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Research Article

Maps Preserving Schatten p-Norms of Convex Combinations

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We study maps ϕ of positive operators of the Schatten p-classes ($1 < p < +\infty$), which preserve the p-norms of convex combinations, that is, $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$, $\forall \rho, \sigma \in \mathcal{S}_p^+(H)_1$, $t \in [0,1]$. They are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary U. In the case $p = 2$, we have the same conclusion whenever it just holds $\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2$ for all the positive Hilbert-Schmidt class operators ρ, σ of norm 1. Some examples are demonstrated.

1. Introduction

The Mazur-Ulam theorem states that every bijective distance preserving map Φ from a Banach space onto another is affine; that is,

$$\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y, 0 \le t \le 1. \tag{1}$$

After translation, we can assume that $\Phi(0) = 0$ and Φ is indeed a surjective real linear isometry. Let us consider another version of this statement. Suppose that Φ is a bijective map from a Hilbert space H onto H and Φ preserves norm of convex combinations:

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in H, \ 0 \le t \le 1. \tag{2}$$

Let us further relax the assumption that (2) holds for just one fixed t in $(0,1)$. By letting $y = x$ in (2), we see that $\|\Phi(x)\| = \|x\|$ for all x in H. Squaring both sides of (2), we will see that the real parts of the inner products coincide; that is,

$$\operatorname{Re}\langle x, y\rangle = \operatorname{Re}\langle \Phi(x), \Phi(y)\rangle, \quad \forall x, y \in H. \tag{3}$$

Then the classical Wigner theorem (see, e.g., [1, Theorem 3]) ensures that there is a surjective real linear isometry $U : H \to H$ such that $\Phi(x) = Ux$ for all x in H.

Characterizing isometries, linear or not, of spaces of operators under various norms has been a fruitful area of research for a long time. See, for example, [2, 3] for good surveys. In particular, the spaces $\mathcal{S}_p(H)$ of the Schatten p-class operators on a (complex) Hilbert space H ($1 \le p < +\infty$) are important objects in both analysis and physics. They are widely used in operator theory and quantum mechanics, for example.

Let $\mathcal{S}_p^+(H)$ be the set of all positive operators in $\mathcal{S}_p(H)$, and let $\mathcal{S}_p^+(H)_1$ be the set of all positive operators in $\mathcal{S}_p^+(H)$ of p-norm one. Recall that an affine automorphism (or S-automorphism in [4] or Kadison automorphism in [5]) is a bijective affine map $\phi : \mathcal{S}_1^+(H)_1 \to \mathcal{S}_1^+(H)_1$; that is,

$$\phi(t\rho + (1-t)\sigma) = t\phi(\rho) + (1-t)\phi(\sigma), \quad \forall \rho, \sigma \in \mathcal{S}_1^+(H)_1, \ t \in [0,1]. \tag{4}$$

It is known (see, e.g., [6]) that affine automorphisms are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary U on H.

Recently, Nagy [7] established a Mazur-Ulam-type result for the Schatten p-class operators. Suppose that $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ ($1 < p < +\infty$) is a bijective map preserving the distance induced by the norm $\|\cdot\|_p$. Then ϕ is implemented by a unitary or an antiunitary operator U such that $\phi(\rho) = U\rho U^*$. In this paper, we will establish a counterpart of Nagy's result similar to the one demonstrated in the first paragraph. More precisely, we will characterize those maps $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfying

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1,\ t \in [0,1]. \tag{5}$$

We will show that they are implemented by a unitary or an antiunitary operator.

Our main theorem follows.

Theorem 1. *Let H be a separable complex Hilbert space of finite or infinite dimension. Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$, which will be assumed to be surjective when* $\dim H = +\infty$. *The following conditions are equivalent.*

(1) *ϕ preserves the Schatten p-norms of convex combinations; that is,*

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1,\ t \in [0,1]. \tag{6}$$

(2) *ϕ preserves the pairings; that is, for all $\rho, \sigma \in \mathcal{S}_p^+(H)_1$, one has $\sigma^{p-1}\rho \in \mathcal{S}_1(H)$, and*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right). \tag{7}$$

(3) *There exists a unitary or antiunitary operator U on H such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_p^+(H)_1. \tag{8}$$

We note that condition (6) becomes a tautology when $p = 1$. On the other hand, the conclusion of Theorem 1 holds again if we replace $\mathcal{S}_p^+(H)_1$ by $\mathcal{S}_p^+(H)$ everywhere. In this case, setting $\sigma = \rho$ in (6), we see that ϕ does map $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$.

The proof of Theorem 1 is given in Section 2. When $p = 2$, we see in Section 3 that for ϕ carrying the expected form stated in Theorem 1(3) it suffices to say that condition (6) held for only one fixed t in $(0,1)$. Finally, we demonstrate some examples in Section 4.

2. Proof of the Main Theorem

In what follows, we fix some notation and definitions used throughout the paper. Let H stand for a separable complex Hilbert space of finite dimension or infinite dimension. Let $B(H)$ denote the algebra of all bounded linear operators on H. For a compact operator T in $B(H)$, let $s_1(T) \ge s_2(T) \ge \cdots \ge 0$ denote the singular values of T, that is, the eigenvalues of $|T| = (TT^*)^{1/2}$ arranged in their decreasing order (repeating according to multiplicity). A compact operator T belongs to the Schatten p-classes $\mathcal{S}_p(H)$ $(1 \le p < +\infty)$ if

$$\|T\|_p := \left(\sum_{i=1}^{\infty} s_i(T)^p\right)^{1/p} = \left(\operatorname{tr}|T|^p\right)^{1/p} < +\infty, \tag{9}$$

where tr denotes the trace functional. We call $\|T\|_p$ the Schatten p-norm of T. In particular, $\mathcal{S}_1(H)$ is the trace class and $\mathcal{S}_2(H)$ is the Hilbert-Schmidt class. The cone of positive operators in $\mathcal{S}_p(H)$ is denoted by $\mathcal{S}_p^+(H)$, and the set of rank one projections in $\mathcal{S}_p^+(H)$ is denoted by $P_1(H)$.

Recall that the norm of a normed space is Fréchet differentiable at $x \ne 0$ if $\lim_{t\to 0}((\|x+ty\| - \|x\|)/t)$ exists and uniform for all norm one vectors y.

Lemma 2 (see [8, Theorem 2.3]). *Let $1 < p < +\infty$ and ρ in $\mathcal{S}_p^+(H)$ be nonzero. The norm of $\mathcal{S}_p^+(H)$ is Fréchet differentiable at ρ. For any σ in $\mathcal{S}_p^+(H)$, one has*

$$\left.\frac{d\|\rho + t\sigma\|_p}{dt}\right|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right). \tag{10}$$

Lemma 3. *Suppose $\rho, \sigma \in \mathcal{S}_p^+(H)$ $(1 < p < +\infty)$. The following conditions are equivalent.*

(1) *$\rho = \sigma$.*

(2) *$\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ for all P in $P_1(H)$ and all t in $[0,1]$.*

(3) *$\operatorname{tr}(P\rho) = \operatorname{tr}(P\sigma)$ for all P in $P_1(H)$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): Differentiating both sides of $\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ at $t = 0^+$, we have $\operatorname{tr} P\rho = \operatorname{tr} P^{p-1}\rho = \operatorname{tr} P^{p-1}\sigma = \operatorname{tr} P\sigma$ by Lemma 2.

(3) $\Rightarrow$ (1): Since ρ and σ are positive, $\rho - \sigma$ is Hermitian. There exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho - \sigma = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$. Choosing $P_i = e_i \otimes e_i$, we have $\lambda_i = \operatorname{tr}(P_i(\rho - \sigma)) = 0$ for all $i = 1, 2, \ldots$. It follows that $\rho - \sigma = 0$. □

We say that two self-adjoint operators ρ, σ in $B(H)$ are orthogonal if $\rho\sigma = 0$, which is equivalent to the property that they have mutually orthogonal ranges.

Lemma 4. *Suppose that $\rho, \sigma \in \mathcal{S}_p^+(H)$ for $1 < p < +\infty$. The following conditions are equivalent.*

(1) *ρ, σ are orthogonal; that is, $\rho\sigma = 0$.*

(2) *$\|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p\|\rho\|_p^p + (1-\alpha)^p\|\sigma\|_p^p$ for any (and thus all) α in $(0,1)$.*

(3) *$\operatorname{tr}(\rho\sigma) = 0$.*

(4) *$\|\rho + t\sigma\|_p \ge \|\rho\|_p$ for all t in $\mathbb{R}$; that is, $\rho \perp \sigma$ in Birkhoff's sense.*

(5) *$\operatorname{tr}(\rho^{p-1}\sigma) = 0$.*

Proof. (1) $\Leftrightarrow$ (2): From [9, Lemma 2.6], we know that for any two positive operators A, B in $\mathcal{S}_p^+(H)$, we have

$$\operatorname{tr}(A+B)^p \ge \operatorname{tr} A^p + \operatorname{tr} B^p. \tag{11}$$

Here, the equality holds if and only if $AB = 0$. Setting $A = \alpha\rho$ and $B = (1-\alpha)\sigma$, we get

$$\begin{aligned}\rho\sigma = 0 &\Longleftrightarrow (\alpha\rho)((1-\alpha)\sigma) = 0\\ &\Longleftrightarrow \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^p = \operatorname{tr}(\alpha\rho)^p + \operatorname{tr}((1-\alpha)\sigma)^p\\ &\Longleftrightarrow \|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p\|\rho\|_p^p + (1-\alpha)^p\|\sigma\|_p^p.\end{aligned} \tag{12}$$

(1) $\Leftrightarrow$ (3): One direction is obvious. For the other, because ρ, σ are positive,

$$\begin{aligned}&\operatorname{tr}\left[\left(\rho^{1/2}\sigma^{1/2}\right)\left(\rho^{1/2}\sigma^{1/2}\right)^*\right]\\ &\quad= \operatorname{tr}\left(\rho^{1/2}\sigma^{1/2}\sigma^{1/2}\rho^{1/2}\right) = \operatorname{tr}(\rho\sigma) = 0.\end{aligned} \tag{13}$$

This forces $\rho^{1/2}\sigma^{1/2} = 0$, and thus $\rho\sigma = \rho^{1/2}(\rho^{1/2}\sigma^{1/2})\sigma^{1/2} = 0$.

(1) $\Rightarrow$ (4): Since $\rho\sigma = 0$, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho = \sum_{i=1}^{\infty}\lambda_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^{\infty}\mu_i e_i \otimes e_i$, $\lambda_i \geq 0$, $\mu_i \geq 0$, and $\lambda_i\mu_i = 0$ for all $i = 1, 2, \ldots$. Hence,

$$\begin{aligned}\|\rho + t\sigma\|_p^p &= \operatorname{tr}|\rho + t\sigma|^p\\ &= \sum_{i=1}^{\infty}(\lambda_i + |t|\mu_i)^p \geq \sum_{i=1}^{\infty}\lambda_i^p = \|\rho\|_p^p.\end{aligned} \tag{14}$$

(4) $\Rightarrow$ (5): Without loss of generality, we can assume that $\rho \neq 0$. Define $f(t) = \|\rho + t\sigma\|_p \geq \|\rho\|_p$. Then $f(t)$ is differentiable and attains its minimum at $t = 0$. From Lemma 2,

$$0 = \left.\frac{d\|\rho + t\sigma\|_p}{dt}\right|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right), \tag{15}$$

and assertion (5) follows.

(5) $\Rightarrow$ (1): As in proving (3) $\Rightarrow$ (1), we have $\rho^{p-1}\sigma = 0$. Then, there exists an orthonormal basis $\{e_i'\}_{i=1}^{\infty}$ of H such that $\rho^{p-1} = \sum_{i=1}^{\infty}\xi_i e_i' \otimes e_i'$, $\sigma = \sum_{i=1}^{\infty}\eta_i e_i' \otimes e_i'$, with $\xi_i \geq 0$, $\eta_i \geq 0$, and $\xi_i\mu_i = 0$ for all $i = 1, 2, \ldots$. Thus, $\operatorname{tr}(\rho\sigma) = \sum_{i=1}^{\infty}\xi_i^{1/(p-1)}\eta_i = 0$. □

Lemma 5. *Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$ preserving the Schatten p-norms of convex combinations; that is,* (6) *holds. Then, one has*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right). \tag{16}$$

Proof. Differentiating both sides of (6) with respect to t and evaluating at $t = 0$, we have

$$\begin{aligned}\left.\frac{d\|t\rho + (1-t)\sigma\|_p}{dt}\right|_{t=0} &= \left.\frac{d\|\sigma + t(\rho - \sigma)\|_p}{dt}\right|_{t=0}\\ &= \operatorname{tr}\left(\frac{\sigma^{p-1}(\rho - \sigma)}{\|\sigma\|_p^{p-1}}\right)\\ &= \frac{\operatorname{tr}\left(\sigma^{p-1}\rho\right)}{\|\sigma\|_p^{p-1}} - \|\sigma\|_p\\ &= \operatorname{tr}\left(\sigma^{p-1}\rho\right) - 1,\\ \left.\frac{d\|t\phi(\rho) + (1-t)\phi(\sigma)\|_p}{dt}\right|_{t=0} &= \frac{\operatorname{tr}\left(\phi(\sigma)^{p-1}\rho\right)}{\|\phi(\sigma)\|_p^{p-1}} - \|\phi(\sigma)\|_p\\ &= \operatorname{tr}\left(\phi(\sigma)^{p-1}\rho\right) - 1.\end{aligned} \tag{17}$$

Since (6) holds for t in $(0, 1]$, these derivatives agree. Therefore, $\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho))$. □

Proposition 6. *Suppose that $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfies*

$$\operatorname{tr}\left(\sigma^{p-1}\rho\right) = \operatorname{tr}\left(\phi(\sigma)^{p-1}\phi(\rho)\right), \quad \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1. \tag{18}$$

Then the following assertions hold.

(1) *ϕ preserves orthogonality in both directions; that is*

$$\rho\sigma = 0 \Longleftrightarrow \phi(\rho)\phi(\sigma) = 0, \quad \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1. \tag{19}$$

(2) *When* $\dim H < +\infty$, *ϕ maps rank-one projections to rank-one projections. This also holds when* $\dim H = +\infty$ *and ϕ is surjective.*

(3) *When* $\dim H < +\infty$, *one has*

$$\operatorname{tr} PQ = \operatorname{tr}\phi(P)\phi(Q), \quad \forall P, Q \in P_1(H). \tag{20}$$

This also holds when $\dim H = +\infty$ *and ϕ is surjective.*

Proof. (1) follows from Lemma 4.

(2) First, we assume that $\dim H = n < +\infty$. Suppose ρ is a rank-one projection. We can find $n-1$ pairwise orthogonal rank-one projections $\rho_1, \ldots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for $1 \leq i \leq n-1$. From (1), we know that $\phi(\rho), \phi(\rho_1), \ldots, \phi(\rho_{n-1})$ are nonzero and pairwise orthogonal. This forces that $\phi(\rho)$ has rank one since $\dim H = n$. By (18), taking $\sigma = \rho$, we see that $\operatorname{tr}\phi(\rho)^p = \operatorname{tr}\rho^p = \operatorname{tr}\rho = 1$. Therefore, the rank-one positive operator $\phi(\rho)$ is a projection.

Next, we consider the case $\dim H = +\infty$ and ϕ is surjective. Suppose that there exists a rank-one projection ρ in $\mathcal{S}_p^+(H)$ such that $\phi(\rho)$ has rank greater than one. Then, there are two nonzero orthogonal operators T_1 and T_2 in $\mathcal{S}_p^+(H)$ such that $\phi(\rho) = T_1 + T_2$. Since ϕ is surjective and preserves orthogonality in both directions, there are two nonzero orthogonal operators ρ_1 and ρ_2 in $\mathcal{S}_p^+(H)_1$ such that $\phi(\rho_1) = T_1/\|T_1\|_p$ and $\phi(\rho_2) = T_1/\|T_2\|_p$. For any σ in $\mathcal{S}_p^+(H)$ with $\sigma\rho = 0$, we have

$$\begin{aligned} &\phi(\sigma)\left(\|T_1\|_p\phi(\rho_1) + \|T_2\|_p\phi(\rho_2)\right) \\ &\quad = \phi(\sigma)(T_1 + T_2) = \phi(\sigma)\phi(\rho) = 0. \end{aligned} \tag{21}$$

It forces that

$$\|T_1\|_p\phi(\sigma)\phi(\rho_1)\phi(\sigma) = -\|T_2\|_p\phi(\sigma)\phi(\rho_2)\phi(\sigma) = 0, \tag{22}$$

and hence $\phi(\sigma)\phi(\rho_1) = \phi(\sigma)\phi(\rho_2) = 0$, because $\phi(\sigma)$, $\phi(\rho_1)$, and $\phi(\rho_2)$ are all positive. This implies $\sigma\rho_1 = \sigma\rho_2 = 0$. Therefore, $\rho_1 = \lambda_1\rho$ and $\rho_2 = \lambda_2\rho$ for some nonzero λ_1, λ_2. This contradicts the fact that $\rho_1\rho_2 = 0$.

(3) From (2), we know that $\phi(P)$, $\phi(Q)$ are rank-one projections in $P_1(H)$. Therefore, $P^{p-1} = P$, $\phi(P)^{p-1} = \phi(P)$. Using (18) with $\sigma = P$, $\rho = Q$, we have

$$\operatorname{tr} PQ = \operatorname{tr}\left(P^{p-1}Q\right) = \operatorname{tr}\left(\phi(P)^{p-1}\phi(Q)\right) = \operatorname{tr}\phi(P)\phi(Q). \tag{23}$$

□

Proof of Theorem 1. (1) ⇒ (2) follows from Lemma 5.

(3) ⇒ (1) is obvious.

(2) ⇒ (3): From Proposition 6, we obtain that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ satisfies $\operatorname{tr} PQ = \operatorname{tr}\phi(P)\phi(Q)$ for all rank-one projections P, Q in $P_1(H)$. From a nonsurjective version of Wigner's theorem, cf. [6, Theorem 2.1.4], there exists an isometry or anti-isometry U on H such that

$$\phi(P) = UPU^*, \quad \forall P \in P_1(H). \tag{24}$$

Note that U is indeed surjective even when H is of infinite dimension, since ϕ is assumed to be surjective in this case.

For any rank-one projection P in $P_1(H)$, setting $\sigma = P$ in (7), we have

$$\begin{aligned} \operatorname{tr}(P\rho) &= \operatorname{tr}\left(P^{p-1}\rho\right) = \operatorname{tr}\left(\phi(P)^{p-1}\phi(\rho)\right) = \operatorname{tr}(\phi(P)\phi(\rho)) \\ &= \operatorname{tr}(UPU^*\phi(\rho)U) = \operatorname{tr}(PU^*\phi(\rho)U). \end{aligned} \tag{25}$$

We have $U^*\phi(\rho)U = \rho$ by Lemma 3. This gives $\phi(\rho) = U\rho U^*$. □

3. Maps Preserving Norms of Just a Special Convex Combination

A careful look at the proof of Lemma 5 tells us that the condition $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ suffices to hold for the members of any sequence in $(0, 1]$ converging to 0 rather than for any point t in $[0, 1]$. Indeed, in order to get some good properties of ϕ stated in the previous section, we only need to assume that ϕ preserves the Schatten p-norm of convex combination with a given system of coefficients.

Proposition 7. *Let $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ $(1 < p < +\infty)$. Let α in $(0, 1)$ be arbitrary but fixed. Suppose*

$$\begin{aligned} &\|\alpha\rho + (1-\alpha)\sigma\|_p = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p, \\ &\qquad \forall\rho, \sigma \in \mathcal{S}_p^+(H)_1. \end{aligned} \tag{26}$$

The following properties are satisfied.

(1) *ϕ is injective.*

(2) *ϕ preserves orthogonality in both directions.*

(3) *When $\dim H < +\infty$, ϕ maps rank-one projections to rank-one projections. This also holds when $\dim H = +\infty$ and ϕ is surjective.*

Proof. (1) Assume $\phi(\rho) = \phi(\sigma)$. We have $\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p = 1$. From (26), we get $\|\alpha\rho + (1-\alpha)\sigma\|_p = 1$. Hence,

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \alpha\|\rho\|_p + (1-\alpha)\|\sigma\|_p. \tag{27}$$

This forces $\rho = \sigma$ since the norm $\|\cdot\|_p$ is strictly convex for $1 < p < +\infty$.

(2) Assume $\rho\sigma = 0$. From Lemma 4, we have

$$\begin{aligned} \|\alpha\rho + (1-\alpha)\sigma\|_p^p &= \alpha^p\|\rho\|^p + (1-\alpha)^p\|\sigma\|^p \\ &= \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p. \end{aligned} \tag{28}$$

Together with (26), we have

$$\begin{aligned} &\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p^p \\ &\quad = \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p. \end{aligned} \tag{29}$$

Hence, we have $\phi(\rho)\phi(\sigma) = 0$ from Lemma 4 again. The other implication follows similarly.

(3) The proof is similar to that of Proposition 6(2). □

When $p = 2$, we get an improvement of Theorem 1.

Theorem 8. *Let H be a separable complex Hilbert space. Suppose that $\phi : \mathcal{S}_2^+(H)_1 \to \mathcal{S}_2^+(H)_1$, which needs to be surjective when $\dim H = +\infty$. The following conditions are equivalent.*

(1) *ϕ preserves the Hilbert-Schmidt norms of all convex combinations; that is,*

$$\begin{aligned} &\|t\rho + (1-t)\sigma\|_2 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_2, \\ &\qquad \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1,\ t \in [0, 1]. \end{aligned} \tag{30}$$

(2) *For any (and thus all) α in $(0, 1)$ one has*

$$\begin{aligned} &\|\alpha\rho + (1-\alpha)\sigma\|_2 = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2, \\ &\qquad \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1. \end{aligned} \tag{31}$$

A special case states that

$$\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2, \quad \forall\rho, \sigma \in \mathcal{S}_2^+(H)_1. \tag{32}$$

(3) $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$ *for all* ρ, σ *in* $\mathcal{S}_2^+(H)_1$.

(4) *There exists a unitary or antiunitary operator* U *such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_2^+(H)_1. \tag{33}$$

Proof. We prove (2) ⇒ (3) only. Observe

$$\begin{aligned}\|\alpha\rho + (1-\alpha)\sigma\|_2^2 &= \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^2 \\ &= \alpha^2 \operatorname{tr}\rho^2 + 2\alpha(1-\alpha)\operatorname{tr}(\rho\sigma) \\ &\quad + (1-\alpha)^2 \operatorname{tr}\sigma^2, \\ \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2^2 &= \alpha^2 \operatorname{tr}\phi(\rho)^2 \\ &\quad + 2\alpha(1-\alpha)\operatorname{tr}(\phi(\rho)\phi(\sigma)) \\ &\quad + (1-\alpha)^2 \operatorname{tr}\phi(\sigma)^2.\end{aligned} \tag{34}$$

We have $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$. □

4. Examples

We remark that all results in previous sections hold for a map $\phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$ which satisfies instead of (6) the condition

$$\begin{gathered}\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \\ \forall \rho, \sigma \in \mathcal{S}_p^+(H), \ t \in [0,1].\end{gathered} \tag{35}$$

The proofs go in exactly the same ways.

The following example shows that a norm preserver of $\mathcal{S}_p^+(H)$ might not be affine.

Example 1. Let H be a finite dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$. Let $1 < p < +\infty$. Define a map ϕ from $\mathcal{S}_p^+(H)$ into itself by

$$\phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\ \dfrac{\|\rho\|_p}{\left\|\sum_{i=1}^n P_i\rho P_i\right\|_p} \displaystyle\sum_{i=1}^n P_i\rho P_i, & \text{if } \rho \neq 0, \end{cases} \tag{36}$$

where $P_i = e_i \otimes e_i$ is a rank-one projection for $i = 1, \ldots, n$. Obviously, $\phi(\rho)$ is positive and $\|\phi(\rho)\|_p = \|\rho\|_p$ for all ρ in $\mathcal{S}_p^+(H)$. However, ϕ does not preserve the Schatten p-norms of convex combinations, as the eigenvalues of ρ and $\phi(\rho)$ can be different from each other.

Our theorems are about the Schatten p-norms for $1 < p < +\infty$. Here is an example of a map of $\mathcal{S}_1^+(H)$ which preserves trace norms of convex combinations. However, it is not implemented by a unitary or antiunitary.

Example 2. Consider Example 1 in the case where $p = 1$. In this case,

$$\phi(\rho) = \sum_{i=1}^n P_i\rho P_i. \tag{37}$$

It is easy to see that the map ϕ satisfies the condition

$$\begin{gathered}\|t\rho + (1-t)\sigma\|_1 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_1, \\ \forall \rho, \sigma \in \mathcal{S}_1^+(H), \ t \in [0,1].\end{gathered} \tag{38}$$

But there does not exist a unitary or antiunitary U such that $\phi(\rho) = U\rho U^*$ for all ρ in $\mathcal{S}_1^+(H)$.

Example 3. Let H be a separable Hilbert space of infinite dimension, and let $\{e_n : n = 1, 2, \ldots\}$ be a basis of H. Let S be the unilateral shift on H defined by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$. Let ϕ be defined by $\phi(\rho) = S\rho S^*$ for ρ in $\mathcal{S}_p^+(H)$. The map ϕ is not surjective, as $e_1 \otimes e_1$ is not in its range. It is easy to see that $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ holds for all ρ, σ in $\mathcal{S}_p^+(H)$ and t in $[0,1]$. However, ϕ is not implemented by a unitary or antiunitary.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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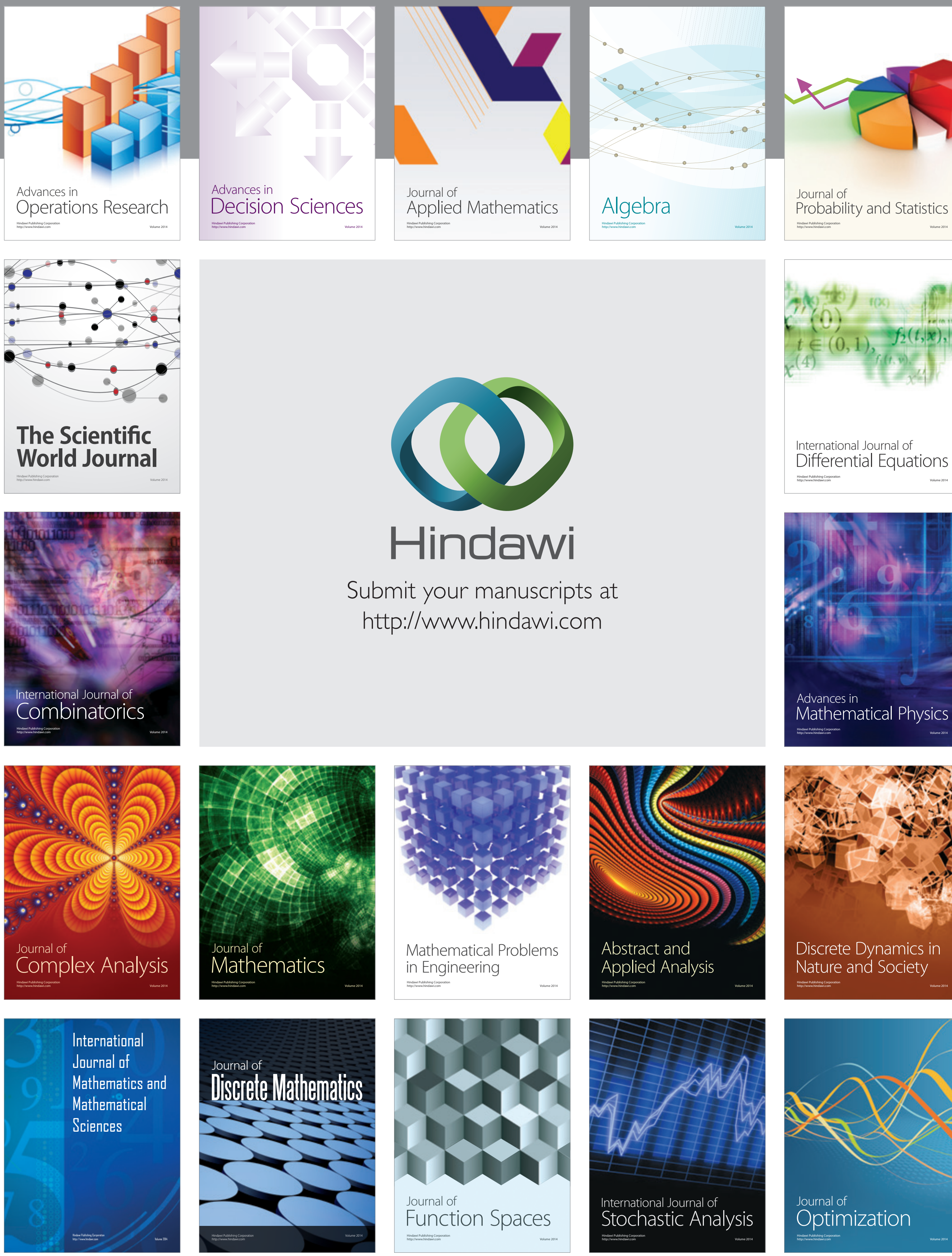
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