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Fixed point theorems for nonlinear non-self mappings in Hilbert spaces and applications

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Abstract

Recently, Kawasaki and Takahashi (J. Nonlinear Convex Anal. 14:71-87, 2013) defined a broad class of nonlinear mappings, called widely more generalized hybrid, in a Hilbert space which contains generalized hybrid mappings (Kocourek *et al.* in Taiwan. J. Math. 14:2497-2511, 2010) and strict pseudo-contractive mappings (Browder and Petryshyn in J. Math. Anal. Appl. 20:197-228, 1967). They proved fixed point theorems for such mappings. In this paper, we prove fixed point theorems for widely more generalized hybrid non-self mappings in a Hilbert space by using the idea of Hojo *et al.* (Fixed Point Theory 12:113-126, 2011) and Kawasaki and Takahashi fixed point theorems (J. Nonlinear Convex Anal. 14:71-87, 2013). Using these fixed point theorems for non-self mappings, we proved the Browder and Petryshyn fixed point theorem (J. Math. Anal. Appl. 20:197-228, 1967) for strict pseudo-contractive non-self mappings and the Kocourek *et al.* fixed point theorem (Taiwan. J. Math. 14:2497-2511, 2010) for super hybrid non-self mappings. In particular, we solve a fixed point problem.

MSC: Primary 47H10; secondary 47H05

Keywords: Hilbert space; nonexpansive mapping; nonspreading mapping; hybrid mapping; fixed point; non-self mapping

1 Introduction

Let $\mathbb{R}$ be the real line and let $[0, \frac{\pi}{2}]$ be a bounded, closed and convex subset of $\mathbb{R}$. Consider a mapping $T : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by

$$Tx = \left(1 + \frac{1}{2}x\right) \cos x - \frac{1}{2}x^2$$

for all $x \in [0, \frac{\pi}{2}]$. Such a mapping T has a unique fixed point $z \in [0, \frac{\pi}{2}]$ such that $\cos z = z$. What kind of fixed point theorems can we use to find such a unique fixed point z of T?

Let H be a real Hilbert space and let C be a non-empty subset of H. Kocourek, Takahashi and Yao [1] introduced a class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [2] and hybrid mappings [3]. A mapping $T : C \to H$ is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, *i.e.*,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, *i.e.*,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, *i.e.*,

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon type [4] for generalized hybrid mappings; see also Kohsaka and Takahashi [5] and Iemoto and Takahashi [6]. Very recently, Kawasaki and Takahashi [7] introduced a broader class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping T from C into H is called widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. In particular, an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [1] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is strict pseudo-contractive in the sense of Browder and Petryshyn [8] if $\alpha = 1$, $\beta = \gamma = 0$, $\delta = -1$, $\varepsilon = \zeta = 0$, $\eta = -k$, where $0 \leq k < 1$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive in general even if it has a fixed point. In [7], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon type [4] for such widely more generalized hybrid mappings in a Hilbert space. In particular, they proved directly the Browder and Petryshyn fixed point theorem [8] for strict pseudo-contractive mappings and the Kocourek, Takahashi and Yao fixed point theorem [1] for super hybrid mappings by using their fixed point theorems. However, we cannot use Kawasaki and Takahashi fixed point theorems to solve the above problem. For a nice synthesis on metric fixed point theory, see Kirk [9].

In this paper, motivated by such a problem, we prove fixed point theorems for widely more generalized hybrid non-self mappings in a Hilbert space by using the idea of Hojo, Takahashi and Yao [10] and Kawasaki and Takahashi fixed point theorems [7]. Using these fixed point theorems for non-self mappings, we prove the Browder and Petryshyn fixed point theorem [8] for strict pseudo-contractive non-self mappings and the Kocourek, Takahashi and Yao fixed point theorem [1] for super hybrid non-self mappings. In particular, we solve the above problem by using one of our fixed point theorems.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a (real) Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. From [11], we know the following basic equality: For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore, we know that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

Let C be a non-empty, closed and convex subset of H and let T be a mapping from C into H. Then we denote by $F(T)$ the set of fixed points of T. A mapping $S : C \to H$ is called super hybrid [1, 12] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha\|Sx-Sy\|^2 + (1-\alpha+\gamma)\|x-Sy\|^2 \\
&\quad\le \bigl(\beta+(\beta-\alpha)\gamma\bigr)\|Sx-y\|^2 + \bigl(1-\beta-(\beta-\alpha-1)\gamma\bigr)\|x-y\|^2 \\
&\qquad + (\alpha-\beta)\gamma\|x-Sx\|^2 + \gamma\|y-Sy\|^2
\end{aligned} \tag{2.3}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-super hybrid mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is (α, β)-generalized hybrid. Thus the class of super hybrid mappings contains generalized hybrid mappings. The following theorem was proved in [12]; see also [1].

Theorem 2.1 ([12]) *Let C be a non-empty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \ne -1$. Let S and T be mappings of C into H such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then S is (α, β, γ)-super hybrid if and only if T is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$. In particular, let C be a nonempty, closed and convex subset of H and let α, β and γ be real numbers with $\gamma \ge 0$. If a mapping $S : C \to C$ is (α, β, γ)-super hybrid, then the mapping $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ is an (α, β)-generalized hybrid mapping of C into itself.*

In [1], Kocourek, Takahashi and Yao also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

Theorem 2.2 ([1]) *Let C be a non-empty, bounded, closed and convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \ge 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping. Then S has a fixed point in C. In particular, if $S : C \to C$ is an (α, β)-generalized hybrid mapping, then S has a fixed point in C.*

A super hybrid mapping is not quasi-nonexpansive in general even if it has a fixed point. There exists a class of nonlinear mappings in a Hilbert space defined by Kawasaki and Takahashi [13] which covers contractive mappings and generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 \\
&\quad + \max\bigl\{\varepsilon\|x-Tx\|^2, \zeta\|y-Ty\|^2\bigr\} \le 0
\end{aligned}$$

for any $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid. Kawasaki and Takahashi [13] proved the following fixed point theorem.

Theorem 2.3 ([13]) *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *and* (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $\varepsilon + \alpha + \gamma > 0$, *or* $\zeta + \alpha + \beta > 0$.

Then T *has a fixed point if and only if there exists* $z \in C$ *such that* $\{T^n z \mid n = 0, 1, \dots\}$ *is bounded. In particular, a fixed point of* T *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$ *under the condition* (1).

Very recently, Kawasaki and Takahashi [7] also proved the following fixed point theorem which will be used in the proofs of our main theorems in this paper.

Theorem 2.4 ([7]) *Let* H *be a Hilbert space, let* C *be a non-empty, closed and convex subset of* H *and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping from* C *into itself, i.e., there exist* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ *such that*

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 + \varepsilon \|x - Tx\|^2 + \zeta \|y - Ty\|^2 + \eta \big\|(x - Tx) - (y - Ty)\big\|^2 \leq 0$$

for all $x, y \in C$. *Suppose that it satisfies the following condition* (1) *or* (2):
(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then T *has a fixed point if and only if there exists* $z \in C$ *such that* $\{T^n z \mid n = 0, 1, \dots\}$ *is bounded. In particular, a fixed point of* T *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$ *under the conditions* (1) *and* (2).

In particular, we have the following theorem from Theorem 2.4.

Theorem 2.5 *Let* H *be a Hilbert space, let* C *be a non-empty, bounded, closed and convex subset of* H *and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping from* C *into itself which satisfies the following condition* (1) *or* (2):
(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then T *has a fixed point. In particular, a fixed point of* T *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$ *under the conditions* (1) *and* (2).

3 Fixed point theorems for non-self mappings

In this section, using the fixed point theorem (Theorem 2.5), we first prove the following fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.1 *Let* C *be a non-empty, bounded, closed and convex subset of a Hilbert space* H *and let* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$. *Let* $T : C \to H$ *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2):
(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\alpha + \beta + \zeta + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\alpha + \gamma + \varepsilon + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

Assume that there exists a positive number $m > 1$ *such that for any* $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point in C. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) *and* (2).

Proof We give the proof for the case of (1). By the assumption, we have that for any $x \in C$, there exist $y \in C$ and t with $0 < t \leq m$ such that $Tx = x + t(y - x)$. From this, we have $Tx = ty + (1 - t)x$ and hence

$$y = \frac{1}{t}Tx + \frac{t-1}{t}x.$$

Define $Ux \in C$ as follows:

$$Ux = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}y = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}\left(\frac{1}{t}Tx + \frac{t-1}{t}x\right) = \frac{1}{m}Tx + \frac{m-1}{m}x.$$

Taking $\lambda > 0$ with $m = 1 + \lambda$, we have that

$$Ux = \frac{1}{1+\lambda}Tx + \frac{\lambda}{1+\lambda}x$$

and hence

$$T = (1 + \lambda)U - \lambda I. \tag{3.1}$$

Since $T : C \to H$ is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we have from (3.1) and (2.1) that for any $x, y \in C$,

$$\begin{aligned}
&\alpha \left\|(1+\lambda)Ux - \lambda x - \big((1+\lambda)Uy - \lambda y\big)\right\|^2 \\
&\quad + \beta \left\|x - \big((1+\lambda)Uy - \lambda y\big)\right\|^2 + \gamma \left\|(1+\lambda)Ux - \lambda x - y\right\|^2 + \delta\|x - y\|^2 \\
&\quad + \varepsilon \left\|x - \big((1+\lambda)Ux - \lambda x\big)\right\|^2 + \zeta \left\|(1+\lambda)Uy - \lambda y - y\right\|^2 \\
&\quad + \eta \left\|x - \big((1+\lambda)Ux - \lambda x\big) - \big(y - \big((1+\lambda)Uy - \lambda y\big)\big)\right\|^2 \\
&= \alpha \left\|(1+\lambda)(Ux - Uy) - \lambda(x - y)\right\|^2 \\
&\quad + \beta \left\|(1+\lambda)(x - Uy) - \lambda(x - y)\right\|^2 + \gamma \left\|(1+\lambda)(Ux - y) - \lambda(x - y)\right\|^2 \\
&\quad + \delta\|x - y\|^2 + \varepsilon \left\|(1+\lambda)(x - Ux)\right\|^2 + \zeta \left\|(1+\lambda)(y - Uy)\right\|^2 \\
&\quad + \eta \left\|(1+\lambda)(x - Ux) - (1+\lambda)(y - Uy)\right\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 - \alpha\lambda\|x - y\|^2 + \alpha\lambda(1+\lambda)\left\|x - y - (Ux - Uy)\right\|^2 \\
&\quad + \beta(1+\lambda)\|x - Uy\|^2 - \beta\lambda\|x - y\|^2 + \beta\lambda(1+\lambda)\|y - Uy\|^2 \\
&\quad + \gamma(1+\lambda)\|Ux - y\|^2 - \gamma\lambda\|x - y\|^2 + \gamma\lambda(1+\lambda)\|x - Ux\|^2 + \delta\|x - y\|^2 \\
&\quad + \varepsilon(1+\lambda)^2\|x - Ux\|^2 + \zeta(1+\lambda)^2\|y - Uy\|^2 \\
&\quad + \eta(1+\lambda)^2\left\|x - Ux - (y - Uy)\right\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 + \beta(1+\lambda)\|x - Uy\|^2 + \gamma(1+\lambda)\|Ux - y\|^2 \\
&\quad + (-\alpha\lambda - \beta\lambda - \gamma\lambda + \delta)\|x - y\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ (\gamma\lambda + \varepsilon\lambda + \varepsilon)(1+\lambda)\|x - Ux\|^2 + (\beta\lambda + \zeta\lambda + \zeta)(1+\lambda)\|y - Uy\|^2 \\
&+ (\alpha\lambda + \eta\lambda + \eta)(1+\lambda)\big\|x - y - (Ux - Uy)\big\|^2 \leq 0.
\end{aligned}
$$

This implies that U is widely more generalized hybrid. Since $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, $\alpha+\beta+\zeta+\eta \geq 0$ and $\zeta+\eta \geq 0$, we obtain that

$$
\begin{aligned}
&\alpha(1+\lambda) + \beta(1+\lambda) + \gamma(1+\lambda) - \alpha\lambda - \beta\lambda - \gamma\lambda + \delta = \alpha+\beta+\gamma+\delta \geq 0, \\
&\alpha(1+\lambda) + \gamma(1+\lambda) + (\gamma\lambda + \varepsilon\lambda + \varepsilon)(1+\lambda) + (\alpha\lambda + \eta\lambda + \eta)(1+\lambda) \\
&\quad = (1+\lambda)\big(\alpha+\gamma+\varepsilon+\eta + \lambda(\gamma+\varepsilon+\alpha+\eta)\big) \\
&\quad = (1+\lambda)^2(\alpha+\gamma+\varepsilon+\eta) > 0, \\
&(\beta\lambda + \zeta\lambda + \zeta)(1+\lambda) + (\alpha\lambda + \eta\lambda + \eta)(1+\lambda) \\
&\quad = \big((\alpha+\beta+\zeta+\eta)\lambda + \zeta + \eta\big)(1+\lambda) \geq 0.
\end{aligned}
$$

By Theorem 2.5, we obtain that $F(U) \neq \emptyset$. Therefore, we have from $F(U) = F(T)$ that $F(T) \neq \emptyset$. Suppose that $\alpha+\beta+\gamma+\delta > 0$. Let p_1 and p_2 be fixed points of T. We have that

$$
\begin{aligned}
&\alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\
&\qquad + \varepsilon\|p_1 - Tp_1\|^2 + \zeta\|p_2 - Tp_2\|^2 + \eta\big\|(p_1 - Tp_1) - (p_2 - Tp_2)\big\|^2 \\
&\quad = (\alpha+\beta+\gamma+\delta)\|p_1 - p_2\|^2 \leq 0
\end{aligned}
$$

and hence $p_1 = p_2$. Therefore, a fixed point of T is unique.

Similarly, we can obtain the desired result for the case when $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $\alpha+\gamma+\varepsilon+\eta \geq 0$ and $\varepsilon+\eta \geq 0$. This completes the proof. □

The following theorem is a useful extension of Theorem 3.1.

Theorem 3.2 *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2):

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, $\alpha+\beta+\zeta+\eta \geq 0$ *and* $[0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda + \zeta + \eta \geq 0\} \neq \emptyset$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $\alpha+\gamma+\varepsilon+\eta \geq 0$ *and* $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$.

Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ under the conditions (1) *and* (2).

Proof Let $\lambda \in [0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda + \zeta + \eta \geq 0\}$ and define $S = (1-\lambda)T + \lambda I$. Then S is a mapping from C into H. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T =$

$\frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned}
&\alpha\left\|\left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|^2 \\
&\quad + \beta\left\|x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|^2 + \gamma\left\|\left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - y\right\|^2 + \delta\|x-y\|^2 \\
&\quad + \varepsilon\left\|x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right)\right\|^2 + \zeta\left\|y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|^2 \\
&\quad + \eta\left\|\left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right)\right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right)\right\|^2 \\
&= \alpha\left\|\frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x-y)\right\|^2 \\
&\quad + \beta\left\|\frac{1}{1-\lambda}(x - Sy) - \frac{\lambda}{1-\lambda}(x-y)\right\|^2 \\
&\quad + \gamma\left\|\frac{1}{1-\lambda}(Sx - y) - \frac{\lambda}{1-\lambda}(x-y)\right\|^2 + \delta\|x-y\|^2 \\
&\quad + \varepsilon\left\|\frac{1}{1-\lambda}(x - Sx)\right\|^2 + \zeta\left\|\frac{1}{1-\lambda}(y - Sy)\right\|^2 \\
&\quad + \eta\left\|\frac{1}{1-\lambda}(x - Sx) - \frac{1}{1-\lambda}(y - Sy)\right\|^2 \\
&= \frac{\alpha}{1-\lambda}\|Sx - Sy\|^2 + \frac{\beta}{1-\lambda}\|x - Sy\|^2 \\
&\quad + \frac{\gamma}{1-\lambda}\|Sx - y\|^2 + \left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma) + \delta\right)\|x-y\|^2 \\
&\quad + \frac{\varepsilon + \gamma\lambda}{(1-\lambda)^2}\|x - Sx\|^2 + \frac{\zeta + \beta\lambda}{(1-\lambda)^2}\|y - Sy\|^2 \\
&\quad + \frac{\eta + \alpha\lambda}{(1-\lambda)^2}\big\|(x - Sx) - (y - Sy)\big\|^2 \le 0.
\end{aligned}$$

Therefore S is an $(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma) + \delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2})$-widely more generalized hybrid mapping. Furthermore, we obtain that

$$\begin{aligned}
&\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\gamma}{1-\lambda} - \frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma) + \delta = \alpha+\beta+\gamma+\delta \ge 0, \\
&\frac{\alpha}{1-\lambda} + \frac{\gamma}{1-\lambda} + \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} = \frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2} > 0, \\
&\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\zeta+\beta\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} = \frac{\alpha+\beta+\zeta+\eta}{(1-\lambda)^2} \ge 0, \\
&\frac{\zeta+\beta\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} = \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2} \ge 0.
\end{aligned}$$

Furthermore, from the assumption, there exists $m > 1$ such that for any $x \in C$,

$$Sx = (1-\lambda)Tx + \lambda x = (1-\lambda)\big(x + t(y-x)\big) + \lambda x = t(1-\lambda)(y-x) + x,$$

where $y \in C$ and $0 < t \leq m$. From $0 \leq \lambda < 1$, we have $0 < t(1-\lambda) \leq m$. Putting $s = t(1-\lambda)$, we have that there exists $m > 1$ such that for any $x \in C$,

$$Sx = x + s(y - x)$$

for some $y \in C$ and s with $0 < s \leq m$. Therefore, we obtain from Theorem 3.1 that $F(S) \neq \emptyset$. Since $F(S) = F(T)$, we obtain that $F(T) \neq \emptyset$.

Next, suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of T. As in the proof of Theorem 3.1, we have $p_1 = p_2$. Therefore a fixed point of T is unique.

In the case of $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\alpha + \gamma + \varepsilon + \eta \geq 0$ and $[0,1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$, we can obtain the desired result by replacing the variables x and y. □

Remark 1 We can also prove Theorems 3.1 and 3.2 by using the condition

$$-\beta - \delta + \varepsilon + \eta > 0, \quad \text{or} \quad -\gamma - \delta + \varepsilon + \eta > 0$$

instead of the condition

$$\alpha + \gamma + \varepsilon + \eta > 0, \quad \text{or} \quad \alpha + \beta + \zeta + \eta > 0,$$

respectively. In fact, in the case of the condition $-\beta - \delta + \varepsilon + \eta > 0$, we obtain from $\alpha + \beta + \gamma + \delta \geq 0$ that

$$0 < -\beta - \delta + \varepsilon + \eta \leq \alpha + \gamma + \varepsilon + \eta.$$

Thus we obtain the desired results by Theorems 3.1 and 3.2. Similarly, in the case of $-\gamma - \delta + \varepsilon + \eta > 0$, we can obtain the results by using the case of $\alpha + \beta + \zeta + \eta > 0$.

4 Fixed point theorems for well-known mappings

Using Theorem 3.1, we first show the following fixed point theorem for generalized hybrid non-self mappings in a Hilbert space; see also Kocourek, Takahashi and Yao [1].

Theorem 4.1 *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a generalized hybrid mapping from C into H, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Suppose $\alpha - \beta \geq 0$ and assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point.

Proof An (α, β)-generalized hybrid mapping T from C into H is an $(\alpha, 1-\alpha, -\beta, -(1-\beta), 0, 0, 0)$-widely more generalized hybrid mapping. Furthermore, $\alpha + (1-\alpha) - \beta - (1-\beta) =$

0, $\alpha + (1-\alpha) + 0 + 0 = 1 > 0$, $\alpha - \beta + 0 + 0 = \alpha - \beta \geq 0$ and $0 + 0 = 0$, that is, it satisfies the condition (2) in Theorem 3.1. Furthermore, since there exists $m \geq 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$, we obtain the desired result from Theorem 3.1. □

Using Theorem 3.1, we can also show the following fixed point theorem for widely generalized hybrid non-self mappings in a Hilbert space; see Kawasaki and Takahashi [13].

Theorem 4.2 *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon > 0$ *and* $\alpha + \beta \geq 0$;

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ *and* $\alpha + \gamma \geq 0$.

Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and $t \in \mathbb{R}$ with $0 < t \leq m$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) *and* (2).

Proof Since T is $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid, we obtain that

$$\begin{aligned} &\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \\ &\quad + \max\{\varepsilon \|x - Tx\|^2, \zeta \|y - Ty\|^2\} \leq 0 \end{aligned}$$

for any $x, y \in C$. In the case of $\alpha + \gamma + \varepsilon > 0$, from

$$\varepsilon \|x - Tx\|^2 \leq \max\{\varepsilon \|x - Tx\|^2, \zeta \|y - Ty\|^2\},$$

we obtain that

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 + \varepsilon \|x - Tx\|^2 \leq 0,$$

that is, it is an $(\alpha, \beta, \gamma, \delta, \varepsilon, 0, 0)$-widely more generalized hybrid mapping. Furthermore, we have that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + 0 = \alpha + \gamma + \varepsilon > 0$, $\alpha + \beta + 0 + 0 = \alpha + \beta \geq 0$ and $0 + 0 = 0$, that is, it satisfies the condition (1) in Theorem 3.1. Furthermore, since there exists $m \geq 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$, we obtain the desired result from Theorem 3.1. In the case of $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ and $\alpha + \gamma \geq 0$, we can obtain the desired result by replacing the variables x and y. □

We know that an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping with $\alpha = 1$, $\beta = \gamma = \varepsilon = \zeta = 0$, $\delta = -1$ and $\eta = -k \in (-1,0]$ is a strict pseudo-contractive mapping in the sense of Browder and Petryshyn [8]. We also define the following mapping: $T : C \to H$ is called a generalized strict pseudo-contractive mapping if there exist $r,k \in \mathbb{R}$ with $0 \le r \le 1$ and $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le r\|x - y\|^2 + k\big\|(x - Tx) - (y - Ty)\big\|^2$$

for any $x,y \in C$. Using Theorem 3.2, we can show the following fixed point theorem for generalized strict pseudo-contractive non-self mappings in a Hilbert space.

Theorem 4.3 *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a generalized strict pseudo-contractive mapping from C into H, that is, there exist $r,k \in \mathbb{R}$ with $0 \le r \le 1$ and $0 \le k < 1$ such that*

$$\|Tx - Ty\|^2 \le r\|x - y\|^2 + k\big\|(x - Tx) - (y - Ty)\big\|^2$$

for all $x,y \in C$. Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and $t \in \mathbb{R}$ with $0 < t \le m$. Then T has a fixed point. In particular, if $0 \le r < 1$, then T has a unique fixed point.

Proof A generalized strict pseudo-contractive mapping T from C into H is a $(1,0,0,-r,0,0,-k)$-widely more generalized hybrid mapping. Furthermore, $1 + 0 + 0 + (-r) \ge 0$, $1 + 0 + 0 + (-k) = 1 - k > 0$, $1 + 0 + 0 + (-k) = 1 - k > 0$ and $[0,1) \cap \{\lambda \mid (1+0)\lambda + 0 - k \ge 0\} = [k,1) \ne \emptyset$, that is, it satisfies the condition (1) in Theorem 3.2. Furthermore, since there exists $m \ge 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \le m$, we obtain the desired result from Theorem 3.2. In particular, if $0 \le r < 1$, then $1 + 0 + 0 + (-r) > 0$. We have from Theorem 3.2 that T has a unique fixed point. □

Let us consider the problem in the Introduction. A mapping $T : [0,\frac{\pi}{2}] \to \mathbb{R}$ was defined as follows:

$$Tx = \left(1 + \frac{1}{2}x\right)\cos x - \frac{1}{2}x^2 \tag{4.1}$$

for all $x \in [0,\frac{\pi}{2}]$. We have that

$$\begin{aligned} & Tx = \left(1 + \frac{1}{2}x\right)\cos x - \frac{1}{2}x^2 \\ \iff \quad & \frac{1}{1 + \frac{1}{2}x}Tx + \frac{\frac{1}{2}x}{1 + \frac{1}{2}x}x = \cos x. \end{aligned}$$

Thus we have that for any $x \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} &\frac{1+\frac{1}{2}x}{1+\pi}\left(\frac{1}{1+\frac{1}{2}x}Tx + \frac{\frac{1}{2}x}{1+\frac{1}{2}x}\right) + \left(1 - \frac{1+\frac{1}{2}x}{1+\pi}\right)x \\ &\quad = \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \left(1 - \frac{1+\frac{1}{2}x}{1+\pi}\right)x, \end{aligned}$$

and hence

$$\frac{1}{1+\pi}Tx + \frac{\pi}{1+\pi}x = \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi - \frac{1}{2}x}{1+\pi}x.$$

Using this, we also have from (2.1) that for any $x, y \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} &\left|\frac{1}{1+\pi}Tx + \frac{\pi}{1+\pi}x - \left(\frac{1}{1+\pi}Ty + \frac{\pi}{1+\pi}y\right)\right|^2 \\ &\quad = \left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi - \frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi - \frac{1}{2}y}{1+\pi}y\right)\right|^2 \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{1+\pi}|Tx - Ty|^2 + \frac{\pi}{1+\pi}|x-y|^2 - \frac{\pi}{(1+\pi)^2}\left|x - y - (Tx - Ty)\right|^2 \\ &\quad = \left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi - \frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi - \frac{1}{2}y}{1+\pi}y\right)\right|^2. \end{aligned} \tag{4.2}$$

Define a function $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ as follows:

$$f(x) = \frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi - \frac{1}{2}x}{1+\pi}x$$

for all $x \in [0, \frac{\pi}{2}]$. Then we have

$$f'(x) = \frac{\frac{1}{2}}{1+\pi}\cos x - \frac{1+\frac{1}{2}x}{1+\pi}\sin x + \frac{\pi}{1+\pi} - \frac{x}{1+\pi}$$

and

$$f''(x) = -\frac{1}{1+\pi}\sin x - \frac{1+\frac{1}{2}x}{1+\pi}\cos x - \frac{1}{1+\pi}.$$

Since

$$f'(0) = \frac{\frac{1}{2}+\pi}{1+\pi}, \qquad f'\left(\frac{\pi}{2}\right) = \frac{-1+\frac{1}{4}\pi}{1+\pi}$$

and $f''(x) < 0$ for all $x \in [0, \frac{\pi}{2}]$, we have from the mean value theorem that there exists a positive number r with $0 < r < 1$ such that

$$\left|\frac{1+\frac{1}{2}x}{1+\pi}\cos x + \frac{\pi - \frac{1}{2}x}{1+\pi}x - \left(\frac{1+\frac{1}{2}y}{1+\pi}\cos y + \frac{\pi - \frac{1}{2}y}{1+\pi}y\right)\right|^2 \le r|x-y|^2$$

for all $x, y \in [0, \frac{\pi}{2}]$. Therefore, we have from (4.2) that

$$\frac{1}{1+\pi}|Tx-Ty|^2 + \frac{\pi}{1+\pi}|x-y|^2 \leq r|x-y|^2 + \frac{\pi}{(1+\pi)^2}\bigl|x-y-(Tx-Ty)\bigr|^2$$

for all $x, y \in [0, \frac{\pi}{2}]$. Furthermore, we have from (4.1) that

$$Tx = \left(1 + \frac{1}{2}x\right)(\cos x - x) + x$$

for all $x \in [0, \frac{\pi}{2}]$. Take $m = 1 + \pi$ and let $t = 1 + \frac{1}{2}x$ and $y = \cos x$ for all $x \in [0, \frac{\pi}{2}]$. Then we have that

$$Tx = t(y-x) + x, y = \cos x \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad 0 < t = 1 + \frac{1}{2}x \leq 1 + \pi.$$

Using Theorem 3.2, we have that T has a unique fixed point $z \in [0, \frac{\pi}{2}]$. We also know that $z = Tz$ is equivalent to $\cos z = z$. In fact,

$$\begin{aligned} z = Tz \quad &\Longleftrightarrow \quad z = \left(1 + \frac{1}{2}z\right)(\cos z - z) + z \\ &\Longleftrightarrow \quad 0 = \left(1 + \frac{1}{2}z\right)(\cos z - z) \\ &\Longleftrightarrow \quad 0 = \cos z - z. \end{aligned}$$

Using Theorem 3.2, we can also show the following fixed point theorem for super hybrid non-self mappings in a Hilbert space; see [1].

Theorem 4.4 *Let H be a Hilbert space, let C be a non-empty, bounded, closed and convex subset of H and let T be a super hybrid mapping from C into H, that is, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that*

$$\begin{aligned} &\alpha\|Tx-Ty\|^2 + (1-\alpha+\gamma)\|x-Ty\|^2 \\ &\quad \leq \bigl(\beta + (\beta-\alpha)\gamma\bigr)\|Tx-y\|^2 + \bigl(1-\beta-(\beta-\alpha-1)\gamma\bigr)\|x-y\|^2 \\ &\qquad + (\alpha-\beta)\gamma\|x-Tx\|^2 + \gamma\|y-Ty\|^2 \end{aligned}$$

for all $x, y \in C$. Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y-x)$$

for some $y \in C$ and t with $0 < t \leq m$. Suppose that $\alpha - \beta \geq 0$ or $\gamma \geq 0$. Then T has a fixed point.

Proof An (α, β, γ)-super hybrid mapping T from C into H is an $(\alpha, 1-\alpha+\gamma, -\beta-(\beta-\alpha)\gamma, -1+\beta+(\beta-\alpha-1)\gamma, -(\alpha-\beta)\gamma, -\gamma, 0)$-widely more generalized hybrid mapping. Furthermore, $\alpha + (1-\alpha+\gamma) + (-\beta-(\beta-\alpha)\gamma) + (-1+\beta+(\beta-\alpha-1)\gamma) = 0$, $\alpha + (1-\alpha+\gamma) + (-\gamma) + 0 = 1 > 0$ and $\alpha - \beta - (\beta-\alpha)\gamma - (\alpha-\beta)\gamma + 0 = \alpha - \beta \geq 0$, that is, it satisfies the conditions $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\alpha + \gamma + \varepsilon + \eta \geq 0$ in (2) of Theorem 3.2. Moreover, we have that

$$\begin{aligned}&[0,1) \cap \bigl\{\lambda \mid \bigl(\alpha + \bigl(-\beta - (\beta - \alpha)\gamma\bigr)\bigr)\lambda + \bigl(-(\alpha - \beta)\gamma\bigr) + 0 \geq 0\bigr\} \\ &\quad = [0,1) \cap \bigl\{\lambda \mid (\alpha - \beta)\bigl((1+\gamma)\lambda - \gamma\bigr) \geq 0\bigr\}.\end{aligned}$$

If $\alpha - \beta > 0$, then

$$\begin{aligned}[0,1) \cap \bigl\{\lambda \mid (\alpha - \beta)\bigl((1+\gamma)\lambda - \gamma\bigr) \geq 0\bigr\} &= [0,1) \cap \bigl\{\lambda \mid (1+\gamma)\lambda - \gamma \geq 0\bigr\} \\ &= \begin{cases} [0,1) & \text{if } \gamma < 0, \\ [\frac{\gamma}{1+\gamma}, 1) & \text{if } \gamma \geq 0 \end{cases} \\ &\neq \emptyset,\end{aligned}$$

that is, it satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. If $\alpha - \beta = 0$, then

$$[0,1) \cap \bigl\{\lambda \mid (\alpha - \beta)\bigl((1+\gamma)\lambda - \gamma\bigr) \geq 0\bigr\} = [0,1) \neq \emptyset,$$

that is, it satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. If $\alpha - \beta < 0$ and $\gamma \geq 0$, then

$$\begin{aligned}&[0,1) \cap \bigl\{\lambda \mid (\alpha - \beta)\bigl((1+\gamma)\lambda - \gamma\bigr) \geq 0\bigr\} \\ &\quad = [0,1) \cap \bigl\{\lambda \mid (1+\gamma)\lambda - \gamma \leq 0\bigr\} \\ &\quad = \left[0, \frac{\gamma}{1+\gamma}\right] \neq \emptyset,\end{aligned}$$

that is, it again satisfies the condition $[0,1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ in (2) of Theorem 3.2. Then we obtain the desired result from Theorem 3.2. Similarly, we obtain the desired result from Theorem 3.2 in the case of (1). □

We remark that some recent results related to this paper have been obtained in [14–17].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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