A new complete Calabi–Yau metric on $\mathbb{C}^3$

Yang Li[1]

Abstract Motivated by the study of collapsing Calabi–Yau 3-folds with a Lefschetz K3 fibration, we construct a complete Calabi–Yau metric on $\mathbb{C}^3$ with maximal volume growth, which in the appropriate scale is expected to model the collapsing metric near the nodal point. This new Calabi–Yau metric has singular tangent cone at infinity $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, and its Riemannian geometry has certain non-standard features near the singularity of the tangent cone, which are more typical of adiabatic limit problems. The proof uses an existence result in H-J. Hein's Ph.D. thesis to perturb an asymptotic approximate solution into an actual solution, and the main difficulty lies in correcting the slowly decaying error terms.

1 Motivations

This work grows out of the attempt to model the collapsing behaviour of Calabi–Yau metrics on a K3 fibred compact Calabi–Yau manifold (*i.e.* Kähler Ricci-flat with parallel nonvanishing holomorphic volume form) over a Riemann surface, where the Kähler class has very small volume on the K3 fibres. Assuming the only singularities in the fibration are nodal, we wish to understand the metric near the critical points of the fibration, in the standard local model of the Lefschetz fibration $f : \mathbb{C}^3 \to \mathbb{C}$ with $f = z_1^2 + z_2^2 + z_3^2$. Knowledge of the model metric is often useful in gluing constructions. The prototype

✉ Yang Li
   yang.li15@imperial.ac.uk

1   Imperial College London, London, UK

examples are the Kummer construction [3] and the Gross-Wilson construction [4] of CY metrics on K3 surfaces, modelled on the Eguchi-Hanson metric and the Ooguri-Vafa metric, respectively.

The main result of this paper is:

Theorem 1.1 *There is a complete Calabi–Yau metric of maximal volume growth on $\mathbb{C}^3$ with the standard Euclidean holomorphic volume form, whose tangent cone at infinity is the singular cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.*

Remark The leading order asymptote and the error estimate will be given in the course of proof (*cf.* (14) and proposition 6.8).

We briefly summarize the method. It is natural to write down the so called semi-Ricci flat metric as a first approximation, which restricts to the Stenzel metrics on the quadric fibres of $f = z_1^2 + z_2^2 + z_3^2 : \mathbb{C}^3 \to \mathbb{C}$. This is a singular metric admitting a large symmetry group, suggesting that the construction of Calabi–Yau metric is a cohomogeneity two problem. After appropriate scaling and some modifications to the ansatz, we construct a metric whose volume form is asymptotic to a multiple of the standard Euclidean volume form on $\mathbb{C}^3$. We carefully examine the error terms involved in the approximate metric, and then apply standard techniques for the Monge–Ampère equation on noncompact manifolds to perturb this into a genuine Calabi–Yau metric.

This Calabi–Yau metric is interesting from several other viewpoints. One context is Gromov-Hausdorff convergence theory in Riemannian geometry. Our metric is a nontrivial Ricci-flat metric on $\mathbb{C}^3$ with maximal volume growth, whose tangent cone at infinity is the singular variety $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$; this falsifies a conjecture [14] in the field. The neighbourhood of vanishing cycles in the Lefschetz fibration are asymptotically collapsed to the nodal line, similar to the picture in the unpublished work of Hein and Naber on A_k-type singularities. From the viewpoints of Euclidean geometry this is counterintuitive as the collapsed region apparently carries a large amount of Lebesgue measure. The behaviour of the Laplacian is likewise non-standard in this region; in fact the cohomogeneity two problem of inverting the Laplacian effectively reduces to solving ODEs, in contrast to the Euclidean intuition, which suggests a genuine dependence of the Laplacian on both variables.

Another context in which our metric is expected to arise is related to Joyce's construction of G_2 manifolds [7]. This typically involves a G_2 orbifold with singularities along an associative submanifold, whose normal direction is modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Joyce and Karigiannis [8] consider desingularising the orbifold, using a family of Eguchi-Hanson metrics to replace $\mathbb{C}^2/\mathbb{Z}_2$. A more subtle question then is to understand what happens if some Eguchi-Hanson fibres are allowed to be singular. In the dimensionally reduced case where the associative submanifold is just the product of S^1 with a complex curve, it

seems plausible that our metric with suitable scaling conventions will model this situation.

It should be pointed out that this work overlaps to some extent with the substantial literature on Calabi–Yau metrics on non-compact manifolds, due to many authors, notably Yau, Tian, Hein, Joyce, Kovalev, Haskins and Nordström, among others. One closely related work is Joyce's construction of Quasi-ALE Calabi–Yau metrics [7], since our asymptotic cone is a finite group quotient of $\mathbb{C}^3$. Joyce's construction is more general and combinatorial in spirit, but the decay rate to the asymptotic flat metric is much faster in his case, and we do not use his result in a direct way. The announced work of Hein and Naber about A_k-type singularities contains many common geometric features, and may use similar strategies. In our construction of Calabi–Yau metric, the work we use substantially is Hein's PhD thesis [6], which gives a criterion for perturbing an approximate Calabi–Yau metric into a genuine one.

This work has since been generalised to higher dimensions by the recent independent work of R. Conlon and F. Rochon [1], and G. Székelyhidi [12], where the fibration f is replaced by more complicated fibrations $\mathbb{C}^n \to \mathbb{C}$ defined using a large class of weighted homogenous polynomials. The difficulty comes from the computational complexity of precisely understanding the main error terms in the ansatz, and developing the linear theory to correct these errors which tend to have very slow decay rate near infinity. Some informal discussions on the moduli question of classifying complete Calabi–Yau metrics on $\mathbb{C}^n$ with maximal volume growth will be given at the end.

2 The Stenzel metric

We review the Stenzel construction of Ricci-flat metrics on the smoothings of ordinary double point ('ODP') singularities [11], to set up some notations and to derive some useful formulae. Let $n \geq 3$. We start with the standard quadratic function

$$f = \sum_1^n z_i^2 : \mathbb{C}^n \to \mathbb{C},$$

and consider the fibres $X_y = \{\sum z_i^2 = y\}$. For later reference, let us establish a few conventions. The standard Euclidean metric on $\mathbb{C}^n$ is $\omega_E = \frac{1}{2}\sqrt{-1}\sum dz_i \wedge d\bar{z}_i$; we will sometimes also denote by ω_E its restriction to fibres. The volume form is $vol_E = \frac{1}{n!}\omega_E^n = \frac{1}{2^n}\widetilde{vol_E}$, where the normalised volume form is $\widetilde{vol_E} = \prod_i(\sqrt{-1}dz_i \wedge d\bar{z}_i)$. The pullback of the standard metric from the base $\mathbb{C}$ is $\omega_0 = \frac{1}{2}\sqrt{-1}df \wedge d\bar{f}$. The norm-squared function on $\mathbb{C}^n$ is $H = \sum |z_i|^2$. The holomorphic n-form on $\mathbb{C}^n$ is $\Omega = dz_1 \ldots dz_n$.

We consider the ansatz $\phi = F(H)$ for the Kähler potential on X_y, where F is locally smooth; this amounts to imposing $SO(n, \mathbb{R})$ symmetry. We compute the metric

$$\bar{\partial}\phi = F'(H)\bar{\partial}H,$$
$$\sqrt{-1}\partial\bar{\partial}\phi = F'\sqrt{-1}\partial\bar{\partial}H + F''\sqrt{-1}\partial H \wedge \bar{\partial}H.$$

Using the binomial theorem, we have

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = F'^{n-1}(\sqrt{-1}\partial\bar{\partial}H)^{n-1}$$
$$+(n-1)F'^{n-2}F''(\sqrt{-1}\partial\bar{\partial}H)^{n-2} \wedge (\sqrt{-1}\partial H \bar{\partial}H).$$

To understand the terms, we compute the forms over $\mathbb{C}^n$:

$$(\sqrt{-1}\partial\bar{\partial}H)^{n-1} = (n-1)! \sum_{i=1}^{n} \prod_{j\neq i}(\sqrt{-1}dz_j \wedge d\bar{z}_j),$$

$$\sqrt{-1}df \wedge d\bar{f} = 4 \sum_{i,j} \sqrt{-1}z_i\bar{z}_j dz_i \wedge d\bar{z}_j,$$

$$\sqrt{-1}\partial H \wedge \bar{\partial}H = \sqrt{-1} \sum_{i,j} z_j\bar{z}_i dz_i \wedge d\bar{z}_j,$$

$$\frac{1}{4}(\sqrt{-1}\partial\bar{\partial}H)^{n-1} \wedge (\sqrt{-1}df \wedge d\bar{f}) = (n-1)!H\widetilde{vol_E}.$$

$$\frac{1}{4}(\sqrt{-1}\partial\bar{\partial}H)^{n-2} \wedge (\sqrt{-1}\partial H \wedge \bar{\partial}H) \wedge (\sqrt{-1}df \wedge d\bar{f})$$
$$= (n-2)!(H^2 - |y|^2)\widetilde{vol_E}.$$

Thus

$$\frac{1}{4}(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} \wedge (\sqrt{-1}df \wedge d\bar{f}) = (n-1)!\widetilde{vol_E}\{HF'^{n-1}$$
$$+(H^2 - |y|^2)F'^{n-2}F''\}. \quad (1)$$

By the adjunction formula the quadric fibres also have trivial canonical bundles, and we can write down the holomorphic volume form Ω_y on X_y by $\Omega = df \wedge \Omega_y$. This gives

$$\widetilde{vol_E} = \sqrt{-1}^{n^2}\Omega \wedge \overline{\Omega} = \sqrt{-1}df \wedge d\bar{f} \wedge (\sqrt{-1}^{(n-1)^2}\Omega_y \wedge \overline{\Omega_y}). \quad (2)$$

2.1 The homogeneous Monge–Ampère equation

The homogeneous Monge–Ampère equation on a fixed fibre X_y with $y \neq 0$ is

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = 0. \tag{3}$$

By (1), this can be written as

$$H F'^{n-1} + (H^2 - |y|^2) F'^{n-2} F'' = 0$$

We can integrate this to obtain

$$F' = \frac{\text{Const}}{(H^2 - |y|^2)^{1/2}}.$$

By a change of variable $H = |f| \cosh \tau$, we obtain the solution up to an affine transformation

$$\phi = F(H) = \tau.$$

This is well defined away from the locus $\{H = |y|\}$.

2.2 The inhomogeneous Monge–Ampère equation

The equation defining the Calabi–Yau metric on X_y is

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = \sqrt{-1}^{(n-1)^2} \Omega_y \wedge \overline{\Omega_y}. \tag{4}$$

We can wedge both sides by $\sqrt{-1}df \wedge d\bar{f}$, and use (1) with (2) to see the following identity holds in $\Omega_{\mathbb{C}^n}^{n,n}|_{X_y}$:

$$4(n-1)!\{H F'^{n-1} + (H^2 - |y|^2) F'^{n-2} F''\} = 1.$$

We can simplify the equation further by a scaling observation: let $F = |y|^{\frac{n-2}{n-1}} \tilde{F}(\frac{H}{|y|})$, then $\tilde{F}$ satisfies the ODE

$$x \left(\frac{d\tilde{F}}{dx} \right)^{n-1} + (x^2 - 1) \left(\frac{d\tilde{F}}{dx} \right)^{n-2} \frac{d^2\tilde{F}}{dx^2} = \frac{1}{4(n-1)!}.$$

Now make the change of variable $x = \frac{H}{|y|} = \cosh \tau$. Then the equation becomes

$$\frac{d}{d\tau}\left[\left(\frac{d\tilde{F}}{d\tau}\right)^{n-1}\right] = \frac{1}{4(n-2)!}\sinh^{n-2}\tau.$$

If we impose the initial condition $\frac{d\tilde{F}}{d\tau}(0) = 0$ (to ensure smooth extension of the potential across the locus $\{H = |y|\}$), then the solution defines up to a numerical factor the Stenzel metric.

Example 2.1 For $n = 3$, the above equation implies $\frac{d\tilde{F}}{d\tau} = \frac{1}{\sqrt{2}}\sinh(\frac{\tau}{2})$, $\tilde{F} = \sqrt{2}\cosh(\frac{\tau}{2}) = \sqrt{\frac{H}{|y|} + 1}$. Thus the Kähler potential is $\phi = F(H) = \sqrt{H + |y|}$. The Stenzel metric in this special case is also known as the Eguchi-Hanson metric.

3 A general symmetric ansatz

We consider Kähler potentials on $\mathbb{C}^n$ of the form $\phi = F(H, \eta)$ where $\eta = |f|^2$. This amounts to imposing the symmetry under $SO(n, \mathbb{R}) \times U(1)$. Notice that this is not a cohomogeneity-one problem, so it may be hard to write down explicit solutions to interesting equations. We shall nevertheless exhibit solutions whose asymptotic behaviour resembles a Calabi–Yau metric. This symmetry reduction technique is very common, and in particular used by Hein and Naber in their work on A_k-type singularities.

We imitate the computations about the Stenzel metric, but now we need to keep track of all partial derivatives, because we are now working on the entire $\mathbb{C}^n$ instead of quadric fibres. Using that f is holomorphic,

$$\begin{aligned}
\bar{\partial}\phi &= F_H\bar{\partial}H + F_\eta f d\bar{f}, \\
\sqrt{-1}\partial\bar{\partial}\phi &= F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H \\
&\quad + \sqrt{-1}F_{H\eta}(\bar{f}df \wedge \bar{\partial}H + f\partial H \wedge d\bar{f}) \\
&\quad + (\eta F_{\eta\eta} + F_\eta)\sqrt{-1}df \wedge d\bar{f}.
\end{aligned}$$

Now using the binomial theorem, one can expand

$$\begin{aligned}
(\sqrt{-1}\partial\bar{\partial}\phi)^n &= (F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^n \\
&\quad + n(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-1} \wedge \\
&\quad \{(\eta F_{\eta\eta} + F_\eta)\sqrt{-1}df \wedge d\bar{f} \\
&\quad + \sqrt{-1}F_{H\eta}(\bar{f}df \wedge \bar{\partial}H + f\partial H \wedge d\bar{f})\}
\end{aligned}$$

$$+ \binom{n}{2}(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-2}$$
$$\wedge (2\eta F_{H\eta}^2 df d\bar{f} \wedge \partial H \bar{\partial}H).$$

Now we compute individual terms

$$(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^n$$
$$= F_H^n n! \widetilde{vol_E} + n F_H^{n-1} F_{HH} H (n-1)! \widetilde{vol_E}$$
$$= n! \widetilde{vol_E} \{F_H^n + F_H^{n-1} F_{HH} H\}.$$
$$(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-1}$$
$$\wedge \sqrt{-1} df \wedge d\bar{f} = 4(n-1)! \widetilde{vol_E}$$
$$\{H F_H^{n-1} + (H^2 - \eta) F_H^{n-2} F_{HH}\}.$$
$$(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-1}$$
$$\wedge \sqrt{-1}(\bar{f} df \wedge \bar{\partial}H + f \partial H \wedge d\bar{f})$$
$$= F_H^{n-1}(\sqrt{-1}\partial\bar{\partial}H)^{n-1}\sqrt{-1}(\bar{f} df \wedge \bar{\partial}H + f \partial H \wedge d\bar{f})$$
$$= 4\eta F_H^{n-1}(n-1)! \widetilde{vol_E}.$$
$$(F_H\sqrt{-1}\partial\bar{\partial}H + F_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-2}$$
$$\wedge df \wedge d\bar{f} \wedge \partial H \wedge \bar{\partial}H$$
$$= F_H^{n-2}(\sqrt{-1}\partial\bar{\partial}H)^{n-2} df \wedge d\bar{f} \wedge \partial H \wedge \bar{\partial}H$$
$$= -4(n-2)! F_H^{n-2}(H^2 - \eta)\widetilde{vol_E}.$$

Combining these computations, we arrive at a formula for the volume form:

Proposition 3.1 *For a symmetric Kähler potential $\phi = F(H, \eta)$, we have a formula*

$$(\sqrt{-1}\partial\bar{\partial}\phi)^n = n! \widetilde{vol_E} \{\{F_H^n + F_H^{n-1} F_{HH} H\}$$
$$+ 4(\eta F_{\eta\eta} + F_\eta)\{H F_H^{n-1} + (H^2 - \eta) F_H^{n-2} F_{HH}\} \quad (5)$$
$$+ 4 F_{H\eta}\eta F_H^{n-1} - 4 F_{H\eta}^2 \eta F_H^{n-2}(H^2 - \eta)\}.$$

Example 3.2 If the potential does not depend on η, then

$$(\sqrt{-1}\partial\bar{\partial}\phi)^n = n! \widetilde{vol_E} \{F_H^n + F_H^{n-1} F_{HH} H\}.$$

For example, for $F = \frac{1}{2} H$, one recovers the Euclidean metric and its volume form. For $F = \log H$, one recovers a solution to the homogeneous Monge–Ampère equation.

Example 3.3 Let $n = 3$. Let us specialise the ansatz to the form

$$\phi = F(H, \eta) = \frac{1}{2}\eta + \sqrt{H + a(\eta)} \tag{6}$$

defined by a non-negative smooth function $a(\eta)$, with the property that $\sqrt{-1}\partial\bar{\partial}\phi$ is positive, and $a(\eta)$ is asymptotic to $\eta^{1/2}$ for large η up to higher orders, so in particular $a' \sim \frac{1}{2}\eta^{-1/2}$, $a'' \sim -\frac{1}{4}\eta^{-3/2}$, etc. An example of this is $a(\eta) = \sqrt{\eta + 1}$.

We compute the various quantities involved in the general formula (5).

$$F_H = \frac{1}{2}\frac{1}{\sqrt{H+a}}, \quad F_{HH} = \frac{-1}{4}\frac{1}{(\sqrt{H+a})^3},$$

$$HF_H^2 + (H^2 - a^2)F_H F_{HH} = \frac{1}{8}, \quad (a^2 - \eta)F_H F_{HH} = -\frac{1}{8}\frac{1}{(H+a)^2}(a^2 - \eta),$$

$$F_H^3 + F_H^2 H F_{HH} = \frac{H + 2a}{16}\frac{1}{(\sqrt{H+a})^5},$$

$$F_\eta = \frac{1}{2} + \frac{1}{2\sqrt{H+a}}a', \quad F_{\eta\eta} = \frac{a''}{2(\sqrt{H+a})} - \frac{a'^2}{4(\sqrt{H+a})^3},$$

$$F_\eta + \eta F_{\eta\eta} = \frac{1}{2} + \frac{a' + \eta a''}{2(\sqrt{H+a})} - \frac{\eta a'^2}{4(\sqrt{H+a})^3}, \quad F_{H\eta} = -\frac{1}{4}\frac{a'}{(\sqrt{H+a})^3}.$$

Combining the computations, we get the volume form of the ansatz

$$(\sqrt{-1}\partial\bar{\partial}\phi)^3$$
$$= \frac{3}{2}\widetilde{vol_E}\left\{1 + (a' + \eta a'')\frac{1}{\sqrt{H+a}} + O\left(\frac{1}{(\sqrt{H+a})^3}\right)\right\}, \quad H \to \infty. \tag{7}$$

We comment that $a' + \eta a'' \sim \frac{1}{4\eta^{1/2}}$ for large η. In particular, to the leading order the volume measure is a constant multiple of the Lebesgue measure. But the leading order error is quite large.

Main Ansatz 3.4 We modify the above ansatz to the form

$$\phi = F(H, \eta) = \frac{1}{2}\eta + \sqrt{H + a(\eta, H)}, \quad a(\eta, H) = \sqrt{\eta + \sqrt{H + 1}}. \tag{8}$$

This is the main ansatz we shall use in the rest of the paper, and some motivations will be discussed in the next section. The computation follows very similar lines, and can be viewed as a perturbation of the previous example. The technical complication comes from incorporating the partial derivatives

of $a(\eta, H)$ with respect to H, which has some minor effects on the secondary asymptotes. We give some intermediate steps:

$$a_\eta = \frac{1}{2a}, \quad a_{\eta\eta} = -\frac{1}{4a^3}, \quad a_{\eta H} = -\frac{1}{8a^3\sqrt{H+1}},$$

$$a_H = \frac{1}{4a\sqrt{H+1}}, \quad a_{HH} = -\frac{1}{16a^3(H+1)} - \frac{1}{8a(H+1)^{3/2}},$$

$$F_H = \frac{1}{2\sqrt{H+1}}(1+a_H), \quad F_{HH} = -\frac{1}{4(H+a)^{3/2}}(1+a_H)^2 + \frac{a_{HH}}{2\sqrt{H+a}},$$

$$F_\eta = \frac{1}{2} + \frac{1}{2\sqrt{H+a}}a_\eta, \quad F_{\eta\eta} = \frac{1}{2\sqrt{H+a}}a_{\eta\eta} - \frac{1}{4(H+a)^{3/2}}a_\eta^2,$$

$$F_{H\eta} = \frac{1}{2\sqrt{H+a}}a_{H\eta} - \frac{1}{4(H+a)^{3/2}}a_\eta(1+a_H).$$

To aid computation, let us introduce the quantities

$$Q_1 = F_H^3 + F_H^2 H F_{HH}, \quad Q_2 = \eta F_{\eta\eta} + F_\eta,$$
$$Q_3 = H F_H^2 + (H^2 - \eta) F_H F_{HH}, \quad Q_4 = F_{H\eta} F_H^2 \eta - F_{H\eta}^2 \eta F_H (H^2 - \eta).$$

Our general formula (5) for the volume form reads

$$\frac{(\sqrt{-1}\partial\bar{\partial}\phi)^3}{6\widetilde{vol}_E} = Q_1 + 4Q_2 Q_3 + 4Q_4.$$

We are interested in the asymptote as $H \to \infty$. It is elementary but useful to notice the basic inequalities for a, H, η, such as $H \geq \eta^{1/2}$, $a \geq \eta^{1/2}$, $a \geq H^{1/4}$, and $a = O(H)$ for $H > 1$. After substituting in the expressions for derivatives, and extracting leading order terms,

$$Q_1 = -\frac{H}{16(H+a)^{5/2}} + \frac{1}{8(H+a)^{3/2}} + O\left(\frac{1}{H^2 a}\right),$$

$$Q_2 = \frac{1}{2} + \frac{1}{8a\sqrt{H+a}} - \frac{1}{16(H+a)^{3/2}} + O\left(\frac{1}{a^3}\right),$$

$$Q_3 = \frac{1}{8} + \frac{H}{32(H+a)a\sqrt{H+1}} - \frac{\sqrt{H+1}}{8(H+a)^2}$$
$$+ \frac{3}{32(H+a)\sqrt{H+1}} - \frac{H-a}{32a(H+1)^{3/2}} + O\left(\frac{1}{a^3}\right)$$
$$= \frac{1}{8} + \frac{H^{1/2}}{32a(H+a)} - \frac{H^{1/2}}{8(H+a)^2} + \frac{3}{32(H+a)\sqrt{H}} - \frac{H-a}{32aH^{3/2}}$$
$$+ \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{64(H+a)^2 H^{5/2}a} + O\left(\frac{1}{a^3}\right),$$

$$Q_4 = -\frac{1}{32}\frac{a}{(H+a)^{5/2}} - \frac{H-a}{128(H+a)^{5/2}} + O\left(\frac{1}{a^3}\right).$$

Combining these computations, we see that

Lemma 3.5 *The volume form of the metric given by the ansatz (8) is*

$$\frac{(\sqrt{-1}\partial\bar{\partial}\phi)^3}{\widetilde{vol}_E} = \frac{3}{2}\left\{1 + \frac{1}{4a(H+a)^{1/2}} + \frac{H^{1/2}}{4a(H+a)}\right.$$
$$- \frac{H^{1/2}}{(H+a)^2} + \frac{3}{4(H+a)\sqrt{H}}$$
$$\left. - \frac{H-a}{4aH^{3/2}} + \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{8(H+a)^2 H^{5/2}a}\right\} + O\left(\frac{1}{a^3}\right). \tag{9}$$

It is worth pointing out for later purposes that the terms we are discarding also have higher order bounds, which can be obtained inductively using the following principle: write all contributions in terms of products of power law expressions in $H + a$, H, a etc; each H-derivative picks up a multiplicative factor $O(\frac{1}{H})$ (the factor coming from a_H are further suppressed); each η-derivative involves differentiating an expression of a, so the derivative picks up a multiplicative factor $O(\frac{1}{a^2})$. The result is that the error term $E = O(\frac{1}{a^3})$ in formula (9) has higher order bounds

$$\left|\frac{\partial^{i+j}}{\partial H^i \partial \eta^j}E\right| = O\left(\frac{1}{H^i a^{3+2j}}\right).$$

.

A cruder version of the asymptotic formula as $H \to \infty$ is:

$$(\sqrt{-1}\partial\bar{\partial}\phi)^3 = \frac{3}{2}\widetilde{vol}_E\left\{1 + \frac{1}{4H^{1/2}a} + E'\right\},$$
$$\frac{\partial^{i+j}}{\partial H^i \partial \eta^j}E' = O\left(\frac{1}{H^{3/2+i}a^{2j}}\right) + O\left(\frac{1}{H^i a^{3+2j}}\right). \tag{10}$$

For small values of η, the $O(\frac{1}{H^{1/2}a})$ decay rate at infinity is an improvement compared to the $O(\frac{a'+\eta a''}{\sqrt{H+a}})$ decay in the previous Example (7).

To take two extreme cases, in the region where $\eta^{1/2} \sim H^{1/4} \gg 1$ then $a \sim H^{1/4}$, and $O(\frac{1}{H^{3/2}}) = O(\frac{1}{a^6})$ is insignificantly small. In the region where $\eta^{1/2} \sim H \gg 1$, then $a \sim H$, and $O(\frac{1}{H^{3/2}}) = O(\frac{1}{a^{3/2}})$ is the dominating error term.

4 Relation to the collapsing metric problem

We shall now informally explain the origin of the main ansatz (8).

Motivating question Consider a Calabi–Yau 3-fold which admits a Lefschetz fibration over a Riemann surface, namely the singularity in the fibration is complex analytically modelled by the standard quadric fibration on $\mathbb{C}^3$. If we vary the Kähler class so that the fibres have area of order $t \ll 1$, we obtain some highly collapsed Calabi–Yau metric on the total space. Can we construct a local model on $\mathbb{C}^3$ describing what happens near the ODP points in the fibration?

Let $n = 3$. A 'semi-Ricci flat form' is a closed (1,1) form $\sqrt{-1}\partial\bar{\partial}\phi$ whose restriction to each fibre agrees with the Calabi–Yau metric on the fibre. This involves some ambiguity because one can always adjust ϕ by a potential pulled back from the base. Motivated by the above question, we can naïvely write down the following ansatz, as a first approximation to the collapsed Calabi–Yau metric, in the local region described by the standard model of the Lefschetz fibration:

$$\phi = F(H, \eta) = \frac{1}{2}\eta + t\sqrt{H + \eta^{1/2}}. \tag{11}$$

Here $\frac{1}{2}\eta$ gives ω_0, the pullback of the standard metric on the base, and the fibrewise restriction is just the Stenzel metric for $n = 3$, rescaled to size $t \ll 1$.

This naïve ansatz is not smooth at the central fibre $\eta = 0$. But at least we know ϕ is PSH. To see this, recall the elementary fact that if ψ, ψ' are PSH, then $\log(e^\psi + e^{\psi'})$ is PSH. Apply this to $\psi = \log H$ and $\psi' = \frac{1}{2}\log\eta$, we get $\log(H + \eta^{1/2})$ is PSH, so ϕ is then easily seen to be PSH.

We shall use the formula (5) as a test for the validity of this naïve ansatz as an approximate model for a Calabi–Yau metric. The semi-Ricci flat condition gives

$$HF_H^2 + (H^2 - \eta)F_H F_{HH} = \frac{1}{8}t^2.$$

To leading order, we expect $\eta F_{\eta\eta} + F_\eta \sim \frac{1}{2}$, by ignoring all terms with t dependence. We notice that all terms with H dependence necessarily also carry a t factor, so the correction terms $F_H^3 + F_H^2 F_{HH} H$ and $4F_{H\eta}\eta F_H^2 - 4F_{H\eta}^2 \eta F_H(H^2 - \eta)$ are both of order t^3. This means the leading order term is $(\sqrt{-1}\partial\bar{\partial}\phi)^3 \sim \frac{3}{2}t^2 \widetilde{vol_E}$, so this metric is approximately Calabi–Yau at the scale where both H and η are comparable to 1, and the relative error of the volume form is of order $O(t)$.

Now $F_H \sim \frac{t}{H^{1/2}}$, so at the scale $H \sim t^{2/3}$, we have $F_H{}^3 \sim t^2$, which is comparable to the leading order volume form, and the approximation breaks down.

Similarly $F_\eta - \frac{1}{2} \sim \frac{t}{H^{1/2}\eta^{1/2}}$, so if $H\eta \sim t^2$, the approximation fails.

The source of the first failure has an interesting geometric interpretation. Think about the fibred Calabi–Yau 3-fold with the collapsing CY metric which we are trying to model. At the nodal point, *i.e.* the origin, the CY metric should be locally Euclidean, but the volume form has order t^2. Therefore it seems reasonable to expect that near the origin, this collapsing CY metric is uniformly equivalent to $t^{2/3}\omega_E$. This uniform equivalence is expected to hold up to the scale $H \sim t^{2/3}$, at which the Calabi–Yau metric matches up with the semi-Ricci flat approximation ansatz. We may call the scale $H \sim t^{2/3}$ the 'quantisation scale'. The reason for choosing this terminology is that, when H is larger than this scale, the CY metric approximates a rescaled semi-Ricci flat metric. This picture is formally similar to the semiclassical limit in quantum mechanics. The quantisation scale is where this approximation breaks down, which can be imagined as quantum fluctuation effects.

The second failure due to the smallness of η seems to be an artefact of our non-smooth choice of ansatz. There is no canonical choice for a smoothing. We make the following choice to achieve good decay properties:

$$\phi = \frac{1}{2}\eta + t\sqrt{H + (\eta + t\sqrt{H + t^{2/3}})^{1/2}}. \tag{12}$$

This is still PSH by similar arguments. The previous discussions essentially apply to the modified situation, with $H\eta \sim t^2$ replaced by $H(\eta + t^{4/3}) \sim t^2$, and now the sole source of failure is due to $H \sim t^{2/3}$.

We zoom in to the quantisation scale. This amounts to using the coordinates $\frac{1}{t^{1/3}}z_i$ instead of z_i, and scale the metric by a factor $t^{-4/3}$. Starting from (12), we get

$$\phi = \frac{1}{2}\eta + \sqrt{H + (\eta + \sqrt{H + 1})^{1/2}}. \tag{13}$$

which is our main ansatz.

5 Riemannian geometry of the main ansatz (8)

We study the asymptotic properties of the metric ansatz (8). As a preliminary remark, the volume form of the metric (8) is asymptotic to the volume form of the Euclidean metric. At one stage we will make use of a non-holomorphic parametrisation, so we shall distinguish the Kähler form ω and the associated Riemannian metric tensor g.

5.1 Asymptotic metric

Now we study the properties of the metric for $H >> 1$, only to the leading order. The asymptotic expression is

$$
\begin{aligned}
\omega = \sqrt{-1}\partial\bar\partial\phi \sim {} & \frac{1}{2}\sqrt{-1}df \wedge d\bar f + \frac{\sqrt{-1}}{2\sqrt{H+a}}\partial\bar\partial H \\
& - \frac{\sqrt{-1}}{4(\sqrt{H+a})^3}\partial H \wedge \bar\partial H \\
\sim {} & \frac{1}{2}\sqrt{-1}df \wedge d\bar f + \frac{\sqrt{-1}}{2\sqrt{H+\eta^{1/2}}}\partial\bar\partial H \\
& - \frac{\sqrt{-1}}{4(\sqrt{H+\eta^{1/2}})^3}\partial H \wedge \bar\partial H
\end{aligned}
\tag{14}
$$

This follows readily from differentiating (8), and discarding the terms coming from differentiating a since these do not affect the leading asymptote.

We want to understand the qualitative features of the distance function.

Lemma 5.1 *For P in the region $\{H > 1\}$, there is a uniform equivalence*

$$
C(\eta^{1/2} + H^{1/4}) \leq d(0, P) \leq C'(\eta^{1/2} + H^{1/4}).
\tag{15}
$$

In other words, $d(0, P)$ is uniformly equivalent to the function a for $P \in \{H > 1\}$.

Proof Using the asymptotic formula, one sees $C\omega \geq \sqrt{-1}df \wedge d\bar f$, so $\eta^{1/2} = |f| \leq Cd(0, P)$. Similarly one sees for $H > 1$, $C\omega \geq \frac{\sqrt{-1}}{\sqrt{H}}\partial\bar\partial H$, so along any path parametrised by the arclength,

$$
\frac{d|z|}{ds} \leq C|z|^{1/2},
$$

which integrates to give $H^{1/4} \leq Cd(0, P)$.

Now for the reverse direction, we take the radial path joining the origin and the point P, and compute the length. This gives $d(0, P) \leq C(\eta^{1/2} + H^{1/4})$. □

Lemma 5.2 *Let $r >> 1$. We have the maximal volume growth condition:*

$$
Cr^6 \leq Vol(B_g(0, r)) \leq C'r^6.
\tag{16}
$$

Proof The description of the distance function implies that

$$
\{H^{1/4} \leq Cr\} \cap \{\eta^{1/2} \leq Cr\} \subset B_g(0, r) \subset \{H^{1/4} \leq C'r\} \cap \{\eta^{1/2} \leq C'r\}
$$

for some appropriate constants C and C'. Thus we can consider instead the set $\{H \leq r^4\} \cap \{\eta \leq r^2\}$ and estimate its volume. This is an exercise in Lebesgue integration theory.

Since the asymptotic volume measure is comparable to the Lebesgue measure,

$$Vol_\omega(\{H \leq r^4\} \cap \{\eta \leq r^2\}) \sim Vol_{Lebesgue}(\{H \leq r^4\} \cap \{\eta \leq r^2\}).$$

By the homogeneity of the Lebesgue measure, the RHS is

$$r^{12} Vol_{Lebesgue}(\{H \leq 1\} \cap \{\eta \leq r^{-6}\}).$$

Now for very large r, the set $\{H \leq 1\} \cap \{\eta \leq r^{-6}\}$ is approximately the product of $\{H \leq 1\} \cap X_0$ with the disc $\{\eta \leq r^{-6}\} \subset \mathbb{C}$ where X_0 is the nodal central fibre, so the above

$$\sim r^{12} Area(\{H \leq 1\} \cap X_0) Area(\{\eta \leq r^{-6}\} \subset \mathbb{C}) \sim r^6.$$

Here the area of $\{H \leq 1\} \cap X_0$ is induced from the restriction of the Euclidean metric. The claimed result follows. $\qquad\square$

We define a family of cones in $\mathbb{C}^3$ by $E_\epsilon = \{\eta^{1/2} \geq \epsilon H\}$, for $0 < \epsilon < 1$. In practice ϵ is a fixed small number. Our next result says these cones are highly squashed in the metric g.

Lemma 5.3 *Take $r > 1$. Let $P \in B_g(r) \cap E_\epsilon$, then the distance of P to the complement of E_ϵ has an upper bound*

$$d_g(P, \mathbb{C}^3 \backslash E_\epsilon) \leq \frac{C}{\epsilon^{1/4}} r^{1/4}. \tag{17}$$

Proof We start by observing that on the affine quadric $\{f = 1\}$ with the Stenzel metric (*cf.* Sect. 2), there is a uniform diameter bound $\text{diam}(\{H \leq \frac{1}{\epsilon}\}) \leq \frac{C}{\epsilon^{1/4}}$ for $0 < \epsilon < 1$, because at large distance the Stenzel metric is asymptotic to the orbifold metric on X_0, and the distance to the origin on the flat orbifold X_0 is $\sqrt{2} H^{1/4}$. By scaling, on the fibre X_f with the Stenzel metric, the diameter bound reads

$$\text{diam}\left(\left\{H \leq \frac{\eta^{1/2}}{\epsilon}\right\} \subset X_f\right) \leq \frac{C}{\epsilon^{1/4}} \eta^{1/8}.$$

Now if $\eta \geq 1$, then the metric g restricted to the fibre X_f is uniformly equivalent to the Stenzel metric, because g is by construction essentially the semi-Ricci flat approximation, for large values of η. Thus if $P \in X_f \cap E_\epsilon$,

we can move the point inside X_f and exit the region E_ϵ within distance $\frac{C}{\epsilon^{1/4}}(\eta^{1/8} + 1) \le \frac{C}{\epsilon^{1/4}}(r^{1/4} + 1)$. Here we add in the constant 1 to make the argument work also for $\eta \le 1$. This means $d(P, \mathbb{C}^3 \backslash E_\epsilon) \le \frac{C}{\epsilon^{1/4}}(r^{1/4} + 1) \le \frac{C}{\epsilon^{1/4}} r^{1/4}$. $\qquad\square$

Remark If we normalise the metric by a factor $\frac{1}{r^2}$, then $P \in B_{\frac{1}{r^2}g}(1)$, and this estimate says $d_{\frac{1}{r^2}g}(P, \mathbb{C}^3 \backslash E_\epsilon) \le \frac{C}{\epsilon^{1/4} r^{3/4}}$. Intuitively, for large r and fixed ϵ, this means the region E_ϵ has very small solid angle.

Now we concentrate on the region $(\mathbb{C}^3 \backslash E_\epsilon) \cap \{H \gg 1\}$, with ϵ a fixed small number. In this region, up to $O(\epsilon)$ error, the asymptote of ω as $H \to \infty$ simplifies further to

$$\omega \sim \frac{1}{2}\sqrt{-1}df \wedge d\bar{f} + \frac{\sqrt{-1}}{2\sqrt{H}}\partial\bar\partial H - \frac{\sqrt{-1}}{4(\sqrt{H})^3}\partial H \wedge \bar\partial H.$$

Ehresmann's theorem (*cf.* Theorem 9.3 [15]) suggests the fibration f restricted to the region $(\mathbb{C}^3 \backslash E_\epsilon) \cap \{H \gg 1\}$ is a smooth fibration over $\mathbb{C}$. For example, if we calculate the monodromy (namely integrating the vector field $\sum_i \frac{\bar{z}_i}{|z|^2}\frac{\partial}{\partial z_i}$ on $\mathbb{C}^3$) along the standard vanishing paths, we can obtain an explicit trivialisation:

Lemma 5.4 *We have a diffeomorphism* $\mathbb{C}^3 \backslash \{H = \eta^{1/2}\} \simeq (X_0 \backslash \{0\}) \times \mathbb{C}$ *over* $\mathbb{C}$, *given by*

$$z \in \mathbb{C}^3 \backslash \{H = \eta^{1/2}\} \mapsto (w, f) \in (X_0 \backslash \{0\}) \times \mathbb{C},$$

$$\begin{aligned}
w = &\frac{1}{2}\left\{\left(\frac{H - \eta^{1/2}}{H + \eta^{1/2}}\right)^{1/4} + \left(\frac{H + \eta^{1/2}}{H - \eta^{1/2}}\right)^{1/4}\right\} z \\
&+ \frac{1}{2}\left\{\left(\frac{H - \eta^{1/2}}{H + \eta^{1/2}}\right)^{1/4} - \left(\frac{H + \eta^{1/2}}{H - \eta^{1/2}}\right)^{1/4}\right\} \frac{f\bar{z}}{\eta^{1/2}}.
\end{aligned} \qquad (18)$$

When $f = 0$, *then it is understood that* $w = z$. *The inverse map can be described as follows:*

$$z = \frac{1}{2}(e^s + e^{-s})w + \frac{1}{2}\left(\frac{2}{e^s + e^{-s}}\right)\frac{f\bar{w}}{|w|^2}, \quad e^{-2s} - e^{2s} = \frac{2\eta^{1/2}}{|w|^2}.$$

The above formulae are quite tedious to work with. But in the region $\{\eta^{1/2} \le \epsilon H\}$, where ϵ is small, then s is almost zero, and the above formulae are well approximated by a much simpler parametrisation over the base $\mathbb{C}$:

$$\chi : X_0 \times \mathbb{C} \to \mathbb{C}^3, \quad z = w + \frac{f\bar{w}}{2|w|^2}. \tag{19}$$

The region $\{\eta^{1/2} \leq \epsilon H, H \gg 1\}$ corresponds roughly to $\{\eta^{1/2} \leq \epsilon |w|^2, |w|^2 \gg 1\}$. We notice χ is not exactly holomorphic, but only approximately so up to an error of order $\frac{\partial z}{\partial \bar{w}} = O(\epsilon)$. Another caveat is that $w \in X_0$, so in coordinates $\sum w_i^2 = 0$.

We now study the asymptotic formula of the metric tensor associated to ω via the parametrisation χ. The base term $\frac{1}{2}\sqrt{-1}df \wedge d\bar{f}$ pulls back exactly to $\frac{1}{2}\sqrt{-1}df \wedge d\bar{f}$, because χ is defined over the base. The function H pulls back to $|w|^2(1 + O(\epsilon))$. To calculate the other terms we need to be careful about the failure of holomorphicity. This means we need to directly pull back the Riemmanian metric tensor.

The term $\sqrt{-1}\partial\bar{\partial}H$ defines a two-tensor $\sum dz_i \otimes d\bar{z}_i = \sum d\bar{z}_i \otimes dz_i$, where to save writing we have passed to the symmetrization. The Riemannian metric tensor is two times its real part. Via χ, we can write $dz_i = dw_i + df\frac{\bar{w}_i}{2|w|^2} + \frac{f}{2|w|^2}d\bar{w}_i - \frac{f\bar{w}_i}{2|w|^4}d|w|^2$. We compute

$$Re\left(\sum_i dz_i \otimes d\bar{z}_i\right) = Re\left\{\left(\sum dw_i \otimes d\bar{w}_i\right)\left(1 + \frac{\eta}{4|w|^4}\right)\right.$$
$$+ \sum_i \frac{\bar{f}}{|w|^2}dw_i \otimes dw_i$$
$$\left. - \sum_i \frac{f}{2|w|^4}d\bar{f} \otimes \bar{w}_i dw_i + \frac{df \otimes d\bar{f}}{4|w|^2}\right\}.$$

Up to $O(\epsilon)$ relative error, this can be simplified as

$$Re\left\{\sum dw_i \otimes d\bar{w}_i - \sum \frac{f}{2|w|^4}d\bar{f} \otimes \bar{w}_i dw_i + \frac{df \otimes d\bar{f}}{4|w|^2}\right\}.$$

Since the metric includes the pullback of the base metric $Re(df \otimes d\bar{f})$, we have $|df| \sim 1$, so in the region $\{\eta^{1/2} \leq \epsilon H\}$ where $|f| \ll |w|^2$, we can drop more terms to get the asymptote up to $O(\epsilon)$ error:

$$H^{-1/2}Re\sum dz_i \otimes d\bar{z}_i \sim |w|^{-1}Re\sum dw_i \otimes d\bar{w}_i.$$

Similarly the term $\sqrt{-1}\partial H \wedge \bar{\partial}H$ defines a symmetric two-tensor $\sum \bar{z}_i dz_i \otimes z_j d\bar{z}_j$. We have the asymptote up to $O(\epsilon)$ error

$$H^{-3/2} Re \sum \bar{z}_i dz_i \otimes z_j d\bar{z}_j \sim |w|^{-3} Re \sum \bar{w}_i dw_i \otimes w_j d\bar{w}_j.$$

To summarize,

Lemma 5.5 *The Riemmanian metric g associated to ω admits the approximate formula in the region $\{\eta^{1/2} \leq \epsilon H, H \gg 1\}$:*

$$\chi^* g = Re \left(df \otimes d\bar{f} + |w|^{-1} \sum dw_i \otimes d\bar{w}_i \right.$$
$$\left. - \frac{1}{2}|w|^{-3} \sum \bar{w}_i dw_i \otimes w_j d\bar{w}_j \right) (1 + O(\epsilon)) \qquad (20)$$

i.e. *up to relative error of order $O(\epsilon)$, the metric g is asymptotic to the product metric $g_{product}$ on $X_0 \times \mathbb{C}$, where the X_0 factor is equipped with the Stenzel metric.*

The information at our disposal now allows us to identify the tangent cone at infinity for the metric g associated to ω.

Proposition 5.6 *The tangent cone of $(\mathbb{C}^3, g)$ at infinity is isometrically $X_0 \times \mathbb{C}$ with the product metric.*

Proof Clearly only asymptotic properties of the metric are relevant. Fix a small number ϵ. Consider large values of r, and apply the scaling factor $\frac{1}{r^2}$ to the metric g. In the ball $B_{\frac{1}{r^2}g}(0, D)$ of fixed radius D, if there is a point P in the region E_ϵ, then by the lemma 5.3, its distance to $\mathbb{C}^3 \backslash E_\epsilon$ is controlled above by $\frac{CD^{1/4}}{\epsilon^{1/4} r^{3/4}} < C\epsilon$ for large r. Thus the only important contribution comes from the region $(\mathbb{C}^3 \backslash E_\epsilon) \cap \{H \geq C\}$. Up to $O(\epsilon)$ error, $(\mathbb{C}^3 \backslash E_\epsilon) \cap B_{\frac{1}{r^2}g}(0, D)$ is Gromov-Hausdorff close to the metric space

$$(B_{r^{-2}g_{product}}(0, D) \cap \{\eta^{1/2} \leq \epsilon|w|^2\}, \frac{1}{r^2}g_{product}) \subset X_0 \times \mathbb{C}.$$

Now we observe that $X_0 \times \mathbb{C}$ with the product metric is conical, and scaling features imply that for large r the region $\{\eta^{1/2} \geq \epsilon|w|^2\}$ is unimportant. Thus the metric space is Gromov-Hausdorff close to $(B_{r^{-2}g_{product}}(0, D), \frac{1}{r^2}g_{product})$, which is the same as $(B_{g_{product}}(0, rD), g_{product})$. From this we see that as $r \to \infty$, $B_{\frac{1}{r^2}g}(0, D)$ Gromov-Hausdorff converges to $(B_{g_{product}}(0, D), g_{product})$, for any fixed D, so the tangent cone is indeed $X_0 \times \mathbb{C}$ with the product metric. $\square$

Remark (Jumping of complex structure) The complex structure on the tangent cone differs from the original manifold $\mathbb{C}^3$. This can be explained by the flat family of subvarieties of $\mathbb{C}_y \times \mathbb{C}^3_{z_1, z_2, z_3}$, defined as

$$\{(y, z_1, z_2, z_3) | ty = z_1^2 + z_2^2 + z_3^2\},$$

whose generic fibre is just a graph over $\mathbb{C}^3$, and the central fibre is $\mathbb{C} \times X_0$; the computation of Gromov Hausdorff limit involves a scaling procedure, which is reflected here by the parameter t.

Remark (Measure theoretic paradox) The link of the tangent cone is singular. The singularity corresponds to the node in X_0, and the cone E_ϵ collapses to the nodal line in $X_0 \times \mathbb{C}$. This behaviour is counterintuitive as E_ϵ seems to carry a large amount of Lebesgue measure, which disappears in the limit, but general theory of noncollapsing Gromov-Hausdorff convergence implies measure theoretic convergence. But this is not a contradiction because in a large ball with respect to the metric g, the set E_ϵ only occupies a very small portion of the measure.

H-J. Hein pointed out an analogy with the Taub-NUT metric which may help to understand this measure theoretic picture. By a well known observation of LeBrun [9], the Taub-NUT space is biholomorphic to $\mathbb{C}^2$ with the standard holomorphic volume form. In fact the holomorphic symplectic moment map makes $\mathbb{C}^2$ into a fibration of affine quadrics, exactly analogous to our situation. The counterintuitive fact then is that the Taub-NUT metric has cubic volume growth. The explanation again lies in the large distortion of distance. The distinctive feature of $n = 2$ is that the quadric fibres are isomorphic to $\mathbb{C}^*$, whose natural cylindrical metric has linear growth rate, which is not maximal. The Taub-NUT metric is asymptotically described by the semi-Ricci flat metric coming from the quadric fibration; the growth rate is quadratic in the base direction and linear in the fibre direction, thus the overall growth is cubic.

5.2 Topological discussions

The topological situation is related to the general story of Milnor fibrations associated to hypersurface singularities. Let us dispense with the metric geometry for a moment. In the ODP context, the sphere $S^5 = \{H = 1\}$ intersects $\mathbb{C}^3 \backslash X_0$ in an open set, which is just the complement of the link $X_0 \cap S^5$ of X_0. We have a fibration over the circle

$$S^5 \cap (\mathbb{C}^3 \backslash X_0) \to S^1, \quad z \mapsto \frac{f}{|f|}.$$

The fibre is topologically identified with the Milnor fibre X_1. This is because via the radial projection map $z \mapsto \frac{z}{|f|^{1/2}}$, this fibration is identified with

$$f : f^{-1}(S^1) \subset \mathbb{C}^3 \to S^1.$$

The monodromy around the circle is described by the standard Picard-Lefschetz formula (*cf.* Theorem 3.16 [16]).

The analogous picture on the tangent cone $X_0 \times \mathbb{C}$ is the following. We have a fibration

$$\{|w| + |f|^2 = 1\} \cap (X_0 \times \mathbb{C} \backslash X_0 \times \{0\}) \to S^1, \quad (w, f) \mapsto \frac{f}{|f|}.$$

(We write the sphere $\{|w| + |f|^2 = 1\}$ here because the radial distance on $X_0 \times \mathbb{C}$ is uniformly equivalent to $\sqrt{|w| + |f|^2}$.) The fibre is homeomorphic to $X_0 \simeq \mathbb{C}^2/\mathbb{Z}_2$. The effect of the metric geometry is to collapse the vanishing cycle to a point, thereby changing X_1 to X_0.

More explicitly, in our metric g, the vanishing cycle $\{H = \eta^{1/2}\}$ on X_f for large η has diameter proportional to $\eta^{1/8}$, but the distance of the vanishing cycle to the origin is of order $\eta^{1/2}$. The tangent cone only sees the regions whose diameter has at least linear growth rate, so the vanishing cycle disappears in the limit. This phenomenon also appears in the announced work of Hein and Naber on A_k type singularities.

The link $S(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C})$ of the tangent cone $X_0 \times \mathbb{C} \simeq \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ is singular along a circle S^1. The appearance of the circle is explained by the $U(1)$ symmetry of the whole setup. Metrically, the points on the circle correspond to the Gromov Hausdorff limit of the vanishing cycles.

It is also interesting to make some comparisons of our situation with Joyce's QALE spaces (*cf.* [7], and the subsequent generalisation into QAC spaces [2]). The main common feature is that the tangent cone at infinity is a finite group quotient $\mathbb{C}^n/G = \mathbb{C}^3/\mathbb{Z}_2$, where the main complication comes from the non-free action of $\mathbb{Z}_2$. However, our setup does not seem to fit naturally into Joyce's framework, not even at the topological level. Joyce requires the ambient manifold to be a resolution of the quotient $\mathbb{C}^n/G$, which is not a natural structure on $\mathbb{C}^3$. Joyce understands $\mathbb{C}^n/G$ in terms of a stratification by fixed point sets of subgroups of G, and describes the structure of his resolution in a stratified manner in terms of what he calls the local product resolution; the simplest local picture looks like

$$Y \times \mathbb{C}^m \to \mathbb{C}^{n-m}/H \times \mathbb{C}^m,$$

where Y is a resolution of $\mathbb{C}^{n-m}/H$ and H is a subgroup of G. Here Y plays a similar role as X_1. His framework does not typically incorporate the Picard-Lefschetz monodromy behaviour, as his product structures are usually globally trivial over stratified pieces. At the metric level, on the local piece $Y \times \mathbb{C}$, Joyce requires asymptotic convergence of the metric to the product metric (this type of phenomenon also happens for QAC spaces [2]). In our case the vanishing

cycles do not have constant size, but instead grow slowly as $O(a^{1/4})$ with the distance to the origin, so the metric in E_ϵ cannot converge to a product metric.

5.3 The Ricci curvature

Having examined the leading order behaviour of the metric, let us compute the Ricci tensor for $H \gg 1$. The formula for the Ricci form is

$$\text{Ric} = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}).$$

Suppose we write $v = \det(g_{i\bar{j}})$ for the volume density function. Then

$$\text{Ric} = \frac{\sqrt{-1}\partial v \wedge \bar{\partial}v}{v^2} - \frac{\sqrt{-1}\partial\bar{\partial}v}{v}.$$

The nature of our ansatz means v depends only on H, η. We have

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial}v &= v_H\sqrt{-1}\partial\bar{\partial}H + v_{HH}\sqrt{-1}\partial H \wedge \bar{\partial}H \\
&+ \sqrt{-1}v_{H\eta}\left(\bar{f}df \wedge \bar{\partial}H + f\partial H \wedge d\bar{f}\right) \\
&+ \left(\eta v_{\eta\eta} + v_\eta\right)\sqrt{-1}df d\bar{f}. \\
\sqrt{-1}\partial v \wedge \bar{\partial}v &= v_H^2\sqrt{-1}\partial H \wedge \bar{\partial}H + \sqrt{-1}v_H v_\eta\left(\partial H \wedge f d\bar{f} + \bar{f}df \wedge \bar{\partial}H\right) \\
&+ v_\eta^2\eta\sqrt{-1}df \wedge d\bar{f}.
\end{aligned}$$

The asymptotic expression for v is given by (10). To leading order, v is a constant. The worst term in (10) is $\frac{1}{H^{1/2}a}$. It is clear by computation that the terms showing up in the quadratic expression $\sqrt{-1}\partial v \wedge \bar{\partial}v$ are much smaller than the terms in the second derivative $\sqrt{-1}\partial\bar{\partial}v$. This reflects the fact that the approximation is good enough for nonlinear effect to be insignificant in the asymptotic regime. Comparing the computation with the asymptotic formula

$$\omega \sim \frac{1}{2}\sqrt{-1}df \wedge d\bar{f} + \frac{\sqrt{-1}}{2\sqrt{H+a}}\partial\bar{\partial}H - \frac{\sqrt{-1}}{4(\sqrt{H+a})^3}\partial H \wedge \bar{\partial}H,$$

we see $-\frac{C}{aH}\omega \leq \text{Ric} \leq \frac{C}{aH}\omega$. Thus the Ricci tensor is controlled by the metric:

$$|\text{Ricci tensor}| \leq \frac{C}{aH}g.$$

This indicates several features of the approximate ansatz. First, recall the distance to the origin is comparable to a. In the metric g, a typical point has

$H \sim a^4$, so the Ricci tensor decays as $O(\frac{1}{a^5})$, which is quite fast. Second, in the worst region E_ϵ, H is comparable to a, so this ansatz only gives quadratic decay of the Ricci curvature. The scalar curvature likewise decays as $O(\frac{1}{a^2})$ in the region E_ϵ.

The last observation links this discussion to the work of Hein [5] on weighted Sobolev inequalities, and their applications to Monge–Ampère equations (*cf.* also Hein's PhD thesis [6]). The main technical condition there is

Definition 5.7 A complete noncompact Riemannian manifold (M, g) is called $SOB(\beta)$ if there is a point $x_0 \in M$, a fixed small number $\delta > 0$, and $C \geq 1$, such that any two points $p, q \in M$ with $d(p, x_0) = d(q, x_0) = s$ are connected within the annulus $A(x_0, (1 - \delta)s, (1 + \delta)s)$ whenever $s > C$. Morever $|B(x_0, s)| \leq Cs^\beta$ for all $s \geq C$, and $|B(x, (1 - \frac{1}{C})r(x))| \geq \frac{1}{C}r(x)^\beta$, and Ricci $\geq -Cr(x)^{-2}$ if $r(x) = d(x_0, x) \geq C$.

Remark The definition of $SOB(\beta)$ used in [6] requires also the connectedness of the annulus $A(s, t)$ for any $t > s > C$, but this stronger condition is not needed for his weighted Sobolev inequality and his main theorem 6.1 below.

Proposition 5.8 *The metric ansatz* (8) *satisfies the condition* $SOB(6)$.

Proof The main point is that we know the maximal volume growth and the at worst quadratic Ricci curvature decay. The completeness property is clear from the asymptotic formula.

The connectedness property in our definition of $SOB(6)$ is implied by the connectedness of the link of the tangent cone at infinity. □

We take the opportunity to explain another technical condition in Hein's work.

Definition 5.9 (*cf.* [6], Definition 4.2) A $C^{k,\alpha}$ quasi-atlas on a complete Kähler manifold (M, ω) is a collection $\{\Phi_x : x \in A\}$, $A \subset M$ of holomorphic local diffeomorphisms $\Phi_x : B \to M$, $\Phi_x(0) = x$, from $B = B(0, 1) \subset \mathbb{C}^n$ into M which extend smoothly to the closure $\bar{B}$, and such that there exists $C \geq 1$ with $\mathrm{inj}(\Phi_x^* g) \geq 1/C$, $\frac{1}{C}g_{\mathbb{C}^n} \leq \Phi_x^* g \leq Cg_{\mathbb{C}^n}$, and $\left\| \Phi_x^* g \right\|_{C^{k,\alpha}(B, g_{\mathbb{C}^n})} \leq C$ for all $x \in A$, and such that for all $y \in M$ there exists $x \in A$ with $y \in \Phi_x(B)$ and $\mathrm{dist}_g(y, \partial\Phi_x(B)) \geq 1/C$.

Lemma 5.10 *Our metric ansatz* (8) *has a* $C^{k,\alpha}$ *quasi-atlas for any k.*

Proof It is enough to consider $H \gg 1$. For any such point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, without loss of generality $|x_1| = \max |x_i|$. Then we take the local coordinates $z'_1 = f - f(x)$, $z'_2 = \frac{1}{H(x)^{1/4}}(z_2 - x_2)$, $z'_3 = \frac{1}{H(x)^{1/4}}(z_3 - x_3)$. Using the asymptotic formula (14) for the metric g, we see that in a coordinate ball the metric g is uniformly equivalent to the Euclidean metric. In fact the uniform equivalence holds up to distance scale $O(H^{1/4})$. The higher order behaviour is clear. □

5.4 The asymptotic Laplacian

It is important in the study of the Monge–Ampère equation to have a well controlled linear theory of the Laplace operator. We want a formula for the main terms in the Laplacian of a general locally smooth function $b(H, \eta)$ depending only on H, η.

Lemma 5.11 *For the metric (8), the formula for the Laplacian is*

$$
\begin{aligned}
\left(\sqrt{-1}\partial\bar{\partial}b\right)&\left(\sqrt{-1}\partial\bar{\partial}\phi\right)^2 \\
&= \widetilde{vol_E}\left\{\left(\frac{3H + a}{\sqrt{H + a}} + O\left(\frac{1}{a}\right)\right) b_H\right. \\
&\quad + \left(\frac{2\left(H^2 - a^2\right)}{\sqrt{H + a}} + O\left(\frac{H}{a}\right)\right) b_{HH} \\
&\quad \left. + \left(O\left(\frac{\eta}{a}\right)\right) b_{H\eta} + \left(1 + O\left(\frac{1}{aH^{1/2}}\right)\right)\left(\eta b_{\eta\eta} + b_\eta\right)\right\}.
\end{aligned}
\tag{21}
$$

Proof We take the first variation of our formula for the volume form (5), namely to change ϕ to $\phi + tb$, and differentiate with respect to t at $t = 0$. The LHS becomes $3(\sqrt{-1}\partial\bar{\partial}\phi)^2 \wedge (\sqrt{-1}\partial\bar{\partial}b)$. The RHS becomes a linear combination of $b_H, b_{HH}, b_{H\eta}, b_{\eta\eta}, b_\eta$. For example, the coefficient of b_{HH} is

$$
\begin{aligned}
6\widetilde{vol_E}&\left\{F_H^2 H + 4(\eta F_{\eta\eta} + F_\eta)(H^2 - \eta)F_H\right\} \\
&= 6\widetilde{vol_E}\left\{\left(\frac{(H^2 - a^2)}{\sqrt{H + a}}\right) + O\left(\frac{H}{a}\right)\right\},
\end{aligned}
$$

which after cancelling the factor 3 gives a term in (21). The other contributions are similar.   □

The interesting feature of this expression is its inhomogeneity. Think about the region E_ϵ, where $\eta^{1/2}$ and H are comparable. If we take a homogeneous expression b, such as $(\eta^{1/2} + H)^\alpha$, and recall that H and η are homogeneous in z_i of degree 2 and 4 respectively, then each H differentiation brings down the degree by 2, and each η differentiation brings down the degree by 4. Then purely on the ground of dimensional analysis, only the terms with b_H and b_{HH} dominate. There is an additional subtlety that $H^2 - a^2$ can have zeros. If we ignore this issue, then the insight we gain from this formula is that if we just want to invert the Laplacian approximately in the region $E_\epsilon \cap \{H \gg 1\}$, then we only need to solve a second order ODE rather than a PDE. The fact that

the η derivative is suppressed, can be interpreted as saying the fibres behave as if they were independent in the region E_ϵ. This phenomenon is typical in adiabatic limit problems, where the metric ansatz ultimately comes from.

One explanation for this phenomenon is that in the intrinsic geometry of the metric g, one should trade η for $H^{1/2}$. With this new homogeneity convention, in the region where $\eta \sim H^{1/2}$, the quantity $H^{1/2}b_H$ should be comparable to b_η, so the Laplacian genuinely depends on two variables in that region. The nonclassical nature of the Laplacian in the region E_ϵ testifies to the singularity of the tangent cone, which breaks down most of the intuitions based on Euclidean geometry.

Lemma 5.12 *The function a is comparable to the distance to the origin for $H > 1$, and globally satisfies the estimate*

$$|da| + a|\sqrt{-1}\partial\bar{\partial}a| \leq C.$$

Proof The gradient bound is easy. To see the bound on $\partial\bar{\partial}a$, we notice that a is PSH, so it suffices to prove $a|\Delta a| \leq C$. This follows from applying formula (21) and the partial derivative computations leading to (9). □

6 Calabi–Yau metric

We aim to improve the approximate solution into a genuine Calabi–Yau metric. Our strategy to achieve this relies on the following existence result for the Monge–Ampère equation on noncompact manifolds, taken from Hein's PhD thesis [6], which builds on previous work of Tian and Yau [13].

Theorem 6.1 ([6], proposition 4.1) *Let (M, ω) be a complete noncompact Kähler manifold with a $C^{3,\alpha}$ atlas, which satisfies the condition $SOB(\beta)$, where $\beta > 2$. Let $f \in C^{2,\alpha}(M)$ satisfy $|f| \leq Cr^{-\mu}$ on $\{r > 1\}$ for some $\mu > 2$. Then there exists $\bar{\alpha} \in (0, \alpha]$ and $u \in C^{4,\bar{\alpha}}(M)$, such that $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^f\omega^n$. If in addition $f \in C^{k,\bar{\alpha}}(M)$ for some $k \geq 3$, then all such solutions u belong to $C^{k+2,\bar{\alpha}}_{loc}(M)$.*

Remark The proof considers a family of viscosity equations of the shape

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{f+\epsilon u_\epsilon}\omega^n,$$

and derives uniform L^∞ estimate for all small $\epsilon > 0$. The desired solution follows by taking a subsequential limit, which is not a priori known to be unique.

The condition $SOB(\beta)$ is chiefly used in the application of certain weighted Sobolev inequalities. The condition $\beta > 2$ reflects a basic distinction between

parabolic and non-parabolic manifolds. The reason to impose $\mu > 2$ is that the method typically only allows us to construct solutions with decaying potential, and if the solution decays with some power law $O(r^{2-\mu})$, then standard potential theory on Euclidean spaces makes us expect the Laplacian to decay with order $O(r^{-\mu})$. As a side remark, since the distance function in our ansatz involves two competing factors $\eta^{1/2}$ and $H^{1/4}$, the intuition from standard Euclidean behaviour is not always very effective.

6.1 Approximately inverting the Laplacian

The problem with applying Hein's result to construct a genuine Calabi–Yau metric is that the error terms to the volume form (*cf.* (9)) have slower than quadratic decay. More precisely, in the region where H is comparable to a^4, all error terms are of order at most $O(\frac{1}{a^3})$, so the approximation is already good enough. But in the region E_ϵ, the worst error term has order $O(\frac{1}{aH^{1/2}})$. Notice also its square is of order $O(\frac{1}{a^2 H}) = O(\frac{1}{a^3})$ and in particular decays faster than quadratically, which indicates that nonlinear effects are asymptotically unimportant.

We shall attempt to remove the leading errors by explicitly inverting the Laplacian up to an admissible amount of error. The result is

Lemma 6.2 *There exists a function $b = \frac{1}{a}\tilde{b}(\frac{H}{a})$ defined for $H \gg 1$, where $\tilde{b}$ is smooth with growth control*

$$\tilde{b}(x) = O(\log x), \quad \frac{d^k\tilde{b}}{dx^k} = O(x^{-k}), \quad \forall k \geq 1,$$

such that

$$\left(\sqrt{-1}\partial\bar{\partial}b\right) \wedge \left(\sqrt{-1}\partial\bar{\partial}\phi\right)^2$$
$$= \frac{1}{2}\widetilde{vol_E}\left\{\frac{1}{4a\,(H+a)^{1/2}} + \frac{H^{1/2}}{4a\,(H+a)} - \frac{H^{1/2}}{(H+a)^2}\right.$$
$$\left. + \frac{3}{4\,(H+a)\,\sqrt{H}} - \frac{H-a}{4a\,H^{3/2}} + O\left(\frac{\log(2H/a)}{a^3}\right)\right\}.$$

Lemma 6.3 *There exists a function $\mathfrak{b} = \frac{1}{a^2}\tilde{\mathfrak{b}}(\frac{H}{a})$ defined for $H \gg 1$, where $\tilde{\mathfrak{b}}$ is smooth with growth control*

$$\tilde{\mathfrak{b}}(x) = O(x^{-1/2}), \quad \frac{d^k\tilde{\mathfrak{b}}}{dx^k} = O(x^{-k}), \quad \forall k \geq 1,$$

such that

$$\left(\sqrt{-1}\partial\bar{\partial}\mathfrak{b}\right) \wedge \left(\sqrt{-1}\partial\bar{\partial}\phi\right)^2$$
$$= \frac{1}{2}\widetilde{vol_E} \left\{ \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{8\left(H+a\right)^2 H^{5/2}a} + O\left(\frac{1}{a^4}\right) \right\}.$$

We begin with examples of treating individual terms before describing the general algorithm.

Example 6.4 We treat the error term $\frac{1}{4a(H+a)^{1/2}}$ in detail. The leading order terms for the Laplacian of a function $b(H, \eta)$ is

$$\left(\sqrt{-1}\partial\bar{\partial}\mathfrak{b}\right)\left(\sqrt{-1}\partial\bar{\partial}\phi\right)^2 \sim \widetilde{vol_E} \left\{ \frac{3H + a}{\sqrt{H+a}}b_H + \frac{2\left(H^2 - a^2\right)}{\sqrt{H+a}}b_{HH} \right\}$$

As an intuitive guideline, although a and H are not independent variables, in the region E_ϵ they behave as if they are, because a is $\eta^{1/2}$ up to lower order corrections. Now we solve a related ODE with variable x:

$$\frac{3x + y}{\sqrt{x+y}}h' + \frac{2(x^2 - y^2)}{\sqrt{x+y}}h'' = \frac{1}{4y(x+y)^{1/2}}.$$

We can rewrite the ODE as

$$\frac{d}{dx}\left\{ 2(x - y)\sqrt{x+y}\frac{dh}{dx} \right\} = \frac{1}{4y(x+y)^{1/2}}.$$

Thus the ODE is reducible to elementary integration. The first integration gives

$$\frac{dh}{dx} = \frac{1}{4y(x - y)\sqrt{x+y}}(\sqrt{x+y} - \sqrt{2y}),$$

where we choose the normalisation that for $x = y$ the derivative $\frac{dh}{dx}$ remains bounded. This is to ensure the smooth dependence of h on x. Integrate further to get

$$h(x) = \frac{1}{2y}\log\left(\frac{\sqrt{x+y} + \sqrt{2y}}{\sqrt{y}}\right).$$

Here the normalisation is chosen to display homogeneity behaviour.

We then substitute H for x and a for y. This gives an approximate solution

$$b^{(1)}(H, \eta) = \frac{1}{2a} \log \left(\frac{\sqrt{H + a} + \sqrt{2a}}{\sqrt{a}} \right).$$

By construction, the expression

$$\frac{3H + a}{\sqrt{H + a}} b_H^{(1)} + \frac{2(H^2 - a^2)}{\sqrt{H + a}} b_{HH}^{(1)} = \frac{1}{4a(H + a)^{1/2}} + O \left(\frac{\log(\frac{2H}{a})}{a^3} \right)$$

The error term $O(\frac{\log(\frac{2H}{a})}{a^3})$ appears because a also depends on H, but the relative error is quite small, so the situation is improved. Now we use the formula (21) to see that in fact

$$(\sqrt{-1}\partial\bar{\partial}b^{(1)})(\sqrt{-1}\partial\bar{\partial}\phi)^2 = \widetilde{vol}_E \left\{ \frac{1}{4a(H + a)^{1/2}} + O \left(\frac{\log(2H/a)}{a^3} \right) \right\}.$$

This means we can invert the Laplacian on this error term approximately, as desired.

Remark In Euclidean potential theory, if the forcing term decays slower than quadratically near infinity, then in general the solution to the Poisson equation will grow at infinity. One explanation that we obtain a decaying potential in our context, is that our forcing term is $O(\frac{1}{a^3})$ in the region where $H \sim \eta^2$, and the slow decay occurs only in a very small solid angle.

Example 6.5 As a second example, we treat the term $\frac{1}{4H^{3/2}}$. We first write down the associated ODE:

$$\frac{d}{dx} \left\{ 2(x - y)\sqrt{x + y}\frac{dh}{dx} \right\} = \frac{3x + y}{\sqrt{x + y}}h' + \frac{2(x^2 - y^2)}{\sqrt{x + y}}h'' = \frac{1}{4x^{3/2}}.$$

The first integration gives

$$\frac{dh}{dx} = \frac{1}{4(x - y)\sqrt{x + y}} \left(\frac{1}{y^{1/2}} - \frac{1}{x^{1/2}} \right),$$

where the normalisation is chosen to make $\frac{dh}{dx}$ bounded at $y = x$. The RHS clearly exhibits homogeneity features, so we can find an integral in the form

$$h(x, y) = \frac{1}{y}\tilde{h}\left(\frac{x}{y} \right),$$

where

$$\frac{d\tilde{h}}{dx} = \frac{1}{4(x-1)\sqrt{x+1}} \left(1 - \frac{1}{x^{1/2}}\right).$$

The detailed expression of the RHS is less important than its qualitative features, namely that it is a smooth expression in x, and its asymptotic growth at large x is $O(\frac{1}{x^{3/2}})$. This implies $\tilde{h}(x) = O(\frac{1}{x^{1/2}})$. We then define

$$b^{(2)}(H, \eta) = \frac{1}{a} \tilde{h}\left(\frac{H}{a}\right),$$

so by construction

$$\frac{3H+a}{\sqrt{H+a}} b_H^{(2)} + \frac{2(H^2-a^2)}{\sqrt{H+a}} b_{HH}^{(2)} = \frac{1}{4H^{3/2}} + O\left(\frac{1}{a^{5/2}H^{1/2}}\right).$$

Using the Laplacian formula (21), one can see further that

$$(\sqrt{-1}\partial\bar{\partial}b^{(2)})(\sqrt{-1}\partial\bar{\partial}\phi)^2 = \widetilde{vol_E}\left\{\frac{1}{4H^{3/2}} + O\left(\frac{1}{a^{5/2}H^{1/2}}\right)\right\}.$$

This means the singular term can be approximately cancelled in this example.

The features of these examples mostly carry over to the other error terms in (9) with minor changes. We can write down with obvious modifications an ODE for each error term. Explicit integrals in most other cases are more difficult to compute. However, by the homogeneity of the ODE, in the case of Lemma 6.2 we can always write

$$h(x, y) = \frac{1}{y} \tilde{h}\left(\frac{x}{y}\right),$$

where $\tilde{h}$ solves the ODE with parameter $y = 1$. The algorithm is that we perform the first integral with the requirement that $\frac{d\tilde{h}}{dx}$ stays bounded near $x = 1$, integrate to find h, substitute H for x and a for y. It is clear that in each case the result is a smooth function $b(H, \eta)$ for $H \gg 1$.

It remains to control the behaviour of the function $\tilde{h}(x)$ for large x, in order to estimate all the errors. For the terms

$$\frac{1}{4a(H+a)^{1/2}}, \quad \frac{H^{1/2}}{4a(H+a)}, \quad -\frac{1}{4aH^{1/2}},$$

the relevant $\tilde{h}$ has leading order growth (up to constant factors) $\tilde{h} \sim \log x$, $\frac{d\tilde{h}}{dx} \sim x^{-1}$. For the terms

$$-\frac{H^{1/2}}{(H+a)^2}, \quad \frac{3}{4(H+a)\sqrt{H}}, \quad \frac{1}{4H^{3/2}},$$

$\tilde{h}$ has leading order growth (up to constant factors) $\tilde{h} \sim x^{-1/2}$, $\frac{d\tilde{h}}{dx} \sim x^{-3/2}$.

It follows that in the first three cases, we can cancel the error terms up to $O(\frac{\log(H/a)}{a^3})$, and in the last three cases, we can cancel the error terms up to $O(\frac{1}{H^{1/2}a^{5/2}})$. In summary, we have the lemma 6.2.

The situation of Lemma 6.3 is analogous. The main modification is that the homogeneity behaviour is slightly different, so we should instead write

$$h(x, y) = \frac{1}{y^2} \tilde{h}\left(\frac{x}{y}\right).$$

Remark It should be pointed out that this ODE method improves the error only in the bad region with $\eta \gg H^{1/2}$, *i.e.* the region collapsing to the singularity of the tangent cone. It is typical in adiabatic problems involving a local fibration structure, that one can improve an approximate solution by solving a related equation along the fibres. This is indeed what happens here.

In the region where η is comparable to $H^{1/2}$, the Laplacian genuinely depends on both H and η derivatives, so ODE reduction is not possible in general, but the error there is already acceptably small.

Remark The form $\sqrt{-1}\partial\bar{\partial}(b + \mathfrak{b})$ compared to $\sqrt{-1}\partial\bar{\partial}\phi$ is of order $O(\frac{1}{aH^{1/2}})$ $+ O(\frac{\log(2H/a)}{a^3})$, so its square has order $O(\frac{1}{a^3})$. This again confirms the fact that the nonlinear effect is asymptotically weak.

6.2 Perturbation into the Calabi–Yau metric

We improve the metric ansatz to allow for the application of Hein's existence result. For this we define a metric

$$\omega' = \sqrt{-1}\partial\bar{\partial}(\phi - (b + \mathfrak{b})\chi). \tag{22}$$

where b is defined in Lemma 6.2 and $\mathfrak{b}$ in Lemma 6.3, $0 \leq \chi \leq 1$ is a cutoff function, vanishes in the ball $\{H \leq R^2\}$, and equals to 1 outside $\{H \leq 4R^2\}$. If we choose the parameter R large enough, we can ensure ω' is a genuine Kähler metric.

For $H \gg 1$, the volume form of this modified ansatz is

$$\omega'^3 = \left(\sqrt{-1}\partial\bar{\partial}\phi\right)^3 - 3(\sqrt{-1}\partial\bar{\partial}\phi)^2 \wedge \sqrt{-1}\partial\bar{\partial}(b + \mathfrak{b}) + O(\frac{1}{a^3})$$
$$= \frac{3}{2}\widetilde{vol_E}\left\{1 + O\left(\frac{\log(2H/a)}{a^3}\right)\right\}.$$

Crudely speaking, the error is now at most $O(\frac{1}{a^{3-\epsilon}})$ for any small $\epsilon > 0$.

Now since ω' deviates from $\sqrt{-1}\partial\bar{\partial}\phi$ by only a small smooth perturbation of order $O(\frac{1}{aH^{1/2}}) + O(\frac{\log(2H/a)}{a^3})$, the condition $SOB(6)$ and the existence of $C^{k,\alpha}$ quasi-atlas still hold for ω' (compare with Sect. 5.3). So Hein's theorem gives us

Theorem 6.6 *There is a smooth bounded potential ψ on $\mathbb{C}^3$ with bounded derivatives to all orders, solving the Monge–Ampère equation*

$$(\omega' + \sqrt{-1}\partial\bar{\partial}\psi)^3 = \frac{3}{2}\widetilde{vol_E}. \tag{23}$$

In particular the metric $\omega_{CY} = \omega' + \sqrt{-1}\partial\bar{\partial}\psi$ is Calabi–Yau, and is uniformly equivalent to ω'.

Since our setup is symmetric under the action of $SO(3, \mathbb{R}) \times U(1)$, and Hein's method works equivariantly with respect to the group action, we can assume the solution is symmetric.

6.3 Decay estimates

We now improve the decay rate of the potential ψ. This can be extracted from Hein's work, and the method is standard. We observe first

Proposition 6.7 *The potential ψ decays as $O(\frac{1}{a^{1-\epsilon}})$ for any $\epsilon > 0$.*

Proof This follows from Hein's thesis [6], the proof of proposition 3.16 (ii), and his comments on non-parabolic manifolds in his Sect. 4.5, where the reader can find more details. The main idea is weighted Moser iteration. The lemma 5.12 is technically required in his argument. The general conclusion is that if the forcing term decays as $O(a^{-\mu})$, for $2 < \mu < 6$, then the potential decays as $O(a^{2-\mu+\epsilon})$. □

Remark We think the decay rate estimate is not far from optimal because our correction term to the potential is $\mathfrak{b} = O(\frac{\log(2H/a)}{a})$. It may be of interest to compare this with Joyce's work [7]. As discussed in Sect. 5.2, our setup does not technically fit into Joyce's definition of QALE manifolds. So we shall just

make the crude remark that for Joyce's ALE manifold of real dimension $2n$, the decay rate of the metric is $O(a^{-2n})$, and the decay rate of the potential is $O(a^{2-2n})$, which in our context is $O(a^{-4})$. In any case it is clear that our decay rate is much slower.

Proposition 6.8 *The potential ψ satisfies the decay bound*

$$\left\| \nabla_{\omega'}^{(k)} \psi \right\|_{L^\infty} \leq \frac{C(k)}{a^{1-\epsilon}(H+1)^{k/4}}. \tag{24}$$

Proof By the smooth estimates in Hein's result, it is enough to consider $H \gg 1$ and prove the decay property. Recall in our proof of the existence of the $C^{k,\alpha}$ quasi-atlas, we introduced a set of coordinates z_1', z_2', z_3' around the point x (see lemma 5.10), which displays that the metric is smoothly equivalent to the standard Euclidean metric on a ball of size $cH^{1/4}(x)$, for some small fixed c. Now we scale both the coordinates and the metric, writing $z_j'' = \frac{z_j'}{cH(x)^{1/4}}$, so $\frac{1}{H(x)^{1/2}}\omega'$ and $\frac{1}{H(x)^{1/2}}\omega_{CY}$ are uniformly equivalent to the standard Euclidean metric in the unit coordinate ball. The Calabi–Yau condition reads

$$\left(\frac{1}{H(x)^{1/2}}\omega' + \frac{1}{H(x)^{1/2}}\sqrt{-1}\partial\bar{\partial}\psi \right)^3 - \left(\frac{1}{H(x)^{1/2}}\omega' \right)^3$$
$$= \frac{1}{H(x)^{3/2}} \left(\omega_{CY}^3 - \omega'^3 \right).$$

The $k-$th derivatives of the forcing function $\frac{\omega'^3-\omega_{CY}^3}{\omega_{CY}^3}$ is of order $O(\frac{\log(2H/a)}{a^3}) = O(\frac{1}{a^{1-\epsilon}H^{1/2}})$, for any $k \geq 0$. The potential $\frac{\psi}{H(x)^{1/2}} = O(\frac{1}{a^{1-\epsilon}H^{1/2}})$. By elliptic bootstrap, we obtain that the $k-$th derivative of $\frac{\psi}{H(x)^{1/2}}$ is estimated by $O(\frac{1}{a^{1-\epsilon}H^{1/2}})$. Reverting back to the z_i' coordinates, this implies the claim. $\square$

Remark In the region where $\eta^{1/2}$ is comparable to $H^{1/4}$, the $k = 2$ case of this estimate implies $\omega_{CY} - \omega = O(\frac{1}{a^{3-\epsilon}})$, where we recall ω is the ansatz (8). The decay estimates are much slower in E_ϵ, which can be expected from the singularity of the tangent cone at infinity.

As a consequence of the decay estimate 6.8 on the metric, the leading order asymptote of ω agrees with that of ω_{CY}, hence

Corollary 6.9 *The asymptotic properties of ω, including maximal volume growth and the tangent cone at infinity, are also true for ω_{CY}.*

Remark This CY metric is a counterexample to a previous conjecture, which says any complete CY metric on $\mathbb{C}^n$ with maximal growth is flat (*cf.* [14], Remark 5.3).

6.4 Riemannian submersion property

We consider the trace $\mathrm{Tr}_{\omega_{CY}} \omega_0$, where $\omega_0 = \frac{1}{2}\sqrt{-1}df \wedge d\bar{f}$ is the standard Euclidean metric on $\mathbb{C}$. We notice that for $H \gg 1$, the asymptotic behaviour of ω_{CY} implies f is almost a Riemannian submersion, or equivalently by considering the pointwise simultaneous diagonalisation of ω_{CY} and ω_0,

$$\mathrm{Tr}_{\omega_{CY}} \omega_0 \sim 1.$$

For the region $H \ll 1$, this is no longer true due to quantising effects. But quite remarkably we have a one sided global estimate:

Proposition 6.10 $\mathrm{Tr}_{\omega_{CY}} \omega_0 \leq 1.$

Proof We use the Chern-Lu formula for the fibration f (*cf.* [10] proposition 7.1), which in this case essentially reduces to the Bochner formula. Notice ω_{CY} is Ricci flat and ω_0 is flat, so

$$\Delta_{\omega_{CY}} \log \mathrm{Tr}_{\omega_{CY}} \omega_0 \geq 0.$$

The subharmonic function $\log \mathrm{Tr}_{\omega_{CY}} \omega_0$ tends to limiting value zero at infinity, hence the maximum principle gives the claim. □

Remark It may be noticed that this argument only depends on the Calabi–Yau condition and the leading asymptote. Establishing a priori properties of the metric may have some bearing on the open question whether the leading asymptote uniquely determines the Calabi–Yau metric.

7 Remarks on moduli questions

7.1 Moduli questions and possible generalisations

Our Calabi–Yau metric ω_{CY} is a nontrivial Ricci flat metric on $\mathbb{C}^3$. This is interesting in relation to the following fundamental question: on a given non-compact complex manifold with a global holomorphic volume form, can we classify the complete Calabi–Yau metrics? What if we impose maximal growth condition? How about the case of $\mathbb{C}^n$?

For instance, one may ask if we can build other examples using more complicated fibrations on $\mathbb{C}^n$, starting from some ansatz which desingularises a semi-Ricci flat metric; optimistically this gives an abundant supply of nontrivial Ricci flat metrics on $\mathbb{C}^n$. Existence results in this direction have now been obtained independently by [12] and [1], shortly after the appearance of the present paper on arXiv. The difficulty for this question is more severe, because

we lose both the explicit nature of the Stenzel metric and the cohomogeneity-two property.

This question is also very relevant for understanding what happens near the (isolated) critical points of a very collapsed Calabi–Yau manifold fibred over some Riemann surface, assuming the singular fibre is a normal Calabi–Yau variety. The expectation is that there should be a complete Calabi–Yau metric on a model space fibred over $\mathbb{C}$, possibly biholomorphic to $\mathbb{C}^n$, whose asymptotic behaviour is a semi-Ricci flat metric, such that after appropriate scaling it describes the collapsing metric near the critical points on the total space of the fibration. (The normality condition is imposed for the fibrewise Calabi–Yau metric to make sense; if we drop this condition, a prototype example is the Ooguri-Vafa metric, which is an incomplete metric modelling the collapsing elliptically fibred K3 surface near a singular fibre (*cf.* Gross and Wilson [4]), and whose local behaviour near the node is modelled on the Taub-NUT metric which does not have maximal growth.) A description of the tangent cone at infinity is likely to involve stability issues, as understood for A_k-type singularities by Hein and Naber.

On the negative side, any meaningful classification program must deal with singular tangent cones, which can be a major source of difficulty.

7.2 Deformation and uniqueness questions

We can also ask the weaker question of classifying Calabi–Yau metrics arising as deformations of a given one, which I believe is more tractable. This depends on the class of deformations, which in turn depends on the function spaces in which one sets up the Monge–Ampère equation. Hein's viscosity solution method surprisingly does not give any suggestion on the appropriate function space, nor does it make any uniqueness assertions. Let us formulate a number of more concrete questions:

Question 7.1 Can we classify all Calabi–Yau metrics with the given holomorphic volume form which are uniformly equivalent to our example?

Instead of giving a complete answer, we remark that $SL(3, \mathbb{C})$ gives automorphisms of $\mathbb{C}^3$ as a complex manifold. But the uniform equivalence forces a compatibility condition with the fibration structure, so reduces this symmetry to $SO(3, \mathbb{C}) \times \langle e^{2\pi i/3} \rangle$. Clearly $SO(3, \mathbb{R}) \times \langle e^{2\pi i/3} \rangle$ is a subgroup preserving also the metric structure. So the moduli space contains at least the homogeneous space $SO(3, \mathbb{C})/SO(3, \mathbb{R}) \simeq SL(2, \mathbb{C})/SU(2)$. These examples are different as Calabi–Yau metrics on the fixed complex manifold $\mathbb{C}^3$, but they are isometric as Riemannian manifolds by a linear change of coordinates.

A more subtle deformation of our metric can be obtained by a simultaneous scaling of our metric and the coordinates, such that the volume form is

preserved. Viewed alternatively, if we start from the ansatz

$$\phi = \frac{1}{2\lambda^{2/3}}\eta + \lambda^{1/3}\sqrt{H + (\eta + \lambda\sqrt{H + \lambda^{2/3}})^{1/2}},$$

where $\lambda > 0$, and solve the Monge–Ampère equation as before, we would get a new Calabi–Yau metric with the same volume form. This deformation can be anticipated from our discussions in Sect. 4.

Question 7.2 Given the leading order asymptote for the Kähler form, then is our Calabi–Yau metric unique?

The uniqueness is clear once we assume sufficient decay of the perturbation term, and the question is about the optimal decay condition we need.

Question 7.3 What are the formal deformations of our example?

This involves linearising the Calabi–Yau equation. We obtain formally an equation on a closed $(1, 1)$ form β:

$$\omega_{CY}^2 \wedge \beta = 0. \tag{25}$$

This says β is primitive. On the space $\mathbb{C}^3$, closedness and exactness are the same. It is clear from the Hodge identities that β is harmonic. For example, the nontrivial deformations discussed above provide particular solutions. The answer in general depends on the function space. In particular, we may ask what are the bounded solutions, and how they relate to moduli questions.

Question 7.4 (Rigidity) If ω_{CY}' is a Calabi–Yau metric on a large ball $\{|z| \leq R\}$ with the same volume form as ω_{CY}, and near the boundary $(1 - \epsilon)\omega_{CY} \leq \omega_{CY}' \leq (1 + \epsilon)\omega_{CY}$ for sufficiently small ϵ, then can we deduce effective estimates of the shape $(1 - C\epsilon)\omega_{CY} \leq \omega_{CY}' \leq (1 + C\epsilon)\omega_{CY}$ on the whole ball, where C is independent of R?

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