INNER PRODUCTS AND MODULE MAPS OF HILBERT C^*-MODULES

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Abstract. Let E and F be two Hilbert C^*-modules over C^*-algebras A and B, respectively. Let T be a surjective linear isometry from E onto F and φ a map from A into B. We will prove in this paper that if the C^*-algebras A and B are commutative, then T preserves the inner products and T is a module map, i.e., there exists a $*$-isomorphism φ between the C^*-algebras such that

$$\langle Tx, Ty\rangle = \varphi(\langle x, y\rangle),$$

and

$$T(xa) = T(x)\varphi(a).$$

In case A or B is noncommutative C^*-algebra, T may not satisfy the equations above in general. We will also give some condition such that T preserves the inner products and T is a module map.

1. Introduction

A (right) Hilbert C^*-module over a C^*-algebra A is a right A-module E equipped with A-valued inner product $\langle \cdot, \cdot\rangle$ which is conjugate A-linear in the first variable and A-linear in the second variable such that E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

Let X be a locally compact Hausdorff space and H a Hilbert space, the Banach space $C_0(X, H)$ of all continuous H-valued functions vanishing at infinity is a Hilbert C^*-module over the C^*-algebra $C_0(X)$ with inner product $\langle f, g\rangle(x) := \langle f(x), g(x)\rangle$ and module operation $(f\phi)(x) = f(x)\phi(x)$, for all $f \in C_0(X, H)$ and $\phi \in C_0(X)$. Every C^*-algebra A is a Hilbert C^*-module over itself with inner product $\langle a, b\rangle := a^*b$.

Let X and Y be two locally compact Hausdorff spaces, the Banach-Stone theorem states that every surjective linear isometry between $C_0(X)$ and $C_0(Y)$ is a weighted composition operator. More precisely, let T be a surjective linear isometry from $C_0(X)$ onto $C_0(Y)$, then there exists a continuous function $h \in C_0(Y)$ with $|h(y)| = 1$, for all y in Y, and a homeomorphism φ from Y onto X such that T is of the form:

$$Tf(y) = h(y)f(\varphi(y)), \forall f \in C_0(X), \forall y \in Y. \tag{1}$$

Let H_1 and H_2 be two Hilbert spaces. In [7], Jerison characterizes surjective linear isometries between $C_0(X, H_1)$ and $C_0(Y, H_2)$, see also [12, 6]. It is said that every surjective linear isometry T from $C_0(X, H_1)$ onto $C_0(Y, H_2)$ is also of the form (1)

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in which $h(y)$ is a unitary operator from H_1 onto H_2 and h is continuous from Y into $(B(H_1, H_2), SOT)$, the space of all bounded linear operators with the strong operator topology. In this case, we can find a relationship of inner products of $C_0(X, H_1)$ and $C_0(Y, H_2)$ by a simple calculation:

$$\begin{aligned} \langle Tf, Tg\rangle(y) &= \langle Tf(y), Tg(y)\rangle = \langle h(y)(f(\varphi(y))), h(y)(f(\varphi(y)))\rangle \\ &= \langle f(\varphi(y)), f(\varphi(y))\rangle = \langle f, g\rangle \circ \varphi(y). \end{aligned}$$

i.e.

$$\langle Tf, Tg\rangle = \langle f, g\rangle \circ \varphi.$$

Let $R_\varphi : C_0(X) \to C_0(Y)$ be the $*$-isomorphism defined by $R_\varphi(\phi) = \phi \circ \varphi$. Then T preserves the inner products with respect to R_φ, i.e.,

$$\langle Tf, Tg\rangle = R_\varphi(\langle f, g\rangle).$$

By (1), it is easy to see that T is a module map with respect to R_φ in the sense

$$T(f\phi) = T(f)R_\varphi(\phi), \text{ for all } f \in C_0(X, H_1) \text{ and } \phi \in C_0(X).$$

It is natural to ask if these properties are true for surjective linear isometries between Hilbert C^*-modules over C^*-algebras. We will show in this paper that the answer is yes if the C^*-algebras are commutative. Unfortunately, if one of the C^*-algebras is noncommutative, the answer is more complicated. We will give an example (see Example 3) to explain this is not true in general. And we will give a condition on T (see Theorem 9) such that T is a module map and preserves the inner products.

2. Preliminaries

Let E be a Hilbert C^*-module over C^*-algebra A. We set $\langle E, E\rangle$ to be the linear span of elements of the form $\langle x, y\rangle$, $x, y \in E$. E is said to be *full* if the closed two-sided ideal $\overline{\langle E, E\rangle}$ equal A.

A *JB^*-triple* is a complex vector space V with a continuous mapping $V^3 \to V$, $(x, y, z) \to \{x, y, z\}$, called a *Jordan triple product*, which is symmetric and linear in x, z and conjugate linear in y such that for x, y, z, u, v in V, we have

(1) $\{x, y, \{z, u, v\}\} = \{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\}$;
(2) the mapping $z \to \{x, x, z\}$ is hermitian and has non-negative spectrum;
(3) $\|\{x, x, x\}\| = \|x\|^3$.

In [5], J. M. Isidro shows that every Hilbert C^*-module is a JB*-triple with the Jordan triple product

$$\{x, y, z\} = \frac{1}{2}(x\langle y, z\rangle + z\langle y, x\rangle).$$

A well-known theorem of Kaup [10] (see also [1]) states that every surjective linear isometry between JB*-triples is a *Jordan triple homomorphism*, i.e., it preserves the Jordan triple product

$$T\{x, y, z\} = \{Tx, Ty, Tz\}, \forall x, y, z \in E.$$

Hence, if T is a surjective linear isometry between Hilbert C^*-modules, then

$$T(x\langle y, z\rangle + z\langle y, x\rangle) = Tx\langle Ty, Tz\rangle + Tz\langle Ty, Tx\rangle, \forall x, y, z \in E. \tag{2}$$

The equation (2) holds if and only if

$$T(x\langle x,x\rangle) = Tx\langle Tx,Tx\rangle, \forall x \in E \tag{3}$$

by triple polarization

$$2\{x,y,z\} = \frac{1}{8}\sum_{\alpha^4=\beta^2=1} \alpha\beta\langle x+\alpha y+\beta z, x+\alpha y+\beta z\rangle(x+\alpha y+\beta z).$$

A ternary ring of operators (TRO) between two Hilbert spaces H and K is a linear subspace $\mathfrak{R}$ of $B(H,K)$, the space of all bounded linear operators from H into K, satisfying $AB^*C \in \mathfrak{R}$. Zettl shows in [17] that every Hilbert C^*-module is isomorphic to a norm closed TRO. In this case, Hilbert C^*-modules have another *triple product*, i.e.,

$$\{x,y,z\} := x\langle y,z\rangle.$$

A map between TROs is said to be a *triple homomorphism* if it preserves the triple products. In the case of Hilbert C^*-modules, a map T is a triple homomorphism if it satisfies

$$T(x\langle y,z\rangle) = Tx\langle Ty,Tz\rangle, \forall x,y,z. \tag{4}$$

We have known every surjective linear isometry is a Jordan triple homomorphism, but it could not be a *triple homomorphism*, see Example 3.

Let $\mathcal{R}$ be a TRO. Then $M_n(\mathcal{R})$, the space of all $n\times n$ matrices whose entries are in $\mathcal{R}$, has a TRO-structure. Let T be a map between TROs $\mathcal{R}_1$ and $\mathcal{R}_2$. For all positive integer n, define maps $T^{(n)} : M_n(\mathcal{R}_1) \to M_n(\mathcal{R}_2)$ by $T^{(n)}((x_{ij})_{ij}) = (T(x_{ij}))_{ij}$. We call T n-isometry if $T^{(n)}$ is isometric and *complete isometry* if each $T^{(n)}$ is isometric for all n. It has been shown that a surjective linear isometry between TROs is a triple homomorphism if and only if it is completely isometric. More details about TROs mentioned above, we refer to [17], see also [14, 3]. In fact, Solel shows in [16] that every surjective 2-isometry between two full Hilbert C^*-modules is necessarily completely isometric.

3. Results

Note that in the case of a commutative C^*-algebra $A = C_0(X)$, for some locally compact Hausdorff space X, Hilbert C^*-modules over $C_0(X)$ are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces, over X.

We showed the following theorem in [4].

Theorem 1. *Let E and F be two Hilbert C^*-modules over commutative C^*-algebras $C_0(X)$ and $C_0(Y)$, respectively. Then every surjective linear isometry from E onto F is a weighted composition operator*

$$Tf(y) = h(y)(f(\varphi(y))), \forall f \in E, \forall y \in Y$$

Here, φ is a homeomorphism from Y onto X, $h(y)$ is a unitary operator between the corresponding fibers of E and F, for all y in Y.

By the similar argument discussed in the introduction, we have

Corollary 2. *Every surjective linear isometry between Hilbert C^*-modules over commutative C^*-algebras preserves the inner products and is a module map.*

Now we discuss the case of noncommutative C^*-algebras. From equation (4), it seems that a surjective linear isometry T indicates that T preserves inner products and that T is a module map. We explain this could be not true in general by a example.

Example 3. Given a positive integer n. The Hilbert column space H_c^n is the subspace of $M_n(\mathbb{C})$ consisting of all matrices whose non-zero entries are only in the first column. Similarly, the Hilbert row space is the subspace consisting of matrices whose non-zero entries are only in the first row. Clearly, H_c and H_r are right Hilbert C^*-modules over C^*-algebras $\mathbb{C}$ and $M_n(\mathbb{C})$, respectively, with the inner product $\langle A, B\rangle := A^*B$. Define a surjective linear isometry $T : H_r^n \to H_c^n$ by $T(A) = A^t$, the transpose of A. Then $\langle T(A), T(B)\rangle = tr\langle A, B\rangle$, the trace of $\langle A, B\rangle$, but T is not a module map with respect to the trace. For the surjective linear isometry $T : H_c^n \to H_r^n$, $T(A) = A^t$. Let $\varphi : \mathbb{C} \to M_n(\mathbb{C})$ be defined by $\varphi(\alpha) = \alpha I$. Then T is a module map with respect to φ, but the equation $\langle TA, TB\rangle = \varphi(\langle A, B\rangle)$ does not hold. It is clear that T does not satisfy the equation (4).

Remark 4. In fact, the corollary above says that there exists a $*$-isomorphism φ between the C^*-algebras such that

$$\langle Tx, Ty\rangle = \varphi(\langle x, y\rangle)$$

and

$$T(xa) = T(x)\varphi(a).$$

We have seen in the Example 3 that even if T is a module map or preserves the inner products, the map φ might be just a linear map.

In the following, E and F stand for two Hilbert C^*-modules over C^*-algebras A and B, respectively. T is a map from E into F and φ is a map from A into B. The following lemmas explain the relations of T, φ, when T preserves the inner products and when T is a module map, see also [8].

Lemma 5. *If φ is linear, every map T from E into F which preserves the inner products with respect to φ is linear.*

Proof. Since T preserves the inner products with respect to φ. Then for all x, y and z in E, α in $\mathbb{C}$,

$$\langle T(\alpha x + y), Tz\rangle = \varphi(\langle \alpha x + y, z\rangle) = \alpha\varphi(\langle x, z\rangle) + \varphi(\langle y, z\rangle) = \langle \alpha Tx + Ty, Tz\rangle.$$

Similarly, we have

$$\langle Tx, T(\alpha y + z)\rangle = \langle Tx, \alpha Ty + Tz\rangle.$$

It is easy to show that

$$\langle T(\alpha x + y) - (\alpha Tx + Ty), T(\alpha x + y) - (\alpha Tx + Ty)\rangle = 0.$$

This proves $T(\alpha x + y) = \alpha Tx + Ty$ and hence T is linear. □

Lemma 6 ([8])**.** *Let T be a surjective linear map which preserves the inner products and is a module map w.r.t. φ. If F is full, then φ is a $*$-homomorphism.*

Proof. Let a_1, a_2 in A and α in $\mathbb{C}$. It is easy to show that

$$\begin{aligned} & T(x)(\varphi(\alpha a_1 + a_2) - \alpha\varphi(a_1) - \varphi(a_2)) \\ = \ & T(x)\varphi(\alpha a_1 + a_2) - \alpha T(x)\varphi(a_1) - T(x)\varphi(a_2) \\ = \ & T(\alpha x a_1 + x a_2) - \alpha T(x a_1) - T(x a_2) = 0. \end{aligned}$$

and

$$\begin{aligned} & T(x)(\varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)) \\ = \ & T(x)\varphi(a_1 a_2) - T(x)\varphi(a_1)\varphi(a_2) \\ = \ & T(x a_1 a_2) - T(x a_1 a_2) = 0. \end{aligned}$$

Since T is surjective and F is full, we have $\varphi(\alpha a_1 + a_2) = \alpha\varphi(a_1) + \varphi(a_2)$ and $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$.

Let x, y in A, we have

$$\varphi(\langle x, y\rangle^*) = \varphi(\langle y, x\rangle) = \langle Ty, Tx\rangle = \langle Tx, Ty\rangle^* = \varphi(\langle x, y\rangle)^*.$$

For a in A,

$$\begin{aligned} & \langle T(x)(\varphi(a^*) - \varphi(a)^*), T(x)(\varphi(a^*) - \varphi(a)^*)\rangle \\ = \ & \varphi(a^*)^*\varphi(\langle x, x\rangle)\varphi(a^*) - \varphi(a^*)^*\varphi(\langle x, x\rangle)\varphi(a)^* - \varphi(a)\varphi(\langle x, x\rangle)\varphi(a^*) + \varphi(a)\varphi(\langle x, x\rangle)\varphi(a)^* \\ = \ & (\varphi(\langle xa^*, x\rangle)\varphi(a^*))^* - (\varphi(a)\varphi(\langle x, x\rangle)\varphi(a^*))^* - \varphi(\langle xa^*, xa^*\rangle) + (\varphi(a)\varphi(\langle x, xa^*\rangle))^* \\ = \ & 0. \end{aligned}$$

Hence, $T(x)(\varphi(a^*) - \varphi(a)^*) = 0$ for all x in E. Since T is surjective and F is full, we have $\varphi(a^*) = \varphi(a)^*$. $\square$

Lemma 7. *If φ is a $*$-homomorphism, then every map T which preserves the inner products w.r.t. φ is a module map w.r.t. φ.*

Proof. Let x and y in E and a in A. Then

$$\langle T(xa), Ty\rangle = \varphi(\langle xa, y\rangle) = \varphi(a)^*\varphi(\langle x, y\rangle) = \langle T(x)\varphi(a), Ty\rangle.$$

Similarly, we have

$$\langle T(x), T(ya)\rangle = \langle T(x), T(y)\varphi(a)\rangle.$$

It is easy to show that

$$\langle T(xa) - T(x)\varphi(a), T(xa) - T(x)\varphi(a)\rangle = 0.$$

Hence, $T(xa) = T(x)\varphi(a)$. $\square$

Lemma 8 ([13])**.** *Let T be a surjective linear isometry and φ a $*$-isomorphism. If T is a module map w.r.t. φ, then T preserves the inner products with respect to φ.*

Proof. It suffices to prove that $\langle Tx, Tx\rangle = \varphi(\langle x, x\rangle)$ for all x in E. Note that $|a| := (a^*a)^{1/2}$. For all b in B, let $\varphi(a) = b$, then

$$\begin{aligned} & \||Tx|b\|^2 = \|b^*|Tx|^2 b\| = \|\langle T(x)\varphi(a), T(x)\varphi(a)\rangle\| \\ = \ & \|\langle T(xa), T(xa)\rangle\| = \|\langle xa, xa\rangle\| = \||x|a\|^2 = \|\varphi(|x|a)\|^2 = \|\varphi(|x|)b\|^2. \end{aligned}$$

By Lemma 3.5 in [11], we get $|Tx| = (\varphi(|x|)$ and hence $\langle Tx, Tx\rangle = \varphi(\langle x, x\rangle)$. $\square$

Theorem 9. *Let T be a surjective linear 2-isometry from E onto F. Then there exists a $*$-isomorphism φ from $\overline{\langle E,E\rangle}$ onto $\overline{\langle F,F\rangle}$ such that, for all x,y in E, and a in A,*

$$\langle Tx,Ty\rangle = \varphi(\langle x,y\rangle)$$

and

$$T(xa) = T(x)\varphi(a).$$

Proof. We can regard E and F as full modules over $\langle E,E\rangle$ and $\langle F,F\rangle$, respectively. In this case, as we mentioned above, T is completely isometric and hence it preserves the triple products

$$T(z\langle x,y\rangle) = Tz\langle Tx,Ty\rangle, \forall x,y,z\in E.$$

Define $\varphi : \langle E,E\rangle \to \langle F,F\rangle$ by

$$\varphi(\sum_{i=i}^{n}\alpha_i\langle x_i,y_i\rangle) := \sum_{i=i}^{n}\alpha_i\langle Tx_i,Ty_i\rangle,\ x_i,y_i\in E,\ \alpha_i\in\mathbb{C},\ i=1,\cdots,n.$$

Let x_i, y_i and $z\in E$, $\alpha_i\in\mathbb{C}$, $i=1,\cdots,n$. Then $\sum_{i=i}^{n}\alpha_i\langle x_i,y_i\rangle = 0$ if and only if $z(\sum_{i=i}^{n}\alpha_i\langle x_i,y_i\rangle) = 0$ for all z if and only if $T(z)(\sum_{i=i}^{n}\alpha_i\langle Tx_i,Ty_i\rangle) = \sum_{i=i}^{n}\alpha_i Tz\langle Tx_i,Ty_i\rangle = \sum_{i=i}^{n}\alpha_i T(z\langle x_i,y_i\rangle) = T(z(\sum_{i=i}^{n}\alpha_i\langle x_i,y_i\rangle)) = 0$ for all z if and only if $\sum_{i=i}^{n}\alpha_i\langle Tx_i,Ty_i\rangle = 0$ since T is injective, $\sum_{i=i}^{n}\alpha_i\langle x_i,y_i\rangle\in\langle E,E\rangle$ and $\sum_{i=i}^{n}\alpha_i\langle Tx_i,Ty_i\rangle\in\langle F,F\rangle$. This shows that φ is well-defined and injective. From the defintion of φ, since T is surjective, it is clear that φ is a surjective $*$-homomorphism and T preserves the inner products w.r.t. φ. By lemma 7, T is a module map w.r.t φ. $\square$

Corollary 10. *Every surjective linear 2-isometry between two full Hilbert C^*-modules preserves the inner products and is a module map with respect to some $*$-isomorphism of underlying C^*-algebras.*

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