

CONTINUOUS SELECTIONS AND FIXED POINTS  
OF MULTI-VALUED MAPPINGS  
ON NONCOMPACT OR NONMETRIZABLE SPACES

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ABSTRACT. In this paper, we obtain several new continuous selection theorems for multi-valued mappings on completely regular spaces and fixed point theorems for multi-valued maps on nonmetrizable spaces. They, in particular, provide a partial solution of a conjecture of X. Wu.

1. INTRODUCTION

In [4, 5], Browder first used a continuous selection theorem to prove the Fan-Browder fixed point theorem. Later, Yannelis and N. D. Prabhakar [17], Ben-El-Mechaiekh [2, 3], Ding, Kim and Tan [8], Horvath [11], Wu [16, 15], Park [12, 13], and many others, established several continuous selection theorems with applications. We note that in all the continuous selection theorems studied by the above authors, the multi-valued maps are defined on a compact or paracompact space. In [18], Yu and Lin studied continuous selections of multi-valued mappings defined on noncompact spaces, but they assume some kind of coercivity conditions instead.

In this paper, we establish a continuous selection theorem for a multi-valued map defined on a completely regular topological space. We do not assume the compactness of its domain.

In the second part of this paper, we discuss collectively fixed points of lower semicontinuous multi-valued maps. Recently, many authors studied fixed point theorems of lower semicontinuous multi-valued maps; see for example [14, 6, 15, 1]. In particular, Wu established the following one.

**Theorem 1.1** ([15]). *Let  $X$  be a nonempty subset of a Hausdorff locally convex topological vector space, let  $D$  be a nonempty compact metrizable subset of  $X$ , and let  $T : X \rightarrow 2^D$  be a multi-valued mapping with the following properties:*

- (a)  $T(x)$  is a nonempty closed convex set for each  $x$  in  $X$ ;
- (b)  $T$  is lower semicontinuous.

*Then there exists a point  $\bar{x}$  in  $D$  such that  $\bar{x} \in T(\bar{x})$ .*

Wu conjectured in [15] that the conclusion of Theorem 1.1 remains true even if the metrizability condition of  $D$  is dropped. In this paper, we shall use the

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approximate continuous selection theorem of Deutsch and Kenderov [7] (see also [20]) to establish an approximate fixed point theorem for a sub-lower semicontinuous multi-valued map. This gives rise to a partial solution of the conjecture of Wu [15]. We shall also provide a simple proof of a Himmelberg type collectively fixed point theorem. We remark that our results differ from the approximate fixed point theorem recently established by Park [13].

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## 2. PRELIMINARIES

Let  $X$  and  $Y$  be topological spaces. A multi-valued map  $T : X \rightarrow 2^Y$  is a map from  $X$  into the power set  $2^Y$  of  $Y$ . Let  $T^{-1} : Y \rightarrow 2^X$  be defined by the condition that  $x \in T^{-1}y$  if and only if  $y \in T(x)$ . Recall that

- (a)  $T$  is said to be *closed* if its graph  $G_r(T) = \{(x, y) : x \in X, y \in T(x)\}$  is closed in the product space  $X \times Y$ ;
- (b)  $T$  is said to be *upper semicontinuous* (in short, u.s.c.) at  $x$  if for every open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists a neighborhood  $W(x)$  of  $x$  such that  $T(W(x)) \subset V$ ;  $T$  is said to be u.s.c. on  $X$  if  $T$  is u.s.c. at every point of  $X$ ;
- (c)  $T$  is said to be *lower semicontinuous* (in short, l.s.c.) at  $x$  if for every open neighborhood  $V(y)$  of every  $y$  in  $T(x)$ , there exists a neighborhood  $W(x)$  of  $x$  such that  $T(u) \cap V(y) \neq \emptyset$  for all  $u$  in  $W(x)$ ;  $T$  is said to be l.s.c. on  $X$  if  $T$  is l.s.c. at every point of  $X$ ;
- (d) in case  $Y$  is a topological linear space,  $T$  is said to be *sub-lower semicontinuous* (see, e.g., [20]) at an  $x$  in  $X$  if for each neighborhood  $V$  of 0 in  $Y$ , there is a  $z$  in  $T(x)$  and a neighborhood  $U(x)$  of  $x$  in  $X$  such that  $z \in T(y) + V$  for each  $y$  in  $U(x)$ ;  $T$  is said to be sub-lower semicontinuous on  $X$  if  $T$  is sub-lower semicontinuous at every point of  $X$ . It is plain that if  $T$  is lower semicontinuous at  $x$ , then  $T$  is sub-lower semicontinuous at  $x$ .

The following lemmas are needed in this paper.

**Lemma 2.1** (Deutsch and Kenderov [7]). *Let  $X$  be a paracompact topological space, let  $Y$  be a locally convex topological linear space, and let  $F : X \rightarrow 2^Y$ . Then  $F$  is sub-lower semicontinuous if and only if for each neighborhood  $V$  of 0 in  $Y$ , there is a continuous function  $f : X \rightarrow Y$  such that  $f(x) \in F(x) + V$  for each  $x$  in  $X$ .*

**Lemma 2.2** (Yuan [19]). *Let  $X$  be a topological space, let  $Y$  be a nonempty subset of a topological vector space with a base  $\mathcal{B}$  for the zero neighborhoods, and let  $F : X \rightarrow 2^Y$ . For each  $V$  in  $\mathcal{B}$ , define  $F_V : X \rightarrow 2^Y$  by*

$$F_V(x) = (F(x) + V) \cap Y, \quad \forall x \in X.$$

*Write  $\bar{y} \in \overline{F}(\bar{x})$  if  $(\bar{x}, \bar{y}) \in \overline{G_r F}$ . Then for any  $\bar{x}$  in  $X$  and  $\bar{y}$  in  $Y$ , we have*

$$\bar{y} \in \overline{F}(\bar{x}) \quad \text{whenever} \quad \bar{y} \in \bigcap_{V \in \mathcal{B}} \overline{F_V}(\bar{x}).$$

**Lemma 2.3** (Himmelberg [10]). *Let  $X$  be a nonempty convex subset of a locally convex topological vector space. Let  $T : X \rightarrow 2^X$  be a u.s.c. multi-valued map*

with nonempty closed convex values such that  $T(X) = \bigcup_{x \in X} T(x)$  is contained in a compact subset of  $X$ . Then there exists an  $\bar{x}$  in  $X$  such that  $\bar{x} \in T(\bar{x})$ .

**Lemma 2.4** (Granas [9]; see also Ding, Kim and Tan [8]). *Let  $D$  be a nonempty compact subset of a topological vector space. Then the convex hull  $\text{co}D$  of  $D$  is  $\sigma$ -compact and hence is paracompact.*

### 3. CONTINUOUS SELECTION THEOREMS

Note that the set  $S^{-1}(y) = \{x \in X : y \in S(x)\}$  below can have empty interior for some  $y$  in  $K$ .

**Theorem 3.1.** *Let  $X$  be a completely regular space and let  $K$  be a nonempty subset of a Hausdorff topological vector space  $E$ . Assume a multi-valued function  $S : X \rightarrow 2^K$  satisfies the following conditions:*

- (a) *For each  $x$  in  $X$ , the set  $S(x)$  is convex.*
- (b)  $X = \bigcup \{\text{int } S^{-1}(y) : y \in K\}$ .

*Then for any compact subset  $F$  of  $X$  there is an open dense subset  $U$  of  $X$  containing  $F$  such that  $S$  has a continuous selection  $f : U \rightarrow K$ , that is,  $f(x) \in S(x)$  for all  $x$  in  $U$ .*

*Proof.* By assumption (b), there are finitely many points  $y_1, \dots, y_n$  in  $K$  such that

$$F \subseteq \text{int } S^{-1}(y_1) \cup \dots \cup \text{int } S^{-1}(y_n).$$

For each  $k = 1, \dots, n$  and  $x$  in  $F \cap \text{int } S^{-1}(y_k)$ , there is a continuous function  $g_x$  on  $X$  such that  $0 \leq g_x \leq 1$ ,  $g_x(x) = 1$  and  $g_x$  vanishes outside  $\text{int } S^{-1}(y_k)$ . By the compactness of  $F$ , there are finitely many  $g_x$  such that for every point in  $F$  at least one of them assumes a value not less than  $1/2$ . Summing them in an appropriate way, we will have nonnegative continuous functions  $g_1, \dots, g_n$  on  $X$  such that  $g_k$  vanishes outside  $\text{int } S^{-1}(y_k)$ , and  $\sum_{k=1}^n g_k(x) \geq 1/2$  for all  $x$  in  $F$ . Let  $V = \{x \in X : \sum_{k=1}^n g_k(x) > 1/3\}$ . Set  $f_j(x) = g_j(x) / \sum_{k=1}^n g_k(x)$  on  $V$ , and  $f_j(x) = 3g_j(x)$  on  $X \setminus V$ . Define a continuous function  $f_V : X \rightarrow E$  by

$$f_V(x) = \sum_{k=1}^n f_k(x)y_k, \quad \forall x \in X.$$

For each  $x$  in  $V$  and for each  $k$  with  $f_k(x) \neq 0$ , we have  $x \in \text{int } S^{-1}(y_k)$ . Hence,  $y_k \in S(x)$ . Consequently,  $f_V(x) \in \text{co}(S(x)) = S(x) \subseteq K$  for all  $x$  in  $V$ . In other words, the restriction of  $f_V$  to  $V$  gives rise to a continuous selection of  $S$  on the open set  $V$  which contains  $F$ .

Denote by

$\mathcal{W} = \{(f_W, W) : \text{where } W \text{ is an open subset of } X \text{ containing } F \text{ and}$

$f_W : W \rightarrow K \text{ gives rise to a continuous selection of } S \text{ on } W\}$ .

Then  $\mathcal{W}$  is not empty as  $(f_V, V) \in \mathcal{W}$ . Order  $\mathcal{W}$  by extension and we get a nonempty partially ordered set. In other words,  $(f_W, W) \leq (f_V, V)$  if  $W \subseteq V$  and  $f_V|_W = f_W$ . Applying Zorn's Lemma, we get a maximal element  $(f_U, U)$  of  $\mathcal{W}$ .

The last step is to verify that  $U$  is dense in  $X$ . Suppose not and there were an  $x$  in  $X$  outside the closure of  $U$ . Let  $x \in \text{int } S^{-1}(y)$  for some  $y$  in  $K$ . By setting  $f|_W \equiv y$ , we get a continuous selection of  $S$  on an open neighborhood  $W$  of  $x$

disjoint from  $U$  by restriction. Then the union  $f_{U \cup W} : U \cup W \rightarrow K$  defined in a natural way provides a contradiction to the maximality of  $(f_U, U)$ .  $\square$

We call a topological space  $X$  *residually paracompact* if for every open dense subset  $U$  of  $X$  the complement  $X \setminus U$  is paracompact.

**Theorem 3.2.** *In addition to conditions (a) and (b) in Theorem 3.1, if we assume further that*

(c)  $X$  is residually paracompact,

then there is a continuous function  $f : X \rightarrow K$  such that  $f(x) \in S(x)$  for all  $x$  in  $X$ .

*Proof.* It follows from Theorem 3.1 that there is a continuous function  $f_U : U \rightarrow K$  defined on an open dense subset  $U$  of  $X$  with  $f_U(x) \in S(x)$  for all  $x$  in  $U$ . For each  $z$  in  $X \setminus U$ , there is a  $y$  in  $K$  such that  $z \in \text{int } S^{-1}(y)$  by condition (b). By setting  $f|_{W_z} \equiv y$  we get a continuous selection of  $S$  on an open neighborhood  $W_z$  of  $z$ . The paracompactness of  $X \setminus U$  ensures it has a locally finite covering by open sets in  $X$ , each of which is contained in some  $W_z$ . Adding one more open set  $U$ , we have a locally finite open covering of  $X$ . This provides us with a family  $\{g_\lambda\}_\lambda$  of nonzero continuous functions from  $X$  into  $[0, 1]$  dominated by the open sets  $W_z$  and  $U$  such that  $g_\lambda(x) = 0$  for all but finitely many  $\lambda$ 's and  $\sum_\lambda g_\lambda(x) = 1$  for all  $x$  in  $X$ . If  $g_\lambda$  vanishes outside  $U$ , we set  $f_\lambda = f_U$ . Otherwise, we fix a choice of  $z$  such that  $g_\lambda$  vanishes outside  $W_z$ , and set  $f_\lambda = f_{W_z}$ . Define  $f : X \rightarrow K$  by

$$f(x) = \sum_\lambda g_\lambda(x) f_\lambda(x), \quad \forall x \in X.$$

For each  $x$  in  $X$ , only finitely many  $g_\lambda(x)$ 's are nonzero in the sum, and the nonzero terms give rise to a convex combination of points in the convex set  $S(x)$ . Thus  $f(x) \in S(x)$  for all  $x$  in  $X$ .  $\square$

It is easy to see that the following corollary follows from Theorem 3.1.

**Corollary 3.3.** *The conclusion of Theorem 3.1 remains true if conditions (a) and (b) are replaced by*

- (a)' for each  $x$  in  $X$ , the set  $S(x)$  is a nonempty convex set;
- (b)' for each  $y$  in  $K$ , the set  $S^{-1}(y)$  is open.

*Remark 3.4.* Corollary 3.3 implies Theorem 3.1.

*Proof.* Let  $T : X \rightarrow 2^K$  be defined by

$$T(x) = \{y \in K : x \in \text{int } S^{-1}(y)\}.$$

Then  $T^{-1}(y) = \text{int } S^{-1}(y)$  is open for each  $y$  in  $K$ . By (b), for each  $x$  in  $X$ , there exists  $y$  in  $K$  such that  $x \in \text{int } S^{-1}(y)$ . Therefore  $y \in T(x) \neq \emptyset$  for each  $x$  in  $X$ . Let  $H : X \rightarrow 2^K$  be defined by  $H(x) = \text{co } T(x)$ . Then  $H(x)$  is nonempty for each  $x$  in  $X$ , and  $H^{-1}(y)$  is open for each  $y$  in  $K$ . By Corollary 3.3, there is an open dense subset  $U$  of  $X$ , containing any but a fixed compact set  $D$ , and there is a continuous function  $f : U \rightarrow K$  such that  $f(x) \in H(x) = \text{co } T(x) \subset S(x)$  for all  $x$  in  $U$ .  $\square$

## 4. FIXED POINT THEOREMS

**Theorem 4.1.** *For each  $i$  in a nonempty index set  $I$ , let  $X_i$  be a nonempty convex subset of a Hausdorff locally convex topological vector space  $E_i$ , and let  $D_i$  be a compact subset of  $X_i$ . Let  $X = \prod_{i \in I} X_i$  be the product space. Let  $F_i : X \rightarrow 2^{D_i}$  be sub-lower semicontinuous with nonempty convex values. Then for every neighborhood  $V_i$  of 0 in  $E_i$ , there exists a point  $\bar{x}_V = (x_{V_i})$  in  $D = \prod_{i \in I} D_i$  such that  $(\bar{x}_{V_i} + V_i) \cap F_i(\bar{x}_V) \neq \emptyset$  for all  $i$  in  $I$ .*

*Proof.* Let  $V_i$  be a neighborhood of zero in  $E_i$  for each  $i$  in  $I$ . Fix any  $i$  in  $I$ . There exists an absolutely convex neighborhood  $W_i$  of 0 such that  $W_i \subset V_i$ . Note that  $D$  is a compact subset of  $X$ . By Lemma 2.4,  $\text{co } D$  is a paracompact subset of  $X$ . Since  $F_i : X \rightarrow 2^{D_i}$  is a sub-lower semicontinuous multi-valued map with nonempty convex values, by Lemma 2.1 there exists a continuous function  $f_i : \text{co } D \rightarrow D_i$  such that

$$f_i(x) \in (F_i(x) + W_i) \cap D_i \text{ for each } x \in \text{co } D.$$

Define  $f : \text{co } D \rightarrow D$  by  $f(x) = \prod_{i \in I} f_i(x)$  for  $x$  in  $\text{co } D$ . By the Himmelberg fixed point theorem (Lemma 2.3), there exists an  $\bar{x}_V = (\bar{x}_{V_i})_{i \in I}$  in  $\text{co } D$  such that  $\bar{x}_V = f(\bar{x}_V) = \prod_{i \in I} f_i(\bar{x}_V)$ . That is,  $\bar{x}_{V_i} = f_i(\bar{x}_V) \in (F_i(\bar{x}_V) + W_i) \cap D_i$ . Thus,  $\bar{x}_{V_i} \in D_i$  and  $(\bar{x}_{V_i} + W_i) \cap F_i(\bar{x}_V) \neq \emptyset$  for all  $i$  in  $I$ . Since  $W_i \subset V_i$ , we have  $(\bar{x}_i + V_i) \cap F_i(\bar{x}_V) \neq \emptyset$  for all  $i$  in  $I$ .  $\square$

**Theorem 4.2.** *Suppose in Theorem 4.1 we assume further that for each  $x = (\bar{x}_i)_{i \in I} \in X$ , its coordinates  $x_i \notin \overline{F_i(x)} \setminus F_i(x)$  for all  $i$  in  $I$ . Then there exists a point  $\bar{x} = (\bar{x}_i)_{i \in I} \in D = \prod_{i \in I} D_i$  such that  $\bar{x}_i \in F_i(\bar{x})$  for each  $i$  in  $I$ .*

*Proof.* For each  $i$  in  $I$ , let  $\mathcal{B}_i$  be the collection of all absolutely convex open neighborhoods of zero in  $E_i$  and  $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$ . Given any  $V = \prod_{i \in I} V_i$  in  $\mathcal{B}$ , let  $Q_V = \{x \in D : x_i \in \overline{F_{V_i}(x)} \text{ for all } i \text{ in } I\}$ . Then  $Q_V$  is a nonempty closed subset of  $D$  for each  $V$  in  $\mathcal{B}$  by Theorem 4.1. Let  $\{V^{(1)}, \dots, V^{(n)}\}$  be any finite subset of  $\mathcal{B}$ . Write  $V^{(i)} = \prod_{j \in I} V_j^{(i)}$ , where  $V_j^{(i)} \in \mathcal{B}_j$  for each  $i = 1, \dots, n$ . Let  $V' = \prod_{j \in I} (\bigcap_{i=1}^n V_j^{(i)}) \in \mathcal{B}$ . Clearly,  $\emptyset \neq Q_{V'} \subseteq \bigcap_{i=1}^n Q_{V^{(i)}}$ . Therefore, the family  $\{Q_V : V \in \mathcal{B}\}$  has the finite intersection property. Since  $Q_V \subset D$  for all  $V$  in  $\mathcal{B}$  and  $D$  is compact,  $\bigcap_{V \in \mathcal{B}} Q_V \neq \emptyset$ . Let  $\bar{x} = (\bar{x}_i)_{i \in I} \in \bigcap_{V \in \mathcal{B}} Q_V$ . Then  $\bar{x}_i \in \overline{F_{V_i}(\bar{x})}$  for all  $i$  in  $I$  and all  $V_i$  in  $\mathcal{B}_i$ , i.e.,  $\bar{x}_i \in \bigcap_{V_i \in \mathcal{B}_i} \overline{F_{V_i}(\bar{x})}$  for all  $i$  in  $I$ . It follows from Lemma 2.2 that  $\bar{x}_i \in \overline{F_i(\bar{x})}$  for all  $i$  in  $I$ . By assumption,  $\bar{x}_i \in F_i(\bar{x})$  for all  $i$  in  $I$ .  $\square$

We remark that if  $F_i$  is closed, then  $x_i \notin \overline{F_i(x)} \setminus F_i(x)$  for each  $x = (x_i)_{i \in I}$  in  $X$ . As a special case of Theorem 4.2, we have the following collectively Himmelberg type fixed point theorem.

**Corollary 4.3.** *For each  $i$  in a nonempty index set  $I$ , let  $X_i$  be a nonempty convex subset of a locally convex topological vector space  $E_i$ , let  $D_i$  be a nonempty compact subset of  $X_i$ , and let  $f_i : X = \prod_{i \in I} X_i \rightarrow D_i$  be a continuous function. Then there exists  $\bar{x} = (\bar{x}_i)_{i \in I} \in D = \prod_{i \in I} D_i$  such that  $\bar{x}_i = f_i(\bar{x})$  for each  $i$  in  $I$ .*

If the index set  $I$  is a singleton, then Theorem 4.2 reduces to the following corollary, which provides a partial solution to a conjecture of Wu [15].

**Corollary 4.4.** *Let  $X$  be a nonempty convex subset of a locally convex topological vector space  $E$ , let  $D$  be a nonempty compact subset of  $X$ , and let  $F : X \rightarrow 2^D$  be sub-lower semicontinuous with nonempty convex values. Suppose  $x \notin \overline{F}(x) \setminus F(x)$  for each  $x$  in  $X$ . Then there exists a point  $\bar{x}$  in  $D$  such that  $\bar{x} \in F(\bar{x})$ .*

By Theorem 4.1, we have the following almost fixed point theorem.

**Corollary 4.5.** *The conclusions of Theorems 4.1 and 4.2 remain valid if the condition “ $F_i : X \rightarrow 2^{D_i}$  is sub-lower semicontinuous for each  $i$  in  $I$ ” is replaced by “ $F_i^{-1}(y_i)$  is open for each  $y_i$  in  $D_i$  and each  $i$  in  $I$ .”*

Finally we remark that in case  $I$  is a singleton, Theorem 4.1 provides a different result from [12, Theorem 3].

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