SOME GEOMETRIC PROPERTIES INHERITED BY THE POSITIVE TENSOR PRODUCTS OF ATOMIC BANACH LATTICES

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ABSTRACT. For Banach lattices X and Y, let $X\hat{\otimes}_{|\pi|}Y$ and $X\check{\otimes}_{|\varepsilon|}Y$ denote the positive projective and injective tensor products of X and Y, respectively. In this paper, we characterize the inheritance of reflexivity and containment of copies of c_0, ℓ_1, ℓ_∞ by $X\hat{\otimes}_{|\pi|}Y$ and $X\check{\otimes}_{|\varepsilon|}Y$ from X and Y, when one of them is an atomic Banach lattice. In this case, we also give an affirmative answer to an open question of Jeurnink.

1. Introduction.

Fremlin [9, 10] and Wittstock [18, 19] in 1970's introduced and investigated the positive projective tensor product $X\hat{\otimes}_{|\pi|}Y$ and the positive injective tensor product $X\check{\otimes}_{|\varepsilon|}Y$ of Banach lattices X and Y, respectively. One of the interesting questions about $X\hat{\otimes}_{|\pi|}Y$ and $X\check{\otimes}_{|\varepsilon|}Y$ is what geometric properties of the Banach lattices X and Y are inherited by $X\hat{\otimes}_{|\pi|}Y$ and $X\check{\otimes}_{|\varepsilon|}Y$. Fremlin [10] in 1974 showed that $L_2[0,1]\hat{\otimes}_{|\pi|}L_2[0,1]$ is not Dedekind complete and then Schep [17] in 1984 showed that $L_p[0,1]\hat{\otimes}_{|\pi|}L_q[0,1]$ $(1 < p, q < \infty, 1/p + 1/q = 1)$ is not Dedekind complete. Thus the Radon-Nikodym property is not inherited by $L_p[0,1]\hat{\otimes}_{|\pi|}L_q[0,1]$ $(1 < p, q < \infty, 1/p + 1/q = 1)$. Nevertheless, Bu and Buskes [2] in 2009 showed that the Radon-Nikodym property is inherited by $X\hat{\otimes}_{|\pi|}Y$ whenever one of X and Y is an atomic Banach lattice. It seems that in this case some geometric properties of Banach lattices can be inherited by their positive tensor products. For instance, the reflexivity, the property of containment of copies of c_0, ℓ_1, ℓ_∞, and etc., are inherited by $\ell_\varphi\hat{\otimes}_{|\pi|}X$ and $\ell_\varphi\check{\otimes}_{|\varepsilon|}X$ where ℓ_φ is an Orlicz sequence space (see [3, 4, 13]).

Let E be an atomic Banach lattice and X be any Banach lattice. The sequential representation $\lambda_{\pi,0}(X)$ of the positive projective tensor product $E\hat{\otimes}_{|\pi|}X$ was given in [2]. In section 5 of this paper, we give a sequential representation $\lambda_{\varepsilon,0}(X)$ of the positive injective

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tensor product $E\check{\otimes}_{|\varepsilon|}X$. After discussing the reflexivity and the containment of copies of c_0, ℓ_1, ℓ_∞ of $\lambda_{\pi,0}(X)$ and $\lambda_{\varepsilon,0}(X)$ in sections 3 and 4, we obtain characterizations of the inheritance of the reflexivity and the containment of copies of c_0, ℓ_1, ℓ_∞ by $E\hat{\otimes}_{|\pi|}X$ and $E\check{\otimes}_{|\varepsilon|}X$ in section 6. As a consequence, we obtain the interesting example which shows that $\mathcal{K}^r(\ell_p, \ell_q)(1 < p \le q < \infty)$, the space generated by positive compact operators from ℓ_p into ℓ_q, contains a sublattice isomorphic to c_0 but does not contain a sublattice isomorphic to ℓ_∞.

Jeurnink pointed out in [12, chapter 4] that in $X^* \otimes Y^*$, the positive projective tensor norm $\|\cdot\|_{|\pi|}$ is greater than or equal to the dual norm of the positive injective tensor norm $\|\cdot\|^*_{|\varepsilon|}$, and he asked the question whether $\|\cdot\|_{|\pi|}$ is equal to $\|\cdot\|^*_{|\varepsilon|}$ in $X^* \otimes Y^*$. In section 6 of this paper, we give an affirmative answer to this question in the case that one of X and Y is an atomic Banach lattice.

For a vector lattice X, let X^+ be its positive cone. The X-valued sequence space $X^{\mathbb{N}}$ is a vector lattice with the order and the lattice operations defined coordinatewise. For each $\bar{x} = (x_i)_i \in X^{\mathbb{N}}$ and each $n \in \mathbb{N}$, let

$$\bar{x}(\le n) = (x_1, \cdots, x_n, 0, 0, \cdots) \quad \text{and} \quad \bar{x}(\ge n) = (0, \cdots, 0, x_n, x_{n+1}, \cdots).$$

For a Banach lattice X, let X^* be its topological dual and B_X be its closed unit ball. For Banach lattices X and Y, let $\mathcal{L}^r(X, Y)$ denote the space of all regular linear operators from X to Y with the usual regular operator norm $\|\cdot\|_r$. If, in addition, Y is Dedekind complete then $\mathcal{L}^r(X, Y)$ is a Banach lattice with $\|T\|_r = \|\,|T|\,\|$ for every $T \in \mathcal{L}^r(X, Y)$. Let $\mathcal{K}^r(X, Y)$ denote the linear span of positive compact operators from X to Y.

2. Banach Lattice-valued Sequence Spaces.

Let λ be a solid sequence space, that is, a subspace of $\mathbb{R}^{\mathbb{N}}$ such that $(a_i)_i \in \lambda$ whenever $|a_i| \le |b_i|$ for all $i \in \mathbb{N}$ and $(b_i)_i \in \lambda$. The *Köthe dual* of λ is defined by

$$\lambda' = \Big\{(b_i)_i \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |a_i b_i| < +\infty, \ \forall\, (a_i)_i \in \lambda\Big\}.$$

In addition, if λ is a Banach lattice then $\lambda' \subseteq \lambda^*$. Thus λ' with the norm induced by λ^* is also a Banach lattice. From now on we always assume that λ is a Banach sequence lattice such that

$$\lambda'' = \lambda \quad \text{and} \quad \|e_i\|_\lambda = 1, \ \forall i \in \mathbb{N}.$$

Here, e_i's are the standard unit vectors in the sequence space λ. Note that if λ is reflexive then both λ and λ' are σ-order continuous.

Define

$$\lambda_\varepsilon(X) = \Big\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : (x^*(|x_i|))_i \in \lambda, \ \ \forall\, x^* \in X^{*+}\Big\}$$

and

$$\|\bar{x}\|_{\lambda_\varepsilon(X)} = \sup\Big\{\|(x^*(|x_i|))_i\|_\lambda : \ x^* \in B_{X^{*+}}\Big\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_\varepsilon(X).$$

Then $\lambda_\varepsilon(X)$ is a Banach lattice (see [5]). (By the Principle of Local Reflexivity for Banach lattices due to Bernau [1] mentioned in section 3 below, and similar to the proof of Proposition 4.3 in [2], we can prove that $\lambda_\varepsilon(X^*)$ defined here is the same as that $\lambda_\varepsilon(X^*)$ defined in [2] if λ' is replaced by λ there.) Let $\lambda_{\varepsilon,0}(X)$ denote the closed sublattice of $\lambda_\varepsilon(X)$ consisting of all such elements of $\lambda_\varepsilon(X)$ whose tails converge to 0, i.e.

$$\lambda_{\varepsilon,0}(X) = \Big\{\bar{x} \in \lambda_\varepsilon(X) : \ \lim_n \|\bar{x}(\geq n)\|_{\lambda_\varepsilon(X)} = 0\Big\}.$$

Then $\lambda_{\varepsilon,0}(X)$ is an ideal of $\lambda_\varepsilon(X)$. We denote $\lambda_{\varepsilon,0}(\mathbb{R})$ by λ_0.

For each $\bar{x} = (x_i)_i \in \lambda_\varepsilon(X)$, define a linear operator $T_{\bar{x}}$ from $(\lambda')_0$ to X by

$$T_{\bar{x}}(a) = \sum_{i=1}^{\infty} a_i x_i, \qquad \forall\, a = (a_i)_i \in (\lambda')_0.$$

Then we have the following proposition due to [5].

Proposition 2.1. *If X is Dedekind complete then $\lambda_\varepsilon(X)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r((\lambda')_0, X)$ under the mapping $\bar{x} \mapsto T_{\bar{x}}$. Moreover, if λ is σ-order continuous then $T_{\bar{x}} \in \mathcal{K}^r((\lambda')_0, X)$ if and only if $\bar{x} \in \lambda_{\varepsilon,0}(X)$.*

Define

$$\lambda_\pi(X) = \Big\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} x_i^*(|x_i|) < +\infty, \ \forall (x_i^*)_i \in \lambda'_\varepsilon(X^*)^+\Big\}$$

and

$$\|\bar{x}\|_{\lambda_\pi(X)} = \sup\Big\{\sum_{i=1}^{\infty} x_i^*(|x_i|) : \ (x_i^*)_i \in B_{\lambda'_\varepsilon(X^*)^+}\Big\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_\pi(X).$$

Then $\lambda_\pi(X)$ is a Banach lattice (see [2]). Let $\lambda_{\pi,0}(X)$ denote the closed sublattice of $\lambda_\pi(X)$ consisting of all such elements of $\lambda_\pi(X)$ whose tails converge to 0, i.e.

$$\lambda_{\pi,0}(X) = \Big\{\bar{x} \in \lambda_\pi(X) : \ \lim_n \|\bar{x}(\geq n)\|_{\lambda_\pi(X)} = 0\Big\}.$$

Then $\lambda_{\pi,0}(X)$ is an ideal of $\lambda_\pi(X)$.

Bu and Buskes [2] asked the question whether $\lambda_{\pi,0}(X) = \lambda_\pi(X)$. We will give an affirmative answer to this question in the following Proposition 2.2. To prove this proposition, we need the following vector-valued sequence spaces $\lambda_w(X)$ and $\lambda_s(X)$ introduced in [2]. Define

$$\lambda_w(X) = \lambda_{weak}(X) = \Big\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : (x^*(x_i))_i \in \lambda,\ \ \forall\, x^* \in X^*\Big\}$$

and

$$\|\bar{x}\|_{\lambda_w(X)} = \sup\Big\{\|(x^*(x_i))_i\|_\lambda :\ x^* \in B_{X^*}\Big\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_w(X).$$

Then $\lambda_w(X)$ is a Banach space (it may not be a Banach lattice). Define

$$\lambda_s(X) = \lambda_{strong}(X) = \Big\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^\infty |x_i^*(x_i)| < +\infty,\ \forall (x_i^*)_i \in \lambda'_w(X^*)\Big\}$$

and

$$\|\bar{x}\|_{\lambda_s(X)} = \sup\Big\{\Big|\sum_{i=1}^\infty x_i^*(x_i)\Big| :\ (x_i^*)_i \in B_{\lambda'_w(X^*)}\Big\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_s(X).$$

Then $\lambda_s(X)$ is a Banach space (it may not be a Banach lattice).

Proposition 2.2. *If λ is σ-order continuous then $\lambda_{\pi,0}(X) = \lambda_\pi(X)$.*

Proof. Take any $\bar{x} = (x_i)_i \in \lambda_\pi(X)$. Assume, without loss of generality, that $\bar{x}$ is positive. For each $n \in \mathbb{N}$ we have $\bar{x}(\leq n) = (x_1, \cdots, x_n, 0, 0, \cdots) \in \lambda_s(X)$. Since $\bar{x}(\leq n)$ is positive,

$$\|\bar{x}(\leq n)\|_{\lambda_s(X)} = \|\bar{x}(\leq n)\|_{\lambda_\pi(X)} \leq \|\bar{x}\|_{\lambda_\pi(X)}, \quad n = 1, 2, \cdots.$$

For any $\bar{x}^* = (x_i^*)_i \in \lambda'_w(X^*)$ and any $n \in \mathbb{N}$,

$$\sum_{i=1}^n |x_i^*(x_i)| \leq \|\bar{x}^*(\leq n)\|_{\lambda'_w(X^*)} \cdot \|\bar{x}(\leq n)\|_{\lambda_s(X)} \leq \|\bar{x}^*\|_{\lambda'_w(X^*)} \cdot \|\bar{x}\|_{\lambda_\pi(X)}.$$

Then $\sum_{i=1}^\infty |x_i^*(x_i)| < \infty$ and hence $\bar{x} \in \lambda_s(X)$. It follows from [2, Proposition 5.2] that $\bar{x} \in \lambda_{\pi,0}(X)$. □

3. Duality and Reflexivity.

To characterize the dual of $\lambda_{\varepsilon,0}(X)$ and the dual of $\lambda_{\pi,0}(X)$, we need the following Principle of Local Reflexivity for Banach lattices due to Bernau [1].

Principle of Local Reflexivity [1]. *Let X be a Banach lattice and J be the canonical injection of X into X^{**}. Suppose $\varepsilon > 0$ and V is a weak* neighborhood of 0 in X^{**}. If G is a finite dimensional sublattice of X^{**} then there is a lattice isomorphism $T : G \longrightarrow T[G] \hookrightarrow X$ such that $\|T\| < 1+\varepsilon$, $\|T^{-1}\| < 1+\varepsilon$, and $x^{**} - J(Tx^{**}) \in \|x^{**}\|V$ for all $x^{**} \in G$.*

Theorem 3.1. *$\lambda_{\varepsilon,0}(X)^*$ is isometrically isomorphic and lattice homomorphic to $\lambda'_\pi(X^*)$.*

Proof. Define $\psi : \lambda'_\pi(X^*) \longrightarrow \lambda_{\varepsilon,0}(X)^*$ by

$$\langle \bar{x}, \psi(\bar{x}^*)\rangle = \sum_{i=1}^{\infty} x_i^*(x_i) \tag{3.1}$$

for each $\bar{x} = (x_i)_i \in \lambda_{\varepsilon,0}(X)$ and each $\bar{x}^* = (x_i^*)_i \in \lambda'_\pi(X^*)$. Then ψ is linear and $\|\psi(\bar{x}^*)\| \le \|\bar{x}^*\|_{\lambda'_\pi(X^*)}$.

On the other hand, take any $\xi \in \lambda_{\varepsilon,0}(X)^*$. For each $i \in \mathbb{N}$, define a linear functional x_i^* on X by

$$x_i^*(x) = \langle (0, \cdots, 0, \overset{i\text{-th place}}{x}, 0, 0, \cdots), \xi\rangle$$

for each $x \in X$. Then $x_i^* \in X^*$. Moreover, for each $x \in X^+$ we have

$$\begin{aligned}
|x_i^*|(x) &= \sup\Big\{|x_i^*(y)| : 0 \le y \le x\Big\} \\
&= \sup\Big\{|\langle (0, \cdots, 0, \overset{i\text{-th place}}{y}, 0, 0, \cdots), \xi\rangle| : 0 \le y \le x\Big\} \\
&= \sup\Big\{|\langle \bar{y}, \xi\rangle| : 0 \le \bar{y} \le (0, \cdots, 0, \overset{i\text{-th place}}{x}, 0, 0, \cdots)\Big\} \\
&= \langle (0, \cdots, 0, \overset{i\text{-th place}}{x}, 0, 0, \cdots), |\xi|\rangle.
\end{aligned}$$

Thus, for each $\bar{x} = (x_i)_i \in \lambda_{\varepsilon,0}(X)^+$ we have

$$\sum_{i=1}^{n} x_i^*(x_i) = \langle (x_i)_1^n, \xi\rangle \quad \text{and} \quad \sum_{i=1}^{n} |x_i^*|(x_i) = \langle (x_i)_1^n, |\xi|\rangle. \tag{3.2}$$

Now take any $(x_i^{**})_i \in \lambda_\varepsilon(X^{**})^+$. For each $\varepsilon > 0$ and each $n \in \mathbb{N}$, let G be the sublattice of X^{**} generated by $\{x_i^{**} : i = 1, 2, \cdots, n\}$ and

$$V = \Big\{x^{**} \in X^{**} : \Big|x^{**}(|x_i^*|)\Big| < \varepsilon/a,\ i = 1, 2, \cdots, n\Big\},$$

where $a = \sum_{i=1}^n \|x_i^{**}\|$. By the Principle of Local Reflexivity, there is a lattice isomorphism $T : G \longrightarrow X$ such that $\|T\| < 1+\varepsilon$ and

$$\Big|x_i^{**}(|x_i^*|) - |x_i^*|(Tx_i^{**})\Big| < \varepsilon\|x_i^{**}\|/a, \qquad i = 1, 2, \cdots, n.$$

Then

$$\begin{aligned}
\left\|\left(Tx_i^{**}\right)_1^n\right\|_{\lambda_{\varepsilon,0}(X)} &= \sup\left\{\left\|\left(x^*(Tx_i^{**})\right)_1^n\right\|_\lambda : x^* \in B_{X^{*+}}\right\}\\
&\le \sup\left\{\left\|T\right\|\cdot\left\|\left(x_i^{**}(x^*)\right)_1^n\right\|_\lambda : x^* \in B_{X^{*+}}\right\}\\
&\le (1+\varepsilon)\cdot\left\|(x_i^{**})_i\right\|_{\lambda_\varepsilon(X^{**})}.
\end{aligned}$$

By (3.2) we have

$$\begin{aligned}
\sum_{i=1}^n \left|x_i^{**}(|x_i^*|)\right| &\le \sum_{i=1}^n \left|x_i^{**}(|x_i^*|) - |x_i^*|(Tx_i^{**})\right| + \sum_{i=1}^n |x_i^*|(Tx_i^{**})\\
&\le \varepsilon + \langle (Tx_i^{**})_1^n, |\xi|\rangle\\
&\le \varepsilon + \|\xi\|\cdot\left\|(Tx_i^{**})_1^n\right\|_{\lambda_{\varepsilon,0}(X)}\\
&\le \varepsilon + (1+\varepsilon)\|\xi\|\cdot\left\|(x_i^{**})_i\right\|_{\lambda_\varepsilon(X^{**})}.
\end{aligned}$$

Therefore

$$\sum_{i=1}^\infty \left|x_i^{**}(|x_i^*|)\right| \le \varepsilon + (1+\varepsilon)\|\xi\|\cdot\ \left\|(x_i^{**})_i\right\|_{\lambda_\varepsilon(X^{**})}.$$

It follows that $\bar{x}^* := (x_i^*)_i \in \lambda'_\pi(X^*)$ and $\|\bar{x}^*\|_{\lambda'_\pi(X^*)} \le \|\xi\|$. For each $\bar{x} = (x_i)_i \in \lambda_{\varepsilon,0}(X)^+$, by (3.1) and (3.2) we have

$$\langle \bar{x}, \psi(\bar{x}^*)\rangle = \lim_n \sum_{i=1}^n x_i^*(x_i) = \lim_n \langle (x_i)_1^n, \xi\rangle = \langle \bar{x}, \xi\rangle.$$

Thus $\psi(\bar{x}^*) = \xi$, and hence ψ is surjective. Moreover,

$$\|\psi(\bar{x}^*)\| \le \|\bar{x}^*\|_{\lambda'_\pi(X^*)} \le \|\xi\| = \|\psi(\bar{x}^*)\|$$

and ψ is an isometry. Furthermore, by (3.1) and (3.2) again we have

$$\langle \bar{x}, |\psi(\bar{x}^*)|\rangle = \langle \bar{x}, |\xi|\rangle = \lim_n \langle (x_i)_1^n, |\xi|\rangle = \lim_n \sum_{i=1}^n |x_i^*|(x_i) = \langle \bar{x}, \psi(|\bar{x}^*|)\rangle.$$

Therefore $|\psi(\bar{x}^*)| = \psi(|\bar{x}^*|)$ and ψ is a lattice homomorphism. □

Similar to the proof of Theorem 3.1 we have the following Theorem 3.2.

Theorem 3.2. $\lambda_{\pi,0}(X)^*$ *is isometrically isomorphic and lattice homomorphic to* $\lambda'_\varepsilon(X^*)$.

Theorem 3.3. *Let* λ *and* X *be reflexive. Then*

(i) $\lambda_{\pi,0}(X)$ *is reflexive if and only if* $\lambda'_\varepsilon(X^*) = \lambda'_{\varepsilon,0}(X^*)$.

(ii) $\lambda_{\varepsilon,0}(X)$ *is reflexive if and only if* $\lambda_\varepsilon(X) = \lambda_{\varepsilon,0}(X)$.

Proof. By Proposition 2.2 we have $\lambda_{\pi,0}(X) = \lambda_\pi(X)$ and $\lambda'_{\pi,0}(X^*) = \lambda'_\pi(X^*)$. (i) If $\lambda'_\varepsilon(X^*) = \lambda'_{\varepsilon,0}(X^*)$ then by Theorems 3.1 and 3.2,

$$\lambda_{\pi,0}(X)^{**} = [\lambda'_\varepsilon(X^*)]^* = [\lambda'_{\varepsilon,0}(X^*)]^* = \lambda''_\pi(X^{**}) = \lambda_\pi(X) = \lambda_{\pi,0}(X).$$

It follows that $\lambda_{\pi,0}(X)$ is reflexive. On the other hand, if $\lambda_{\pi,0}(X)$ is reflexive then by Theorems 3.1 and 3.2 again,

$$\lambda'_{\varepsilon,0}(X^*)^* = \lambda''_\pi(X^{**}) = \lambda_\pi(X) = \lambda_{\pi,0}(X) = \lambda_{\pi,0}(X)^{**} = \lambda'_\varepsilon(X^*)^*.$$

It follows that $\lambda'_\varepsilon(X^*) = \lambda'_{\varepsilon,0}(X^*)$. (ii) follows from the following fact:

$$\lambda_{\varepsilon,0}(X)^{**} = \lambda'_\pi(X^*)^* = \lambda'_{\pi,0}(X^*)^* = \lambda''_\varepsilon(X^{**}) = \lambda_\varepsilon(X).$$

□

4. Containment of copies of c_0, ℓ_∞, and ℓ_1.

It is known from [14, p. 92, Theorem 2.4.12] that a Banach lattice contains no sublattice isomorphic to c_0 if and only if it is a KB-space. In this case, it is also σ-order continuous. For completeness we mention [2, Theorem 5.5] and [5, Theorem 7] as the following proposition.

Proposition 4.1. (i) $\lambda_{\pi,0}(X)$ *contains no sublattice isomorphic to* c_0 *if and only if both* λ *and* X *contain no sublattice isomorphic to* c_0.

(ii) $\lambda_{\varepsilon,0}(X)$ *contains no sublattice isomorphic to* c_0*, if and only if* $\lambda_\varepsilon(X)$ *contains no sublattice isomorphic to* c_0*, if and only if both* λ *and* X *contain no sublattice isomorphic to* c_0 *and* $\lambda_\varepsilon(X) = \lambda_{\varepsilon,0}(X)$.

Note that $((\lambda')_0)^* = ((\lambda')_0)' = \lambda'' = \lambda$. Then the fact that λ contains no sublattice isomorphic to ℓ_∞ is equivalent to the fact that λ contains no sublattice isomorphic to c_0. In this case, λ is σ-order continuous and hence $\lambda_{\pi,0}(X^*) = \lambda_\pi(X^*)$ by Proposition 2.2. It follows from Theorems 3.1 and 3.2 that $\lambda_\pi(X^*) = \lambda'_{\varepsilon,0}(X)^*$ and $\lambda_\varepsilon(X^*) = \lambda'_{\pi,0}(X)^*$. Thus by Proposition 4.1 we have the following.

Theorem 4.2. (i) $\lambda_{\pi,0}(X^*)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if both* λ *and* X^* *contain no sublattice isomorphic to* ℓ_∞.

(ii) $\lambda_\varepsilon(X^*)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if both* λ *and* X^* *contain no sublattice isomorphic to* ℓ_∞ *and* $\lambda_\varepsilon(X^*) = \lambda_{\varepsilon,0}(X^*)$.

It is known from [14, p. 83, Proposition 2.3.12] that a Banach lattice contains a sublattice isomorphic to ℓ_1 if and only if its dual contains a sublattice isomorphic to ℓ_∞. Note that $(\lambda_0)^* = \lambda'$, and by Theorems 3.1 and 3.2, that $\lambda_{\pi,0}(X)^* = \lambda'_\varepsilon(X^*)$ and $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$. By Theorem 4.2 we have the following.

Theorem 4.3. (i) $\lambda_{\pi,0}(X)$ *contains no sublattice isomorphic to* ℓ_1 *if and only if both* λ_0 *and* X *contain no sublattice isomorphic to* ℓ_1 *and* $\lambda'_\varepsilon(X^*) = \lambda'_{\varepsilon,0}(X^*)$.

(ii) $\lambda_{\varepsilon,0}(X)$ *contains no sublattice isomorphic to* ℓ_1 *if and only if both* λ_0 *and* X *contain no sublattice isomorphic to* ℓ_1.

For an infinite subset M of $\mathbb{N}$, let $\ell_\infty(M)$ denote the subspace of ℓ_∞ consisting of all $(\xi_n)_n \in \ell_\infty$ with $\xi_n = 0$ for $n \notin M$. It is known from [6, p. 13, Remark 1.3.2] that if an operator $T : \ell_\infty \longrightarrow Z$ is weakly compact then for all $\xi = (\xi_n)_n \in \ell_\infty$, the series $\sum_n \xi_n T(e_n)$ converges in norm in Z. But its limit $\sum_{n=1}^\infty \xi_n T(e_n)$ and $T(\xi)$ may not coincide. To get the main result in this section, we need the following result due to Drewnowski [8] (also see [6, p. 14, Corollary 1.3.3]).

Lemma 4.4 [8]**.** *Let* Z *be a Banach space and let* $T_i : \ell_\infty \longrightarrow Z$ *be weakly compact operators for each* $i \in \mathbb{N}$. *Then there exists an infinite subset* M *of* $\mathbb{N}$ *such that*

$$T_i(\xi) = \sum_{n=1}^{\infty} \xi_n T_i(e_n), \quad \forall \xi = (\xi_n)_n \in \ell_\infty(M),\ \forall i \in \mathbb{N}.$$

Theorem 4.5. *Let* λ' *be* σ*-order continuous. Then* $\lambda_{\varepsilon,0}(X)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if* X *contains no sublattice isomorphic to* ℓ_∞.

Proof. Since X is a closed sublattice of $\lambda_{\varepsilon,0}(X)$, it follows that $\lambda_{\varepsilon,0}(X)$ contains a sublattice isomorphic to ℓ_∞ whenever X contains a sublattice isomorphic to ℓ_∞. Now suppose that X contains no sublattice isomorphic to ℓ_∞ but $\lambda_{\varepsilon,0}(X)$ contains a sublattice isomorphic to ℓ_∞. Then there is an isomorphism $T : \ell_\infty \longrightarrow T(\ell_\infty) \hookrightarrow \lambda_{\varepsilon,0}(X)$. For each $i \in \mathbb{N}$, define a bounded linear operator $T_i : \ell_\infty \longrightarrow X$ by $T_i(\xi) = T(\xi)_i$ for each $\xi \in \ell_\infty$, where $T(\xi)_i$ denotes the i-th coordinate of $T(\xi)$. Since X contains no sublattice isomorphic to ℓ_∞, by Rosenthal's ℓ_∞-theorem (see [15] or [6, p. 12, Theorem 1.3.1]), each T_i is weakly compact. Moreover, by Lemma 4.4 there exists an infinite subset M of $\mathbb{N}$ such that for all $\xi = (\xi_n)_n \in \ell_\infty(M)$,

$$T(\xi)_i = T_i(\xi) = \sum_{n=1}^{\infty} \xi_n T_i(e_n) = \sum_{n=1}^{\infty} \xi_n T(e_n)_i, \qquad \forall i \in \mathbb{N}. \tag{4.1}$$

Note that for each $m \in \mathbb{N}$,

$$\begin{aligned} \Big\| \sum_{n=1}^{m} \xi_n T(e_n) \Big\|_{\lambda_\varepsilon(X)} &= \Big\| T\Big((\xi_1, \cdots, \xi_m, 0, 0, \cdots) \Big) \Big\|_{\lambda_\varepsilon(X)} \\ &\leq \|T\| \cdot \|(\xi_1, \cdots, \xi_m, 0, 0, \cdots)\|_{\ell_\infty} \\ &\leq \|T\| \cdot \|\xi\|_{\ell_\infty}. \end{aligned}$$

Thus for each $\bar{x}^* = (x_i^*)_i \in \lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ and for each $m, k \in \mathbb{N}$,

$$\begin{aligned} & \Big| \langle \sum_{n=1}^{m} \xi_n T(e_n) - T(\xi), \bar{x}^* \rangle \Big| \\ = & \Big| \langle \sum_{n=1}^{m} \xi_n T(e_n) - T(\xi), \bar{x}^*(\leq k) \rangle + \langle \sum_{n=1}^{m} \xi_n T(e_n) - T(\xi), \bar{x}^*(> k) \rangle \Big| \\ \leq & \sum_{i=1}^{k} x_i^* \Big(\sum_{n=m+1}^{\infty} \xi_n T(e_n)_i \Big) + \|\bar{x}^*(> k)\|_{\lambda'_\pi(X^*)} \cdot \Big(\Big\| \sum_{n=1}^{m} \xi_n T(e_n) \Big\|_{\lambda_\varepsilon(X)} + \Big\| T(\xi) \Big\|_{\lambda_\varepsilon(X)} \Big) \\ \leq & \sum_{i=1}^{k} x_i^* \Big(\sum_{n=m+1}^{\infty} \xi_n T(e_n)_i \Big) + 2\|T\| \cdot \|\xi\|_{\ell_\infty} \cdot \|\bar{x}^*(> k)\|_{\lambda'_\pi(X^*)}. \end{aligned} \tag{4.2}$$

By Proposition 2.2 we have $\lim_k \|\bar{x}^*(> k)\|_{\lambda'_\pi(X^*)} = 0$. It follows from (4.1) and (4.2) that the series $\sum_n \xi_n T(e_n)$ converges to $T(\xi)$ weakly in $\lambda_{\varepsilon,0}(X)$ for all $\xi \in \ell_\infty(M)$. Thus the series $\sum_{n \in M} T(e_n)$ is weakly subseries convergent and hence subseries convergent in $\lambda_{\varepsilon,0}(X)$. Therefore $T(e_n) \longrightarrow 0$ in $\lambda_{\varepsilon,0}(X)$ as $n \in M$ and $n \to \infty$. But for each $n \in \mathbb{N}$, $\|T(e_n)\|_{\lambda_\varepsilon(X)} \geq \|e_n\|_{\ell_\infty} / \|T^{-1}\| = 1/\|T^{-1}\|$. This contradiction shows that $\lambda_{\varepsilon,0}(X)$ contains no sublattice isomorphic to ℓ_∞. □

5. Positive Tensor Products.

For Banach lattices X and Y, let $X \otimes Y$ denote the algebraic tensor product of X and Y. For each $u = \sum_{k=1}^{m} x_k \otimes y_k \in X \otimes Y$, define $T_u : X^* \longrightarrow Y$ by

$$T_u(x^*) = \sum_{k=1}^{m} x^*(x_k) y_k, \quad \forall x^* \in X^*.$$

The *injective cone* of $X \otimes Y$ is defined by

$$C_i = \Big\{ u \in X \otimes Y : T_u \geq 0 \Big\},$$

and the *positive injective tensor norm* of $X \otimes Y$ is defined by

$$\|u\|_{|\varepsilon|} = \|T_u\|_r.$$

Let $X \check{\otimes}_{|\varepsilon|} Y$ denote the completion of $X \otimes Y$ with respect to $\|\cdot\|_{|\varepsilon|}$. Then $X \check{\otimes}_{|\varepsilon|} Y$ with C_i as its positive cone is a Banach lattice (see [18, 19] or see [14, section 3.8]), called the *positive injective tensor product* of X and Y. It follows from [14, Theorem 3.8.6 and Proposition 3.8.7] that the mapping $(u \mapsto T_u) : X \otimes Y \to \mathcal{L}^r(X^*, Y) \hookrightarrow \mathcal{L}^r(X^*, Y^{**})$ extends isometrically to a lattice homomorphism $X \check{\otimes}_{|\varepsilon|} Y \to \mathcal{L}^r(X^*, Y^{**})$. That is, every $v \in X \check{\otimes}_{|\varepsilon|} Y$ corresponds to $T_v \in \mathcal{L}^r(X^*, Y^{**})$ such that $\|T_v\|_r = \|v\|_{|\varepsilon|}$ and $T_{|v|} = |T_v|$.

The *projective cone* of $X \otimes Y$ is defined by

$$C_p = \Big\{ \sum_{k=1}^{n} x_k \otimes y_k : n \in \mathbb{N}, x_k \in X^+, y_k \in Y^+ \Big\},$$

and the *positive projective tensor norm* on $X \otimes Y$ is defined by

$$\|u\|_{|\pi|} = \sup \Big\{ \Big| \sum_{k=1}^{n} \phi(x_k, y_k) \Big| : u = \sum_{k=1}^{n} x_k \otimes y_k \in X \otimes Y, \phi \in M \Big\},$$

where M is the set of all positive bilinear functionals ϕ on $X \times Y$ with $\|\phi\| \leq 1$. Let $X \hat{\otimes}_{|\pi|} Y$ denote the completion of $X \otimes Y$ with respect to $\|\cdot\|_{|\pi|}$. Then $X \hat{\otimes}_{|\pi|} Y$ with C_p as its positive cone is a Banach lattice (see [9, 10] or see [14, section 3.8]), called the *positive projective tensor product* of X and Y. The positive projective tensor norm $\|\cdot\|_{|\pi|}$ has another equivalent form:

$$\|u\|_{|\pi|} = \inf \Big\{ \sum_{k=1}^{n} \|x_k\| \cdot \|y_k\| : x_k \in X^+, y_k \in Y^+, |u| \leq \sum_{k=1}^{n} x_k \otimes y_k \Big\}.$$

Bu and Buskes [2] gave a sequential representation of the positive projective tensor product $\lambda \hat{\otimes}_{|\pi|} X$. That is, if λ is σ-order continuous then $\lambda \hat{\otimes}_{|\pi|} X$ is isometrically isomorphic and lattice homomorphic to $\lambda_{\pi,0}(X)$. Next in Theorem 5.2 we will give a sequential representation of the positive injective tensor product $\lambda \check{\otimes}_{|\varepsilon|} X$.

Lemma 5.1. *Let X and Y be vector lattices such that Y is Dedekind complete, and let $T \in \mathcal{L}^r(X, Y)$. If e is an atom in X then $|T|(e) = |T(e)|$.*

Proof. Recall that if e is an atom in X then $x \in X$ with $0 \leq x \leq e$ implies that $x = \alpha e$ for some $\alpha \in \mathbb{R}^+$. Thus

$$|T|(e) = \sup\{|T(x)| : 0 \leq x \leq e\} = \sup\{|T(\alpha e)| : 0 \leq \alpha e \leq e\} = |T(e)|.$$

□

Theorem 5.2. *If λ is σ-order continuous then $\lambda \check{\otimes}_{|\varepsilon|} X$ is isometrically isomorphic and lattice homomorphic to $\lambda_{\varepsilon,0}(X)$.*

Proof. Since λ is σ-order continuous, it follows from [2, Lemma 3.4] that $\lambda_0 = \lambda$ and hence, $\lambda^* = \lambda'$. Thus every $v \in \lambda \check{\otimes}_{|\varepsilon|} X$ corresponds to $T_v \in \mathcal{L}^r(\lambda', X^{**})$ such that

$$\|T_v\|_r = \|v\|_{|\varepsilon|} \quad \text{and} \quad T_{|v|} = |T_v|. \tag{5.1}$$

Let ϕ denote the linear map from $\lambda \otimes X$ into $X^{\mathbb{N}}$ induced by the natural map: $\lambda \times X \longrightarrow X^{\mathbb{N}}$ with $(t, x) \mapsto (t_i x)_i$ for every $t = (t_i)_i \in \lambda$ and every $x \in X$. That is, for every $u \in \lambda \otimes X$ with a representation $u = \sum_{k=1}^m t^{(k)} \otimes x_k$, we have

$$\phi(u) = \Big(\sum_{k=1}^m t_i^{(k)} x_k \Big)_i. \tag{5.2}$$

For every $x^* \in X^{*+}$ and every $s = (s_i)_i \in \lambda'^+$,

$$\sum_{i=1}^\infty s_i x^* \Big(| \sum_{k=1}^m t_i^{(k)} x_k | \Big) \leq \sum_{i=1}^\infty \sum_{k=1}^m s_i |t_i^{(k)}| x^*(|x_k|) \leq \|x^*\| \cdot \|s\|_{\lambda'} \cdot \sum_{k=1}^m \|x_k\| \cdot \|t^{(k)}\|_\lambda.$$

Thus $\phi(u) \in \lambda_\varepsilon(X)$. Moreover, by [2, Lemma 3.4], $\lim_n \|t^{(k)}(\geq n)\|_\lambda = 0$ for $k = 1, 2, \cdots, m$ and hence, $\phi(u) \in \lambda_{\varepsilon,0}(X)$.

For every $s = (s_i)_i \in (\lambda')_0$ and every $x^* \in X^*$, we have

$$\langle T_u(s), x^* \rangle = \sum_{k=1}^m \sum_{i=1}^\infty s_i t_i^{(k)} x^*(x_k) = \langle \phi(u), (s_i x^*)_i \rangle, \tag{5.3}$$

and by Lemma 5.1 we have

$$|T_u|(s) = |T_u| \Big(\sum_{i=1}^\infty s_i e_i \Big) = \sum_{i=1}^\infty s_i |T_u|(e_i) = \sum_{i=1}^\infty s_i |T_u(e_i)| = \sum_{i=1}^\infty s_i \Big| \sum_{k=1}^m t_i^{(k)} x_k \Big|,$$

and thus we have

$$\langle |\phi(u)|, (s_i x^*)_i \rangle = \langle \Big(| \sum_{k=1}^m t_i^{(k)} x_k | \Big)_i, (s_i x^*)_i \rangle = \sum_{i=1}^\infty s_i x^* \Big(| \sum_{k=1}^m t_i^{(k)} x_k | \Big) = \langle |T_u|(s), x^* \rangle. \tag{5.4}$$

If $\phi(u)_i$ denotes the i-th coordinate of $\phi(u)$ then

$$\begin{aligned}
\Big\|\phi(u)\Big\|_{\lambda_\varepsilon(X)} &= \sup\Big\{\Big\|\Big(x^*(|\phi(u)_i|)\Big)_i\Big\|_\lambda : x^*\in B_{X^{*+}}\Big\}\\
&= \sup\Big\{\sum_{i=1}^\infty s_i x^*(|\phi(u)_i|) : s=(s_i)_i\in B_{(\lambda')_0^+}, x^*\in B_{X^{*+}}\Big\}\\
&= \sup\Big\{\langle|\phi(u)|,(s_ix^*)_i\rangle : s=(s_i)_i\in B_{(\lambda')_0^+}, x^*\in B_{X^{*+}}\Big\}\\
&= \sup\Big\{\langle|T_u|(s),x^*\rangle : s=(s_i)_i\in B_{(\lambda')_0^+}, x^*\in B_{X^{*+}}\Big\}\\
&= \|\,|T_u|\,\| = \|T_u\|_r = \|u\|_{|\varepsilon|}
\end{aligned}$$

and hence, ϕ is an isometry. Extend ϕ isometrically from $(\lambda\otimes X, \|\cdot\|_{|\varepsilon|})$ to its completion $\lambda\check{\otimes}_{|\varepsilon|}X$, denoted by $\tilde{\phi}$.

Now take any $\bar{x}=(x_i)_i\in\lambda_{\varepsilon,0}(X)$ and let $w_n=\sum_{i=1}^n e_i\otimes x_i$ for each $n\in\mathbb{N}$. Then for every $m,n\in\mathbb{N}$ with $m>n$,

$$\begin{aligned}
\|w_m-w_n\|_{|\varepsilon|} &= \Big\|\sum_{i=n+1}^m e_i\otimes x_i\Big\|_{|\varepsilon|} = \Big\|\tilde{\phi}(\sum_{i=n+1}^m e_i\otimes x_i)\Big\|_{\lambda_\varepsilon(X)}\\
&= \|(0,\cdots,0,x_{n+1},\cdots,x_m,0,0,\cdots)\|_{\lambda_\varepsilon(X)}\\
&\to 0 \text{ as } m,n\to\infty.
\end{aligned}$$

Thus $\{w_n\}_1^\infty$ is a Cauchy sequence in $\lambda\check{\otimes}_{|\varepsilon|}X$ and hence, there exists $w\in\lambda\check{\otimes}_{|\varepsilon|}X$ such that $w=\lim_n w_n$. Note that

$$\tilde{\phi}(w)=\tilde{\phi}(\lim_n w_n)=\lim_n\tilde{\phi}(w_n)=\lim_n\bar{x}(\le n)=\bar{x}.$$

Therefore, $\tilde{\phi}$ is surjective.

Finally we show that $\tilde{\phi}$ is a lattice homomorphism. Note that $\tilde{\phi}$ and the map $v\mapsto T_v$ are continuous and $\lambda\otimes X$ is norm dense in $\lambda\check{\otimes}_{|\varepsilon|}X$. By (5.3) for every $v\in\lambda\check{\otimes}_{|\varepsilon|}X$, every $s=(s_i)_i\in(\lambda')_0$, and every $x^*\in X^*$ we have

$$\langle T_v(s),x^*\rangle=\langle\tilde{\phi}(v),(s_ix^*)_i\rangle. \tag{5.5}$$

Thus for every $u\in\lambda\otimes X$, every $s=(s_i)_i\in(\lambda')_0$, and every $x^*\in X^*$, it follows from (5.1), (5.4), and (5.5) that

$$\langle|\tilde{\phi}(u)|,(s_ix^*)_i\rangle=\langle|T_u|(s),x^*\rangle=\langle T_{|u|}(s),x^*\rangle=\langle\tilde{\phi}(|u|),(s_ix^*)_i\rangle,$$

which implies that $|\tilde{\phi}(u)|=\tilde{\phi}(|u|)$. Since $\tilde{\phi}$ is continuous and $\lambda\otimes X$ is norm dense in $\lambda\check{\otimes}_{|\varepsilon|}X$, it follows that $|\tilde{\phi}(v)|=\tilde{\phi}(|v|)$ for every $v\in\lambda\check{\otimes}_{|\varepsilon|}X$ and hence, $\tilde{\phi}$ is a lattice homomorphism. □

6. Properties inherited by Positive Tensor Products.

In this section, we will verify some geometric properties that can be inherited by the positive projective tensor product $E\hat{\otimes}_{|\pi|}X$ and the positive injective tensor product $E\check{\otimes}_{|\varepsilon|}X$, provided that E is an atomic Banach lattice and X is any Banach lattice.

6.1. Reflexivity. A little bit modifying Proposition 3.4 in [11] gives the following lemma.

Lemma 6.1 [11]**.** *Let E be a Banach lattice and F a separable closed sublattice of E. Then there exists an ideal G of E containing F such that G is generated by a separable closed sublattice of E, and there exists a lattice isometric embedding $\varphi : G^* \longrightarrow E^*$ such that $\varphi(g^*)(g) = g^*(g)$ for every $g \in G$ and every $g^* \in G^*$. In particular, $\varphi[G^*]$ is norm one positive complemented in E^*.*

In the previous lemma, if E is reflexive then for every $x \in E$ we define $g : G^* \longrightarrow \mathbb{R}$ by

$$\langle g, g^*\rangle = \langle x, \varphi(g^*)\rangle, \qquad \forall\, g^* \in G^*.$$

Then

$$\begin{aligned}\|g\| &= \sup\Big\{|\langle g, g^*\rangle| : g^* \in G^*, \|g^*\| \le 1\Big\}\\ &= \sup\Big\{|\langle x, \varphi(g^*)\rangle| : \|\varphi(g^*)\| = \|g^*\| \le 1\Big\}\\ &\le \|x\|.\end{aligned}$$

Thus $g \in G^{**} = G$ and hence the map $x \to g$ is a norm one positive projection from E onto G. Therefore we have the following.

Lemma 6.2. *Let E be a reflexive Banach lattice and F a separable closed sublattice of E. Then there exists an ideal G of E containing F such that G is generated by a separable closed sublattice of E and G is norm one positive complemented in E.*

We need the following result (due to Wolff [20]) to transform an atomic Banach lattice into a sequence Banach lattice.

Lemma 6.3 [20]**.** *Let E be a Dedekind complete separable Banach lattice. Then E is atomic if and only if there is an order continuous and injective lattice homomorphism from E into a sublattice of $\mathbb{R}^{\mathbb{N}}$.*

Let G be a Dedekind complete separable atomic Banach lattice. By Lemma 6.3 there is an order continuous and injective lattice homomorphism ϕ from G onto $\phi[G]$, a sublattice of $\mathbb{R}^{\mathbb{N}}$. Define a norm on $\phi[G]$ by $\|\phi(g)\| = \|g\|$ for all $g \in G$. Then ϕ is also an isometry, and hence $\lambda := \phi[G]$ is a Banach sequence lattice such that $G\hat{\otimes}_{|\pi|}X$ and $G\check{\otimes}_{|\varepsilon|}X$ are isometrically isomorphic and lattice homomorphic to $\lambda\hat{\otimes}_{|\pi|}X$ and $\lambda\check{\otimes}_{|\varepsilon|}X$, respectively. Note that $\lambda\hat{\otimes}_{|\pi|}X = \lambda_{\pi,0}(X)$ and $\lambda\check{\otimes}_{|\varepsilon|}X = \lambda_{\varepsilon,0}(X)$ isometrically. Thus by Proposition 2.1 and Theorem 3.3 we have the following lemma.

Lemma 6.4. *Let G be a separable atomic reflexive Banach lattice and X be a reflexive Banach lattice.*

(i) *$G\hat{\otimes}_{|\pi|}X$ is reflexive if and only if every positive linear operator from G into X^* is compact.*

(ii) *$G\check{\otimes}_{|\varepsilon|}X$ is reflexive if and only if every positive linear operator from G^* into X is compact.*

In the following theorem we will remove the separability of G in the previous lemma.

Theorem 6.5. *Let E be an atomic reflexive Banach lattice and X be a reflexive Banach lattice.*

(i) *$E\hat{\otimes}_{|\pi|}X$ is reflexive if and only if every positive linear operator from E into X^* is compact.*

(ii) *$E\check{\otimes}_{|\varepsilon|}X$ is reflexive if and only if every positive linear operator from E^* into X is compact.*

Proof. (i) Suppose that every positive linear operator from E into X^* is compact. To show that $E\hat{\otimes}_{|\pi|}X$ is reflexive, it suffices to show that every separable closed sublattice S of $E\hat{\otimes}_{|\pi|}X$ is reflexive. By the proof of [2, Proposition 6.3] and by Lemma 6.2, there exists an ideal G of E such that S is a closed sublattice of $G\hat{\otimes}_{|\pi|}X$, where G is generated by a separable closed sublattice of E and G is norm one positive complemented in E. Thus every positive linear operator from G to X^* is compact. Note that an ideal generated by a separable closed sublattice in an atomic KB-space is also separable. Then G is separable. It follows from Lemma 6.4 that $G\hat{\otimes}_{|\pi|}X$ is reflexive. Therefore S, as a closed sublattice of $G\hat{\otimes}_{|\pi|}X$, is also reflexive.

On the other hand, suppose that $E\hat{\otimes}_{|\pi|}X$ is reflexive and there exists a positive linear operator $T : E \longrightarrow X^*$ that is not compact. That is, there is a sequence $(x_n)_1^\infty$ in B_E such that the sequence $(Tx_n)_1^\infty$ has no convergent subsequence in X^*. Let F be the separable sublattice generated by all x_n's. By Lemma 6.2, there exists an ideal G of E containing F such that G is generated by a separable closed sublattice of E and G is norm one positive complemented in E. Thus $G\hat{\otimes}_{|\pi|}X$ is a closed sublattice of $E\hat{\otimes}_{|\pi|}X$, and hence $G\hat{\otimes}_{|\pi|}X$ is also reflexive. It follows from Lemma 6.4 that every positive linear operator from G into X^* is compact. But $T|_G$ is not compact. This contradiction shows that if $E\hat{\otimes}_{|\pi|}X$ is reflexive then every positive linear operator from E into X^* must be compact.

(ii) Suppose that every positive linear operator from E^* into X is compact. To show that $E\check{\otimes}_{|\varepsilon|}X$ is reflexive, it suffices to show that every separable closed sublattice S of $E\check{\otimes}_{|\varepsilon|}X$ is reflexive. Since S is separable, there is a separable closed sublattice F of E such that $S \subseteq F\check{\otimes}_{|\varepsilon|}X$. Let G be the ideal in Lemma 6.1. Then $S \subseteq G\check{\otimes}_{|\varepsilon|}X$. Note that $\varphi[G^*]$ is norm one positive complemented in E^*. Every positive linear operator from G^* into X is compact. It follows from Lemma 6.4 that $G\check{\otimes}_{|\varepsilon|}X$ is reflexive. Note that if E_1 is a closed sublattice of E then $E_1\check{\otimes}_{|\varepsilon|}X$ is also a closed sublattice of $E\check{\otimes}_{|\varepsilon|}X$. Thus S, as a closed sublattice of $E\check{\otimes}_{|\varepsilon|}X$, is also a closed sublattice of $G\check{\otimes}_{|\varepsilon|}X$ and hence is reflexive. The proof of the second part is the same as the proof of the second part in (i). □

6.2. Jeurnink's open question. For a norm $\|\cdot\|$ in a vector space Z, let $\|\cdot\|^*$ denote the dual norm of $\|\cdot\|$ in the dual space $(Z, \|\cdot\|)^*$. For Banach lattices E and X, it follows from [16, p. 204, Theorem 3.2] that $(E\hat{\otimes}_{|\pi|}X)^* = \mathcal{L}^r(E, X^*)$ isometrically. Note that $E^*\check{\otimes}_{|\varepsilon|}X^*$ is a sublattice of $\mathcal{L}^r(E, X^*)$. Thus $\|\cdot\|_{|\varepsilon|} = \|\cdot\|_{|\pi|}^*$ in the vector space $E^* \otimes X^*$. On the other hand, Jeurnink pointed out in [12, chapter 4] that $\|\cdot\|_{|\pi|} \geq \|\cdot\|_{|\varepsilon|}^*$ in the vector space $E^* \otimes X^*$. He also asked the question whether $\|\cdot\|_{|\pi|} \leq \|\cdot\|_{|\varepsilon|}^*$ also holds. The following theorem gives an affirmative answer to this question in the case that E is a reflexive atomic Banach lattice.

Theorem 6.6. *If E is a reflexive atomic Banach lattice then $\|\cdot\|_{|\pi|} = \|\cdot\|_{|\varepsilon|}^*$ in the vector space $E^* \otimes X^*$.*

Proof. It follows from [12, chapter 4] that $\|\cdot\|_{|\pi|} \geq \|\cdot\|_{|\varepsilon|}^*$. Next we show that $\|\cdot\|_{|\pi|} \leq \|\cdot\|_{|\varepsilon|}^*$.

Take any $u \in E^* \otimes X^*$. Then u admits a representation $u = \sum_{k=1}^n z_k^* \otimes x_k^*$ where $z_k^* \in E^*, x_k^* \in X^*, k = 1, 2, \cdots, n$. Let G be the ideal generated by $\{z_k^*\}_1^n$. Then G is separable since E^* is atomic. By Lemma 6.3, there is an order continuous and injective lattice homomorphism ϕ from G onto $\phi[G]$, a sublattice of $\mathbb{R}^{\mathbb{N}}$. Define a norm on $\phi[G]$ by

$\|\phi(g)\| = \|g\|$ for all $g \in G$. Then ϕ is also an isometry and hence $\phi[G]$ is a reflexive Banach sequence lattice. Let $\lambda = \phi[G]^*$. Then λ is reflexive and hence, both λ and λ' are σ-order continuous, and thus $\lambda' = \lambda^* = \phi[G]^{**} = \phi[G]$. By Proposition 2.2 and Theorems 3.1 and 5.2, we have

$$(\lambda\check{\otimes}_{|\varepsilon|}X)^* = \lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*) = \lambda'_{\pi,0}(X^*) = \lambda'\hat{\otimes}_{|\pi|}X^*.$$

It follows that $\|\cdot\|_{|\pi|} = \|\cdot\|^*_{|\varepsilon|}$ in the vector space $\lambda' \otimes X^*$, and hence $\|\cdot\|_{|\pi|} = \|\cdot\|^*_{|\varepsilon|}$ in the vector space $G \otimes X^*$. Note that G^* is a sublattice of E, and hence $G^* \otimes_{|\varepsilon|} X$ is a sublattice of $E \otimes_{|\varepsilon|} X$. It follows from the equivalent form of the positive projective tensor norm $\|\cdot\|_{|\pi|}$ given at the beginning of section 5 that in the vector space $E^* \otimes X^*$,

$$\|u\|_{E^*\otimes_{|\pi|}X^*} \leq \|u\|_{G\otimes_{|\pi|}X^*} = \|u\|^*_{G^*\otimes_{|\varepsilon|}X} \leq \|u\|^*_{E\otimes_{|\varepsilon|}X}.$$

□

6.3. Copies of c_0, ℓ_∞, and ℓ_1. Recall that the properties of the containment of copies of c_0, ℓ_1, ℓ_∞ are separably determined. That is, a Banach space has these properties if and only if every its separable closed subspace has the same properties. With the help of Lemma 6.3 and using the same argument as the proof of Theorem 7.5 in [2], we can modify Proposition 4.1(i) and Theorems 4.2(i) and 4.3(ii) to get the following results.

Theorem 6.7. *Let E be an atomic Banach lattice and X be a Banach lattice.*

(i) *$E\hat{\otimes}_{|\pi|}X$ contains no sublattice isomorphic to c_0 if and only if neither E nor X contains a sublattice isomorphic to c_0 (*[2, Theorem 7.5(i)]*).*

(ii) *$E\hat{\otimes}_{|\pi|}X^*$ contains no sublattice isomorphic to ℓ_∞ if and only if neither E nor X contains a sublattice isomorphic to ℓ_∞.*

(iii) *$E\check{\otimes}_{|\varepsilon|}X$ contains no sublattice isomorphic to ℓ_1 if and only if neither E nor X contains a sublattice isomorphic to ℓ_1.*

With the help of Lemma 6.3 and using the same argument as the proof of Theorem 6.5, we can modify Proposition 4.1(ii) and Theorems 4.3(i) and 4.5 to get the following results.

Theorem 6.8. *Let E be a reflexive atomic Banach lattice and X be a Banach lattice.*

(i) *$E\check{\otimes}_{|\varepsilon|}X$ contains no sublattice isomorphic to c_0 if and only if X contains no sublattice isomorphic to c_0 and every positive linear operator from E^* into X is compact.*

(ii) *$E\hat{\otimes}_{|\pi|}X$ contains no sublattice isomorphic to ℓ_1 if and only if X contains no sublattice isomorphic to ℓ_1 and every positive linear operator from E into X^* is compact.*

(iii) $E\check{\otimes}_{|\varepsilon|}X$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if* X *contains no sublattice isomorphic to* ℓ_∞.

Recall that if λ is reflexive then both λ and λ' are σ-order continuous. Thus by Proposition 2.1 and Theorems 4.2(ii) and 4.5 we have the following.

Theorem 6.9. *Let* λ *be a reflexive Banach sequence lattice and* X *be a Dedekind complete Banach lattice.*

(i) $\mathcal{K}^r(\lambda, X)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if* X *contains no sublattice isomorphic to* ℓ_∞.

(ii) $\mathcal{L}^r(\lambda, X^*)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if* X^* *contains no sublattice isomorphic to* ℓ_∞ *and every positive linear operator from* λ *to* X^* *is compact.*

Corollary 6.10. *Let* $1 < p, q < \infty$ *and* E_q *be an infinite dimensional reflexive* L_q*-space.*

(i) $\mathcal{K}^r(\ell_p, E_q)$ *contains no sublattice isomorphic to* ℓ_∞ *for every* p, q *with* $1 < p, q < \infty$.

(ii) $\mathcal{K}^r(\ell_p, E_q)$ *contains no sublattice isomorphic to* c_0 *if and only if* $p > q$.

(iii) $\mathcal{L}^r(\ell_p, E_q)$ *contains no sublattice isomorphic to* ℓ_∞ *if and only if* $p > q$.

Proof. (i) follows from Theorem 6.9 (i). By [7, Theorem 4.9], every positive linear operator from ℓ_p to E_q is compact if and only if $p > q$. Thus (iii) follows from Theorem 6.9 (ii). By [5, Theorem 7], $\mathcal{K}^r(\ell_p, E_q)$ contains no sublattice isomorphic to c_0 if and only if every positive linear operator from ℓ_p to E_q is compact. Thus (ii) follows. □

From Corollary 6.10 we have the following interesting example. If $1 < p \le q < \infty$ then $\mathcal{K}^r(\ell_p, \ell_q)$ contains a sublattice isomorphic to c_0 but does not contain a sublattice isomorphic to ℓ_∞.

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