BDDC and FETI-DP methods with enriched coarse spaces for elliptic problems with oscillatory and high contrast coefficients

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1 Introduction

BDDC (Balancing Domain Decomposition by Constraints) and FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms with adaptively enriched coarse spaces are developed and analyzed for second order elliptic problems with high contrast and random coefficients. Among many approaches to form adaptive coarse spaces, we consider an approach using eigenvectors of generalized eigenvalues problems defined on each subdomain interface, see Mandel and Sousedík [2007], Galvis and Efendiev [2010], Spillane et al. [2011, 2013], Klawonn et al. [2015].

The main contribution of the current work is to extend the methods in Dohrmann and Pechstein [2013], Klawonn et al. [2014] to three-dimensional problems. In three dimensions, there are three types of equivalence classes on the subdomain interfaces, i.e., faces, edges, and vertices. A face is shared by two subdomains. An edge is shared by more than two subdomains. Vertices are end points of edges. In addition to the generalized eigenvalue problems on faces, which are already considered in Dohrmann and Pechstein [2013], Klawonn et al. [2014] for two-dimensional problems, generalized eigenvalues problems on edges are proposed.

Equipped with the coarse space formed by using the selected eigenvectors, the condition numbers of the resulting algorithms are determined by the user defined tolerance value λ_{TOL} that is used to select the eigenvectors. An estimate of condition numbers is obtained as $C\lambda_{TOL}$, where the constant C is independent of coefficients and any mesh parameters. We note that a

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full version of the current paper was submitted to a journal. We also note that an adaptive BDDC algorithm for three-dimensional problems was considered and numerically tested in Mandel et al. [2012] for difficult engineering applications.

This paper is organized as follows. A brief description of BDDC and FETI-DP algorithms is given in Section 2. Adaptive selection of coarse spaces is presented in Section 3 and the estimate of condition numbers of the both algorithms is provided in Section 4.

2 BDDC and FETI-DP algorithms

To present BDDC and FETI-DP algorithms, we introduce a finite element space $\widehat{X}$ for a given domain Ω, where the model elliptic problem is defined as

$$-\nabla \cdot (\rho(x)\nabla u(x)) = f(x) \tag{1}$$

with a boundary condition on $u(x)$ and with $\rho(x)$ highly varying and random. The domain is then partitioned into non-overlapping subdomains $\{\Omega_i\}$ and X_i are the restrictions of $\widehat{X}$ to Ω_i. The subdomain interfaces are assumed to be aligned to the given triangles in X. In three dimensions, the subdomain interfaces consist of faces, edges, and vertices. We introduce W_i as the restriction of X_i to the subdomain interface unknowns, W and X as the product of the local finite element spaces W_i and X_i, respectively. We note that functions in W or X are decoupled across the subdomain interfaces. We then select some primal unknowns among the decoupled unknowns on the interfaces and enforce continuity on them and denote the corresponding spaces $\widetilde{W}$ and $\widetilde{X}$.

The preconditioners in BDDC and FETI-DP algorithms will be developed based on the partially coupled space $\widetilde{W}$ and appropriate scaling matrices. We refer to Dohrmann [2003], Farhat et al. [2001], Li and Widlund [2006] for general introduction of these algorithms. The unknowns at subdomain vertices will first be included in the set of primal unknowns. Additional set of primal unknowns will be selected by solving generalized eigenvalue problems on faces and edges. In the BDDC algorithm, they are enforced just like unknowns at subdomain vertices after a change of basis, while in the FETI-DP algorithm they are enforced by using a projection, see Klawonn et al. [2015].

We next define the matrices K_i and S_i. The matrices K_i are obtained from the Galerkin approximation of

$$a(u,v) = \int_{\Omega_i} \rho(x)\nabla u \cdot \nabla v \, dx$$

by using finite element spaces X_i and S_i are the Schur complements of K_i, that are obtained after eliminating unknowns interior to Ω_i. Let $\widetilde{R}_i : \widetilde{W} \to W_i$

be the restriction into $\partial\Omega_i \setminus \partial\Omega$ and let $\widetilde{S}$ be the partially coupled matrix defined by

$$\widetilde{S} = \sum_{i=1}^{N} \widetilde{R}_i^T S_i \widetilde{R}_i.$$

Let $\widetilde{R}$ be the restriction from $\widehat{W}$ to $\widetilde{W}$. The discrete problem of (1) is then written as

$$\widetilde{R}^T \widetilde{S} \widetilde{R} = \widetilde{R}^T \widetilde{g},$$

where $\widetilde{g}$ is the vector given by the right hand side $f(x)$. The above matrix equation can be solved iteratively by using preconditioners. The BDDC preconditioner is then given by

$$M_{BDDC}^{-1} = \widetilde{R}^T \widetilde{D} \widetilde{S}^{-1} \widetilde{D}^T \widetilde{R},$$

where $\widetilde{D}$ is a scaling matrix of the form

$$\widetilde{D} = \sum_{i=1}^{N} \widetilde{R}_i^T D_i \widetilde{R}_i.$$

Here the matrices D_i are defined for unknowns in W_i and they are introduced to resolve heterogeneity in $\rho(x)$ across the subdomain interface. In more detail, D_i consists of blocks $D_F^{(i)}$, $D_E^{(i)}$, $D_V^{(i)}$, where F denotes an equivalence class shared by two subdomains, i.e., Ω_i and its neighboring subdomain Ω_j, E denotes an equivalence class shared by more than two subdomains, and V denotes the end points of E, respectively. We note that those blocks should satisfy the partition of unity for a given F, E, and V, respectively, and call them faces, edges, and vertices, respectively. We refer to Klawonn and Widlund [2006] for these definitions.

The FETI-DP preconditioner is a dual form of the BDDC preconditioner. In our case, the unknowns at subdomain vertices are chosen as the initial set of primal unknowns and the algebraic system of the FETI-DP algorithm is obtained as

$$B\widetilde{S}^{-1}B^T\lambda = d,$$

where $\widetilde{S}$ is the partially coupled matrix at subdomain vertices and B is a matrix with entries 0, -1, and 1, which is used to enforce continuity at the decoupled interface unknowns. The above algebraic system is then solved by an iterative method with the following projected preconditioner

$$M_{FETI}^{-1} = (I-P)B_D \widetilde{S} B_D^T (I - P^T),$$

where B_D is defined by

$$B_D = \begin{pmatrix} B_{D,\Delta} & 0 \end{pmatrix} = \begin{pmatrix} B_{D,\Delta}^{(1)} & \cdots & B_{D,\Delta}^{(i)} & 0 \end{pmatrix}.$$

In the above, $B_{D,\Delta}^{(i)}$ is a scaled matrix of $B_{\Delta}^{(i)}$ where rows corresponding to Lagrange multipliers to the unknowns $w^{(i)} \in W_i$ are multiplied with a scaling matrix $(D_C^{(j)})^T$ when the Lagrange multipliers connect $w^{(i)}$ to $w^{(j)} \in W_j$ and Ω_j is the neighboring subdomain sharing the interface C of $\partial\Omega_i$. The interface C can be F, faces, or E, edges. The matrix P is a projection operator related to the additional primal constraints and it is given by

$$P = U(U^T F_{DP} U)^{-1} U^T F_{DP},$$

where $F_{DP} = B\widetilde{S}^{-1}B^T$ and U consists of columns related to the additional primal constraints on the decoupled interface unknowns.

3 Adaptively enriched coarse spaces

With the standard choice of primal unknowns, values at subdomain vertices, edge averages, and face averages, the performance of BDDC and FETI-DP preconditioners can often deteriorate for bad arrangements of the coefficient $\rho(x)$. The preconditioner can be enriched by using adaptively chosen primal constraints. The adaptive constraints will be selected by considering generalized eigenvalue problems on each equivalence class. The idea is originated from the upper bound estimate of BDDC and FETI-DP preconditioners. In the estimate of condition numbers of BDDC and FETI-DP preconditioners, the average and jump operators are defined as

$$E_D = \widetilde{R}\widetilde{R}^T\widetilde{D}, \quad P_D = B_D^T B.$$

When adaptive constraints are introduced, they are enforced strongly just like unknowns at vertices after a change of basis formulation in the BDDC algorithm. In contrast, in the FETI-DP algorithm the additional constraints are enforced weakly by using a projection P. In general, $E_D + P_D = I$ does not hold when adaptively enriched constraints are included in the preconditioners. Thus the analysis of BDDC and FETI-DP algorithms requires the following estimates, respectively,

$$\langle \widetilde{S}(I - E_D)\widetilde{w}_a, (I - E_D)\widetilde{w}_a \rangle \le C \langle \widetilde{S}\widetilde{w}_a, \widetilde{w}_a \rangle,$$

$$\langle \widetilde{S} P_D \widetilde{w}, P_D \widetilde{w} \rangle \le C \langle \widetilde{S}\widetilde{w}, \widetilde{w} \rangle.$$

In the above, $\widetilde{w}_a$ is strongly coupled at the initial set of primal unknowns and the adaptively enriched primal unknowns after the change of basis while $\widetilde{w}$ is strongly coupled at the initial set of primal unknowns and satisfies the adaptive constraints across the subdomain interfaces, $v^T(w_i - w_j) = 0$ with v a vector of an adaptive constraint.

For a face F, shared by two subdomains Ω_i and Ω_j, we restrict the operator $I - E_D$ to $F \subset \partial\Omega_i$ and obtain

$$((I - E_D)\widetilde{w}_a)|_F = D_F^{(j)}(\widetilde{w}_{F,\Delta}^{(i)} - \widetilde{w}_{F,\Delta}^{(j)}), \tag{2}$$

where $\widetilde{w}_{F,\Delta}^{(i)}$ denotes the vector of unknowns on $F \subset \partial\Omega_i$ with zero primal unknowns and the dual unknowns identical to $\widetilde{w}_a$. Similarly, for an edge $E \subset \partial\Omega_i$,

$$((I - E_D)\widetilde{w}_a)|_E = \sum_{m \in E(i)} D_E^{(m)}(\widetilde{w}_{E,\Delta}^{(i)} - \widetilde{w}_{E,\Delta}^{(m)}),$$

where $E(i)$ denotes the set of subdomain indices sharing the edge E with Ω_i. We now introduce a Schur complement matrix $\widetilde{S}_C^{(i)}$ of S_i, which are obtained after eliminating unknowns except those interior to C. Here C can be an equivalence class, F or E. For semi-positive definite matrices A and B, we introduce a parallel sum defined as, see Anderson and Duffin [1969],

$$A : B = A(A + B)^{+}B,$$

where $(A + B)^{+}$ denotes a pseudo inverse. The parallel sum satisfies the following properties

$$A : B = B : A, \quad A : B \le A, \quad A : B \le B, \tag{3}$$

and it was first used in forming generalized eigenvalues problems by Dohrmann and Pechstein [2013]. We note that a similar approach was considered by Klawonn et al. [2014] in a more general form. Both are limited to the two-dimensional problems with only face equivalence classes. In this work, generalized eigenvalue problems for edge equivalence classes will be introduced to extend the previous approaches to three dimensions.

For a face F, the following generalized eigenvalue problem is considered

$$A_F v_F = \lambda \widetilde{A}_F v_F,$$

where

$$A_F = (D_F^{(j)})^T S_F^{(i)} D_F^{(j)} + (D_F^{(i)})^T S_F^{(j)} D_F^{(i)}, \; \widetilde{A}_F = \widetilde{S}_F^{(i)} : \widetilde{S}_F^{(j)},$$

and $S_F^{(i)}$ denote block matrix of S_i to the unknowns interior to F. The eigenvalues are all positive and we select eigenvectors $v_{F,l}$, $l \in N(F)$ with associated eigenvalues λ_l larger than a given λ_{TOL}. The following constraints will then be enforced on the unknowns in F,

$$(A_F v_{F,l})^T (w_F^{(i)} - w_F^{(j)}) = 0, \; l \in N(F).$$

After a change of unknowns, the above constraints can be transformed into explicit unknowns and they are added to the initial set of primal unknowns and denoted by $w_{F,\Pi}^{(i)}$. The remaining unknowns are called dual unknowns and denoted by $w_{F,\Delta}^{(i)}$. Using (2), for the two-dimensional case we obtain that

$$\begin{aligned}
&\langle \widetilde{S}(I-E_D)\widetilde{w}_a, (I-E_D)\widetilde{w}_a\rangle \leq C\sum_F (\langle A_F \widetilde{w}_{F,\Delta}^{(i)}, \widetilde{w}_{F,\Delta}^{(i)}\rangle + \langle A_F \widetilde{w}_{F,\Delta}^{(j)}, \widetilde{w}_{F,\Delta}^{(j)}\rangle)\\
&\leq C\lambda_{TOL}\sum_F (\langle \widetilde{A}_F \widetilde{w}_{F,\Delta}^{(i)}, \widetilde{w}_{F,\Delta}^{(i)}\rangle + \langle \widetilde{A}_F \widetilde{w}_{F,\Delta}^{(j)}, \widetilde{w}_{F,\Delta}^{(j)}\rangle)\\
&\leq C\lambda_{TOL}\sum_F (\langle S^{(i)} w_i, w_i\rangle + \langle S^{(j)} w_j, w_j\rangle),
\end{aligned}$$

where the estimate on the dual unknowns are bounded by λ_{TOL} in the second inequality, and (3) and the minimum energy property of $\widetilde{S}_F^{(i)}$ are used in the last inequality.

For an edge E, shared by more than two subdomains, we introduce the following generalized eigenvalue problem,

$$A_E v_E = \lambda \widetilde{A}_E v_E,$$

where

$$A_E = \sum_{m\in I(E)} \sum_{l\in I(E)\setminus\{m\}} (D_E^{(l)})^T S_E^{(m)} D_E^{(l)}, \quad \widetilde{A}_E = \prod_{m\in I(E)} \widetilde{S}_E^{(m)},$$

and $I(E)$ denotes the set of subdomain indices sharing E in common, and $\prod_{m\in I(E)} \widetilde{S}_E^{(m)}$ is the parallel sum applied to those matrices $\widetilde{S}_E^{(m)}$. For a given λ_{TOL}, the eigenvectors with their eigenvalues larger than λ_{TOL} will be selected and denoted by $v_{E,l}$, $l \in N(E)$. The following constraints will then be enforced on the unknowns in E,

$$(A_E v_{E,l})^T (w_E^{(i)} - w_E^{(m)}) = 0,\ l \in N(E),\ m \in I(E)\setminus\{i\}.$$

Using the adaptively selected primal unknowns on each face F and edge E, we can obtain the following estimate

$$\langle \widetilde{S}(I-E_D)\widetilde{w}_a, (I-E_D)\widetilde{w}_a\rangle \leq C\lambda_{TOL}\langle \widetilde{S}\widetilde{w}_a, \widetilde{w}_a\rangle,$$

where C is a constant depending on the maximum number of edges and faces per subdomain, and the maximum number of subdomains sharing an edge but is independent of the coefficient $\rho(x)$.

4 Condition number estimate

Using the adaptively enriched primal constraints described in Section 3, we can obtain the following bound of the condition numbers for the given λ_{TOL}:

Theorem 1. *The BDDC algorithm with the change of basis formulation for the adaptively chosen set of primal unknowns with a given tolerance λ_{TOL} has the following bound of condition numbers,*

$$\kappa(M_{BDDC,a}^{-1}\widetilde{R}^T\widetilde{S}_a\widetilde{R}) \leq C\lambda_{TOL},$$

and the FETI-DP algorithm with the projector preconditioner M_{FETI}^{-1} has the bound

$$\kappa(M_{FETI}^{-1}F_{DP}) \leq C\lambda_{TOL},$$

where C is a constant depending only on $N_{F(i)}$, $N_{E(i)}$, $N_{I(E)}$, which are the number of faces per subdomain, the number of edges per subdomain, and the number of subdomains sharing an edge E, respectively.

In the above $M_{BDDC,a}$ and $\widetilde{S}_a$ denote the BDDC preconditioner and the partially assembled matrix of S_i after the change of unknowns for the adaptive primal constraints. We refer to Kim et al. [2015] for detailed proofs of the above theorem. We note that for the FETI-DP algorithm with the projector preconditioner the approaches in Toselli and Widlund [2005] can be used to obtain the upper bound estimate

$$\langle \widetilde{S}P_D\widetilde{w}, P_D\widetilde{w}\rangle \leq C\lambda_{TOL}\langle \widetilde{S}\widetilde{w}, \widetilde{w}\rangle,$$

where $\widetilde{w}$ is strongly coupled at vertices and the adaptive primal constraints on F and E are enforced on $\widetilde{w}$ by using the projection P.

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