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Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems

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Abstract The purpose of this paper is to investigate the problem of finding a common element of the set of fixed points $F(S)$ of a nonexpansive mapping S and the set of solutions Ω_A of the variational inequality for a monotone, Lipschitz continuous mapping A. We introduce a hybrid extragradient-like approximation method which is based on the well-known extragradient method and a hybrid (or outer approximation) method. The method produces three sequences which are shown to converge strongly to the same common element of $F(S) \cap \Omega_A$. As applications, the method provides an algorithm for finding the common fixed point of a nonexpansive mapping and a pseudocontractive mapping, or a common zero of a monotone Lipschitz continuous mapping and a maximal monotone mapping.

Keywords Hybrid extragradient-like approximation method · Variational inequality · Fixed point · Monotone mapping · Nonexpansive mapping · Demiclosedness principle

Mathematics Subject Classification (2000) 47J20 · 47H09

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and A be a mapping of C into H. Then A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is called α-inverse-strongly monotone (see [4,12]) if there exists a positive constant α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A is called k-Lipschitz continuous if there exists a positive constant k such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

It is clear that if A is α-inverse-strongly monotone, then A is monotone and Lipschitz continuous.

In this paper, we consider the following variational inequality (for short, VI(A, C)): find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the VI(C, A) is denoted by Ω_A. A mapping $S : C \to C$ is called nonexpansive [7] if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(S)$ the set of fixed points of S, i.e., $F(S) = \{u \in C : Su = u\}$. A more restrictive class of maps are the contractive maps, i.e. maps $S : C \to C$ such that for some $\alpha \in (0, 1)$,

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Due to the many applications of the variational inequality problem to several branches of mathematics, but also to mechanics, economics etc, finding its solutions is a very important field of research. In some cases, as for strictly monotone operators A, the solution, if it exists, is unique. More generally the set of solutions Ω_A of a continuous monotone mapping A is a convex subset of C. In such cases one is often interested in finding a solution that has some desirable properties. For instance, Antipin has investigated methods for finding a solution of a variational inequality that satisfies some additional inequality constraints [1,2], in a finite-dimensional space. Takahashi and Toyoda [18] considered the problem of finding a solution of the variational inequality which is also a fixed point of some mapping, in an infinite-dimensional setting. More precisely, given a nonempty, closed and convex set $C \subseteq H$, a nonexpansive mapping $S : C \to C$ and an α-inverse-strongly-monotone mapping $A : C \to H$, they introduced the following iterative scheme in order to find an element of $F(S) \cap \Omega_A$:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \end{cases} \tag{1}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and P_C is the metric projection of H onto C. It is shown in [18] that if $F(S) \cap \Omega_A \neq \emptyset$, then the sequence $\{x_n\}$ generated by Eq. 1 converges weakly to some $z \in F(S) \cap \Omega_A$. Later on, in order to

achieve strong convergence to an element of $F(S) \cap \Omega_A$ under the same assumptions, Iiduka and Takahashi [10] modified the iterative scheme by using a hybrid method as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{2}$$

for all $n \ge 0$, where $0 \le \alpha_n \le c < 1$ and $0 < a \le \lambda_n \le b < 2\alpha$. It is shown in [10] that if $F(S) \cap \Omega_A \ne \emptyset$, then the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap\Omega_A} x$.

The restriction of the above methods to the class of of α-inverse strongly monotone mappings (i.e., mappings whose inverse is strongly monotone) excludes some important classes of mappings, as pointed out by Nadezhkina and Takahashi [14]. The so-called extragradient method, introduced in 1976 by Korpelevich [11] for a finite-dimensional space, provides an iterative process converging to a solution of $\mathrm{VI}(A, C)$ by only assuming that $C \subseteq \mathbb{R}^n$ is closed and convex and $A : C \to \mathbb{R}^n$ is monotone and k-Lipschitz continuous. The extragradient method was further extended to infinite dimensional spaces by many authors; see for instance the recent contributions of He, Yang and Yuan [8], Solodov and Svaiter [17], Ceng and Yao [6,19] etc.

By modifying the extragradient method, Nadezhkina and Takahashi were able to show the following weak convergence result, for mappings A that are only supposed to be monotone and k-Lipschitz:

Theorem 1 [13, Theorem 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega_A \ne \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda A y_n) \end{cases}$$

for all $n \ge 0$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega_A$ where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega_A} x_n$.

Further, inspired by Nadezhkina and Takahashi's extragradient method [13], Ceng and Yao [5] also introduced and considered an extragradient-like approximation method which is based on the above extragradient method and the viscosity approximation method, and proved the following strong convergence result.

Theorem 2 [5, Theorem 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\alpha \in (0, 1)$, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega_A \ne \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = (1-\gamma_n) x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n) x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda A y_n), \end{cases}$$

for all $n \geq 0$, where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $q = P_{F(S)\cap\Omega_A} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\liminf_{n\to\infty} \langle Ax_n, y - x_n\rangle \geq 0$ *for all* $y \in C$.

Very recently, by combining a hybrid-type method with an extragradient-type method, Nadezhkina and Takahashi [14] introduced the following iterative method for finding an element of $F(S) \cap \Omega_A$ and established the following strong convergence theorem.

Theorem 3 [14, Theorem 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega_A \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x, \end{cases} \tag{3}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to the same point $q = P_{F(S)\cap\Omega_A} x$.

In this paper, we introduce a hybrid extragradient-like approximation method which is based on the above extragradient method and the hybrid (or outer approximation) method, i.e.,

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n Ax_n), \\ z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S P_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n)b^2 \|Ax_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0 \end{cases}$$

for all $n \geq 0$, where $\{\lambda_n\}$ is a sequence in $[a, b]$ with $a > 0$ and $b < \frac{1}{2k}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$
(iv) $\lim_n \gamma_n = 1$ and $\gamma_n > 3/4$ for all $n \geq 0$.

It is shown that the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ generated by the above hybrid extragradient-like approximation method are well-defined and converge strongly to the same point $q = P_{F(S)\cap\Omega_A} x$. Using this theorem, we construct an iterative process for finding a common fixed point of two mappings, one of which is nonexpansive and the other taken from the more general class of Lipschitz pseudocontractive mappings.

In the next section we will recall some basic notions and results. In Sect. 3 we present and prove our main theorem. The last section is devoted to some applications.

2 Preliminaries

Given a nonempty, closed and convex subset C of a Hilbert space H, for any $x \in H$ there exists a unique element $P_C x \in C$ which is nearest to x, i.e. for all $y \in C$,

$$\|x - P_C x\| \leq \|x - y\|. \tag{4}$$

The projection operator $P_C : H \to C$ is nonexpansive on H: For every $x, y \in H$,

$$\|P_C x - P_C y\| \leq \|x - y\|.$$

In addition, it has the following properties: For every $x \in H$ and $y \in C$,

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2; \tag{5}$$

also,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{6}$$

Assume that A is monotone and continuous. Then the solutions of the variational inequality $\mathrm{VI}(A, C)$ can be characterized as solutions of the so-called Minty variational inequality:

$$x^* \in \Omega_A \Leftrightarrow \left\langle Ax, x - x^*\right\rangle \geq 0, \quad \forall x \in C. \tag{7}$$

We will also make use of Browder's demiclosedness principle, cf. for instance [16]. Let us denote by I the identity operator in H.

Proposition 4 *Let $C \subseteq H$ be closed and convex. Assume that $S : C \to H$ is nonexpansive. If S has a fixed point, then $I - S$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I - S)x_n\}$ converges strongly to some $y \in H$, it follows that $(I - S)x = y$.*

A mapping $T : C \to C$ is called pseudocontractive if and only if for all $x, y \in C$,

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2.$$

It is clear that any contractive mapping is pseudocontractive. Also, it is easy to see that T is pseudocontractive if and only if the mapping $A = I - T$ is monotone [3].

A multivalued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$, $x^* \in Tx$ and $y^* \in Ty$ one has $\langle y^* - x^*, y - x\rangle \geq 0$. Such a mapping is called maximal monotone if it has no proper monotone extension, i.e., if $B_1 : H \to 2^H$ is monotone and $Bx \subseteq B_1 x$ for all $x \in H$, then $B = B_1$. If B is maximal monotone, then for each $r > 0$ and $x \in H$ there exists a unique element $z \in H$ such that $(I + rB)z = x$. This element is denoted by $J_r^B x$. The mapping J_r^B thus defined is called the resolvent of B [9].

3 The main convergence result

In this section we define an iterative process and prove its convergence to a member of $F(S) \cap \Omega_A$, where S is nonexpansive and A is monotone and k-Lipschitz continuous.

Theorem 5 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega_A \neq \emptyset$. We define inductively the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ by*

$$\begin{aligned} x_0 &\in C, \\ y_n &= (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n &= (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S P_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n) b^2 \|Ax_n\|^2\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0 \end{aligned}$$

for all $n \ge 0$, where $\{\lambda_n\}$ is a sequence in $[a, b]$ with $a > 0$ and $b < \frac{1}{2k}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \le 1$ *for all* $n \ge 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$
(iv) $\lim_n \gamma_n = 1$ *and* $\gamma_n > 3/4$ *for all* $n \ge 0$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ are well-defined and converge strongly to the same point $q = P_{F(S)\cap\Omega_A} x_0$.

Proof We divide the proof into several steps.

Step 1. Assuming that x_n is a well-defined element of C for some $n \in \mathbb{N}$, we show that $F(S) \cap \Omega_A \subset C_n$.

Since x_n is defined, y_n, z_n are obviously well-defined elements of C. Let $x^* \in F(S) \cap \Omega_A$ be arbitrary. Set $t_n = P_C(x_n - \lambda_n A y_n)$ for all $n \ge 0$. Taking $x = x_n - \lambda_n A y_n$ and $y = x^*$ in (5), we obtain

$$\begin{aligned} \|t_n - x^*\|^2 &\le \|x_n - \lambda_n A y_n - x^*\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n \langle A y_n, x^* - y_n\rangle + 2\lambda_n \langle A y_n, y_n - t_n\rangle - \|x_n - t_n\|^2. \end{aligned}$$

Since by relation (7) we have $\langle A y_n, y_n - x^*\rangle \ge 0$, we deduce

$$\begin{aligned} \|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - x^*\|^2 - \|(x_n - y_n) + (y_n - t_n)\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned} \tag{8}$$

We estimate the last term, using $y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n)$:

$$\begin{aligned} &\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\ &\quad = \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \lambda_n \langle A x_n - A y_n, t_n - y_n\rangle \\ &\quad \le \langle x_n - \lambda_n A x_n - (1-\gamma_n)x_n - \gamma_n P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\ &\qquad + \lambda_n \|A x_n - A y_n\| \|t_n - y_n\| \\ &\quad \le \gamma_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\ &\qquad - (1-\gamma_n)\lambda_n \langle A x_n, t_n - y_n\rangle + \lambda_n k \|x_n - y_n\| \|t_n - y_n\|. \end{aligned} \tag{9}$$

In addition, using properties Eqs. (4) and (6) of the projection $P_C(x_n - \lambda_n Ax_n)$ we obtain

$$\begin{aligned}
&\langle x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n), t_n - y_n\rangle \\
&= \langle x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n), t_n - (1-\gamma_n)x_n - \gamma_n P_C(x_n - \lambda_n Ax_n)\rangle \\
&= (1-\gamma_n)\,\langle x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n), t_n - x_n\rangle \\
&\quad + \gamma_n\,\langle x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n), t_n - P_C(x_n - \lambda_n Ax_n)\rangle \\
&\le (1-\gamma_n)\,\|x_n - \lambda_n Ax_n - P_C(x_n - \lambda_n Ax_n)\|\,\|t_n - x_n\| \\
&\le (1-\gamma_n)\,\lambda_n\,\|Ax_n\|\,(\|t_n - y_n\| + \|y_n - x_n\|)\,. \qquad (10)
\end{aligned}$$

Gathering (8), (9), (10) and using $\gamma_n \le 1$ and $\lambda_n \le b$ we find

$$\begin{aligned}
\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\gamma_n\,(1-\gamma_n)\,b\,\|Ax_n\|\,(\|t_n - y_n\| + \|y_n - x_n\|) \\
&\quad + 2(1-\gamma_n)b\,\|Ax_n\|\,\|t_n - y_n\| + 2bk\,\|x_n - y_n\|\,\|t_n - y_n\| \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + (1-\gamma_n)\left(2b^2\,\|Ax_n\|^2 + \|t_n - y_n\|^2 + \|y_n - x_n\|^2\right) \\
&\quad + (1-\gamma_n)\left(b^2\,\|Ax_n\|^2 + \|t_n - y_n\|^2\right) + bk\left(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\right) \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2(\gamma_n - bk) \\
&\quad - \|y_n - t_n\|^2(2\gamma_n - 1 - bk) + 3(1-\gamma_n)b^2\,\|Ax_n\|^2\,. \qquad (11)
\end{aligned}$$

Using our assumptions $b < \frac{1}{2k}$ and $\gamma_n > 3/4$, we obtain that for all $n \in \mathbb{N}$,

$$\|t_n - x^*\|^2 \le \|x_n - x^*\|^2 + 3(1-\gamma_n)b^2\,\|Ax_n\|^2\,. \qquad (12)$$

Also, using again relation (7) and properties of P_C, we obtain

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &= \|(1-\gamma_n)(x_n - x^*) + \gamma_n\left(P_C(x_n - \lambda_n Ax_n) - x^*\right)\|^2 \\
&\le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|P_C(x_n - \lambda_n Ax_n) - P_C x^*\|^2 \\
&\le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x_n - x^* - \lambda_n Ax_n\|^2 \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\left[\|x_n - x^*\|^2 - 2\lambda_n\langle Ax_n, x_n - x^*\rangle + \lambda_n^2\|Ax_n\|^2\right] \\
&\le \|x_n - x^*\|^2 + b^2\|Ax_n\|^2 \qquad (13)
\end{aligned}$$

Since S is nonexpansive and $x^* \in F(S)$ we have $\|St_n - x^*\| \le \|t_n - x^*\|$. Thus, relations (12) and (13) imply that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n St_n - x^*\|^2 \\
&\le (1-\alpha_n-\beta_n)\,\|x_n - x^*\|^2 + \alpha_n\|y_n - x^*\|^2 + \beta_n\|St_n - x^*\|^2 \qquad (14) \\
&\le (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 + \alpha_n\left[\|x_n - x^*\|^2 + b^2\|Ax_n\|^2\right] \\
&\quad + \beta_n\left[\|x_n - x^*\|^2 + 3(1-\gamma_n)b^2\,\|Ax_n\|^2\right] \\
&= \|x_n - x^*\|^2 + (3 - 3\gamma_n + \alpha_n)b^2\,\|Ax_n\|^2\,. \qquad (15)
\end{aligned}$$

Consequently, $x^* \in C_n$. Hence $F(S) \cap \Omega_A \subset C_n$.

Step 2. We show that the sequence $\{x_n\}$ is well-defined and $F(S) \cap \Omega_A \subset C_n \cap Q_n$ for all $n \ge 0$.

We show this assertion by mathematical induction. For $n = 0$ we have $Q_0 = C$. Hence by Step 1 we obtain $F(S) \cap \Omega_A \subset C_0 \cap Q_0$. Assume that x_k is defined and $F(S) \cap \Omega_A \subset C_k \cap Q_k$

for some $k \geq 0$. Then y_k, z_k are well-defined elements of C. Note that C_k is a closed convex subset of C since

$$C_k = \{z \in C : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle \leq (3 - 3\gamma_k + \alpha_k)b^2 \|Ax_k\|^2\}.$$

Also, it is obvious that Q_k is closed and convex. Thus, $C_k \cap Q_k$ is a closed convex subset, which is nonempty since by assumption it contains $F(S) \cap \Omega_A$. Consequently, $x_{k+1} = P_{C_k \cap Q_k} x_0$ is well-defined.

The definition of x_{k+1} and of Q_{k+1} imply that $C_k \cap Q_k \subseteq Q_{k+1}$. Hence, $F(S) \cap \Omega_A \subseteq Q_{k+1}$. Using Step 1 we infer that $F(S) \cap \Omega_A \subseteq C_{k+1} \cap Q_{k+1}$.

Step 3. We show that the following statements hold:

(1) $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists, and $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$;
(2) $\lim_{n\to\infty}(z_n - x_n) = 0$.

Indeed, take any $x^* \in F(S) \cap \Omega_A$. Using $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $x^* \in F(S) \cap \Omega_A \subset C_n \cap Q_n$, we obtain

$$\|x_{n+1} - x_0\| \leq \|x^* - x_0\|, \quad \forall n \geq 0. \tag{16}$$

Therefore, $\{x_n\}$ is bounded and so is $\{Ax_n\}$ due to the Lipschitz continuity of A. From the definition of Q_n it is clear that $x_n = P_{Q_n} x_0$. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$, we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \quad \forall n \geq 0. \tag{17}$$

In particular, $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ hence $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Then relation (17) implies that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0. \tag{18}$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (3 - 3\gamma_n + \alpha_n)b^2 \|Ax_n\|^2 .$$

Since $\{Ax_n\}$ is bounded and $\lim_{n\to\infty} \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, we deduce that $\lim_{n\to\infty}(z_n - x_{n+1}) = 0$. Combining with Eq. 18 we infer that $\lim_{n\to\infty}(z_n - x_n) = 0$.

Step 4. We show that the following statements hold:

(1) $\lim_{n\to\infty}(x_n - y_n) = 0$;
(2) $\lim_{n\to\infty}(Sx_n - x_n) = 0$.

Indeed, from inequalities 14 and 15 we infer

$$\begin{aligned} \|z_n - x^*\|^2 - \|x_n - x^*\|^2 &\leq (-\alpha_n - \beta_n)\|x_n - x^*\|^2 + \alpha_n \|y_n - x^*\|^2 + \beta_n \|St_n - x^*\|^2 \\ &\leq (3 - 3\gamma_n + \alpha_n)b^2 \|Ax_n\|^2 . \end{aligned} \tag{19}$$

Since $\alpha_n \to 0$, $\gamma_n \to 1$, and $\{x_n\}$, $\{Ax_n\}$, $\{y_n\}$ are bounded, we deduce from (19) that

$$\lim_{n\to+\infty} \beta_n(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) = 0.$$

Using $\liminf_{n\to+\infty} \beta_n > 0$ we get $\lim_{n\to+\infty}(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) = 0$. Then Eq. 12 implies

$$\begin{aligned} \lim_{n\to+\infty}(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) &\leq \lim_{n\to+\infty}(\|t_n - x^*\|^2 - \|x_n - x^*\|^2) \\ &\leq \lim_{n\to+\infty} 3(1 - \gamma_n)b^2 \|Ax_n\|^2 = 0 \end{aligned}$$

thus, $\lim_{n\to+\infty}(\|t_n - x^*\|^2 - \|x_n - x^*\|^2) = 0$. Now we rewrite inequality (11) as

$$\begin{aligned}&\|x_n - y_n\|^2(\gamma_n - bk) + \|y_n - t_n\|^2(2\gamma_n - 1 - bk)\\ &\quad\le \|x_n - x^*\|^2 - \|t_n - x^*\|^2 + 3(1-\gamma_n)b^2\|Ax_n\|^2.\end{aligned}$$

We deduce that

$$\lim_{n\to+\infty}\left[\|x_n - y_n\|^2(\gamma_n - bk) + \|y_n - t_n\|^2(2\gamma_n - 1 - bk)\right] = 0.$$

Our assumptions on λ_n and γ_n imply that $\gamma_n - bk > 1/4$ and $2\gamma_n - 1 - bk > \frac{1}{2} - bk > 0$. Consequently, $\lim_{n\to+\infty}(x_n - y_n) = \lim_{n\to+\infty}(y_n - t_n) = 0$. Hence, $\lim_{n\to+\infty}(x_n - t_n) = 0$. Using that S is nonexpansive, we get $\lim_{n\to+\infty}(Sx_n - St_n) = 0$.

We rewrite the definition of z_n as

$$z_n - x_n = -\alpha_n x_n + \alpha_n y_n + \beta_n(St_n - x_n).$$

From $\lim_{n\to+\infty}(z_n - x_n) = 0$, $\lim_{n\to+\infty}\alpha_n = 0$, the boundedness of x_n, y_n and $\liminf_{n\to+\infty}\beta_n > 0$ we infer that $\lim_{n\to+\infty}(St_n - x_n) = 0$. Thus finally $\lim_{n\to+\infty}(Sx_n - x_n) = 0$.

Step 5. We claim that $\omega_w(x_n) \subset F(S) \cap \Omega_A$, where $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{u \in H : \{x_{n_j}\} \text{ converges weakly to } u \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in C and hence $\omega_w(x_n) \neq \emptyset$. Let $u \in \omega_w(x_n)$ be arbitrary. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to u. Since we also have $\lim_{j\to\infty}(x_{n_j} - Sx_{n_j}) = 0$, from the demiclosedness principle it follows that $(I - S)u = 0$. Thus $u \in F(S)$. We now show that $u \in \Omega_A$.

Since $t_n = P_C(x_n - \lambda_n Ay_n)$, for every $x \in C$ we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - x\rangle \ge 0$$

hence

$$\langle x - t_n, Ay_n\rangle \ge \left\langle x - t_n, \frac{x_n - t_n}{\lambda_n}\right\rangle.$$

Combining with monotonicity of A we obtain

$$\begin{aligned}\langle x - t_n, Ax\rangle &\ge \langle x - t_n, At_n\rangle\\ &= \langle x - t_n, At_n - Ay_n\rangle + \langle x - t_n, Ay_n\rangle\\ &\ge \langle x - t_n, At_n - Ay_n\rangle + \left\langle x - t_n, \frac{x_n - t_n}{\lambda_n}\right\rangle.\end{aligned}$$

Since $\lim_{n\to+\infty}(x_n - t_n) = \lim_{n\to+\infty}(y_n - t_n) = 0$, A is Lipschitz continuous and $\lambda_n > a > 0$ we deduce that

$$\langle x - u, Ax\rangle = \lim_{n_j\to+\infty}\langle x - t_{n_j}, Ax\rangle \ge 0, \quad \forall x \in C.$$

Then relation (7) entails that $u \in F(S) \cap \Omega_A$.

Step 6. We show that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to the same point $q = P_{F(S)\cap\Omega_A}x_0$.

Assume that $\{x_n\}$ does not converge strongly to q. Then there exists $\varepsilon > 0$ and a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\|x_{n_j} - q\| > \varepsilon$ for all j. Without loss of generality we may assume that $\{x_{n_j}\}$ converges weakly to some point u. By Step 5, $u \in F(S) \cap \Omega_A$. Using $q = P_{F(S)\cap\Omega_A}x_0$, the weak lower semicontinuity of $\|\cdot\|$, and relation (16) for $x^* = q$, we obtain

$$\|q - x_0\| \le \|u - x_0\| \le \liminf_{j\to\infty} \|x_{n_j} - x_0\| = \lim_{n\to+\infty} \|x_n - x_0\| \le \|q - x_0\|. \quad (20)$$

It follows that $\|q-x_0\| = \|u-x_0\|$, hence $u = q$ since q is the unique element in $F(S)\cap\Omega_A$ that minimizes the distance from x_0. Also, relation (20) implies $\lim_{j\to\infty} \|x_{n_j} - x_0\| = \|q - x_0\|$. Since $\{x_{n_j} - x_0\}$ converges weakly to $q - x_0$, this shows that $\{x_{n_j} - x_0\}$ converges strongly to $q - x_0$, and hence $\{x_{n_j}\}$ converges strongly to q, a contradiction.

Thus, $\{x_n\}$ converges strongly to q. It is easy to see that $\{y_n\}$, $\{z_n\}$ converge strongly to the same point q. □

4 Applications

If one takes $\alpha_n = 0$, $\beta_n = 1$ and $\gamma_n = 1$ for all $n \in \mathbb{N}$ in Theorem 5, then one finds the following simpler theorem:

Theorem 6 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega_A \neq \emptyset$. We define inductively the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ by*

$$\begin{aligned} x_0 &\in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ z_n &= S P_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n} x_0 \end{aligned}$$

for all $n \ge 0$, where $\{\lambda_n\}$ is a sequence in $[a, b]$ with $a > 0$ and $b < \frac{1}{2k}$. Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are well-defined and converge strongly to the same point $q = P_{F(S)\cap\Omega_A}x_0$.

However, relations like (15) suggest that, as is often the case, taking more general sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ might improve the rate of convergence to a solution.

Taking $S = I$, $\alpha_n = 0$ and $\beta_n = 1$ in Theorem 5, one finds the following theorem providing an algorithm to find the solution of a variational inequality:

Theorem 7 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping such that $\Omega_A \neq \emptyset$. We define inductively the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ by*

$$\begin{aligned} x_0 &\in C, \\ y_n &= (1 - \gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n &= P_C(x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + (3 - 3\gamma_n)b^2\|Ax_n\|^2\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n} x_0 \end{aligned}$$

for all $n \geq 0$, *where* $\{\lambda_n\}$ *is a sequence in* $[a, b]$ *with* $a > 0$ *and* $b < \frac{1}{2k}$, *and* $\{\gamma_n\}$ *is a sequence in* $[0, 1]$ *such that* $\lim_n \gamma_n = 1$ *and* $\gamma_n > 3/4$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *are well-defined and converge strongly to the same point* $q = P_{\Omega_A} x_0$.

Taking $\gamma_n = 1$ and $\alpha_n = 0$ in Theorem 5, one recovers the main result of [14]. If in addition one puts $A = 0$, one recovers the main result of [15] on an algorithm to find the fixed point of a nonexpansive mapping.

Another consequence of Theorem 5 is the following.

Theorem 8 *Let* H *be a real Hilbert space,* $A : H \to H$ *be monotone and* k*-Lipschitz, and* $S : H \to H$ *be nonexpansive, such that* $F(S) \cap A^{-1}\{0\} \neq \emptyset$. *Define the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *by*

$$\begin{cases} x_0 \in H \\ y_n = x_n - \lambda_n A x_n \\ z_n = (1 - \beta_n) x_n - \alpha_n \lambda_n A x_n + \beta_n S(x_n - \frac{\lambda_n}{\gamma_n} A y_n) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n) b^2 \|A x_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

for all $n \geq 0$, *where* $\{\lambda_n\}$ *is a sequence in* $[a, b]$ *with* $a > 0$ *and* $b < \frac{1}{4k}$, *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0, 1]$ *satisfying the conditions:*

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$
(iv) $\lim_n \gamma_n = 1$ *and* $\gamma_n > 3/4$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *are well-defined and converge strongly to the same point* $q = P_{F(S)\cap A^{-1}\{0\}} x_0$.

Proof We set $\lambda'_n = \lambda_n/\gamma_n$. Then $a \leq \lambda'_n < \frac{4}{3}\lambda_n < 2b < \frac{1}{2}k$. Thus we can apply Theorem 5 for this sequence, and for $C = H$. We have $P_H = I$ and $\Omega_A = A^{-1}\{0\}$. Then Theorem 5 guarantees that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to $q = P_{F(S)\cap A^{-1}\{0\}} x_0$, where

$$\begin{aligned} y_n &= x_n - \gamma_n \lambda'_n A x_n = x_n - \lambda_n A x_n, \\ z_n &= (1 - \alpha_n - \beta_n) x_n + \alpha_n y_n + \beta_n S(x_n - \lambda'_n A y_n) \\ &= (1 - \beta_n)\, x_n - \alpha_n \lambda_n A x_n + \beta_n S(x_n - \frac{\lambda_n}{\gamma_n} A y_n). \end{aligned}$$

□

Because of the relations that exist between monotone operators and nonexpansive mappings, Theorem 5 can also be applied for finding the common zeros of two monotone mappings, or the common fixed points of two mappings. For example, suppose that $A : H \to H$ is a monotone, Lipschitz continuous mapping and $B : H \to H$ is a maximal monotone mapping. Assume that the set of common zeros $A^{-1}(0) \cap B^{-1}(0)$ is nonempty. Theorem 5 can be applied to find an element of $A^{-1}(0) \cap B^{-1}(0)$ as follows. It is known that for any $r > 0$ the resolvent J_r^B of B is nonexpansive [9]; also, if we set $C = H$ then $F(J_r^B) = B^{-1}(0)$ while $\Omega_A = A^{-1}(0)$. Thus, by applying Theorem 5 to the mappings A and J_r^B we can find an element of $\Omega_A \cap F(J_r^B) = A^{-1}(0) \cap B^{-1}(0)$.

Likewise, assume that $C \subseteq H$ is nonempty, closed convex, $T : C \to C$ is pseudocontractive and Lipschitz, and $S : C \to C$ is nonexpansive, such that $F(T) \cap F(S) \neq \emptyset$. We can find an element of $F(T) \cap F(S)$ as follows. If we set $A = I - T$ then it is known that A is monotone and Lipschitz [3]. Also, it is easy to see that $\Omega_A = F(T)$. Indeed, if $u \in F(T)$ then $Au = 0$ so that $u \in \Omega_A$. Conversely, if $u \in \Omega_A$ then

$$\langle u - Tu, y - u\rangle \geq 0, \quad \forall y \in C.$$

Setting $y = Tu$ we get $\langle u - Tu, u - Tu\rangle \leq 0$, i.e., $u \in F(T)$. Consequently, Theorem 5 can be applied to the mappings A and S to produce sequences converging to an element of $\Omega_A \cap F(S) = F(T) \cap F(S)$.

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