Therefore, using Parseval's identity, triangle inequality, and Hölder inequality, we see that

$$\|\mathcal{A}_{\Gamma}(f_{1}, f_{2}) - \mathbb{E}[f_{1}]\mathbb{E}[f_{2}]\|_{2} \leq \|\widehat{J}_{2}\|_{l^{2}} + \|\widehat{J}_{3}\|_{l^{2}}$$

$$\lesssim \frac{1}{p^{\theta}}\|f_{1}\|_{2}\|f_{2}\|_{2} + \left(\sum_{s}\left|\sum_{n}\widehat{f}_{1}(s-n)\widehat{f}_{2}(n)K_{\Gamma}(s-n,n)\right|^{2}\right)^{\frac{1}{2}},$$

where K_{Γ} is given by (1.2).

Set $\gamma_0 = \min\{\alpha - \frac{1}{4}, \frac{\beta}{4} - \frac{1}{4}\}$. Hence it remains to show

(2.2)
$$\sum_{s} \left| \sum_{n} \hat{f}_{1}(s-n) \hat{f}_{2}(n) K_{\Gamma}(s-n,n) \right|^{2} \lesssim \frac{1}{p^{2\gamma_{0}}} \|f_{1}\|_{2}^{2} \|f_{2}\|_{2}^{2}$$

Next we choose to employ a TT^* method (Our method and Bourgain-Chang's diverge from here). The left hand side of (2.2) equals

$$\sum_{s} \sum_{n_1, n_2} \hat{f}_1(s - n_1) \overline{\hat{f}_1(s - n_2)} \hat{f}_2(n_1) \overline{\hat{f}_2(n_2)} K_{\Gamma}(s - n_1, n_1) \overline{K_{\Gamma}(s - n_2, n_2)},$$

which, after changing variables $n_1 = v, n_2 = v + h, s = u + v$, can be rewritten as

(2.3)
$$\sum_{h} \left(\sum_{u,v} F_h(u) G_h(v) K_{\Gamma}(u,v) \overline{K_{\Gamma}(u-h,v+h)} \right),$$

where

$$F_h(x) = \hat{f}_1(x)\hat{f}_1(x-h);$$

$$G_h(x) = \hat{f}_2(x)\overline{\hat{f}_2(x+h)}.$$

When h = 0, using condition (2), we see that the inner double sum in (2.3) is bounded by

$$p^{-2\alpha} ||F_0||_{l^1} ||G_0||_{l^1} = p^{-2\alpha} ||f_1||_2^2 ||f_2||_2^2,$$

which is better than $p^{-2\gamma_0} ||f_1||_2^2 ||f_2||_2^2$ as $\alpha > \gamma_0$. Therefore, it remains to handle the case when *h* is nonzero. The tool is the following bilinear form estimate, which may be interesting on its own right (see [11] for applications of some related bilinear forms).

Proposition 2.1. Fix $h \neq 0$. Let $\varphi_1, \varphi_2 : \mathbb{F}_p \to \mathbb{F}_p$ satisfy (2) and (3) (with parameters α and β , resp.) in Theorem 1.1. Let $\gamma_0 = \min\{\alpha - \frac{1}{4}, \frac{\beta}{4} - \frac{1}{4}\}$. Then for any $F, G : \mathbb{F}_p \to \mathbb{C}$,

(2.4)
$$\sum_{u,v} F(u)G(v)K_{\Gamma}(u,v)\overline{K_{\Gamma}(u-h,v+h)} \lesssim \frac{1}{p^{2\gamma_0}} \|F\|_{l^2} \|G\|_{l^2}$$

Once this proposition is proved, one can use (2.4) and apply Cauchy-Schwarz inequality a few times to (2.3) to get the desired estimate (2.2).

By duality, it is easy to see that Proposition 2.1 can be reduced to the following finite field version of Hörmander principle (see Theorem 1.1 in [8] for its continuous counterpart):

Lemma 2.2. Fix $h \neq 0$. Let $\varphi_1, \varphi_2 : \mathbb{F}_p \to \mathbb{F}_p$ satisfy (2) and (3) (with parameters α and β , resp.) in Theorem 1.1. Let $\gamma_0 = \min\{\alpha - \frac{1}{4}, \frac{\beta}{4} - \frac{1}{4}\}$. Define an operator

$$T(g)(x) = \sum_{y} g(y) K_{\Gamma}(x, y) \overline{K_{\Gamma}(x - h, y + h)}.$$

Then

$$||T(g)||_{l^2} \lesssim \frac{1}{p^{2\gamma_0}} ||g||_{l^2}$$

Proof. We will show that

(2.5)
$$||T(g)||_{l^2}^2 \lesssim \frac{1}{p^{4\gamma_0}} ||g||_{l^2}^2.$$

A straightforward calculation gives

$$||T(g)||_{l^{2}}^{2} = \sum_{x,y,y'} g(y)\overline{g(y')}K_{\Gamma}(x,y)\overline{K_{\Gamma}(x-h,y+h)}K_{\Gamma}(x,y')K_{\Gamma}(x-h,y'+h)$$

$$(2.6) \qquad \leq \sum_{(y,y')\in D_{\Gamma,h}} |g(y)||g(y')||I| + \sum_{(y,y')\notin D_{\Gamma,h}} |g(y)||g(y')||I|,$$

where $D_{\Gamma,h}$ is the generalized diagonal set in condition (3) and

$$I_{\Gamma} = \sum_{x} K_{\Gamma}(x, y) \overline{K_{\Gamma}(x - h, y + h)} K_{\Gamma}(x, y') K_{\Gamma}(x - h, y' + h).$$

We estimate the two terms in (2.6) by different methods. Using the definition of generalized diagonal set and the trivial estimate $I_{\Gamma} \leq \frac{p}{p^{4\alpha}}$ from (2), the first term in (2.6) is estimated by

(2.7)
$$\sum_{(y,y')\in D_{\Gamma,h}} |g(y)||g(y')||I_{\Gamma}| \lesssim \sum_{y} |g(y)|^2 \frac{p}{p^{4\alpha}} = \frac{1}{p^{4\alpha-1}} ||g||_{l^2}^2$$

For the second term in (2.6), we use the assumption $I_{\Gamma} \lesssim \frac{1}{p^{\beta}}$ for $(y, y') \notin D_{\Gamma,h}$ and Cauchy-Schwarz inequality to get the estimate

(2.8)
$$\sum_{(y,y')\notin D_{\Gamma,h}} |g(y)||g(y')||I_{\Gamma,h}| \lesssim \frac{\sqrt{p}\sqrt{p}}{p^{\beta}} ||g||_{l^2}^2 = \frac{1}{p^{\beta-1}} ||g||_{l^2}^2.$$

Combining (2.7) and (2.8), we obtain

$$||T(g)||_{l^2}^2 \lesssim \max\left\{\frac{1}{p^{4\alpha-1}}, \frac{1}{p^{\beta-1}}\right\} ||g||_{l^2}^2 = \frac{1}{p^{4\gamma_0}} ||g||_{l^2}^2,$$

which is exactly what we aimed for: (2.5).

3. Proof of Theorem 1.3

To prove Theorem 1.3, first note that we can assume without loss of generality that the two polynomials φ_1 and φ_2 have distinct leading terms. This is because we can rewrite (1.6) in its dual form as

(3.1)
$$|\mathbb{E}_{x,y}f_1(x+\varphi_1(y))f_2(x+\varphi_2(y))f_3(x)-\mathbb{E}[f_1]\mathbb{E}[f_2]\mathbb{E}[f_3]| \lesssim p^{-1/8}||f_1||_2||f_2||_2||f_3||_2,$$

and do a change of variable $x \to x + \varphi_1(y)$ on the left-hand-side of (3.1) if necessary (We are indebted to Sarah Peluse for pointing this out).

We will verify that for linearly independent polynomials $\varphi_1, \varphi_2 \in \mathbb{F}_p[X]$ with distinct leading terms, the conditions (1), (2) and (3) in Theorem 1.1 are satisfied with parameters $\theta = \frac{1}{2}, \alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, resp. and thus prove Theorem 1.3 using Theorem 1.1.

Let d_1 and d_2 denote the degrees of φ_1 and φ_2 , resp. Without loss of generality, we assume that $d_1 \leq d_2$.

Conditions (1) and (2) can be verified in the same way, using the well-known square-root cancellation result of Weil [15] (see also [4]). Therefore, $\theta = \alpha = \frac{1}{2}$. Note that the linearly independence of the two polynomials is crucial to obtain (2).

Now we focus on the verification of condition (3). We will from now on write for simplicity that $K = K_{\Gamma}$ and $I = I_{\Gamma}$. Recall that for $y \neq 0$,

$$K(x,y) = \frac{1}{p} \sum_{z \in \mathbb{F}_p} e_p(x\varphi_1(z) + y\varphi_2(z)).$$

Plug in the definition of K, put the sum over x innermost, and we see that

$$\begin{split} I &= \sum_{x \in \mathbb{F}_p} K(x, y) \overline{K(x - h, y + h)} \overline{K(x, y')} K(x - h, y' + h) \\ &= \frac{1}{p^4} \sum_x \sum_{z_1, z_2, z_3, z_4} e_p [x \varphi_1(z_1) + y \varphi_2(z_1) - (x - h) \varphi_1(z_2) - (y + h) \varphi_2(z_2) - x \varphi_1(z_3) \\ &- y' \varphi_2(z_3) + (x - h) \varphi_1(z_4) + (y' + h) \varphi_2(z_4)] \\ &= \frac{1}{p^3} \sum_{\substack{z_1, z_2, z_3, z_4 \\ G(z_1, z_2, z_3, z_4) = 0}} e_p (F(z_1, z_2, z_3, z_4)), \end{split}$$

where

$$G(z_1, z_2, z_3, z_4) = \varphi_1(z_1) - \varphi_1(z_2) - \varphi_1(z_3) + \varphi_1(z_4),$$

$$F(z_1, z_2, z_3, z_4) = y\varphi_2(z_1) + h\varphi_1(z_2) - (y+h)\varphi_2(z_2) - y'\varphi_2(z_3) - h\varphi_1(z_4) + (y'+h)\varphi_2(z_4)$$

It remains to get the estimate

(3.2)
$$\sum_{\substack{z_1, z_2, z_3, z_4\\G(z_1, z_2, z_3, z_4)=0}} e_p(F(z_1, z_2, z_3, z_4)) \lesssim p^{\frac{3}{2}}.$$

We need machinery of algebraic geometry to prove (3.2). To benefit readers who are not very familiar with algebraic geometry, we first prove (3.2) in a simpler case. We assume $\varphi_1(z) = z$, and consequently φ_2 has degree at least 2 by the linearly independence assumption. In this case, the restriction $G(z_1, z_2, z_3, z_4) = 0$ can be dropped once z_4 is replaced with $z_2 + z_3 - z_1$. Therefore, (3.2) is reduced to

(3.3)
$$\sum_{z_1, z_2, z_3} e_p(F(z_1, z_2, z_3, z_2 + z_3 - z_1)) \lesssim p^{\frac{3}{2}}$$

Such character sum is studied by Deligne in his resolution of Weil conjectures:

Theorem (Theorem 8.4, [5]). Let $f \in \mathbb{F}_p[X_1, \ldots, X_n]$ be a polynomial of degree $d \geq 1$. Suppose that d is prime to p, and the projective hypersurface defined by the highest degree homogeneous term f_d is smooth, i.e., the gradient of f_d is non-zero at any point in $\{f_d = 0\} \setminus \{0\}$. Then

$$\sum_{z_1,\dots,z_n} e_p(f(z_1,\dots,z_n)) \lesssim p^{\frac{n}{2}}$$

For notational convenience, we write $d = d_2$, the degree of φ_2 . Let bz^d denote the leading term of $\varphi_2(z)$. Then the highest degree homogeneous term of $F(z_1, z_2, z_3, z_2 + z_3 - z_1)$ is

$$F_d(z_1, z_2, z_3) = byz_1^d - b(y+h)z_2^d - by'z_3^d + b(y'+h)(z_2+z_3-z_1)^d.$$

We need to verify the smoothness $\{F_d = 0\}$. By straightforward calculations, $\nabla F_d = \mathbf{0}$ implies

$$\begin{cases} z_1 = \left(\frac{y'+h}{y}\right)^{\frac{1}{d-1}} (z_2 + z_3 - z_1) \\ z_2 = \left(\frac{y'+h}{y+h}\right)^{\frac{1}{d-1}} (z_2 + z_3 - z_1) \\ z_3 = \left(\frac{y'+h}{y'}\right)^{\frac{1}{d-1}} (z_2 + z_3 - z_1) \end{cases}$$

The above system has nonzero solutions only when

(3.4)
$$\left(\frac{y'+h}{y+h}\right)^{\frac{1}{d-1}} + \left(\frac{y'+h}{y'}\right)^{\frac{1}{d-1}} - \left(\frac{y'+h}{y}\right)^{\frac{1}{d-1}} = 1$$

Put those pairs (y, y') satisfying (3.4) as a set $D_{\Gamma,h}$, and it is not hard to check that $D_{\Gamma,h}$ is generalized diagonal. By Deligne's Theorem, (3.3) holds for any $(y, y') \notin D_{\Gamma,h}$. This finishes the verification of condition (3) with $\beta = \frac{3}{2}$, assuming $\varphi_1(z) = z$.

Now we turn to the general case. In [9], Katz generalizes Deligne's theorem to exponential sums over smooth affine varieties, and in [10], to singular algebraic varieties. We need the following special case of [10, Theorem 4] (The reader could skip its long proof and use it as a "black box" on an early reading of the paper):

Theorem 3.1. Let $F, G \in \mathbb{F}_p[X_1, \ldots, X_4]$. Assume that the degree of F is indivisible by p, the homogeneous leading term of G defines a smooth projective hypersurface, and the homogeneous leading terms of G and that of F together define a smooth co-dimension-2 variety in the projective space. Then (3.2) holds, i.e.,

$$\sum_{\substack{z_1, z_2, z_3, z_4 \\ G(z_1, z_2, z_3, z_4) = 0}} e_p(F(z_1, z_2, z_3, z_4)) \lesssim p^{\frac{3}{2}}.$$

Proof. We explain in detail how to realize this theorem as a special case of Katz's theorem. We will try to explain this derivation for mathematicians who are not experts in algebraic geometry. (However, Katz's proof requires much more advanced algebraic geometry than we can go into here).

We first restate part of Katz's theorem. Then we will explain Katz's notation and how it applies to our case.

Theorem (Katz, Theorem 4 [10]). Let N and d be natural numbers, let k be a finite field in which d is invertible, let $\psi : k \to \mathbb{C}^{\times}$ be an additive character. Let X be a closed subscheme of \mathbb{P}^N of dimension d. Let L be a section of $H^0(X, \mathcal{O}(1))$ and H a section of $H^0(X, \mathcal{O}(D))$. Let V, f, ϵ , δ be defined as in [10, pp. 878-879]. If assumptions (H1)' and (H2) of [10, pp. 878] hold, and $\epsilon \leq \delta$, then

$$\left|\sum_{x \in V(k)} \psi(f(x))\right| \le C \times (\#k)^{(n+1+\delta)/2}$$

where C is a constant depending only on N, d, and the number and degree of the equations defining X.

We will choose our data so that $k = \mathbb{F}_p$, $V(k) = \{z_1, z_2, z_3, z_4 \in \mathbb{F}_p \mid G(z_1, z_2, z_3, z_4) = 0\}$, $\psi(f(x)) = e_p(F(z_1, z_2, z_3, z_4))$ for $x = (z_1, z_2, z_3, z_4) \in V(k)$, n = 3, and $\epsilon = \delta = -1$. Furthermore *C* will depend only on the degree of *F* and *G*.

Examining Katz's bound, and plugging in these statements, it is clear that if we can in fact choose our data in this way, while verifying Katz's conditions, we obtain exactly our stated bound.

In what remains, we will first explain all of Katz's notation that is needed to choose (X, L, H) so that

$$V(k) = \{z_1, z_2, z_3, z_4 \in \mathbb{F}_p \mid G(z_1, z_2, z_3, z_4) = 0\} \text{ and } \psi(f(x)) = e_p(F(z_1, z_2, z_3, z_4)),$$

and second we will verify (H1)' and (H2) and calculate ϵ, δ , explaining more of Katz's notation along the way.

For the first part, because we are interested in the \mathbb{F}_p -points $V(\mathbb{F}_p)$ of a scheme V, we will describe schemes mostly by their set of \mathbb{F}_p -points (though schemes in fact have more structure than this.) First, we take N = 4, so $\mathbb{P}^N = \mathbb{P}^4$ is the space whose \mathbb{F}_p -points $\mathbb{P}^4(\mathbb{F}_p)$ are the set of quintuples $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{F}_p$, not all zero, up to multiplication by nonzero scalars. We let \tilde{G} be the homogenization of G, where we add additional powers of z_5 to all the non-leading terms of G to make every term have equal degree. Let X be the vanishing set of \tilde{G} , so that $X(\mathbb{F}_p)$ is the subset of $\mathbb{P}^4(\mathbb{F}_p)$ consisting of tuples $(z_1, z_2, z_3, z_4, z_5)$ with $\tilde{G}(z_1, z_2, z_3, z_4, z_5) = 0$. We must choose L as an element of $H^0(X, \mathcal{O}(1))$, which is the space of linear functions in the variables z_1, z_2, z_3, z_4, z_5 , and we choose $L = z_5$. Now Katz defines V to be the locus in X where L is nonzero. Hence $V(\mathbb{F}_p)$ is the set of tuples $(z_1, z_2, z_3, z_4, z_5) = 0$. For each such tuple there exists a unique scalar multiplication that sends z_5 to 1, so we can express it equally as the set of tuples (z_1, z_2, z_3, z_4) with $\tilde{G}(z_1, z_2, z_3, z_4, 1) = 0$. By construction, $\tilde{G}(z_1, z_2, z_3, z_4, 1) = G(z_1, z_2, z_3, z_4)$, so $V(\mathbb{F}_p) = \{z_1, z_2, z_3, z_4 \in \mathbb{F}_p \mid G(z_1, z_2, z_3, z_4) = 0\}$, as desired.

Next, because $e_p : \mathbb{F}_p \to \mathbb{C}^{\times}$ is an additive character, we set $\psi = e_p$. We then need to choose H, a homogeneous form of degree d in the variables z_1, z_2, z_3, z_4, z_5 , so that $f(x) = F(z_1, z_2, z_3, z_4)$. Katz defines f as H/L^d . We take d to be the degree of F and H to be the homogenization \tilde{F} of F, just as we did with G. Because we are using the bijection between 4-tuples and 5-tuples that sends (z_1, z_2, z_3, z_4) to $(z_1, z_2, z_3, z_4, 1)$, we need to check that $f(z_1, z_2, z_3, z_4, 1) = F(z_1, z_2, z_3, z_4)$. This follows because

$$f(z_1, z_2, z_3, z_4, 1) = \frac{\tilde{F}(z_1, z_2, z_3, z_4, 1)}{L(z_1, z_2, z_3, z_4, 1)^d} = \frac{\tilde{F}(z_1, z_2, z_3, z_4, 1)}{1^d} = F(z_1, z_2, z_3, z_4).$$

We have therefore shown how to specialize the left side of Katz's bound to the left side of our own bound. It remains to check Katz's assumptions and also the assumptions we made in applying Katz's bound. These are as follows:

- (1) d is invertible in k.
- (2) Katz's assumption (H1)' holds.
- (3) Katz's assumption (H2) holds.
- (4) $\delta = -1$.
- (5) $\epsilon = -1$.
- (6) n = 3.
- (7) C depends only on the degree of F and G.

The first condition, that d is invertible in k, is easy to interpret, as we set $k = \mathbb{F}_p$ and set d to equal the degree of F, so this is equivalent to the degree of F being prime to p, which we have already assumed in the statement of the theorem.

Katz's assumption (H1)' is that X is Cohen-Macauley and equidimensional of dimension $n \ge 1$. Because H is the hypersurface defined by a single equation $\tilde{G} = 0$ in \mathbb{P}^4 , a smooth variety of dimension 4, it is automatically Cohen-Macauley of dimension 3. This verifies assumptions (2) and (6).

Katz defines C as an explicit function of his numerical data, which consists of N, the number r of equations needed to define X, the degrees of those equations, and d. In our case

N = 4, r = 1, the degree of the unique equation needed to define X is the degree of G, and d is the degree of F. Hence C is some explicit function of those degrees (assumption (7)).

Katz defines ϵ as the dimension of the singular locus of the scheme-theoretic intersection $X \cap L$. For us L is the closed subset of \mathbb{P}^4 where $z_5 = 0$. (Katz abuses notation slightly to use L also to refer to the vanishing locus of L.) So $X \cap L$ is the closed subset where $z_5 = 0$ and $\tilde{G} = 0$. Because $z_5 = 0$, we can ignore z_5 and work in \mathbb{P}^3 with coordinates z_1, z_2, z_3, z_4 . When we do this, because all non-leading monomials of G were multiplied by a positive power of z_5 in \tilde{G} , all non-leading monomials become 0 and we are left with just the zero-locus. So $X \cap L$ is the vanishing locus of the leading term of G in \mathbb{P}^3 , which we assumed in the statement of the theorem is a nonsingular hypersurface, so its singular locus is empty, which by convention Katz assigns dimension -1, verifying $\epsilon = -1$ (assumption (5)).

Katz defines δ as the dimension of the singular locus of the scheme-theoretic intersection $X \cap L \cap H$, and (H2) is his assumption that this has dimension n-2. This is the joint vanishing locus of \tilde{G}, z_5 , and \tilde{F} in \mathbb{P}^4 , which for the same reason as before is the vanishing locus of the leading terms of F and G in \mathbb{P}^3 . Because we assumed this is a smooth subscheme of codimension 2, it has dimension 3-2=n-2, verifying condition (H2), and its singular locus is empty and has dimension -1, verifying $\delta = -1$ (assumptions (3) and (4)).

Now we are ready to prove (3.2) using Theorem 3.1. The first two conditions in the theorem are easy to check. To check the third condition, we handle two cases separately: $d_1 < d_2$ and $d_1 = d_2$.

First assume $d_1 < d_2$. Let az^{d_1} and bz^{d_2} denote the leading term of φ_1 and φ_2 , resp. The homogeneous leading term of G and F are

$$G_{d_1}(z_1, z_2, z_3, z_4) := a z_1^{d_1} - a z_2^{d_1} - a z_3^{d_1} + a z_4^{d_1},$$

and

$$F_{d_2}(z_1, z_2, z_3, z_4) := by z_1^{d_2} - b(y+h) z_2^{d_2} - by' z_3^{d_2} + b(y'+h) z_4^{d_2},$$

resp. We need to show that the Jacobian matrix

$$J = \begin{bmatrix} \nabla G_{d_1} \\ \nabla F_{d_2} \end{bmatrix} = \begin{bmatrix} d_1 a z_1^{d_1 - 1} & -d_1 a z_2^{d_1 - 1} & -d_1 a z_3^{d_1 - 1} & d_1 a z_4^{d_1 - 1} \\ d_2 b y z_1^{d_2 - 1} & -d_2 b (y + h) z_2^{d_2 - 1} & -d_2 b y' z_3^{d_2 - 1} & d_2 b (y' + h) z_4^{d_2 - 1} \end{bmatrix}$$

has full rank at any point in $\{G_{d_1} = F_{d_2} = 0\} \setminus \{\mathbf{0}\}$. When J has rank less than 2, assuming $z_1 z_2 z_3 z_4 \neq 0$, we can solve for each z_i and plug in $G_{d_1} = 0$ to get the equation

(3.5)
$$\left(\frac{1}{y}\right)^{\frac{d_1}{d_2-1}} - \left(\frac{1}{y+h}\right)^{\frac{d_1}{d_2-1}} - \left(\frac{1}{y'}\right)^{\frac{d_1}{d_2-1}} + \left(\frac{1}{y'+h}\right)^{\frac{d_1}{d_2-1}} = 0$$

If one or two of the four variables z_1, z_2, z_3, z_4 are zero, then a new equation can be obtained by deleting the corresponding term(s) in the above equation. The solutions to (3.5) and its variants lie in a generalized diagonal set. So we can apply Theorem 3.1 for pairs (y, y')outside this set.

Secondly consider the case $d_1 = d_2 = d$. The homogeneous leading term of G and F are

$$G_d(z_1, z_2, z_3, z_4) := az_1^d - az_2^d - az_3^d + az_4^d,$$

and

$$F_d(z_1, z_2, z_3, z_4) := byz_1^d - (b(y+h) - ah)z_2^d - by'z_3^d + (b(y'+h) - ah)z_4^d,$$

resp. The Jacobian matrix becomes

$$J = \begin{bmatrix} \nabla G_d \\ \nabla F_d \end{bmatrix} = \begin{bmatrix} daz_1^{d-1} & -daz_2^{d-1} & -daz_3^{d-1} & daz_4^{d-1} \\ dbyz_1^{d-1} & -d(b(y+h)-ah)z_2^{d-1} & -dby'z_3^{d-1} & d(b(y'+h)-ah)z_4^{d-1} \end{bmatrix}$$

When $z_1 z_2 z_3 z_4 \neq 0$, J has rank 1 only when

(3.6)
$$by = b(y+h) - ah = by' = b(y'+h) - ah.$$

One or two terms in the above equation can be dropped if the corresponding variable is zero. Since we assume that φ_1 and φ_2 have distinct leading terms, $a \neq b$. It is then easy to see that the solutions to (3.6) and its variants form a generalized diagonal set. So Theorem 3.1 applies in most cases, and we are done.

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