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Zero product preserving linear maps of CCR C*-algebras with Hausdorff spectrum ☆

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ABSTRACT

In this paper, we try to attack a conjecture of Araujo and Jarosz that every bijective linear map θ between C*-algebras, with both θ and its inverse θ^{-1} preserving zero products, arises from an algebra isomorphism followed by a central multiplier. We show it is true for CCR C*-algebras with Hausdorff spectrum, and in general, some special C*-algebras associated to continuous fields of C*-algebras.

1. Introduction

The theory of general C*-algebras is made easy by observing the interplay between their algebraic and analytical structures. For example, the norm structure can be recovered from the $*$-algebraic structure in a C*-algebra. It is further shown by Gardner [10] (see also [16, Theorem 4.1.20]) that two C*-algebras are $*$-algebraic isomorphic if and only if they are algebraic isomorphic.

Extending results in [18,17], they are shown in [6] for the unital case and in [19, Corollary 2.6] for the general case that two C*-algebras $\mathcal{A}$, $\mathcal{B}$ are algebraic isomorphic if and only if there is a *continuous* bijective linear map θ between them preserving zero products, that is,

$$\theta(a)\theta(b) = 0 \quad \text{in } \mathcal{B} \text{ whenever } ab = 0 \text{ in } \mathcal{A}.$$

In this case,

$$\theta = \theta^{**}(1)\Psi, \tag{1.1}$$

where θ^{**} is the bidual map of θ, and $\theta^{**}(1)$ is an invertible central multiplier of $\mathcal{B}$, while Ψ is an algebra isomorphism form $\mathcal{A}$ onto $\mathcal{B}$. Consequently, the topological, linear and zero product structures determine a C*-algebra.

In [2], Araujo and Jarosz show that every bijective linear map θ between unital standard operator algebras on Banach spaces, with both θ and its inverse θ^{-1} preserving zero products, carries the standard form (1.1). In particular, such maps

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are automatically bounded. Their results apply to those maps between standard C*-algebras, i.e., those containing compact operators. They state a conjecture in [2] to ask whether every such map between two arbitrary C*-algebras carries the standard form (1.1). In other words, they want to know whether the linear and the zero product structures suffice to determine a C*-algebra.

This might be a hard problem, as we do not have suitable functional calculus to use if we do not know in advance the map is bounded. As a matter of facts, the structure of unbounded zero product preserving linear functionals of C*-algebras is quite complicated (see [4]). Furthermore, we know that Banach algebra homomorphisms can be unbounded (see, e.g., [7]). One possible way to attack this problem is to decompose a general C*-algebra into a family of simple C*-algebras, e.g., the ones consist of compact operators. Together with [13], this suggests us to study continuous fields of C*-algebras whose fibers are elementary C*-algebras, which give rise to exactly all CCR C*-algebras with Hausdorff spectrum.

In Section 2, we shall develop a structure theory of zero product preserving linear maps θ between two continuous fields of C*-algebras $(X, \{A_x\}, \mathcal{A})$ and $(Y, \{B_y\}, \mathcal{B})$. These maps carry a standard form

$$\theta(f)(y) = H_y\big(f\big(\varphi(y)\big)\big), \quad \forall f \in \mathcal{A},\ \forall y \in Y, \tag{1.2}$$

where φ is a map from Y into X, and each fiber linear map $H_y : A_{\varphi(y)} \to B_y$ is zero product preserving. In Section 3, we assume, in addition, θ is bijective and its inverse θ^{-1} also preserves zero products. Then, φ is a homeomorphism. Moreover, all fiber linear maps H_y are bounded whenever X (or Y) contains no isolated points, or all the fiber C*-algebras are standard operator algebras. In these cases, θ is bounded and thus, by results in [6,19], carries the standard form (1.1). Eventually, we solve the open problem in affirmative for the CCR C*-algebra case; namely, two CCR C*-algebras with Hausdorff spectrum are $*$-isomorphic if and only if they have the same linear and zero product structures.

It might be worthwhile to mention that the group C*-algebra of a compact group is a direct sum of matrix algebras, and thus a CCR with Hausdorff spectrum (see, e.g., [8, 15.1]). Consequently, results in this paper can be applied. Of course, the most interesting part is to characterize further the group structure through this kind of maps. We hope this will be achieved in coming future.

Finally, we would like to express our deep gratitude to the referee for his/her careful reading and helpful comments.

2. Zero product preservers between continuous fields of Banach algebras

We shall follow [9,8] for notations. Let T be a locally compact Hausdorff space, called *base space*. For each t in T there is a (complex) Banach space E_t. A *vector field* x is an element in the product space $\prod_{t\in T} E_t$, that is, $x(t) \in E_t$, $\forall t \in T$.

Definition 2.1. *A continuous field* $\mathcal{E} = (T, \{E_t\}, \mathcal{A})$ *of Banach spaces over a locally compact space* T is a family $\{E_t\}_{t\in T}$ of Banach spaces, with a set $\mathcal{A}$ of vector fields such that

(i) $\mathcal{A}$ is a (complex) vector subspace of $\prod_{t\in T} E_t$.
(ii) For every t in T, the set of all $x(t)$ with x in $\mathcal{A}$ is dense in E_t.
(iii) For every x in $\mathcal{A}$, the function $t \mapsto \|x(t)\|$ is continuous on T and vanishes at infinity.
(iv) Let x be a vector field. Suppose for every t in T and every $\epsilon > 0$, there is a neighborhood U of t and a y in $\mathcal{A}$ such that $\|x(t) - y(t)\| < \epsilon$ for all t in U. Then $x \in \mathcal{A}$.

Elements in $\mathcal{A}$ are called *continuous vector fields*.

It is not difficult to see that $\mathcal{A}$ becomes a Banach space under the norm $\|x\| = \sup_{t\in T} \|x(t)\|$. If g is in $C_b(T)$, i.e., g is a bounded continuous complex-valued function on T, and x is in $\mathcal{A}$ then $t \mapsto g(t)x(t)$ defines a continuous vector field gx on T. The set $\{x(t)\colon x \in \mathcal{A}\}$ coincides with E_t for every t in T. Moreover, for any distinct points s, t in T and any α in E_s and β in E_t, there is a continuous vector field x such that $x(s) = \alpha$ and $x(t) = \beta$ (see, e.g., [9,14]).

Definition 2.2. A *continuous field of Banach algebras* (resp. C*-*algebras*) $(X, \{A_x\}, \mathcal{A})$ is a continuous field of Banach spaces with Banach algebra (resp. C*-algebra) fibres A_x such that $\mathcal{A}$ becomes a Banach algebra (resp. C*-algebra) under the pointwise algebraic (resp. $*$-algebraic) operations and norm $\|f\| = \sup \|f(x)\|$.

Example 2.3. Recall that a C*-algebra $\mathcal{A}$ is called a *CCR* if every irreducible representation of $\mathcal{A}$ consists of compact operators. The spectrum $\hat{\mathcal{A}}$ of $\mathcal{A}$ is the family of unitary equivalence classes of nonzero irreducible representations under the hull-kernel topology. This topology is always locally compact, and the spectrum of a CCR C*-algebra is T_1. Let $\mathcal{A}$ be a CCR C*-algebras with Hausdorff spectrum $X = \hat{\mathcal{A}}$. According to [8, Theorem 10.5.4], we can represent $\mathcal{A}$ as a continuous field of C*-algebras $(X, \{A_x\}, \mathcal{A})$, where A_x consists of compact linear operators on a Hilbert space H_x for each x in X.

Let $(X, \{A_x\}, \mathcal{A})$ and $(Y, \{B_y\}, \mathcal{B})$ be two continuous fields of C*-algebras, and let $\theta : \mathcal{A} \to \mathcal{B}$ be a zero product preserving linear map. Denote by $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ the one-point compactifications of X and Y, respectively. Note that the point ∞ at infinity will be isolated in X_∞ if X is already compact. Set for each x in X the sets

$$I_x = \{f \in \mathcal{A}\colon\ f \text{ vanishes in a neighborhood in } X_\infty \text{ of } x\},$$
$$M_x = \big\{f \in \mathcal{A}\colon\ f(x) = 0\big\}.$$

In particular,

$$I_\infty = \{f \in \mathcal{A}\colon\ f \text{ has a compact support}\},$$
$$M_\infty = \mathcal{A}.$$

Similar conventions are also made for each y in Y. Furthermore, denote by δ_y the evaluation map at y in Y, i.e.,

$$\delta_y(g) = g(y) \in B_y, \quad \forall g \in \mathcal{B}.$$

We call a Banach algebra A *primitive* if it has an (isometric) faithful irreducible representation $\pi : A \to B(E)$ into the Banach algebra of all bounded linear operators on a Banach space E. We call a linear map between Banach algebras has a *primitive range* if the Banach algebra generated by its range is primitive.

Theorem 2.4. *Let $(X, \{A_x\}, \mathcal{A})$, $(Y, \{B_y\}, \mathcal{B})$ be continuous fields of Banach algebras over locally compact Hausdorff spaces X, Y, respectively. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a zero product preserving linear map such that $\delta_y \circ \theta : \mathcal{A} \to B_y$ has primitive range for every y in Y.*
If we set

$$Y_0 = \{y \in Y_\infty\colon\ \delta_y \circ \theta = 0\},$$

then there is a unique continuous map $\varphi : Y \setminus Y_0 \to X_\infty$ satisfying the condition that

$$\theta(I_{\varphi(y)}) \subseteq M_y.$$

Set

$$Y_1 = \big\{y \in Y \setminus Y_0\colon\ \theta(M_{\varphi(y)}) \subseteq M_y\big\},$$
$$Y_2 = \big\{y \in Y \setminus Y_0\colon\ \theta(M_{\varphi(y)}) \nsubseteq M_y\big\}.$$

Then $\infty \in Y_0$ and Y_0 is compact,

$$\theta(f)|_{Y_0} = 0, \quad \forall f \in \mathcal{A},$$

and Y_2 is open in Y_∞. Moreover, there is a linear map $H_y : A_{\varphi(y)} \to B_y$ for each y in Y_1 such that

$$\theta(f)(y) = H_y\big(f\big(\varphi(y)\big)\big), \quad \forall f \in \mathcal{A},\ \forall y \in Y_1. \tag{2.1}$$

The exceptional set $\varphi(Y_2)$ consists of finitely many non-isolated points in X_∞. Furthermore, θ is bounded if and only if $Y_2 = \emptyset$ and all H_y are bounded. In this case,

$$\|\theta\| = \sup_y \|H_y\|.$$

Finally, the fiber maps H_y are zero product preserving if $(X, \{A_x\}, \mathcal{A})$ is a continuous field of C*-*algebras.*

Composing $\delta_y \circ \theta$ with a faithful irreducible representation of the Banach algebra generated by $\{\theta(f)(y) \in B_y\colon\ f \in \mathcal{A}\}$, we can assume that B_y is an irreducible subalgebra of the algebra $B(E_y)$ of all bounded linear operators on some Banach space E_y and $\delta_y \circ \theta$ is again zero-product preserving with range generating B_y.

It is clear that Y_0 is compact, contains the point at infinity, and

$$\theta(f)|_{Y_0} = 0, \quad \forall f \in \mathcal{A}.$$

On the other hand, for each $y \in Y \setminus Y_0$, the range $\theta(\mathcal{A})$ is not trivial at y. For every open subset U of X, denote by $\mathcal{A}_U$ the subalgebra of all f in $\mathcal{A}$ vanishing outside a compact subset of U. For each y in $Y \setminus Y_0$, denote by

$$S_y = \big\{x \in X_\infty\colon \text{ for every open neighborhood } U \text{ of } x, \text{ there is an } f \text{ in } \mathcal{A}_U \text{ such that } \theta(f)(y) \neq 0\big\}.$$

We divide the proof of Theorem 2.4 into several lemmas.

Lemma 2.5. *The set S_y is nonempty for each y in $Y \setminus Y_0$.*

Proof. Suppose on the contrary that for each x in X_∞ there is an open neighborhood U_x of x in X_∞ such that $\theta(f)(y) = 0$ for all f in $\mathcal{A}_{U_x}$. Let V_x be an open neighborhood of x with compact closure $\overline{V} \subseteq U$. By compactness,

$$X_\infty = V_{x_0} \cup V_{x_1} \cup \cdots \cup V_{x_n}$$

for some points $x_0 = \infty$, $x_1, \ldots, x_n$ in X_∞. Let

$$1 = h_0 + h_1 + \cdots + h_n$$

be a continuous partition of unity such that h_i vanishes outside V_{x_i} for $i = 0, 1, \ldots, n$. For any g in $\mathcal{A}$, observe that

$$(h_i g) \in \mathcal{A}_{U_{x_i}} \quad \text{implies} \quad \theta(h_i g)(y) = 0,$$

and then $\theta(g)(y) = 0$, $\forall g \in \mathcal{A}$. This gives a contradiction $y \in Y_0$. □

Lemma 2.6. *S_y consists of exactly one point for all y in $Y \setminus Y_0$.*

Proof. We shall verify that $x_1, x_2 \in S_y$ implies $x_1 = x_2$. Suppose $x_2 \neq x_1$. Let U_1 and U_2 be disjoint open neighborhoods of x_1 and x_2, respectively. Since

$$f_1 f_2 = f_2 f_1 = 0 \quad \text{for all } f_i \text{ in } \mathcal{A}_{U_i},\ i = 1, 2,$$

we have

$$\theta(f_1)\theta(f_2) = \theta(f_2)\theta(f_1) = 0 \quad \text{in } \mathcal{B}.$$

Let E_1 be the intersection of the kernels of all $\theta(f_1)(y)$ with f_1 in $\mathcal{A}_{U_1}$. Because both $\theta|_{\mathcal{A}_{U_1}}$ and $\theta|_{\mathcal{A}_{U_2}}$ are not trivial at y, we see that E_1 is a proper nontrivial subspace of E_y, that is, $\{0\} \neq E_1 \neq E_y$.

Let V be a nonempty open set in Y such that the compact closure $\overline{V} \subseteq U_1$. For any h in $\mathcal{A}_V$, let g be in $C(X_\infty)$ such that $g = 1$ on the support of h and g vanishes outside V. Then for each f in $\mathcal{A}$, since fg vanishes outside V, we have $\theta(fg)(y)|_{E_1} = 0$. On the other hand, we have $h(f(1-g)) = 0$. This implies $\theta(h)(y)\theta(f)(y)|_{E_1} = \theta(h)(y)\theta(fg)(y)|_{E_1} = 0$, $\forall f \in \mathcal{A}$. Since V is an arbitrary nonempty open set with compact closure contained in U_1, we have $\theta(h)(y)\theta(f)(y)|_{E_1} = 0$ for all $f \in \mathcal{A}$ and for all $h \in \mathcal{A}_{U_1}$. Therefore, $\theta(\mathcal{A})(y)(E_1) \subseteq E_1$. Since $\theta(\mathcal{A})(y)$ generates the irreducible algebra B_y, we see that E_1 could not be proper. This is a contradiction. □

Define a map φ from $Y \setminus Y_0$ into X_∞ by $S_y = \{\varphi(y)\}$.

Lemma 2.7. *The point $\varphi(y)$ is the unique point in X_∞ satisfying the condition that*

$$\theta(I_{\varphi(y)}) \subseteq M_y, \quad \forall y \in Y \setminus Y_0. \tag{2.2}$$

Proof. Let $f \in I_{\varphi(y)}$ vanish in an open neighborhood U of $\varphi(y)$. For all $x \notin U$, by the definition of S_y there is an open neighborhood V_x of x such that $\theta(\mathcal{A}_{V_x})(y) = \{0\}$. By compactness, we can write $X_\infty = U \cup V_{x_1} \cup \cdots \cup V_{x_n}$ for some $x_1, \ldots, x_n$ in $X_\infty \setminus U$. Let $1 = h + h_1 + \cdots + h_n$ be a corresponding continuous partition of unity. Note that $\theta(h_i g)(y) = 0$ for all g in $\mathcal{A}$ and $i = 1, \ldots, n$. Hence, $\theta(g)(y) = \theta(hg)(y)$ for all g in $\mathcal{A}$. As $f(hg) = 0$, we see that $\theta(f)(y)\theta(g)(y) = \theta(f)(y)\theta(hg)(y) = 0$. Since $\delta_y \circ \theta$ has a primitive range, $\theta(f)(y) = 0$, or $\theta(f) \in M_y$. Finally, the uniqueness assertion follows from the definition of S_y. □

It is clear that the map φ is uniquely characterized by (2.2). Now the definitions of the sets Y_1 and Y_2 make sense.

Lemma 2.8. *$\varphi : Y \setminus Y_0 \to X_\infty$ is continuous.*

Proof. Suppose $y_\lambda \to y$ in $Y \setminus Y_0$, but $x_\lambda = \varphi(y_\lambda) \to x \neq \varphi(y)$. By Lemma 2.7, $\theta(I_x) \nsubseteq M_y$. Let U_x, $U_{\varphi(y)}$ be disjoint compact neighborhoods of $x, \varphi(y)$, respectively. Let $g \in C(X_\infty)$ such that $g = 1$ on U_x and $g = 0$ on $U_{\varphi(y)}$. Since $x_\lambda \to x$, $(1-g)f \in I_{x_\lambda}$ eventually. Thus, $\theta((1-g)f) \in M_{y_\lambda}$ eventually. By the continuity of the norm function, $\theta((1-g)f)(y) = 0$. On the other hand, $gf \in I_{\varphi(y)}$ implies $\theta(gf) \in M_y$. Hence, $\theta(f)(y) = 0$, $\forall f \in \mathcal{A}$. This gives $y \in Y_0$, a contradiction. □

Lemma 2.9. *Let $\{y_n\}$ be an infinite sequence in $Y \setminus Y_0$ such that $\varphi(y_n)$ are distinct points in X_∞. Then*

$$\limsup \|\delta_{y_n} \circ \theta\| < +\infty.$$

Proof. Suppose not, by passing to a subsequence if necessary, we can assume that $\|\delta_{y_n} \circ \theta\| > n^4$, and there is an element f_n in $\mathcal{A}$ such that $\|f_n\| \leqslant 1$ and $\|\theta(f_n)(y_n)\| > n^3$, for $n = 1, 2, \ldots$. Let V_n, U_n be compact neighborhoods of x_n in X_∞ such that V_n is contained in the interior of U_n, and $U_n \cap U_m = \emptyset$, for distinct $n, m = 1, 2, \ldots$. Let $g_n \in C(X_\infty)$ such that $g_n = 1$ on V_n and $g_n = 0$ outside U_n for $n = 1, 2, \ldots$. Observe

$$\begin{aligned}\theta(f_n)(y_n) &= \theta(g_n f)(y_n) + \theta\big((1-g_n)f\big)(y_n)\\ &= \theta(g_n f)(y_n), \quad \text{as } (1-g_n)f \in I_{x_n}.\end{aligned}$$

So we can assume f_n is supported in U_n, for $n = 1, 2, \ldots$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \in \mathcal{A}.$$

Since $n^2 f - f_n \in I_{x_n}$, we have $n^2\theta(f)(y_n) = \theta(f_n)(y_n)$ by (2.2), and thus $\|\theta(f)(y_n)\| > n$, for $n = 1, 2, \ldots$. As $\theta(f)$ in $\mathcal{B}$ has a bounded norm, we arrive at a contradiction. $\square$

Lemma 2.10. *$\varphi(Y_2)$ is a finite set of non-isolated points in X_∞.*

Proof. Let $x = \varphi(y)$ with y in Y_2. Then by (2.2) we have

$$\theta(I_x) \subseteq M_y \quad \text{but } \theta(M_x) \nsubseteq M_y.$$

This implies the linear operator $\delta_y \circ \theta$ is unbounded, since I_x is dense in M_x by Urysohn's Lemma. By Lemma 2.9, we can have only finitely many of such x's. So $\varphi(Y_2)$ is a finite set. Moreover, if x is an isolated point in X_∞ then $I_x = M_x$, and thus $x \notin \varphi(Y_2)$. $\square$

Proof of Theorem 2.4. Let $y \in Y_1$, we have $\theta(M_{\varphi(y)}) \subseteq M_y$. Hence, there is a linear operator $H_y : E_{\varphi(y)} \to F_y$ such that

$$\theta(f)(y) = H_y\big(f\big(\varphi(y)\big)\big), \quad \forall f \in \mathcal{A}. \tag{2.3}$$

Next we want to see that Y_2 is open, or equivalently, $Y_0 \cup Y_1$ is closed in Y_∞. Let $y_\lambda \to y$ with y_λ in $Y_0 \cup Y_1$. We want to show that $y \in Y_0 \cup Y_1$. Since Y_0 is compact, we may assume $y_\lambda \in Y_1$ for all λ. Suppose $y \notin Y_0$. By Lemma 2.8, we see that $\varphi(y_\lambda) \to \varphi(y)$. In the case there is a subnet of $\{\varphi(y_\lambda)\}$ consisting of only finitely many points, we can assume $\varphi(y_\lambda) = \varphi(y)$ for all λ. Then for all f in $\mathcal{A}$, $f(\varphi(y)) = 0$ implies $f(\varphi(y_\lambda)) = 0$, and thus $\theta(f)(y_\lambda) = 0$ for all λ by (2.3). By continuity, $\theta(f)(y) = 0$. Consequently, $\theta(M_{\varphi(y)}) \subseteq M_y$, and thus $y \in Y_1$. In the other case, every subnet of $\{\varphi(y_\lambda)\}$ contains infinitely many points. Lemma 2.9 asserts that $M = \limsup \|H_{y_\lambda}\| < +\infty$. This gives

$$\big\|\theta(f)(y)\big\| = \lim \big\|\theta(f)(y_\lambda)\big\| = \lim \big\|H_{y_\lambda}\big(f\big(\varphi(y_\lambda)\big)\big)\big\| \leqslant M\big\|f\big(\varphi(y)\big)\big\|.$$

Thus, if $f(\varphi(y)) = 0$ we have $\theta(f)(y) = 0$. Consequently, $y \in Y_1$.

Now observe that the boundedness of θ implies $Y_2 = \emptyset$. Moreover,

$$\begin{aligned}\|\theta\| &= \sup\big\{\big\|\theta(f)\big\|:\ f \in \mathcal{A} \text{ with } \|f\| = 1\big\}\\ &= \sup\big\{\big\|H_y\big(f\big(\varphi(y)\big)\big)\big\|:\ f \in \mathcal{A} \text{ with } \|f\| = 1,\ y \in Y_1\big\}\\ &\leqslant \sup\big\{\|H_y\|:\ y \in Y_1\big\}.\end{aligned} \tag{2.4}$$

The reverse inequality is plain. Conversely, we suppose $Y_2 = \emptyset$ and all H_y are bounded. We claim that $\sup \|H_y\| < +\infty$. For else, there is a sequence $\{y_n\}$ in Y_1 such that $\lim_{n\to\infty} \|H_{y_n}\| = +\infty$. By Lemma 2.9, we can assume all $\varphi(y_n) = x$ in X. Let $e \in A_x$ and $f \in \mathcal{A}$ such that $f(x) = e$. Then

$$\big\|H_{y_n}(e)\big\| = \big\|\theta(f)(y_n)\big\| \leqslant \big\|\theta(f)\big\|, \quad n = 1, 2, \ldots.$$

It follows from the uniform boundedness principle that $\sup \|H_{y_n}\| < +\infty$, a contradiction. It then follows from (2.4) that θ is bounded.

Finally, suppose $(X, \{A_x\}, \mathcal{A})$ is a continuous field of C*-algebras, and in particular, $\mathcal{A}$ is a C*-algebra. Let $\alpha\beta = 0$ in A_x for some x in $\varphi(Y_1)$. Consider the closed two-sided ideal $I = \{c \in \mathcal{A}:\ c(x) = 0\}$ of $\mathcal{A}$. Let a, b in $\mathcal{A}$ be such that $a(x) = \alpha$, $b(x) = \beta$. Then $ab \in I$. By a result of Akemann and Pedersen [1] (see also [6, Lemma 4.14]), we shall have a', b' in $\mathcal{A}$ such that $a'(x) = \alpha$, $b(x') = \beta$ and $a'b' = 0$. Now $\theta(a')\theta(b') = 0$ implies $H_y(\alpha)H_y(\beta) = 0$. So each H_y preserves zero products. $\square$

3. Zero product preservers between CCR C*-algebras

Recall that an algebra A of continuous linear operators on some locally convex space E is called *standard* if A contains all finite rank operators. Note that we do not assume A contains the identity map on E or A is closed under any topology. The following result belongs to Araujo and Jarosz [2, Theorem 1]. They verify the case of unital standard operator algebras on Banach spaces. The arguments below slightly simplify theirs.

Proposition 3.1. *(See [2].) Let $\theta : A \to B$ be a bijective linear map between standard operator algebras A, B on locally convex spaces M, N, respectively, such that both θ and its inverse θ^{-1} preserve zero products. Then there is a nonzero scalar λ and a weak–weak bi-continuous invertible linear map $S : M \to N$ such that*

$$\theta(a) = \lambda S a S^{-1}, \quad \forall a \in A.$$

In case both M, N are Frechet spaces, S is bi-continuous in the metric topologies. In particular, θ is bounded if both M, N are Banach spaces.

Proof. Put

$$a^{\perp} = \{c \in A\colon\ ca = 0\}, \quad \text{for all nonzero } a \text{ in } A.$$

We see that $a^{\perp} \subseteq b^{\perp}$ if and only if the closure of the range space of a contains that of b. Consequently, $a^{\perp}$ is maximum among all $b^{\perp}$ if and only if a is of rank one. By the zero product preserving property of θ and θ^{-1}, we see that θ preserves the order of $a^{\perp}$'s, and thus sends the maxima onto the maxima. In other words, θ sends rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist linear maps $S : M \to N$ and $T : N \to M$ such that

$$\theta(a) = SaT, \quad \forall a \in \mathcal{F}(M),$$

where $\mathcal{F}(M)$ is the algebra of all continuous finite rank operators on M. In particular,

$$\theta(x \otimes y') = Sx \otimes T'y',$$

for every rank one operator $x \otimes y'$ with x in M, y' in the topological dual space M' of M. Here, T' is the (algebraic) dual map of T, and $(x \otimes y')(z) = y'(z)x$ defines a rank at most one continuous operator on M. Consequently, $T'M' \subseteq N'$ and thus T is weak–weak continuous. Dealing with the inverse θ^{-1}, we see that T^{-1} is also weak–weak continuous. Moreover, if $y_2'(x_1) = 0$ then $(x_2 \otimes y_2')(x_1 \otimes y_1') = 0$. Thus, $\theta(x_2 \otimes y_2')\theta(x_1 \otimes y_1') = 0$. In other words,

$$\begin{aligned} y_2'(x_1) = 0 \quad &\text{implies} \quad \big(T'y_2'\big)(Sx_1)\big(Sx_2 \otimes T'y_1'\big) = 0, \quad \forall x_1, x_2 \in M,\ y_1', y_2' \in M' \\ &\text{implies} \quad y_2'(TSx_1) = 0, \quad \forall x_1 \in M,\ y_2' \in M'. \end{aligned}$$

By linearity, $T = \lambda S^{-1}$ for some nonzero scalar λ, and

$$\theta(a) = \lambda S a S^{-1}, \quad \forall a \in \mathcal{F}(M).$$

In general, let $a \in A$. For any $x \neq 0$ in M, let $x' \in M'$ such that $x'(x) = 1$. Set $b = a - (ax \otimes x')$. Observe $b(x \otimes x') = 0$. Thus,

$$\theta(b)\theta(x \otimes x') = \lambda\big(\theta(b)Sx\big) \otimes \big(S^{-1}\big)'x' = 0.$$

This implies

$$\theta(a)Sx = \lambda\big(Sax \otimes \big(S^{-1}\big)'x'\big)(Sx) = \lambda Sax, \quad \forall x \in M.$$

Hence,

$$\theta(a) = \lambda S a S^{-1}, \quad \forall a \in A.$$

In case M, N are Frechet spaces, the Closed Graph Theorem ensures that both S, S^{-1} are continuous in the metric topology. If they are Banach spaces, then θ is automatically bounded. $\square$

Theorem 3.2. *Let $(X, \{A_x\}, \mathcal{A})$, $(Y, \{B_y\}, \mathcal{B})$ be continuous fields of primitive Banach algebras over locally compact base spaces. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a bijective linear map such that both θ, θ^{-1} preserve zero products. Suppose, in addition, at least one of the following conditions hold.*

(1) *X (or Y) contains no isolated points.*
(2) *All fibers A_x and B_y are standard operator algebras.*

Then θ is automatically bounded and X, Y are homeomorphic. Indeed, θ assumes the standard form (1.2) *with all fiber linear maps being bounded.*

If the case (2) *holds, and $\mathcal{A}$ (resp. $\mathcal{B}$) is a continuous field of standard* C*-*algebras A_x (resp. B_y) on Hilbert spaces H_x (resp. K_y), then there exist a homeomorphism $\varphi : Y \to X$, a bounded and away from zero continuous scalar function λ on Y, a bounded invertible linear map S_y from $H_{\varphi(y)}$ onto K_y for each y in Y such that*

$$\theta(f)(y) = \lambda(y) S_y f\big(\varphi(y)\big) S_y^{-1}, \quad \forall f \in \mathcal{A},\ y \in Y.$$

In other words, the standard form (1.1) *holds*:

$$\theta = \theta^{**}(1)\Psi,$$

where the invertible central multiplier $\theta^{**}(1)$ of $\mathcal{B}$ is represented by the operator field $y \mapsto \lambda(y) I_y$ with I_y being the identity map on each fiber Hilbert space K_y, and the algebra isomorphism Ψ is given by $\Psi(f)(y) = S_y f(\varphi(y)) S_y^{-1}$.

Proof. We first note that $Y_0 = \{\infty\}$. Moreover, it follows from (2.2) that $\varphi(Y) = \varphi(Y_1) \cup \varphi(Y_2)$ is dense in X. Since $\varphi(Y_2)$ is a finite set of non-isolated points in X, we see that $\varphi(Y_1)$ alone is dense in X. On the other hand, let $y \in Y_1$ with $\varphi(y) = x$ in X, and $\psi(x) = z$ in Y_∞. Here, the map $\psi : X \to Y_\infty$, and the decomposition $X = X_1 \cup X_2$ is induced by θ^{-1} in an analogous way. In particular, we have

$$\theta(M_x) \subseteq M_y \quad \text{and} \quad \theta^{-1}(I_z) \subseteq M_x.$$

Consequently, $I_z \subseteq \theta(M_x) \subseteq M_y$ gives $y = z \in \psi(X)$. In case $y \in \psi(X_1)$, we have $\theta(M_x) = M_y$. Since $\psi(X_2)$ is a finite set of non-isolated points in Y, we have $\theta(M_{\varphi(y)}) = M_y$ for all but at most finitely many y in Y_1. Therefore, the linear map H_y is bijective for all but at most finitely many y in Y_1, which are non-isolated points in Y. Hence, if $\theta(f)$ vanishes on Y_1 then f vanishes on the dense set $\varphi(Y_1)$ by (2.1), and thus $f = 0$. Therefore, Y_1 is dense in Y by the surjectivity of θ. The openness of Y_2 forces itself to be empty.

Now, $Y = Y_1$ and $X = X_1$ imply that both θ and θ^{-1} can be written as weighted composition operators:

$$\theta(f)(y) = H_y\big(f\big(\varphi(y)\big)\big), \quad \forall f \in \mathcal{A},\ \forall y \in Y,$$
$$\theta^{-1}(g)(x) = T_x\big(g\big(\psi(x)\big)\big), \quad \forall g \in \mathcal{B},\ \forall x \in X.$$

It is easy to see that the linear map $H_y : E_{\varphi(y)} \to F_y$ has T_y as the inverse for every y in Y, and thus it is bijective. By Lemma 2.9, at most finitely many H_y are unbounded.

Let y be a non-isolated point in Y. We shall show that the linear map H_y is bounded. Suppose not, then for each $n = 1, 2, \ldots$ there is an f_n in $\mathcal{A}$ of norm one such that $\|\theta(f_n)(y)\| = \|H_y(f_n(\varphi(y)))\| > n^4$. By the continuity of the norm of $\theta(f_n)$, there are all distinct points y_n in Y nearby y such that $\|\theta(f_n)(y_n)\| > n^3$. Let $x_n = \varphi(y_n)$ in X for $n = 1, 2, \ldots$. Since φ is a homeomorphism, we can assume also that all x_n are distinct with disjoint compact neighborhoods U_n. By multiplying with a norm one continuous scalar function, we can assume each f_n is supported in U_n. Let $f = \sum_n \frac{1}{n^2} f_n$ in $\mathcal{A}$. Since $n^2 f - f_n \in I_{x_n}$, we have $n^2\theta(f)(y_n) = \theta(f_n)(y_n)$ and thus $\|\theta(f)(y_n)\| > n$ for $n = 1, 2, \ldots$. This absurdity tells us that H_y is bounded for all non-isolated y in Y_1.

For the case (1), if Y (or equivalently, its homeomorphic image X) contains no isolated points then all fiber linear maps H_y are bounded. By Theorem 2.4, we have $\|\theta\| = \sup \|H_y\| < +\infty$.

Suppose now the case (2) holds. By Proposition 3.1, each fiber linear map assumes the form $H_y(a) = \lambda(y) S_y a S_y^{-1}$, and θ is uniformly bounded. To see that λ is continuous on Y, we make use of a result of Lee [14, Lemma 2] which asserts that the multiplier algebras $M(A)$ and $M(B)$ can be represented as families of bounded operator fields in $(X, \{M(A_x)\})$ and $(Y, \{M(B_x)\})$, respectively. By restricting the double dual map of θ to $M(A)$, we see that the invertible central multiplier $\theta^{**}(1)(y) = \lambda(y) I_y$. It follows from the Dauns–Hofmann Theorem (see, e.g., [15, Theorem A.34]) that λ is a continuous function on X. Since $\theta^{**}(1)$ is invertible, we see that λ is bounded and away from zero. It is also plain that the algebra isomorphism $\Psi = \theta^{**}(1)^{-1}\theta$ is given by sending a continuous operator field $\{f(y)\}$ to $\{S_y f(\varphi(y)) S_y^{-1}\}$. □

As a special case of Theorem 3.2(2), here comes

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be CCR* C*-*algebras with Hausdorff spectrum $X = \hat{\mathcal{A}}$ and $Y = \hat{\mathcal{B}}$, respectively. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a bijective linear map such that*

$$ab = 0 \quad \text{in } \mathcal{A} \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \quad \text{in } \mathcal{B}. \tag{3.1}$$

Then θ is automatically bounded. Indeed, $\theta = m\Psi$ where $m = \theta^{**}(1)$ is an invertible central multiplier of $\mathcal{B}$ and Ψ is an algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Corollary 3.4. *Two CCR* C*-*algebras with Hausdorff spectrum are isomorphic as* C*-*algebras if and only if they have the same linear and zero product structures.*

Proof. It follows from Theorem 3.3 that if there is a bijective linear map $\theta : \mathcal{A} \to \mathcal{B}$ between two CCR C*-algebras with Hausdorff spectrum, then $\mathcal{A}$ and $\mathcal{B}$ are algebraically isomorphic (via the map $\Psi = \theta^{**}(1)^{-1}\theta$). As shown in [10] (see also [16, Theorem 4.1.20]), $\mathcal{A}$ and $\mathcal{B}$ are also $*$-isomorphic. On the other hand, the norm of an element a of a C*-algebra equals the square root of the spectral radius of a^*a, which is a $*$-algebraic property. So $\mathcal{A}$ and $\mathcal{B}$ are isometrically $*$-isomorphic. $\square$

Remark 3.5. (a) The two way zero product preserving assumption (3.1) in Theorem 3.3 cannot be dropped easily. For example, abelian C*-algebras $C_0(X)$ are CCR. In [4], there are many examples of unbounded zero product preserving linear functionals of $C_0(X)$, provided X is an infinite set. In [12], an unbounded zero product preserving linear map from c onto ℓ_∞ is given, where both c, the C*-algebra of convergent scalar sequences, and ℓ_∞ are CCR with Hausdorff spectrum.

(b) In [9], Fell defines the notion of a *full algebra of operator fields* $\mathcal{A}$ as those satisfying conditions (i), (ii), (iii) in Definition 2.1 and $\mathcal{A}$ becomes a C*-algebra in Definition 2.2. Fell calls those satisfied in addition condition (iv) in Definition 2.1 a *maximal* full algebra of operator fields. He has pointed out that $\mathcal{A}$ is maximal if and only if for all α_x in a fiber algebra A_x and β_y in another fiber A_y there is a continuous field a in $\mathcal{A}$ such that $a(x) = \alpha_x$ and $a(y) = \beta_y$. This is also equivalent to saying that for all a in $\mathcal{A}$, and for all bounded complex scalar continuous function g on X, we have $ga \in \mathcal{A}$. In our discussion, we follow the usage of notations of Dixmier [8] and simply assume that all continuous fields are maximal.

(c) We note that every C*-algebra with Hausdorff spectrum can be represented as a continuous field of primitive C*-algebras over the spectrum [15, §5.1]. Hence, Theorems 2.4 and 3.2 apply to every zero product preserving linear map between two C*-algebras with Hausdorff spectrum.

(d) It is pointed out by Fell in [9, p. 243] that a CCR C*-algebra has Hausdorff spectrum if and only if it can be represented as a (maximal) continuous field of primitive C*-algebras over some locally compact Hausdorff base space.

(e) One might observe that Theorem 3.3 can be extended to GCR C*-algebras. However, for a GCR C*-algebra $\mathcal{A}$ with Hausdorff spectrum, $\mathcal{A}$ is automatically a CCR, and thus nothing new can be achieved in this plausible generality. Indeed, a separable C*-algebra is a GCR (resp. CCR) if and only if its spectrum is T_0 (resp. T_1); see, e.g., [5]. In general, a GCR C*-algebra is a CCR if and only if its spectrum is T_1 [11, Theorem 4].

To end the paper we present an other example as an evident to support our general conjecture that linear and zero product structures suffice to determine a C*-algebra.

Example 3.6. (See [6].) Let $\mathcal{M}$ be a properly infinite W^*-algebra and θ a zero product preserving linear map from $\mathcal{M}$ onto a unital algebra $\mathcal{N}$. Then

$$\theta(a) = \theta(1)\Psi(a), \quad \text{for all } a \text{ in } \mathcal{M},$$

where $\theta(1)$ is an invertible element in the center of $\mathcal{N}$ and Ψ is an algebra homomorphism from $\mathcal{M}$ onto $\mathcal{N}$. In particular, if $\mathcal{N}$ is a semi-simple Banach algebra then θ is automatically bounded, by, e.g., a result of Aupetit [3] which ensures that every surjective algebra homomorphism between semi-simple Banach algebras is bounded.

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