J. Fixed Point Theory Appl. 16 (2014) 203–227
DOI 10.1007/s11784-014-0208-x
Published online December 31, 2014

Journal of Fixed Point Theory and Applications

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Attractive point and mean convergence theorems for semigroups of mappings without continuity in Banach spaces

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To Professor Andrzej Granas

Abstract. In this paper, we first introduce a broad semigroup of mappings without continuity in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings and semigroups of nonexpansive mappings. Then we prove an attractive point and fixed point theorem for the semigroups of mappings without continuity. Furthermore, we establish a mean convergence theorem for the semigroups of mappings without continuity in a Banach space. Using these results, we obtain well-known and new theorems which are connected with attractive point, fixed point and mean convergence results in Banach spaces.

Mathematics Subject Classification. 47H05, 47H09, 47H20.

Keywords. Attractive point, Banach limit, Banach space, fixed point, generalized nonspreading mapping, invariant mean, mean convergence, nonexpansive semigroup.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. For a mapping T of C into itself, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [27] of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}; \tag{1.1}$$

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\| \ \forall x \in C\}. \tag{1.2}$$

A mapping $T : C \to C$ is called *generalized hybrid* [16] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

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for all $x, y \in C$; see also [1]. We call such a mapping an $(\alpha,\ \beta)$-*generalized hybrid* mapping. A $(1,0)$-generalized hybrid mapping is nonexpansive; i.e.,

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C.$$

A $(2,1)$-generalized hybrid mapping is *nonspreading* [19]; i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C.$$

A $(\frac{3}{2},\ \frac{1}{2})$-generalized hybrid mapping is also *hybrid* [26]; i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see, for instance, [13]. Recently, Takahashi and Takeuchi [27] proved the following attractive point and mean convergence theorem without convexity in a Hilbert space; see also [20].

Theorem 1.1 (See [27]**).** *Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_1 \in C, \quad v_{n+1} = Tv_n, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} v_k$$

for all $n \in \mathbb{N}$. If $\{v_n\}$ is bounded, then the following hold:

(i) *$A(T)$ is nonempty, closed and convex;*
(ii) *$\{b_n\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)}v_n$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

Atsushiba and Takakashi [2] also proved such a theorem for commutative semigroups of nonexpansive mappings in a Hilbert space. Motivated by [2, 27],the authors established in [28] attractive point and mean convergence theorems for semigroups of mappings without continuity in a Hilbert space which unifies the results of [27] and [2]. On the other hand, we know that the class of generalized hybrid mappings in a Hilbert space was extended to the class of generalized nonspreading mappings in a Banach space in the sense of Kocourek, Takahashi and Yao [17]. It is natural to extend the authors' theorems in [28] to Banach spaces.

In this paper, we first introduce a broad semigroup of mappings without continuity in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings and semigroups of nonexpansive mappings. Then we prove an attractive point and fixed point theorem for the semigroups of mappings without continuity. Furthermore, we establish a mean convergence theorem of Baillon's type [3] for the semigroups. Using these results, we obtain well-known and new theorems which are connected with attractive point, fixed point and mean convergence results in Banach spaces.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The *duality* mapping J from E into E^* is defined by

$$Jx = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\} \quad \forall x \in E,$$

where $\langle x, x^* \rangle$ is the value of $x^* \in E^*$ at $x \in E$. Let $S(E)$ be the unit sphere centered at the origin of E. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. It is known that if E is uniformly convex, then E is strictly convex and reflexive. Furthermore, we know from [25] that

(i) if E is smooth, then J is single valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E has a Fréchet differentiable norm, then J is continuous.

Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H,

$$\phi(x, y) = \|x - y\|^2 \quad \forall x, y \in H.$$

Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.2}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.3}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.4}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.5}$$

If E is a smooth, strictly convex and reflexive Banach space, then for any $x, y \in E$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$,

$$\phi\big(x, J^{-1}\big(\lambda Jy + (1 - \lambda)Jz\big)\big) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z). \tag{2.6}$$

Let $\phi_* : E^* \times E^* \to \mathbb{R}$ be a function defined by

$$\phi_*(x^*, y^*) := \|x^*\|^2 - 2\big\langle J^{-1}y^*, x^* \big\rangle + \|y^*\|^2 \quad \forall x^*, y^* \in E^*.$$

We have that

$$\phi(x,y) = \phi_*(Jy,Jx) \quad \forall x,y \in E. \tag{2.7}$$

The following lemmas were proved by Xu [30] and by Kamimura and Takahashi [15].

Lemma 2.1 (See [30]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0,2r] \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|ax + (1-a)y\|^2 \le a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\, g(\|x-y\|)$$

for all $x, y \in B_r$ and $a \in [0,1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.2 (See [15]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0,2r] \to [0,\infty)$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \le \phi(x,y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *generalized nonexpansive* [10] if $F(T) \neq \emptyset$ and

$$\phi(Tx,y) \le \phi(x,y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R\big(Rx + t(x - Rx)\big) = Rx$$

for all $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp., *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp., sunny generalized nonexpansive retraction) R from E onto D; see [9, 10, 11] for more details. The following results are in [10].

Lemma 2.3 (See [10]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.4 (See [10]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x,z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \le 0$ for all $y \in C$;*
(ii) *$\phi(Rx,z) + \phi(x,Rx) \le \phi(x,z)$.*

In 2007, Kohsaka and Takahashi [18] proved the following results.

Lemma 2.5 (See [18]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.6 (See [18]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Inthakon, Dhompongsa and Takahashi [14] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [12].

Lemma 2.7 (See [14]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.5 and 2.7.

Lemma 2.8 (See [14]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at

$$f = (x_1, x_2, x_3, \dots) \in l^\infty.$$

Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if

$$\mu(e) = \|\mu\| = 1,$$

where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if

$$\mu_n(x_{n+1}) = \mu_n(x_n).$$

We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have

$$\mu(f) = \mu_n(x_n) = a.$$

See [25] for the proof of existence of a Banach limit and its other elementary properties.

3. Semitopological semigroups and invariant means

Let S be a semitopological semigroup; i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s+t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if

$$\mu(e) = \|\mu\| = 1,$$

where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s) \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on l^∞. The following theorem is Theorem 1.4.5 in [25].

Theorem 3.1 (See [25]**).** *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$; i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let S be a semitopological semigroup. For any $f \in C(S)$ and $c \in \mathbb{R}$, we write

$$f(s) \to c \quad \text{as } s \to \infty_R$$

if for each $\varepsilon > 0$ there exists an $\omega \in S$ such that

$$|f(tw) - c| < \varepsilon \quad \forall t \in S.$$

We denote $f(s) \to c$, as $s \to \infty_R$, by

$$\lim_{s \to \infty_R} f(s) = c \quad \text{or} \quad \lim_s f(s) = c.$$

When S is commutative, we also denote $s \to \infty_R$ by $s \to \infty$.

Theorem 3.2 (See [25]**).** *Let $f \in C(S)$ and $c \in \mathbb{R}$. If*

$$f(s) \to c \quad \textit{as } s \to \infty_R,$$

then $\mu(f) = c$ for all right invariant mean μ on $C(S)$.

Theorem 3.3 (See [25]**).** *If $f \in C(S)$ fulfills*

$$f(ts) \leq f(s) \quad \forall t, s \in S,$$

then

$$f(t) \to \inf_{w \in S} f(w) \quad \text{as } t \to \infty_R.$$

Let E be a Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of T_s, $s \in S$; i.e.,

$$F(\mathcal{S}) = \cap \{F(T_s) : s \in S\}.$$

The following definition [23] is crucial in the nonlinear ergodic theory of abstract semigroups; see also [6]. Let E be a reflexive Banach space and let E^* be the dual space of E. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y^* \rangle = \langle z_0, y^* \rangle \quad \forall y^* \in E^*. \tag{3.1}$$

In fact, since $\{u(s) : s \in S\}$ is bounded and μ is a mean on $C(S)$, we can define a real-valued function g as follows:

$$g(y^*) = \mu_s \langle u(s), y^* \rangle \quad \forall y^* \in E^*.$$

We have that for any $y^*, z^* \in E^*$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} g(\alpha y^* + \beta z^*) &= \mu_s \langle u(s), \alpha y^* + \beta z^* \rangle \\ &= \alpha \mu_s \langle u(s), y^* \rangle + \beta \mu_s \langle u(s), z^* \rangle \\ &= \alpha g(y^*) + \beta g(z^*). \end{aligned}$$

Then g is a linear functional of E^* into $\mathbb{R}$. Furthermore, we have that for any $y^* \in E^*$,

$$\begin{aligned} |g(y^*)| &= |\mu_s \langle u(s), y^* \rangle| \\ &\leq \|\mu_s\| \sup_s |\langle u(s), y^* \rangle| \\ &\leq \|\mu_s\| \sup_s \|u(s)\| \, \|y^*\| \\ &= \Big(\sup_s \|u(s)\| \Big) \|y^*\|. \end{aligned}$$

Put $K = \sup_s \|u(s)\|$. We have

$$|g(y^*)| \leq K \|y^*\| \quad \forall y^* \in E^*.$$

Then g is bounded. By the Riesz theorem, there exists $z_0 \in E$ such that

$$g(y^*) = \langle z_0, y^* \rangle \quad \forall y^* \in E^*. \tag{3.2}$$

It is obvious that such $z_0 \in E$ is unique. Furthermore, we have

$$z_0 \in \overline{co}\{u(s) : s \in S\}.$$

In fact, if $z_0 \notin \overline{co}\,\{u(s) : s \in S\}$, then there exists $y_0^* \in E^*$ from the separation theorem such that

$$\langle z_0, y_0^* \rangle < \inf \big\{ \langle z, y_0^* \rangle : z \in \overline{co}\,\{u(s) : s \in S\} \big\}.$$

Using the property of a mean, we have

$$\begin{aligned}\langle z_0, y_0^* \rangle &< \inf \big\{ \langle z, y_0^* \rangle : z \in \overline{co}\,\{u(s) : s \in S\} \big\} \\ &\le \inf\{\langle u(s), y_0^* \rangle : s \in S\} \\ &\le \mu_s \langle u(s), y_0^* \rangle \\ &= \langle z_0, y_0^* \rangle.\end{aligned}$$

This is a contradiction. Thus we have $z_0 \in \overline{co}\,\{u(s) : s \in S\}$. We call such z_0 the *mean vector* of u for μ. In particular, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Putting $u(s) = T_s x$ for all $s \in S$, we have that there exists $z_0 \in E$ such tat

$$\mu_s \langle T_s x, y^* \rangle = \langle z_0, y^* \rangle \quad \forall y^* \in E^*.$$

We denote such z_0 by $T_\mu x$. A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

See [4, 25] for more details.

4. Attractive point theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. For a mapping T from C into C, we denote by $A(T)$ the set of attractive points [21] of T; that is,

$$A(T) = \big\{ u \in E : \phi(u, Tx) \le \phi(u, x)\, \forall x \in C \big\}.$$

Let S be a commutative semitopological semigroup with identity. For a continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings of C into itself, we denote the set $A(\mathcal{S})$ of common attractive points of $\mathcal{S} = \{T_s : s \in S\}$ by

$$A(\mathcal{S}) = \cap \{A(T_t) : t \in S\}.$$

We know the following lemma from Lin and Takahashi [21]

Lemma 4.1 (See [21]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then, $A(T)$ is a closed and convex subset of E.*

We have from Lemma 4.1 that $A(\mathcal{S})$ is closed and convex. Using the technique developed by Takahashi [23], we can prove the following attractive point theorem for a family of mappings in a Banach space.

Theorem 4.2. *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative* *semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y)$$

for all $y \in C$ and $t \in S$. Then,

$$A(\mathcal{S}) = \cap \{A(T_t) : t \in S\}$$

is nonempty. In particular, if E is strictly convex and C is closed and convex, then

$$F(\mathcal{S}) = \cap \{F(T_t) : t \in S\}$$

is nonempty.

Proof. Using a mean μ and a bounded set $\{T_s x : s \in S\}$, we define a function $g : E^* \to \mathbb{R}$ as follows:

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle \quad \forall x^* \in E^*.$$

Since E is reflexive, as in Section 3, there exists a unique element z of E such that

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle = \langle z, x^* \rangle \quad \forall x^* \in E^*.$$

Such an element z is in $D = \overline{co}\,\{T_s x : s \in S\}$, where $\overline{co}A$ is the closure of the convex hull of A. Take $t \in S$. We have from (2.3) that for $y \in C$ and $s \in S$,

$$\phi(T_s x, y) = \phi(T_s x, T_t y) + \phi(T_t y, y) + 2\langle T_s x - T_t y, JT_t y - Jy \rangle.$$

Then we have that for any $y \in C$,

$$\begin{aligned} \mu_s \phi(T_s x, y) &= \mu_s \phi(T_s x, T_t y) + \mu_s \phi(T_t y, y) + 2\mu_s \langle T_s x - T_t y, JT_t y - Jy \rangle \\ &= \mu_s \phi(T_s x, T_t y) + \phi(T_t y, y) + 2\langle z - T_t y, JT_t y - Jy \rangle. \end{aligned}$$

Since $\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y)$ by assumption, we have

$$\mu_s \phi(T_s x, y) \leq \mu_s \phi(T_s x, y) + \phi(T_t y, y) + 2\langle z - T_t y, JT_t y - Jy \rangle.$$

This implies that

$$0 \leq \phi(T_t y, y) + 2\langle z - T_t y, JT_t y - Jy \rangle \quad \forall y \in C. \tag{4.1}$$

Then we have from (4.1) and (2.4) that

$$\begin{aligned} 0 &\leq \phi(T_t y, y) + \phi(z, y) + \phi(T_t y, T_t y) - \phi(z, T_t y) - \phi(T_t y, y) \qquad (4.2) \\ &= \phi(z, y) - \phi(z, T_t y). \end{aligned}$$

This implies that $\phi(z, T_t y) \leq \phi(z, y)$ for all $y \in C$ and hence $z \in A(T_t)$. Therefore $z \in A(\mathcal{S}) = \cap \{A(T_t) : t \in S\}$.

In particular, if E is strictly convex and C is closed and convex, then we have from $D = \overline{co}\,\{T_s x : s \in S\} \subset C$ that z is an element of C. Putting $y = z$ in (4.2), we have

$$\phi(z, T_t z) \leq \phi(z, z).$$

Thus we have $\phi(z, T_t z) \leq 0$ and hence $\phi(z, T_t z) = 0$. Since E is strictly convex, we have $T_t z = z$. Therefore $z \in F(\mathcal{S}) = \cap \{F(T_t) : t \in S\}$. This completes the proof. $\square$

Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. A mapping $T : C \to C$ is called *generalized nonspreading* [17] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\qquad \le \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{4.3}$$

for all $x, y \in C$. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (4.3), we obtain

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x) \quad \forall x, y \in C.$$

Such a mapping T is *nonspreading* in the sense of Kohsaka and Takahashi [19]. In the case of $\alpha = 1$ and $\beta = \gamma = \delta = 0$ in (4.3), we obtain

$$\phi(Tx, Ty) \le \phi(x, y) \quad \forall x, y \in C.$$

We call such T a *ϕ-nonexpansive* mapping. If E is a Hilbert space, then we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. Thus from (4.3), we obtain the following:

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\big(\|Ty - Tx\|^2 - \|Ty - x\|^2\big) \\ &\qquad \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\big(\|y - Tx\|^2 - \|y - x\|^2\big) \end{aligned}$$

for all $x, y \in C$. This implies that

$$\begin{aligned} &(\alpha + \gamma)\|Tx - Ty\|^2 + [1 - (\alpha + \gamma)]\,\|x - Ty\|^2 \\ &\qquad \le (\beta + \delta)\|Tx - y\|^2 + [1 - (\beta + \delta)]\,\|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. That is, T is a generalized hybrid mapping in the sense of [16]. Using Theorem 4.2, we have the following attractive point theorem for generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [21].

Theorem 4.3 (See [21]**).** *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a generalized nonspreading mapping of C into itself. Then the following are equivalent:*

(1) *$A(T) \ne \emptyset$;*
(2) *$\{T^n v_0\}$ is bounded for some $v_0 \in C$.*

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(1) *$F(T) \ne \emptyset$;*
(2) *$\{T^n v_0\}$ is bounded for some $v_0 \in C$.*

Proof. If $A(T) \ne \emptyset$, then $\phi(u, Tx) \le \phi(u, x)$ for all $u \in A(T)$ and $x \in C$. So, $\phi(u, T^n x) \le \phi(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$ and hence $\{T^n x : n \in \mathbb{N}\}$ is bounded. We show the reverse. Since $T : C \to C$ is a generalized nonspreading, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\big(\phi(Ty, Tx) - \phi(Ty, x)\big) \\ &\qquad \le \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\big(\phi(y, Tx) - \phi(y, x)\big) \end{aligned} \tag{4.4}$$

for all $x, y \in C$. Replacing x by $T^n v_0$ in inequality (4.4), we get

$$\alpha\phi\big(T^{n+1}v_0, Ty\big) + (1-\alpha)\phi(T^n v_0, Ty) + \gamma\big(\phi(Ty, T^{n+1}v_0) - \phi(Ty, T^n v_0)\big) \\ \leq \beta\phi\big(T^{n+1}v_0, y\big) + (1-\beta)\phi(T^n v_0, y) + \delta\big(\phi(y, T^{n+1}v_0) - \phi(y, T^n v_0)\big).$$

Since $\{T^n v_0\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. We have

$$\mu_n \phi(T^n v_0, Ty) \leq \mu_n \phi(T^n v_0, y)$$

for all $y \in C$. Therefore we have Theorem 4.3 from Theorem 4.2. □

Let E be a smooth Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup. A continuous representation

$$\mathcal{S} = \{T_s : s \in S\}$$

of S as mappings on C is a *ϕ-nonexpansive semigroup* on C if each T_s, $s \in S$, is ϕ-nonexpansive; i.e.,

$$\phi(T_s x, T_s y) \leq \phi(x, y) \quad \forall x, y \in C.$$

In the case when E is a Hilbert space, a ϕ-nonexpansive semigroup on C is called a *nonexpansive semigroup* on C; see Atsushiba and Takakashi [2]. Using Theorem 4.2, we can also prove an attractive point theorem for ϕ-nonexpansive semigroups in a Banach space.

Theorem 4.4. *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then, $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.*

Proof. Since $\mathcal{S} = \{T_s : s \in S\}$ is a ϕ-nonexpansive semigroup on C, we have

$$\phi(T_{t+s}x, T_t y) \leq \phi(T_s x, y)$$

for all $x, y \in C$ and $s, t \in S$. Since $\{T_s z\}$ is bounded for some $z \in C$, we can apply an invariant mean μ to both sides of the inequality. Then, we have that for any $y \in C$ and $t \in S$,

$$\mu_s \phi(T_{t+s}z, T_t y) \leq \mu_s \phi(T_s z, y)$$

and hence

$$\mu_s \phi(T_s z, T_t y) \leq \mu_s \phi(T_s z, y).$$

We have from Theorem 4.2 that $A(\mathcal{S})$ is nonempty. Additionally, if E is strictly convex and C is closed and convex, then we have from Theorem 4.2 that $F(\mathcal{S})$ is nonempty. This completes the proof. □

As a direct consequence of Theorem 4.4, we obtain the following theorem which was proved by Atsushiba and Takakashi [2].

Theorem 4.5 (See [2]**).** *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then $A(\mathcal{S})$ is nonempty. Additionally, if C is closed and convex, then $F(\mathcal{S})$ is nonempty.*

5. Skew-attractive point theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. We denote by $B(T)$ the set of *skew-attractive points* [21] of T; i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \le \phi(x, z)\, \forall x \in C\}.$$

The following lemma was proved by Lin and Takahashi [21].

Lemma 5.1 (See [21]**).** *Let E be a smooth Banach space and let C be a nonempty subset of H. Let T be a mapping from C into E. Then, $B(T)$ is closed.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^* \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. The mapping T^* is called the *duality mapping* of T; see [29] and [7]. If T is a mapping of C into itself, then T^* is a mapping of JC into JC. The following lemma was also proved by Lin and Takahashi [21] and Takahashi and Yao [29].

Lemma 5.2 (See [21, 29]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E and let T^* be the duality mapping of T. Then the following hold:*

(1) $JF(T) = F(T^*)$;
(2) $JB(T) = A(T^*)$;
(3) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Let E be a smooth Banach space and let C be a nonempty subset of E. We denote by $B(\mathcal{S})$ the set of all common skew-attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself; i.e.,

$$B(\mathcal{S}) = \cap\{B(T_s) : s \in S\}.$$

We obtain the following skew-attractive point theorem for semigroups of mappings without continuity in a Banach space.

Theorem 5.3. *Let E be a strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that*

$\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that

$$\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$$

for all $y \in C$ and $t \in S$. Then, $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\}$ is nonempty. In particular, if C is closed and JC is closed and convex, then

$$F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$$

is nonempty.

Proof. Assume that $\{T_s x : s \in S\}$ is a bounded subset of C for some $x \in C$. Put $x^* = Jx$ and $y^* = Jy$, where $y \in C$ and define

$$T_s^* = JT_sJ^{-1} \quad \forall s \in S.$$

Then, we have

$$T_s^*T_t^* = JT_sJ^{-1}JT_tJ^{-1} = JT_sT_tJ^{-1} = JT_{s+t}J^{-1} = T_{s+t}^* \quad \forall s, t \in S.$$

On the other hand, if $s \to t$, then we have from the continuity of J that for any $y^* \in JC$,

$$\|T_s^* y^* - T_t^* y^*\| = \|JT_sJ^{-1}Jy - JT_tJ^{-1}Jy\| = \|JT_s y - JT_t y\| \to 0.$$

Therefore, $\mathcal{S}^* = \{T_s^* : s \in S\}$ is a continuous representation of S as mappings of JC into itself. Furthermore, since $\{T_s x : s \in S\}$ is bounded and

$$\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$$

for all $y \in C$ and $t \in S$, we have

$$\begin{aligned}
\mu_s \phi_*(T_s^* x^*, T_t^* y^*) &= \mu_s \phi_*(JT_sJ^{-1}Jx, JT_tJ^{-1}Jy) \\
&= \mu_s \phi(T_tJ^{-1}Jy, T_sJ^{-1}Jx) \\
&= \mu_s \phi(T_t y, T_s x) \\
&\leq \mu_s \phi(y, T_s x) \\
&= \mu_s \phi_*(JT_s x, Jy) \\
&= \mu_s \phi_*(JT_sJ^{-1}Jx, Jy) \\
&= \mu_s \phi_*(T_s^* x^*, y^*)
\end{aligned}$$

for all $y^* \in JC$ and $t \in S$. Therefore, we have from Theorem 4.2 that

$$A(\mathcal{S}^*) = \cap\{A(T_t^*) : t \in S\}$$

is nonempty. Since $J : E \to E^*$ is a one-to-one and onto mapping, we have from Lemma 5.2 that

$$\begin{aligned}
B(\mathcal{S}) &= \cap\{B(T_t) : t \in S\} \\
&= \cap\{J^{-1}A(T_t^*) : t \in S\} \\
&= J^{-1}\big(\cap\{A(T_t^*) : t \in S\}\big) \\
&= J^{-1}A(\mathcal{S}^*).
\end{aligned}$$

Since $A(\mathcal{S}^*)$ is nonempty, we have that $B(\mathcal{S})$ is nonempty. In particular, if C is closed and JC is closed and convex, then we have from Theorem 4.2 that

$$F(\mathcal{S}^*) = \cap \{F(T_t^*) : t \in S\}$$

is nonempty. We also have from Lemma 5.2 that

$$\begin{aligned} F(\mathcal{S}) &= \cap \{F(T_t) : t \in S\} \\ &= \cap \{J^{-1}F(T_t^*) : t \in S\} \\ &= J^{-1}\big(\cap \{F(T_t^*) : t \in S\}\big) \\ &= J^{-1}F(\mathcal{S}^*). \end{aligned}$$

Therefore,

$$F(\mathcal{S}) = \cap \{F(T_t) : t \in S\}$$

is nonempty. This completes the proof. □

Let E be a smooth Banach space and let J be the duality mapping from E into E^*. Let C be a nonempty subset of E. A mapping $T : C \to E$ is called *skew-generalized nonspreading* [8] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Ty, Tx) + (1-\alpha)\phi(Ty, x) + \gamma\big(\phi(Tx, Ty) - \phi(x, Ty)\big) \\ &\qquad \le \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\big(\phi(Tx, y) - \phi(x, y)\big) \end{aligned} \tag{5.1}$$

for all $x, y \in C$. Using Theorem 5.3, we have the following attractive point theorem for skew-generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [21].

Theorem 5.4 (See [21]**).** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a skew-generalized nonspreading mapping of C into itself. Then the following are equivalent:*

(1) *$B(T) \ne \emptyset$;*
(2) *$\{T^n v_0\}$ is bounded for some $v_0 \in C$.*

Additionally, if C is closed and JC is closed and convex, then the following are equivalent:

(1) *$F(T) \ne \emptyset$;*
(2) *$\{T^n v_0\}$ is bounded for some $v_0 \in C$.*

Proof. If $B(T) \ne \emptyset$, then $\phi(Ty, u) \le \phi(y, u)$ for all $u \in B(T)$ and $y \in C$. So, $\phi(T^n y, u) \le \phi(y, u)$ for all $n \in \mathbb{N}$ and $y \in C$ and then $\{T^n y\}$ is bounded for all $y \in C$. We show the reverse. Replacing x by $T^n v_0$ in (5.1), where $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} &\alpha\phi\big(Ty, T^{n+1}v_0\big) + (1-\alpha)\phi(Ty, T^n v_0) + \gamma\big(\phi(T^{n+1}v_0, Ty) - \phi(T^n v_0, Ty)\big) \\ &\qquad \le \beta\phi\big(y, T^{n+1}v_0\big) + (1-\beta)\phi(y, T^n v_0) + \delta\big(\phi(T^{n+1}v_0, y) - \phi(T^n v_0, y)\big) \end{aligned}$$

for all $y \in C$. Since $\{T^n v_0\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. We have

$$\mu_n \phi(Ty, T^n v_0) \le \mu_n \phi(y, T^n v_0)$$

for all $y \in C$. Therefore we have Theorem 5.4 from Theorem 5.3. □

We know that this theorem (Theorem 5.4) is a generalization of the corresponding result of Dhompongsa et al. [5]. Using Theorem 5.3, we can also prove Theorem 4.4.

6. Nonlinear mean convergence theorems

In this section, we prove a nonlinear mean convergence theorem of Baillon's type [3] for semigroups of mappings without continuity in a Banach space. Before proving it, we need the following four lemmas which are proved by using the ideas of [22, 24].

Lemma 6.1. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let D be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup with identity and let $C(S)$ be the Banach space of all bounded real-valued continuous functions on S with supremum norm. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\} \subset D$ is bounded and let μ be a mean on $C(S)$. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s \phi(u(s), z) \quad \forall z \in D,$$

then the mean vector z_0 of $\{u(s) : s \in S\}$ for μ is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Proof. For a bounded net $\{u(s)\} \subset D$ and a mean μ on $C(S)$, we know that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_s \phi(u(s), z) \quad \forall z \in D$$

is well defined. We also know from the proof of Theorem 4.2 that there exists a mean vector z_0 of $\{u(s)\}$ for μ, that is, there exists $z_0 \in \overline{co}\,\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y^* \rangle = \langle z_0, y^* \rangle \quad \forall y^* \in E^*.$$

Since D is closed and convex and $\{u(s)\} \subset D$, we have $z_0 \in D$. Furthermore, we have from (2.3) and (2.4) that for any $z \in D$,

$$\begin{aligned}
g(z) - g(z_0) &= \mu_s \phi(u(s), z) - \mu_s \phi(u(s), z_0) \\
&= \mu_s \big(\phi(u(s), z) - \phi(u(s), z_0)\big) \\
&= \mu_s \big(\phi(u(s), z) - \phi(u(s), z) - \phi(z, z_0) - 2\langle u(s) - z, Jz - Jz_0\rangle\big) \\
&= \mu_s \big(- \phi(z, z_0) - 2\langle u(s) - z, Jz - Jz_0\rangle\big) \\
&= -\phi(z, z_0) - 2\langle z_0 - z, Jz - Jz_0\rangle \\
&= -\phi(z, z_0) - \phi(z_0, z_0) - \phi(z, z) + \phi(z_0, z) + \phi(z, z_0) \\
&= \phi(z_0, z).
\end{aligned}$$

Then we have

$$g(z) = g(z_0) + \phi(z_0, z) \quad \forall z \in D. \tag{6.1}$$

This implies that $z_0 \in D$ is a minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Furthermore, if $u \in D$ satisfies $g(u) = g(z_0)$, then we have from (6.1) that $\phi(z_0, u) = 0$. Since E is strictly convex, we have that $z_0 = u$ and hence z_0 is a unique minimizer in D such that $g(z_0) = \min\{g(z) : z \in D\}$. This completes the proof. $\square$

Using Lemma 6.1, we obtain the following result.

Lemma 6.2. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that $A(\mathcal{S}) = B(\mathcal{S})$ is nonempty. Then for any $x \in C$, the net $\{T_s x : s \in S\}$ is bounded and the set*

$$\cap_s \overline{co}\,\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$$

consists of one point z_0, where z_0 is a unique minimizer of $A(\mathcal{S})$ such that

$$\lim_s \phi(T_s x, z_0) = \min\Big\{\lim_s \phi(T_s x, z) : z \in A(\mathcal{S})\Big\}.$$

Additionally, if C is closed and convex, then the set

$$\cap_s \overline{co}\,\{T_{t+s}x : t \in S\} \cap F(\mathcal{S})$$

consists of one point z_0.

Proof. Since $A(\mathcal{S}) = B(\mathcal{S})$ is nonempty, then for any $z \in A(\mathcal{S}) = B(\mathcal{S})$ and $x \in C$, we have

$$\phi(T_{t+s}x, z) \le \phi(T_s x, z) \quad \forall s, t \in S.$$

Thus $\{T_s x : s \in S\}$ is bounded. Let μ be an invariant mean on $C(S)$. From Lemma 6.1, a unique minimizer $z_0 \in E$ such that

$$\mu_s \phi(T_s x, z_0) = \min\big\{\mu_s \phi(T_s x, y) : y \in E\big\}$$

is the mean vector $z_0 \in E$ of $\{T_s x : s \in S\}$ for μ, that is, a point $z_0 \in E$ such that $z_0 \in \overline{co}\,\{T_s x : s \in S\}$ and

$$\mu_s \langle T_s x, y^* \rangle = \langle z_0, y^* \rangle \quad \forall y^* \in E^*.$$

We also know from the proof of Theorem 4.2 that $z_0 \in A(\mathcal{S})$. Furthermore, this $z_0 \in A(\mathcal{S})$ satisfies

$$\mu_s \phi(T_s x, z_0) = \min\big\{\mu_s \phi(T_s x, y) : y \in A(\mathcal{S})\big\}.$$

Let us show that $z_0 \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\}$. If not, there exists some $s_0 \in S$ such that $z_0 \notin \overline{co}\,\{T_{t+s_0}x : t \in S\}$. By the separation theorem, there exists $y_0^* \in E^*$ such that

$$\langle z_0, y_0^* \rangle < \inf\big\{\langle z, y_0^* \rangle : z \in \overline{co}\,\{T_{t+s_0}x : t \in S\}\big\}.$$

Using the property of the invariant mean μ, we have

$$\begin{aligned}\langle z_0, y_0^*\rangle &< \inf\left\{\langle z, y_0^*\rangle : z \in \overline{co}\,\{T_{t+s_0}x : t \in S\}\right\}\\ &\le \inf\{\langle T_{t+s_0}x, y_0^*\rangle : t \in S\}\\ &\le \mu_t\langle T_{t+s_0}x, y_0^*\rangle\\ &= \mu_t\langle T_t x, y_0^*\rangle\\ &= \langle z_0, y_0^*\rangle.\end{aligned}$$

This is a contradiction. Thus $z_0 \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\}$. Next, we show that $\cap_s \overline{co}\,\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$ consists of one point z_0. Assume that

$$z_1 \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\} \cap A(\mathcal{S}).$$

Since $z_1 \in A(\mathcal{S}) = B(\mathcal{S})$, we have

$$\phi(T_{t+s}x, z_1) \le \phi(T_s x, z_1) \quad \forall s, t \in S.$$

Then $\lim_s \phi(T_s x, z_1)$ exists. Furthermore, we know from the property of an invariant mean μ that

$$\mu_s \phi(T_s x, z_1) = \lim_s \phi(T_s x, z_1).$$

In general, since $\lim_s \phi(T_s x, z)$ exists for every $z \in A(\mathcal{S})$, we define a function $g : A(\mathcal{S}) \to \mathbb{R}$ as follows:

$$g(z) = \lim_s \phi(T_s x, z) \quad \forall z \in A(\mathcal{S}).$$

Since

$$\phi(z_0, z_1) = \phi(T_s x, z_1) - \phi(T_s x, z_0) - 2\langle T_s x - z_0, Jz_0 - Jz_1\rangle$$

for every $s \in S$, we have

$$\begin{aligned}&\phi(z_0, z_1) + 2\lim_s\langle T_s x - z_0, Jz_0 - Jz_1\rangle\\ &\qquad = \lim_s \phi(T_s x, z_1) - \lim_s \phi(T_s x, z_0)\\ &\qquad \ge 0.\end{aligned}$$

Let $\epsilon > 0$. Then we have

$$2\lim_s\langle T_s x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon.$$

Hence there exists $s_0 \in S$ such that

$$2\langle T_{s+s_0}x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon$$

for every $s \in S$. Since $z_1 \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\}$, we have

$$2\langle z_1 - z_0, Jz_0 - Jz_1\rangle \ge -\phi(z_0, z_1) - \epsilon.$$

We have from (2.4) that

$$\phi(z_1, z_1) + \phi(z_0, z_0) - \phi(z_1, z_0) - \phi(z_0, z_1) \ge -\phi(z_0, z_1) - \epsilon$$

and hence $\phi(z_1, z_0) \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\phi(z_1, z_0) = 0$. Since E is strictly convex, we have $z_0 = z_1$. Therefore,

$$\{z_0\} = \cap_s \overline{co}\,\{T_{t+s}x : n \in S\} \cap A(\mathcal{S}).$$

Additionally, if C is closed and convex, then

$$z_0 \in \cap_s \overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S}).$$

Since $\cap_s \overline{co}\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$ consists of one point z_0, we have

$$\cap_s \overline{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S}) = \{z_0\}.$$

This completes the proof. □

Lemma 6.3. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and*

$$\mu_s \phi(T_s x, T_t y) \le \mu_s \phi(T_s x, y), \quad y \in C, \ t \in S$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $C(S)$; i.e., for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. If a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in E$, then $u \in A(\mathcal{S})$. Additionally, if E is strictly convex and C is closed and convex, then $u \in F(\mathcal{S})$.

Proof. Since $\{\mu_\alpha\}$ is a net of means on $C(S)$, it has a cluster point μ in the weak* topology. We show that μ is an invariant mean on $C(S)$. In fact, since the set

$$\{\lambda \in C(S)^* : \lambda(e) = \|\lambda\| = 1\}$$

is closed in the weak* topology, it follows that μ is a mean on $C(S)$. Furthermore, since $\{\mu_\alpha\}$ is asymptotically invariant, for any $\varepsilon > 0$, $f \in C(S)$ and $s \in S$, there exists α_0 such that

$$\left|\mu_\alpha(f) - \mu_\alpha(l_s f)\right| \le \frac{\varepsilon}{3} \quad \forall \alpha \ge \alpha_0.$$

Since μ is a cluster point of $\{\mu_\alpha\}$, we can choose $\beta \ge \alpha_0$ such that

$$|\mu_\beta(f) - \mu(f)| \le \frac{\varepsilon}{3} \quad \text{and} \quad \left|\mu_\beta(l_s f) - \mu(l_s f)\right| \le \frac{\varepsilon}{3}.$$

Hence we have

$$\begin{aligned} \left|\mu(f) - \mu(l_s f)\right| &\le \left|\mu(f) - \mu_\beta(f)\right| + \left|\mu_\beta(f) - \mu_\beta(l_s f)\right| \\ &\quad + \left|\mu_\beta(l_s f) - \mu(l_s f)\right| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\mu(f) = \mu(l_s f) \quad \forall f \in C(S), \ s \in S.$$

Suppose that a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to some $u \in E$. If λ is also a cluster point of $\{\mu_{\alpha_\beta}\}$ in the weak* topology, then λ is a cluster point of $\{\mu_\alpha\}$. Then λ is an invariant mean on $C(S)$. Without loss of generality, we may assume that $\mu_{\alpha_\beta} \rightharpoonup \lambda$ in the weak* topology. Furthermore, we have from $T_{\mu_{\alpha_\beta}} x \rightharpoonup u$ that

$$\lambda_s \langle T_s x, y^* \rangle = \lim_\beta \, (\mu_{\alpha_\beta})_s \langle T_s x, y^* \rangle = \lim_\beta \, \langle T_{\mu_{\alpha_\beta}} x, y^* \rangle = \langle u, y^* \rangle \quad \forall y^* \in E^*.$$

On the other hand, we have from (2.4) that for $y \in C$ and $s, t \in S$,

$$2\langle T_s x - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(T_s x, T_t y) - \phi(T_s x, y).$$

We apply μ_{α_β} to both sides of the inequality, we get

$$\begin{aligned} 2(\mu_{\alpha_\beta})_s \langle T_s x - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) \\ = (\mu_{\alpha_\beta})_s \phi(T_s x, T_t y) - (\mu_{\alpha_\beta})_s \phi(T_s x, y). \end{aligned}$$

Since $\mu_{\alpha_\beta} \rightharpoonup \lambda$, we have

$$2(\lambda)_s \langle T_s x - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = (\lambda)_s \phi(T_s x, T_t y) - (\lambda)_s \phi(T_s x, y)$$

and hence

$$2\langle u - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = (\lambda)_s \phi(T_s x, T_t y) - (\lambda)_s \phi(T_s x, y).$$

Since $(\lambda)_s \phi(T_s x, T_t y) - (\lambda)_s \phi(T_s x, y) \le 0$ by the assumption, we have

$$2\langle u - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) \le 0.$$

Since $2\langle u - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(u, T_t y) - \phi(u, y)$, we have

$$\phi(u, T_t y) \le \phi(u, y), \quad y \in C, \ t \in S. \tag{6.2}$$

This implies that $u \in A(T_t)$. Therefore $u \in A(\mathcal{S})$.

In particular, if E is strictly convex and C is closed and convex, then u is an element of C. Putting $y = u$ in (6.2), we get $T_t u = u$. Therefore

$$u \in F(\mathcal{S}) = \cap \{F(T_t) : t \in S\}.$$

This completes the proof. □

Lemma 6.4. *Let E be a uniformly convex and smooth Banach space. Let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $B(\mathcal{S})$. Furthermore, for any $x \in C$, $\lim_s RT_s x$ exists in $B(\mathcal{S})$, where $\lim_s RT_s x = q$ means that $\lim_s \|RT_s x - q\| = 0$.*

Proof. We have from Lemmas 5.1 and 5.2 that $B(\mathcal{S})$ is closed and $JB(\mathcal{S})$ is closed and convex. Then from Lemmas 2.3 and 2.5, there exists a unique sunny generalized nonexpansive retraction R of E onto $B(\mathcal{S})$. For an invariant mean μ on $C(S)$, there exists $q \in E$ such that

$$\mu_t \langle RT_t x, y^* \rangle = \langle q, y^* \rangle \quad \forall y^* \in E^*.$$

We also have that for any $s \in S$,

$$\mu_t \langle RT_{t+s} x, y^* \rangle = \mu_t \langle RT_t x, y^* \rangle = \langle q, y^* \rangle \quad \forall y^* \in E^*.$$

Thus

$$q \in \overline{co} \left\{ RT_{t+s} x : t \in S \right\} \quad \forall s \in S. \tag{6.3}$$

From the property of the sunny generalized nonexpansive retraction R, we have

$$0 \leq \langle v - Rv, JRv - Ju\rangle \quad \forall v \in E,\ u \in B(\mathcal{S}). \tag{6.4}$$

We have from (6.4) and (2.4) that

$$\begin{aligned} 0 &\leq 2\langle v - Rv, JRv - Ju\rangle \\ &= \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv) - \phi(Rv,u). \end{aligned}$$

Hence we have

$$\phi(Rv,u) \leq \phi(v,u) - \phi(v,Rv) \quad \forall v \in E,\ u \in B(\mathcal{S}). \tag{6.5}$$

Since $\phi(T_s z, u) \leq \phi(z,u)$ for all $s \in S$, $u \in B(\mathcal{S})$ and $z \in C$, it follows that

$$\phi(T_{t+s}x, RT_{t+s}x) \leq \phi(T_{t+s}x, RT_s x) \leq \phi(T_s x, RT_s x). \tag{6.6}$$

Hence we have from (6.6) and Theorem 3.3 that

$$\phi(T_s x, RT_s x) \to \inf_{w\in S} \phi(T_w x, RT_w x) \quad \text{as } s \to \infty. \tag{6.7}$$

Putting $u = RT_s x$ and $v = T_{t+s}x$ in (6.5), we get

$$\begin{aligned} \phi(RT_{t+s}x, RT_s x) &\leq \phi(T_{t+s}x, RT_s x) - \phi(T_{t+s}x, RT_{t+s}x) \\ &\leq \phi(T_s x, RT_s x) - \phi(T_{t+s}x, RT_{t+s}x) \\ &\leq \phi(T_s x, RT_s x) - \inf_{w\in S} \phi(T_w x, RT_w x). \end{aligned}$$

Since $\phi(\cdot, z)$ is weakly lower semicontinuous for all $z \in E$, we have from (6.3) that

$$\phi(q, RT_s x) \leq \phi(T_s x, RT_s x) - \inf_{w\in S} \phi(T_w x, RT_w x) \quad \forall s \in S.$$

Thus we have from (6.7) that

$$\|RT_s x - q\| \to 0 \quad \text{as } s \to \infty.$$

Since $B(\mathcal{S})$ is closed, then $\{RT_s x\}$ converges strongly to $q \in B(\mathcal{S})$. This completes the proof. $\square$

Now, we can prove the following nonlinear mean convergence theorem of Baillon's type [3] for semigroups of mappings without continuity in a Banach space.

Theorem 6.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Suppose that*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y) \quad \forall x, y \in C,\ t \in S \tag{6.8}$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $C(S)$; i.e., for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0.$$

Then, $\{T_{\mu_\alpha}x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where

$$u = \lim_s R_{B(\mathcal{S})} T_s x.$$

Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where

$$u = \lim_s R_{F(\mathcal{S})} T_s x.$$

Proof. Let $x \in C$. Since $A(\mathcal{S})$ is nonempty, the net $\{T_s x : s \in S\}$ is bounded. So, $\{T_{\mu_\alpha}x\}$ is bounded. In fact, we have

$$\begin{aligned}
\|T_{\mu_\alpha}x\| &= \sup\left\{|\langle T_{\mu_\alpha}x, x^*\rangle| : \|x^*\| = 1\right\} \\
&= \sup\left\{|(\mu_\alpha)_t\langle T_t x, x^*\rangle| : \|x^*\| = 1\right\} \\
&\leq \sup\left\{\|\mu_\alpha\| \cdot \sup_t |\langle T_t x, x^*\rangle| : \|x^*\| = 1\right\} \\
&\leq \sup\left\{\sup_t \|T_t x\| \cdot \|x^*\| : \|x^*\| = 1\right\} \\
&= \sup_t \|T_t x\|.
\end{aligned}$$

We also know from Theorem 6.2 that the set

$$\cap_s \overline{co}\,\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$$

consists of one point. To prove that $\{T_{\mu_\alpha}x\}$ converges weakly to a point z_0 in $A(\mathcal{S})$, it is sufficient to show that if a subnet $\{T_{\mu_{\alpha_\beta}}x\}$ of $\{T_{\mu_\alpha}x\}$ converges weakly to a point $v \in E$, i.e., $T_{\mu_{\alpha_\beta}}x \rightharpoonup v$, then $v \in A(\mathcal{S})$ and

$$v \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\}.$$

From Lemma 6.3, we have that $v \in A(\mathcal{S})$. Next, we show that

$$v \in \cap_s \overline{co}\,\{T_{t+s}x : t \in S\}.$$

Since $\{T_{\mu_{\alpha_\beta}}x\} \rightharpoonup v$, we also know that

$$\lambda_s\langle T_s x, y\rangle = \langle v, y\rangle \quad \forall y \in H$$

for some invariant mean λ on $C(S)$. Then, from Lemma 6.2, we have that

$$v \in \cap_t \overline{co}\,\{T_{t+s}x : s \in S\}.$$

Therefore $\{T_{\mu_\alpha}x\}$ converges weakly to z_0 of $A(\mathcal{S})$. Additionally, if C is closed and convex, then $z_0 \in C$ and hence $z_0 \in F(\mathcal{S})$. Therefore $\{T_{\mu_\alpha}x\}$ converges weakly to $z_0 \in F(\mathcal{S})$. To show that $z_0 = \lim_s P_{F(\mathcal{S})}T_s x$, we may follow the proof of Lemma 6.4. This completes the proof. □

Using Theorem 6.5, we obtain the following nonlinear mean convergence theorem for generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [21].

Theorem 6.6 (See [21]**).** *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$, where $z_0 = \lim_{n\to\infty} RT^n x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to $z_0 \in F(T)$.

Proof. Let $x \in C$. Since $A(T)$ is nonempty, the sequence $\{T^n x\}$ is bounded. Since $T : C \to C$ is a generalized nonspreading, we have

$$\begin{aligned} &\alpha\phi\big(T^{n+1}x, Ty\big) + (1-\alpha)\phi(T^n x, Ty) + \gamma\big(\phi(Ty, T^{n+1}x) - \phi(Ty, T^n x)\big) \\ &\qquad \leq \beta\phi\big(T^{n+1}x, y\big) + (1-\beta)\phi(T^n x, y) + \delta\big(\phi(y, T^{n+1}x) - \phi(y, T^n x)\big) \end{aligned}$$

for all $x, y \in C$. Since $\{T^n x\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. We have

$$\mu_n \phi(T^n x, Ty) \leq \mu_n \phi(T^n x, y)$$

for all $y \in C$. Therefore we have Theorem 6.6 from Theorem 6.5. □

Using Theorem 6.5, we also have the nonlinear mean convergence theorem for ϕ-nonexpansive semigroups in a Banach space.

Theorem 6.7. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $C(S)$; i.e., for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. Then, $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s R_{B(\mathcal{S})} T_s x$. Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s R_{F(\mathcal{S})} T_s x$.*

Proof. Since $\mathcal{S} = \{T_s : s \in S\}$ is a ϕ-nonexpansive semigroup on C, we have

$$\phi(T_{t+s}x, T_t y) \leq \phi(T_s x, y)$$

for all $x, y \in C$ and $s, t \in S$. Since $A(T)$ is nonempty, $\{T_s x\}$ is bounded for all $x \in C$. So, we can apply an invariant mean μ to both sides of the inequality. Then we have that for any $y \in C$ and $t \in S$,

$$\mu_s \phi(T_{t+s}x, T_t y) \leq \mu_s \phi(T_s x, y)$$

and hence

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y).$$

By Theorem 6.5, we have Theorem 6.7. This completes the proof. □

As a direct consequence of Theorem 6.7, we have a mean convergence theorem for commutative semigroups of nonexpansive mappings in a Hilbert space which was proved by Atsushiba and Takakashi [2].

Theorem 6.8 (See [2]**).** *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $A(\mathcal{S})$ is nonempty. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $C(S)$. Then, $\{T_\alpha x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s P_{A(\mathcal{S})} T_s x$. Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s P_{F(\mathcal{S})} T_s x$.*

Acknowledgements

The first author was partially supported by Grant-in-Aid for Scientific Research No. 23540188 from Japan Society for the Promotion of Science. The second and the third authors were partially supported by Taiwan MOST grant 102-2115-M-110-002-MY2 and the grant NSC 102-2115-M-037-002-MY3, respectively.

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