DK CONJECTURE FOR DERIVED CATEGORIES OF GRASSMANNIAN FLIPS

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ABSTRACT. The DK Flip Conjecture of Bondal-Orlov [3] and Kawamata [6] states that there should be an embedding of derived categories for any flip, which is known to be true for toroidal flips. In this paper, we construct new examples of Grassmannian flips which satisfy the DK Flip Conjecture.

1. INTRODUCTION

It is well-known that the birational geometry of an algebraic variety Y is closely related to its bounded derived category of coherent sheaves $D^bCoh(Y)$, or simply $D(Y)$. The DK Conjecture by Bondal-Orlov [3] and Kawamata [6] is one of the most fundamental open problems in this area. Recall that a birational map $f: X_2 \dashrightarrow X_1$ between two smooth varieties X_1 and X_2 is called a $flop$ if there is a third smooth variety X with two birational morphisms $\pi_1: X \to X_1$ and $\pi_2: X \to X_2$ such that $f = \pi_1 \circ \pi_2^{-1}$ and $\pi_2^* K_{X_2} = \pi_1^* K_{X_1}$. f is called a $flip$ if $\pi_2^* K_{X_2} = \pi_1^* K_{X_1} + D$ for some effective divisor D on X.[1]

$$\begin{tikzcd} & X \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ X_2 \arrow[rr, dashed, "f"] & & X_1 \end{tikzcd}$$

DK Flip Conjecture (Bondal-Orlov [3] and Kawamata [6]). *For any flip*

$$X_2 \dashrightarrow X_1,$$

there is a derived embedding:

$$D(X_1) \hookrightarrow D(X_2).$$

Unlike the situations for flops (see [11] for the survey of DK conjecture), there are few examples of flips proven to satisfy the DK conjecture except for some toroidal flips (see [7], [8], [9] and [10]). In this paper, we construct some new examples of flips that satisfy the DK flip conjecture.

Consider the partial flag variety

$$Fl(1,2,N) = \{(V_1,V_2) | V_1 \subset V_2 \subset \mathbb{C}^N, \dim V_1 = 1, \dim V_2 = 2\}$$

which admits two projective space fibrations onto $\mathbb{P}^{N-1}$ and $Gr(2,N)$ respectively:

[1]In some contexts or references, the definition of $flop$ (resp. $flip$) used here is usually called K-equivalence (resp. K-inequivalence).

$$\begin{tikzcd}
 & Fl(1,2,N) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
Gr(2,N) & & \mathbb{P}^{N-1}.
\end{tikzcd}$$

Note that $Fl(1,2,N) \cong \mathbb{P}_{Gr(2,N)}(U) \cong \mathbb{P}_{\mathbb{P}^{N-1}}(Q)$, where U is the tautological rank 2 subbundle on $Gr(2,N)$ and Q is the tautological rank $(N-1)$ quotient bundle on $\mathbb{P}^{N-1}$. Denote the ample generator of $Pic(\mathbb{P}^{N-1})$ (resp. $Pic(Gr(2,N))$) by h (resp. H). Let

$$X = Tot_{Fl(1,2,N)}(\mathscr{O}(H+h)^{\vee}), \quad X_1 = Tot(p_{1*}(\mathscr{O}(H+h))^{\vee}), \quad X_2 = Tot(p_{2*}(\mathscr{O}(H+h))^{\vee}).$$

It is easy to check that

$$p_{2*}\mathscr{O}(h+H) = U^{\vee} \otimes \mathscr{O}(H), \tag{1.1}$$

$$p_{1*}\mathscr{O}(h+H) = Q^{\vee} \otimes \mathscr{O}(2h). \tag{1.2}$$

Then $X_2 = Tot_{Gr(2,N)}(U(-H))$, $X_1 = Tot_{\mathbb{P}^{N-1}}(Q(-2h))$ and X will be isomorphic to the two blowing ups: $Bl_{Gr(2,N)}X_2$ and $Bl_{\mathbb{P}^{N-1}}X_1$ with the same exceptional divisor $E \cong Fl(1,2,V)$:

$$\begin{tikzcd}
 & E = Fl(1,2,N) \arrow[dl, "p_1"'] \arrow[d, hook, "j"] \arrow[dr, "p_2"] & \\
Gr(2,N) \arrow[d, hook] & X \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \mathbb{P}^{N-1} \arrow[d, hook] \\
X_2 \arrow[rr, dashed, "f"] & & X_1.
\end{tikzcd} \tag{1.3}$$

So we get a birational map f from X_2 to X_1. Now an easy computation implies that

$$D = \pi_2^* K_{X_2} - \pi_1^* K_{X_1} = (N-3)E.$$

Hence f is a flip if $N > 3$ and a flop if $N = 3$.

Theorem 1.1. *The flip $f : X_2 \dashrightarrow X_1$ considered above satisfies the DK Flip Conjecture, i.e. there is a fully-faithful embedding of triangulated categories:*

$$D(X_1) \overset{\Phi}{\hookrightarrow} D(X_2).$$

Remark 1.2.

(1) When $N = 3$, (1.3) is the Mukai flop of $\Omega_{\mathbb{P}^2}$.
(2) Theorem 1.1 holds for any flip which locally looks like (1.3).

Convention. In this paper, $\mathbb{P}(V) = Proj(Sym^{\bullet} V^{\vee})$ for any vector bundle V. The derived functors $\mathbb{R}Hom(-,-)$ and $\mathrm{Ext}^{\bullet}(-,-)$ are taken over the total space X. We will omit the natural functors p_1^*, p_2^* and j_*, i_{1*}, i_{2*} if no confusion occurs.

Strategy of Proof. Firstly, we can embed $D(X_1)$ and $D(X_2)$ into $D(X)$ by Orlov's blow up formula [17] so that we have the following two semiorthogonal decompositions (SOD) of $D(X)$:

$$D(X) = \langle \pi_1^* D(X_2), j_* p_2^* D(Gr(2,N)) \rangle, \tag{1.4}$$

$$D(X) = \Big\langle \pi_2^* D(X_1), \langle j_*(p_1^* D(\mathbb{P}^{N-1}))(kH) \rangle_{0 \le k \le N-3} \Big\rangle. \tag{1.5}$$

The left orthogonal complements of $D(X_1)$ and $D(X_2)$ are (copies) of derived categories of Grassmaniann. It is known that both $D(\mathbb{P}^{N-1})$ and $D(Gr(2,N))$ admit full exceptional collections by [1] and [13]. The former one consists of line bundles only while the latter one involves $S^k U$, symmetric powers of the tautological subbundle U on $Gr(2,N)$.

Secondly, we use SOD of $D(Gr(2,N))$ involving $S^k U$ to simplify (1.4) to the form (3.1) :

$$D(X) = \Big\langle \mathscr{D}, \langle \mathscr{O}(\ell h) \rangle_{-1 \le \ell \le n-1}, \mathscr{H}, \langle \mathscr{F}_\ell \rangle_{0 \le \ell \le n-2}, S^{n-1}U(H), \mathscr{A}(\ell H) \rangle_{2 \le \ell \le 2n-2} \Big\rangle$$

via mutation techniques by Kuznetsov in [14] (C.f. also [15], [19] or [16]). However, there are still lots of symmetric powers of U remain in the left orthogonal complement of $\mathscr{D}$, which does not happen for flop situation (see remark 3.7).

Thirdly, to get rid of the remaining $S^k U$'s, we apply the chess-game method introduced in [18] (also in [4]) which is a systematic method to do cancellation of categories and prove Theorem 1.1 then. Roughly speaking, chess-game method is an analogy of the spectral sequence argument in cohomologies.

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2. Vanishing of Cohomology and Mutations

In this section, we list vanishing results and mutations that will be used later in the subsequent sections. For $\ell \ge 0$, let

$$\mathscr{A}^l = \langle \mathscr{O}, U^\vee, \cdots, S^{n-l-1}U^\vee \rangle;;$$
$$\mathscr{A}_l = \langle S^l U^\vee, S^{l+1}U^\vee, \cdots, S^{n-1}U^\vee \rangle$$
$$\mathscr{B}_\ell = \langle \mathscr{O}((\ell+1)h), S^\ell U^\vee(H-h) \rangle;$$
$$\mathscr{C}_\ell = \langle \mathscr{O}(\ell h), S^\ell U^\vee(H-2h) \rangle;$$
$$\mathscr{E}_\ell = \langle \mathscr{O}(\ell h), S^{\ell-1}U^\vee(H-h), \mathscr{O}((\ell+1)(H-h)-h) \rangle;$$

$$\mathscr{F}_\ell = \begin{cases} \langle S^\ell U^\vee(H), \mathscr{O}((\ell+2)(H-h)), \mathscr{O}((\ell+3)(H-h)-h)\rangle & \text{if } \ell \le n-4; \\ \langle S^\ell U^\vee(H), \mathscr{O}((\ell+2)(H-h))\rangle & \text{if } \ell > n-4; \end{cases}$$

$$\mathscr{H} = \begin{cases} \langle \mathscr{O}(H-2h), \mathscr{O}(H-h)\rangle & \text{if } n=2; \\ \langle \mathscr{O}(H-2h), \mathscr{O}(H-h), \mathscr{O}(2H-3h)\rangle & \text{if } n \ge 3; \end{cases}$$

Lemma 2.1 (Kapranov [5] and Kuznetsov [13])**.** *$D(Gr(2,N))$ admits a full exceptional collections:*

$$D(Gr(2,N)) = \begin{cases} \left\langle \langle \mathscr{A}^1(kH)\rangle_{0\le k\le n-1}, \langle \mathscr{A}(\ell H)\rangle_{n-2\le \ell\le 2n-1} \right\rangle & \textit{if } N=2n, \\ \langle \mathscr{A}(kH)\rangle_{0\le k\le N-1} & \textit{if } N=2n+1, \end{cases} \tag{2.1}$$

Lemma 2.2. *(1) For any $0 \le k \le n-1$,*

$$\mathbb{R}Hom(S^{n-k-1}U^\vee(H-h), \mathscr{A}^k) = 0.$$

(2) For $0 \le k \le n-1$,

$$\mathbb{R}Hom(\mathscr{A}_{k+2}(-h), \mathscr{O}(kh)) = 0;$$
$$\mathbb{R}Hom(\mathscr{A}^{n-k-1}(H-h), \mathscr{O}(kh)) = 0.$$

(3) For $1 \le k < \ell \le n-2$,

$$\mathbb{R}Hom(\mathscr{C}_\ell, S^{k-1}U^\vee(H-h)) = 0.$$

(4) For $1 \le k \le n-2$ and $n-k \le \ell \le n-1$,

$$\mathbb{R}Hom(S^{n-2-k}U^\vee(H-h), \mathscr{O}(\ell H)) = 0;$$
$$\mathbb{R}Hom(S^{n-k}U^\vee(H-h), \mathscr{A}^{k+1}(H)) = 0;$$
$$\mathbb{R}Hom(\mathscr{O}((n-k)(H-h)-h), \mathscr{A}^{k+2}(H)) = 0.$$

(5) For any $0 \le \ell < k \le r$, $r = [(n-1)/2]$, i.e., the greatest integer no bigger than $[(n-1)/2]$,

$$\mathbb{R}Hom(\mathscr{A}_{n-2\ell-1}((n+\ell)H), \mathscr{A}^{2k+1}((n+k)H)) = 0.$$

(6) Either (i) $2 \le a-b \le 2n-3$ and $b > 0$ or (ii) $b < 0$,

$$\mathrm{Ext}^\bullet(\mathscr{O}(ah), \mathscr{O}(bH)) = 0.$$

Proof. We give a proof of (1) here and the rest can be obtained using same arguments (See Appendix A1 for detailed proof of others). By adjunction of pullback-pushforward,

$$\mathbb{R}Hom(S^{n-k-1}U^\vee(H-h), \mathscr{A}^k) = \mathbb{R}Hom_E(\mathbb{L}j^* j_* S^{n-k-1}U^\vee(H-h), \mathscr{A}^k).$$

Recall the distinguished triangle associated to the closed immersion $j : E \hookrightarrow X$:

$$\mathscr{O}(H+h)[1] = \mathscr{O}_E(-E)[1] \longrightarrow \mathbb{L}j^* j_* \longrightarrow id \xrightarrow{[1]} \tag{2.2}$$

and hence inducing a distinguished triangle of complex of vector spaces:

$$\mathbb{R}Hom_E(S^{n-k-1}U^\vee(2H)[1],\mathscr{A}^k) \longrightarrow \mathbb{R}Hom(S^{n-k-1}U^\vee(H-h),\mathscr{A}^k) \longrightarrow$$
$$\mathbb{R}Hom_E(S^{n-k-1}U^\vee(H-h),\mathscr{A}^k) \xrightarrow{[1]} .$$

So it is sufficient to show the vanishing of the first and third terms. With the help of projection formula (1.1) or Lemma A3, we have the following for $0 \le a \le n-k-1$:

$$(i)\ \mathrm{Ext}^\bullet_E(S^{n-k-1}U^\vee(2H),S^aU^\vee) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-k-1}U^\vee(2H),S^aU^\vee),$$
$$(ii)\ \mathrm{Ext}^\bullet_E(S^{n-k-1}U^\vee(H-h),S^aU^\vee) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-k-1}U^\vee(H),S^aU^\vee\otimes U^\vee)$$
$$= \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-k-1}U^\vee(H),S^{a+1}U^\vee\oplus S^{a-1}U^\vee(H)).$$

It is noted that all these vanish by SOD of $D(Gr(2,N))$ (2.1) except for the case $k=0, a=n-1$:

$$\mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-1}U^\vee(H),S^nU^\vee) = 0.$$

By Littlewood-Richardson rule,

$$S^{n-1}U^\vee(-H)\otimes S^nU^\vee = \bigoplus_{t=0}^{n-1}\Sigma^{n-t-1,-n+t}U^\vee.$$

Then

$$\mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-1}U^\vee(H),S^nU^\vee) = \bigoplus_{t=0}^{n-1}H^\bullet(Gr(2,N),\Sigma^{n-t-1,-n+t}U^\vee) = 0$$

by checking the criterion of Theorem A2 (BWB). □

For the projection p_2, there exists a relative Euler sequence on $E=\mathbb{P}_{Gr(2,N)}(U)$:

$$0\longrightarrow \mathscr{O}(H-h)\longrightarrow U^\vee \longrightarrow \mathscr{O}(h)\longrightarrow 0.$$

For each k, there are two short exact sequences on E and X by taking k-th symmetric power :

$$0\longrightarrow S^{k-1}U^\vee(H-h)\longrightarrow S^kU^\vee \longrightarrow \mathscr{O}(kh)\longrightarrow 0; \tag{2.3}$$
$$0\longrightarrow \mathscr{O}(k(H-h))\longrightarrow S^kU^\vee \longrightarrow S^{k-1}U^\vee(kh)\longrightarrow 0. \tag{2.4}$$

Not surprisingly, these two induce the following mutations (See Appendix A1 for the proof):

Lemma 2.3. *For any* $1\le k\le n-1$,

$$(1)\ \mathbb{L}_{S^{k-1}U^\vee(H-h)}S^kU^\vee = \mathscr{O}(kh);$$
$$(2)\ \mathbb{R}_{\mathscr{O}(kh)}S^kU^\vee = S^{k-1}U^\vee(H-h);$$
$$(3)\ \mathbb{R}_{S^{k-1}U^\vee(-h)}S^kU^\vee = \mathscr{O}(k(H-h)).$$

3. Mutations on Derived category of X Odd Case

In this section, we simplify SOD of $D(X)$ (1.4) by mutation techniques when $N=2n+1$ ($N=2n$ case will be explained in section 5).[2] The main result of this section is

Proposition 3.1. *For $n\geq 2$,*

$$D(X)=\left\langle \mathscr{D},\langle \mathscr{O}(\ell h)\rangle_{-1\leq \ell\leq n-1},\mathscr{H},\langle \mathscr{F}_\ell\rangle_{0\leq \ell\leq n-2},S^{n-1}U(H),\mathscr{A}(\ell H)\rangle_{2\leq \ell\leq 2n-2}\right\rangle \tag{3.1}$$

where $\mathscr{D}=\mathbb{L}_{\langle \mathscr{A}(-h),\mathscr{A}(H-h)\rangle}\pi_2^*D(X_2)$.

Proof. At first we left mutate $\langle \mathscr{A}((2n-1)H),\mathscr{A}(2nH)\rangle$ to the far left. Note that

$$\mathbb{L}_{\langle \mathscr{A}((2n-1)H),\mathscr{A}(2nH)\rangle^{\perp}}\Big|_{\langle \mathscr{A}((2n-1)H),\mathscr{A}(2nH)\rangle}=-\otimes K_X[dimX]$$

and $K_X\big|_E=\mathscr{O}(-h-(2n-1)H)$ (see Lemma A1). Thus

$$D(X)=\left\langle \mathscr{A}(-h),\mathscr{A}(H-h),\pi_2^*D(X_2),\langle \mathscr{A}(kH)\rangle_{0\leq k\leq 2n-2}\right\rangle.$$

Then left mutate $\pi_2^*D(X_2)$ through $\langle \mathscr{A}(-h),\mathscr{A}(H-h)\rangle$:

$$D(X)=\left\langle \mathscr{D},\mathscr{A}(-h),\mathscr{A}(H-h),\langle \mathscr{A}(kH)\rangle_{0\leq k\leq 2n-2}\right\rangle. \tag{3.2}$$

To arrive (3.1), we need to do mutation on $\langle \mathscr{A}(-h),\mathscr{A}(H-h),\mathscr{A},\mathscr{A}(H)\rangle$ which is involved and consists of the 4 inductive steps.

(1) **First Inductive Step ($n\geq 1$)**: Mutation on $\langle \mathscr{A}(H-h),\mathscr{A}\rangle$.

Lemma 3.2. *For $1\leq k\leq n$,*

$$\langle \mathscr{A}(H-h),\mathscr{A}\rangle=\left\langle \mathscr{A}^k(H-h),\mathscr{A}^{k-1},\langle \mathscr{B}_\ell\rangle_{n-k\leq \ell\leq n-2},S^{n-1}U^\vee(H-h)\right\rangle.$$

Proof. Prove by induction on k.

- Base case ($k=1$). We can exchange $S^{n-1}U^\vee(H-h)$ and $\mathscr{A}$ by Lemma 1 ($k=0$):

$$LHS=\langle \mathscr{A}^1(H-h),S^{n-1}U^\vee(H-h),\mathscr{A}\rangle=\langle \mathscr{A}^1(H-h),\mathscr{A},S^{n-1}U^\vee(H-h)\rangle=RHS.$$

- Assume that we have SOD for case k. Then

$$\begin{aligned}RHS&=\left\langle \mathscr{A}^{k+1}(H-h),\underline{S^{n-k-1}U^\vee(H-h),\mathscr{A}^k},S^{n-k}U^\vee,\langle \mathscr{B}_\ell\rangle_{n-k\leq \ell\leq n-2}\right\rangle\\&=\left\langle \mathscr{A}^{k+1}(H-h),\mathscr{A}^k,\underline{S^{n-k-1}U^\vee(H-h),S^{n-k}U^\vee},\langle \mathscr{B}_\ell\rangle_{n-k\leq \ell\leq n-2}\right\rangle\\&=\left\langle \mathscr{A}^{k+1}(H-h),\mathscr{A}^k,\langle \mathscr{B}_\ell\rangle_{n-k-1\leq \ell\leq n-2}\right\rangle.\end{aligned}$$

This is just the case $k+1$ and the lemma follows. In the second line, we exchange $S^{n-k-1}U^\vee(H-h)$ and $\mathscr{A}^k$ by Lemma 2.2 (1) and the in the last line we left mutate $S^{n-k}U^\vee$ through $S^{n-k-1}U^\vee(H-h)$ (Lemma 2.3).

[2] The readers can refer to Appendix A1 for the preliminaries and background knowledge of left and right mutations.

$\square$

Apply Lemma 3.2 for the final case $k = n$:

$$\langle \mathscr{A}(H-h), \mathscr{A} \rangle = \Big\langle \mathscr{O}, \langle \mathscr{B}_\ell \rangle_{0 \le \ell \le n-2}, S^{n-1}U^\vee(H-h) \Big\rangle. \tag{3.3}$$

Remark 3.3. When $n = 1$, (3.3) is nothing but

$$\langle \mathscr{O}(H-h), \mathscr{O} \rangle = \langle \mathscr{O}, \mathscr{O}(H-h) \rangle.$$

(2) **Second Inductive Step ($n \ge 2$)**: Mutation on $\Big\langle \mathscr{A}_1(-h), \mathscr{O}, \langle \mathscr{B}_\ell \rangle_{0 \le \ell \le n-2} \Big\rangle$.

Lemma 3.4. *For any* $1 \le k \le n-1$,

$$\Big\langle \mathscr{A}_1(-h), \mathscr{O}, \langle \mathscr{B}_\ell \rangle_{0 \le \ell \le n-2} \Big\rangle = \Big\langle \langle \mathscr{C}_\ell \rangle_{0 \le \ell \le k-1}, \mathscr{A}_{k+1}(-h), \mathscr{A}^{n-k+1}(H-h), \langle \mathscr{B}_\ell \rangle_{k-1 \le \ell \le n-2} \Big\rangle.$$

Proof. Prove by induction on k:

- Base case $k = 1$ is equivalent to $\langle U^\vee(-h), \mathscr{A}_2(-h), \mathscr{O} \rangle = \langle \mathscr{O}, \mathscr{O}(H-2h), \mathscr{A}_2(-h) \rangle$, which can be proved by firstly exchange $\mathscr{O}$ and $\mathscr{A}_2(-h)$ using Lemma 2.2 (2) ($k = 0$) and right mutate $U^\vee(-h)$ through $\mathscr{O}$ (Lemma 2.3).
- Assume we have the SOD for case k. Then

$$\begin{aligned} RHS &= \Big\langle \langle \mathscr{C}_\ell \rangle_{0 \le \ell \le k-1}, S^{k+1}U^\vee(-h), \underline{\mathscr{A}_{k+2}(-h), \mathscr{A}^{n-k-1}(H-h), \mathscr{O}(kh)}, S^{k-1}U^\vee(H-h), \langle \mathscr{B}_\ell \rangle_{k \le \ell \le n-2} \Big\rangle \\ &= \Big\langle \langle \mathscr{C}_\ell \rangle_{0 \le \ell \le k-1}, \underline{S^{k+1}U^\vee(-h), \mathscr{O}(kh)}, \mathscr{A}_{k+2}(-h), \mathscr{A}^{n-k}(H-h), \langle \mathscr{B}_l \rangle_{k \le \ell \le n-2} \Big\rangle \\ &= \Big\langle \langle \mathscr{C}_\ell \rangle_{0 \le \ell \le k}, \mathscr{A}_{k+2}(-h), \mathscr{A}^{n-k}(H-h), \langle \mathscr{B}_\ell \rangle_{k \le \ell \le n-2} \Big\rangle. \end{aligned}$$

  This is the case $k+1$. In the second line we exchange $\mathscr{A}_{k+2}(-h)$ and $\langle \mathscr{A}^{n-k-1}(H-h), \mathscr{O}(kh) \rangle$ by Lemma 2.2 (2) and in the third line we right mutate $S^{k+1}U^\vee(-h)$ through $\mathscr{O}(kh)$ (Lemma 2.3).

$\square$

Apply Lemma 3.4 to the final case ($k = n-1$):

$$\Big\langle \mathscr{A}_1(-h), \mathscr{O}, \langle \mathscr{B}_\ell \rangle_{0 \le \ell \le n-2} \Big\rangle = \Big\langle \langle \mathscr{C}_\ell \rangle_{0 \le \ell \le n-2}, \mathscr{A}^2(H-h), \mathscr{B}_{n-2} \Big\rangle.$$

(3) **Third Inductive Step ($n \ge 3$):** Mutation on $\Big\langle \langle \mathscr{C}_\ell \rangle_{1 \le \ell \le n-2}, \mathscr{A}^2(H-h) \Big\rangle$.

Lemma 3.5. *For any* $1 \le k \le n-1$,

$$\Big\langle \langle \mathscr{C}_\ell \rangle_{1 \le \ell \le n-2}, \mathscr{A}^2(H-h) \Big\rangle = \Big\langle \langle \mathscr{E}_\ell \rangle_{1 \le \ell \le k-1}, \langle \mathscr{C}_\ell \rangle_{k \le \ell \le n-2}, \mathscr{A}^2_{k-1}(H-h) \Big\rangle.$$

Proof. Prove by induction on k

- Base case $k = 1$ is trivial.

- Assume that we have the SOD for the case k. Then

$$\begin{aligned}RHS &= \Big\langle \langle \mathscr{E}_\ell\rangle_{1\le \ell\le k-1}, \mathscr{O}(kh), S^kU^\vee(H-2h), \underline{\langle \mathscr{C}_\ell\rangle_{k+1\le\ell\le n-2}, S^{k-1}U^\vee(H-h)}, \mathscr{A}_k^2(H-h)\Big\rangle \\ &= \Big\langle \langle \mathscr{E}_\ell\rangle_{1\le \ell\le k-1}, \mathscr{O}(kh), \underline{S^kU^\vee(H-2h), S^{k-1}U^\vee(H-h)}, \langle \mathscr{C}_\ell\rangle_{k+1\le\ell\le n-2}, \mathscr{A}_k^2(H-h)\Big\rangle \\ &= \Big\langle \langle \mathscr{E}_\ell\rangle_{1\le \ell\le k}, \langle \mathscr{C}_\ell\rangle_{k+1\le\ell\le n-2}, \mathscr{A}_k^2(H-h)\Big\rangle.\end{aligned}$$

  This is the case $k+1$. In the second line we exchange $\langle \mathscr{C}_\ell\rangle_{k+1\le\ell\le n-2}$ and $S^{k-1}U^\vee(H-h)$ by Lemma 2.2 (3) and in the third line we right mutate $S^kU^\vee(H-2h)$ through $S^{k-1}U^\vee(H-h)$ (Lemma 2.3).

$\square$

Apply Lemma 3.5 to the final case ($k=n-1$):

$$\Big\langle \langle \mathscr{C}_\ell\rangle_{1\le\ell\le n-2}, \mathscr{A}^2(H-h)\Big\rangle = \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2}.$$

Lastly in the third inductive process, we do mutation on $\langle S^{n-1}U^\vee(H-h), \mathscr{A}^1(H)\rangle$ by first exchanging $S^{n-1}U^\vee(H-h)$ and $\mathscr{A}^2(H)$ by Lemma 2.2 (4) and then right mutating $S^{n-1}U^\vee(H-h)$ through $S^{n-2}U^\vee(H)$:

$$\langle S^{n-1}U^\vee(H-h), \mathscr{A}^1(H)\rangle = \langle \mathscr{A}^2(H), \mathscr{F}_{n-2}\rangle.$$

(4) **Forth Inductive Process ($n\ge 3$)**:Mutation on $\Big\langle \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2}, \mathscr{B}_{n-2}, \mathscr{A}^2(H)\Big\rangle$.

Lemma 3.6. *For any* $1\le k\le n-2$*,*

$$\begin{aligned}&\Big\langle \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2}, \mathscr{B}_{n-2}, \mathscr{A}^2(H)\Big\rangle \\ =&\Big\langle \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-1-k}, \langle \mathscr{O}(\ell H)\rangle_{n-k\le\ell\le n-1}, \mathscr{A}^{k+2}(H), \langle \mathscr{F}_\ell\rangle_{n-2-k\le\ell\le n-3}\Big\rangle.\end{aligned}$$

Proof. Prove by induction on k.

- Base case $k=1$ is equivalent to

$$\langle \mathscr{B}_{n-2}, \mathscr{A}^2(H)\rangle = \langle \mathscr{O}((n-1)h), \mathscr{A}^3(H), \mathscr{F}_{n-3}\rangle.$$

  We can exchange $S^{n-2}U^\vee(H-h)$ and $\mathscr{A}^3(H)$ by Lemma 2.2(4) and then right mutate $S^{n-2}U^\vee(H-h)$ through $S^{n-3}U^\vee(H)$ (Lemma 2.3). That is,

$$\begin{aligned}\ell HS &= \langle \mathscr{O}((n-1)h), \mathscr{A}^3(H), S^{n-2}U^\vee(H-h), S^{n-3}U^\vee(H)\rangle \\ &= RHS.\end{aligned}$$

- Assume that we have the SOD for case k. Then

$$\begin{aligned} RHS &= \Big\langle \langle \mathscr{E}_\ell \rangle_{1\le \ell\le n-2-k}, \mathscr{O}((n-1-k)h), \underline{S^{n-2-k}U^\vee(H-h), \mathscr{O}((n-k)(H-h)-h),} \\ &\underline{\langle \mathscr{O}(\ell H)\rangle_{n-k\le\ell\le n-1}, \mathscr{A}^{k+3}(H)}, S^{n-k-3}U^\vee(H), \langle \mathscr{F}_\ell\rangle_{n-2-k\le \ell\le n-3}\Big\rangle \\ &= \Big\langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2-k}, \mathscr{O}((n-1-k)h), \langle \mathscr{O}(\ell H)\rangle_{n-k\le \ell\le n-1}, \mathscr{A}^{k+3}(H), \\ &\underline{S^{n-2-k}U^\vee(H-h), S^{n-k-3}U^\vee(H)}, \mathscr{O}((n-1-k)(H-h)-h), \langle \mathscr{F}_\ell\rangle_{n-2-k\le\ell\le n-3}\Big\rangle \\ &= \Big\langle \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2-k}, \langle \mathscr{O}(\ell H)\rangle_{n-1-k\le\ell\le n-1}, \mathscr{A}^{k+3}(H), \langle \mathscr{F}_\ell\rangle_{n-3-k\le\ell\le n-3}\Big\rangle. \end{aligned}$$

  This is just the case $k+1$. It is very similar with the argument in the base case above by Lemma 2.2(4) and Lemma 2.3: (i) Exchange $\langle S^{n-2-k}U^\vee(H-h), \mathscr{O}((n-k)(H-h)-h)\rangle$ and $\langle\langle \mathscr{O}(\ell H)\rangle_{n-k\le\ell\le n-1}, \mathscr{A}^{k+3}(H)\rangle$; (ii) Exchange $\mathscr{O}((n-k)(H-h)-h)$ and $S^{n-k-3}U^\vee(H)$; (iii) Right mutate $S^{n-2-k}U^\vee(H-h)$ through $S^{n-k-3}U^\vee(H)$.

□

Apply Lemma 3.6 to the final case ($k=n-2$):

$$\Big\langle \langle \mathscr{E}_\ell\rangle_{1\le\ell\le n-2}, \mathscr{B}_{n-2}, \mathscr{A}^2(H)\Big\rangle = \Big\langle \mathscr{E}_1, \langle \mathscr{O}(\ell H)\rangle_{2\le\ell\le n-1}, \langle \mathscr{F}_\ell\rangle_{0\le\ell\le n-3}\Big\rangle.$$

In summary, the outcome of inductive step 1-4 is

$$\langle \mathscr{A}(-h), \mathscr{A}(H-h), \mathscr{A}, \mathscr{A}^1(H)\rangle = \Big\langle \langle \mathscr{O}(\ell H)\rangle_{-1\le\ell\le n-1}, \mathscr{H}, \langle \mathscr{F}_\ell\rangle_{0\le\ell\le n-2}\Big\rangle \ (n\ge 2). \tag{3.4}$$

We will get SOD (3.1) after reorganizing the collections (3.4) by Lemma 2.2 (6).

□

Remark 3.7. For $N=3$, the inductive steps 2-4 are not needed and SOD (1.4) becomes the following:

$$D(X) = \langle \mathscr{D}, \mathscr{O}(-h), \mathscr{O}, \mathscr{O}(H-h)\rangle. \tag{3.5}$$

In this case, we left mutate $\mathscr{O}(H-h)$ to the far left and left mutate $\mathscr{D}$ to the far left:

$$D(X) = \langle \mathbb{L}_{\mathscr{O}(-2h)}\mathscr{D}, \mathscr{O}(-2h), \mathscr{O}(-h), \mathscr{O}\rangle. \tag{3.6}$$

Then Theorem 1.1 follows by comparing SOD (3.6) with (1.5) (and actually gives a derived equivalence).

4. Proof of Main theorem by Chess Game Method: Odd Case

In this section we use chess game method developed in [18] and [4] to prove the theorem 1.1. To do this, we need to mutate SOD (3.1) and (1.5) properly.

4.1. **Mutation of SOD (3.1).**

(1) Transposition on $\langle \mathscr{A}(\ell H)\rangle_{n\le\ell\le 2n-2}$.

For any $0\le\ell\le r$, where $r=[(n-1)/2]$, we can write the SOD of $\mathscr{A}((n+\ell)H)$ as

$$\mathscr{A}((n+\ell)H)=\langle \mathscr{A}^{2\ell+1}((n+\ell)H),\mathscr{A}_{n-2\ell-1}((n+\ell)H)\rangle.$$

By Lemma 2.2(5), we can transpose $\langle \mathscr{A}^{2\ell+1}((n+\ell)H)\rangle_{0\le\ell\le r}$ to the far left of $\langle \mathscr{A}(\ell H)\rangle_{n\le\ell\le 2n-2}$ so that we get the following SOD:

$$\langle \mathscr{A}(\ell H)\rangle_{n\le\ell\le 2n-2}=\Big\langle \langle \mathscr{A}^{2\ell+1}((n+\ell)H)\rangle_{0\le\ell\le r},\langle \mathscr{A}_{n-2\ell-1}((n+\ell)H)\rangle_{0\le\ell\le r},\langle \mathscr{A}(\ell H)\rangle_{n+1+r\le\ell\le 2n-2}\Big\rangle.$$

(2) Left mutate $\Big\langle \langle \mathscr{A}_{n-2\ell-1}((n+\ell)H)\rangle_{0\le\ell\le r},\langle \mathscr{A}(\ell H)\rangle_{n+1+r\le\ell\le 2n-2}\Big\rangle$ to the far left and then left mutate $\mathscr{D}$ to the far left:

$$\begin{aligned}D(X)=\Big\langle &\mathscr{D}_2,\langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r},\langle \mathscr{A}((\ell-2n+1)H-h)\rangle_{n+1+r\le\ell\le 2n-2},\\ &\langle \mathscr{O}(\ell H)\rangle_{-1\le\ell\le n-1},\mathscr{H},\langle \mathscr{F}_\ell\rangle_{0\le\ell\le n-2},S^{n-1}U^\vee(H),\langle \mathscr{A}(kH)\rangle_{2\le k\le n-1},\\ &\mathscr{A}^{2\ell+1}((n+\ell)H)\rangle_{0\le\ell\le r}\Big\rangle,\end{aligned}\tag{4.1}$$

where $\mathscr{D}_2=\mathbb{L}_{\Big\langle \langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r},\langle \mathscr{A}((\ell-2n+1)H-h)\rangle_{n+1+r\le\ell\le 2n-2}\Big\rangle}\mathscr{D}$.

4.2. **Mutation on SOD (1.5).** Denote $j_*(p_1^*\mathscr{O}(ah)\otimes\mathscr{O}(bH))$ by $\mathscr{O}(a,b)$ and we use the following SOD for $D(\mathbb{P}^{2n})$:

$$D(\mathbb{P}^{2n})=\langle \mathscr{O}(kh)\rangle_{-1-n\le k\le n-1}\qquad\text{(Cf. Figure 1)}$$

Consider the following staircase shape subcategory $\mathscr{S}_k$ $(0\le k\le n-2)$ of ${}^\perp\pi_1^*D(X_1)$:

$$\left\langle\begin{array}{cccccc}\mathscr{O}(n-1-2k,n+k), & \mathscr{O}(n-2-2k,n+k), & \mathscr{O}(n-3-2k,n+k), & \cdots, & \mathscr{O}(n-2,n+k), & \mathscr{O}(n-1,n+k)\\ & & \mathscr{O}(n-3-2k,n+k-1), & \cdots, & \mathscr{O}(n-2,n+k-1), & \mathscr{O}(n-1,n+k-1)\\ & & \ddots & & \vdots & \vdots\\ & & & \mathscr{O}(n-3,n+1), & \mathscr{O}(n-2,n+1), & \mathscr{O}(n-1,n+1)\\ & & & & & \mathscr{O}(n-1,n)\end{array}\right\rangle.$$

Left mutate $\mathscr{S}_{n-2}$ (blue part in Figure 1) to the far right by applying Lemma 2.2(6).

$$\begin{aligned}D(X)=\Big\langle &\pi_1^*D(X_1),\langle j_*(p_1^*D(\mathbb{P}^{2n}))(kH)\rangle_{0\le k\le n},\langle \mathscr{O}(a,n)\rangle_{-1-n\le a\le n-2},\\ &\Big\langle \mathbb{L}_{\mathscr{S}_{k-1}}\mathscr{O}(-1-n,n+k),\langle \mathscr{O}(a,n+k)\rangle_{-n\le a\le n-2k-2}\Big\rangle_{1\le k\le n-2},\mathscr{S}_{n-3}\Big\rangle.\end{aligned}\tag{4.2}$$

Note that the red part is $\langle \mathbb{L}_{\mathscr{S}_{k-1}}\mathscr{O}(-1-n,n+k)\rangle_{1\le k\le n-2}$ in Figure 2. Then left mutate $\mathscr{S}_{n-2}$ to the far left and left mutate $\pi_1^*D(X_1)$ to the far left (Figure 2):

$$\begin{aligned}D(X)=\Big\langle &\mathscr{D}_1,\mathscr{S}'_{n-2},\langle j_*(p_1^*D(\mathbb{P}^{2n}))(kH)\rangle_{0\le k\le n},\langle \mathscr{O}(a,n)\rangle_{-1-n\le a\le n-2},\\ &\Big\langle \mathbb{L}_{\mathscr{S}_{k-1}}\mathscr{O}(-1-n,n+k),\langle \mathscr{O}(a,n+k)\rangle_{-n\le a\le n-2k-2}\Big\rangle_{1\le k\le n-2}\Big\rangle\end{aligned}\tag{4.3}$$

where

$$\mathscr{S}' = \mathscr{S} \otimes \mathscr{O}(-1, 1-2n), \qquad \mathscr{D}_1 = \mathbb{L}_{\mathscr{S}'_{n-2}} \pi_1^* D(X_1).$$

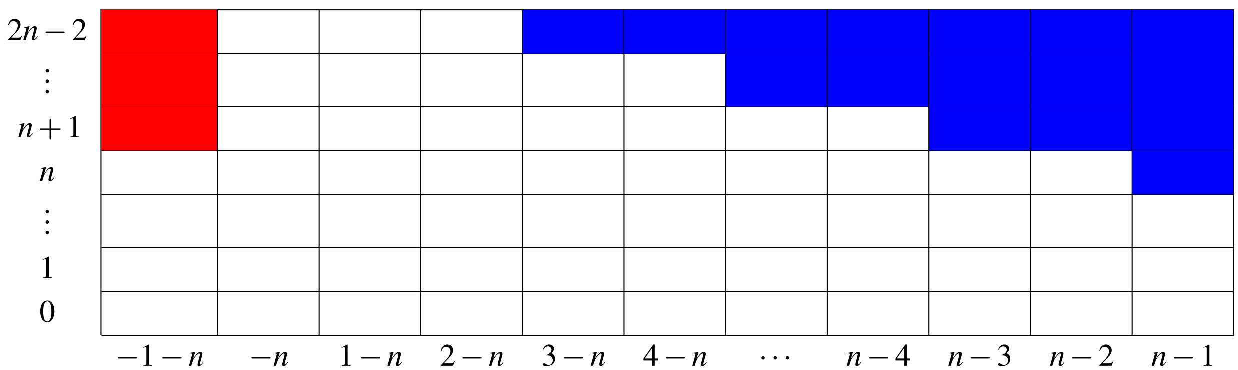

FIGURE 1. SOD of ${}^{\perp}\pi_1^* D(X_1)$(4.2). The horizontal (resp. vertical) direction corresponds to $\mathscr{O}(h)$ (resp. $\mathscr{O}(H)$).

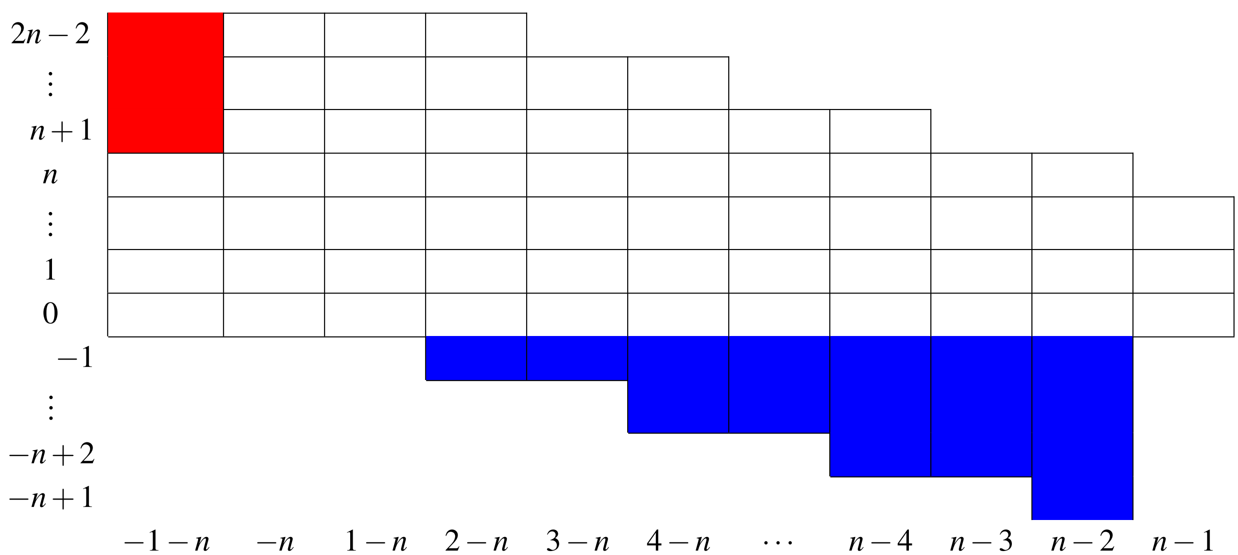

FIGURE 2. Mutated Chessboard (SOD of ${}^{\perp}\mathscr{D}_1$ (4.3))

4.3. **Proof of Theorem 1.1.** To prove that there is a fully-faithful embedding from $D(X_2)$ into $D(X_1)$, it is equivalent to show

$$\phi : \mathscr{D}_1 \xrightarrow{i_{\mathscr{D}_1}} D(X) \xrightarrow{\pi_{\mathscr{D}_2}} \mathscr{D}_2$$

is fully-faithful, where $i_{\mathscr{D}_1}$ is the natural embedding and $\pi_{\mathscr{D}_2}$ is left adjoint to the natural embedding. That is, for any $x, y \in \mathscr{D}_1$,

$$Hom_{\mathscr{D}_2}(\phi(x), \phi(y)) = Hom_{\mathscr{D}_1}(x, y).$$

By adjunction,

$$Hom(\phi(x),\phi(y)) = Hom(\pi_{\mathscr{D}_2} i_{\mathscr{D}_1} x, \pi_{\mathscr{D}_2} i_{\mathscr{D}_1} y) = Hom(i_{\mathscr{D}_1} x, i_{\mathscr{D}_2} \pi_{\mathscr{D}_2} i_{\mathscr{D}_1} y).$$

So it is sufficient to show that

$$Cone(y \to \pi_{\mathscr{D}_2} y) = 0. \tag{4.4}$$

Note that $\pi_{\mathscr{D}_2} = \mathbb{L}_{^\perp\mathscr{D}_2}$ and to achieve (4.4), we will show that $Hom(^\perp\mathscr{D}_2, y) = 0$, which is equivalent to say that $^\perp\mathscr{D}_2$ lies in the whole region in Figure 2. Actually, we will show that $^\perp\mathscr{D}_2$ lies in the subregion consisting of the union of two parts:

(i)

$$\begin{cases} x+2y \le 3n-2; \\ 0 \le y \le 2n-2; \\ -n \le x \le n-1; \end{cases}$$

(ii)

$$\begin{cases} -n \le x+2y \le 3n-4; \\ -n \le x \le n-2. \end{cases}$$

Proposition 4.1. *For any integer k,*

$$S^k U^\vee \in \langle \mathscr{O}(k-2\ell,\ell)\rangle_{0\le\ell\le k}$$

Proof. We prove by induction on k. Note that the RHS forms an exceptional collection by Lemma 2.2(6).

(1) Base case $k=0$ is trivial.

(2) Assume that we have $S^k U^\vee \in \langle \mathscr{O}(k-2\ell,l)\rangle_{0\le\ell\le k}$ and thus

$$S^k U^\vee(H-h) \in \langle \mathscr{O}(k-2\ell-1,\ell+1)\rangle_{0\le\ell\le k} = \langle \mathscr{O}(k-2\ell,\ell)\rangle_{1\le\ell\le k+1}.$$

By short exact sequence (2.3), $S^{k+1}U^\vee \in \langle \mathscr{O}(k-2\ell,\ell)\rangle_{0\le\ell\le k+1}$.

□

Remark 4.2. We can view that $S^k U^\vee$ lies in the "segment" $\{x+2y=k, 0\le y\le k, -k\le x\le k\}$. Hence, we have that for any integers a,b,

(i) $S^a U^\vee(bH)$ lies in $\{x+2y=a+2b, b\le y\le a+b, -a\le x\le a\}$;

(ii) $S^a U^\vee(bH-h)$ lies in $\{x+2y=a+2b-1, b\le y\le a+b, -a-1\le x\le a-1\}$.

Now we divide the objects of $^\perp\mathscr{D}_2$ into two groups:

(1) $\Big\langle \langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r}, \langle \mathscr{A}((\ell-2n+1)H-h)\rangle_{n+1+r\le\ell\le 2n-2} \Big\rangle$;

(2) $\Big\langle \langle \mathscr{O}(\ell H)\rangle_{-1\le\ell\le n-1}, \mathscr{H}, \langle \mathscr{F}_\ell\rangle_{0\le\ell\le n-2}, S^{n-1}U^\vee(H), \langle \mathscr{A}(kH)\rangle_{2\le k\le n-1}, \langle \mathscr{A}^{2\ell+1}((n+\ell)H)\rangle_{0\le\ell\le r} \Big\rangle$.

Claim 4.3. *Group (1) (resp. (2)) lies in region (1) (resp. (2)).*

Proof. For $S^aU^\vee(bH-h) \in \langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r}$, it is easy to check that

$$-n \le a+2b-1 \le 3n-4, \quad -n \le a-1 \le n-2, \quad -a-1 \ge -n$$

under the conditions:

$$b = n-2\ell-1, \quad 0 \le a \le n-1, \quad 0 \le \ell \le r.$$

So $\langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r}$ lies in region (1). The other cases can be proved by the same arguments. $\square$

Therefore (4.4) holds and the fully faithful functor Φ is given by the composition of following functors

$$D(X_1) \xrightarrow{\mathbb{L}_{\mathscr{S}'_{n-2}}\pi_1^*} \mathscr{D}_1 \xrightarrow{\phi} \mathscr{D}_2 \xrightarrow{\mathbb{R}_{\left\langle \langle \mathscr{A}_{n-2\ell-1}((-n+1+\ell)H-h)\rangle_{0\le\ell\le r}, \langle \mathscr{A}((\ell-2n+1)H-h)\rangle_{n+1+r\le\ell\le 2n-2} \right\rangle}} \mathscr{D} \xrightarrow{\pi_{2*}\mathbb{R}_{\langle \mathscr{A}(-h),\mathscr{A}(H-h)\rangle}} D(X_2).$$

5. Even dimensional case: $N = 2n\,(n \ge 2)$

We sketch the process to prove even dimensional cases as follows:

(1) Step 1: Left mutate $\mathscr{A}^1((2n-2)H), \mathscr{A}^1((2n-1)H)$ to the far left and then left mutate $\pi_2^*D(X_2)$ to the far left:

$$D(X) = \left\langle \mathscr{D}', \mathscr{A}^1(-h), \mathscr{A}^1(H-h), \langle \mathscr{A}(\ell H)\rangle_{0\le\ell\le n-1}, \langle \mathscr{A}^1(kH)\rangle_{n\le k\le 2n-3} \right\rangle$$

where

$$\mathscr{D}' = \mathbb{L}_{\langle \mathscr{A}^1(-h),\mathscr{A}^1(H-h)\rangle}\pi_2^*D(X_2).$$

(2) Step 2: Mutations on $\langle \mathscr{A}^1(-h), \mathscr{A}^1(H-h), \mathscr{A}, \mathscr{A}(H)\rangle$.

$$\langle \mathscr{A}(-h), \mathscr{A}(H-h), \mathscr{A}, \mathscr{A}(H)\rangle = \left\langle \langle \mathscr{O}(\ell H)\rangle_{-1\le\ell\le n-1}, \mathscr{H}', \langle \mathscr{F}'_\ell\rangle_{0\le k\le n-3}, S^{n-2}U^\vee(H), S^{n-1}U^\vee(H) \right\rangle.$$

Here

$$\mathscr{H}' = \begin{cases} \langle \mathscr{O}(H-h)\rangle & \text{if } n=2; \\ \langle \mathscr{O}(H-2h), \mathscr{O}(H-h)\rangle & \text{if } n=3; \\ \langle \mathscr{O}(H-2h), \mathscr{O}(H-h), \mathscr{O}(2H-3h)\rangle & \text{if } n \ge 4' \end{cases}$$

$$\mathscr{F}'_\ell = \begin{cases} \langle S^\ell U^\vee(H), \mathscr{O}((\ell+2)(H-h)), \mathscr{O}((l+3)(H-h)-h)\rangle & \text{if } \ell \le n-5; \\ \langle S^\ell U^\vee(H), \mathscr{O}((\ell+2)(H-h))\rangle & \text{if } \ell > n-5. \end{cases}$$

(3) Step 3: mutation on $\langle \mathscr{A}^1(\ell H)\rangle_{n\le\ell\le 2n-3}$. Let $r' = [(n-1)/2]-1$ and consider the following SOD for $\mathscr{A}(aH)$ where $2n-2-r' \le a \le 2n-2$,

$$\mathscr{A}^1((n+\ell)H) = \langle \mathscr{A}^{2\ell+3}((n+\ell)H), \mathscr{A}^1_{n-2\ell-3}((n+\ell)H)\rangle.$$

Next left mutate $\left\langle S^{n-1}U^{\vee}((n-1)H), \langle \mathscr{A}^1_{n-2\ell-3}((n+\ell)H)\rangle_{0\leq \ell \leq r'}\right\rangle$ to the far left and then left mutate $\mathscr{D}'$ to the far left:

$$\begin{aligned} D(X) = \Big\langle & \mathscr{D}'_2, S^{n-1}U^{\vee}(-H-h), \langle \mathscr{A}^1_{n-2\ell-3}(-(\ell+2)H-h)\rangle_{0\leq \ell\leq r'}, \\ & \langle \mathscr{A}^1((\ell-2n+3)H-h)\rangle_{n+r'+1\leq \ell\leq 2n-3}, \langle \mathscr{O}(\ell H)\rangle_{-1\leq \ell\leq n-1}, \\ & \mathscr{H}', \langle \mathscr{F}'_\ell\rangle_{0\leq \ell\leq n-3}, S^{n-2}U^{\vee}(H), S^{n-1}U^{\vee}(H), \langle \mathscr{A}(\ell H)\rangle_{2\leq \ell\leq n-2}, \\ & \langle \mathscr{A}^{2\ell-3}((n+\ell)H)\rangle_{-1\leq \ell\leq r'}\Big\rangle, \end{aligned}$$

where

$$\mathscr{D}'_2 = \mathbb{L}_{\left\langle S^{n-1}U^{\vee}((n-1)H), \langle \mathscr{A}^1_{n-2\ell-3}((n+\ell)H)\rangle_{0\leq \ell\leq r'}\right\rangle} \mathscr{D}'.$$

(4) Step 4: mutation on the SOD (1.5) of $D(X)$: mutate blue part to the far left and mutate $D(X_1)$ to the far left (Figure 3).
(5) Step 5: Conclude the main theorem via analyzing ${}^{\perp}\mathscr{D}'_2$.

Remark 5.1. It should be noted that when $n=2, N=4$, there is no $U^{\vee}(H)$ in the final SOD of $D(X)$ since we have mutated it to the far left in the beginning of step 3:

$$D(X) = \langle \mathbb{L}_{U^{\vee}(-H-h)}\mathscr{D}', U^{\vee}(-H-h), \mathscr{O}(-h), \mathscr{O}, \mathscr{O}(h), \mathscr{O}(H-h), \mathscr{O}(H)\rangle.$$

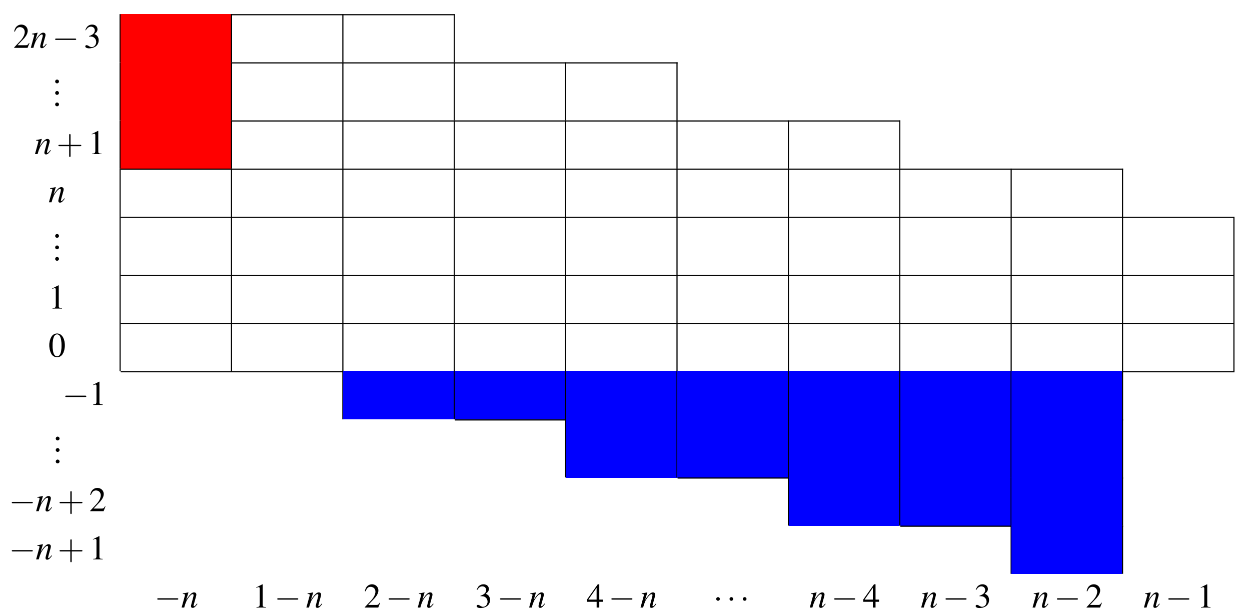

FIGURE 3. Mutated Chessboard (${}^{\perp}\mathscr{D}'_1$)

A1. **Semiorthogonal decompositions and mutations.** A *semiorthogonal decomposition* (SOD) of a triangulated category $\mathscr{T}$, written as:

$$\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n \rangle, \tag{.1}$$

is formed by a sequence of full triangulated subcategories $\mathscr{T}_1, \dots, \mathscr{T}_n$ of $\mathscr{T}$ such that

(1) the natural inclusion functor $\iota_{\mathscr{T}_i} : \mathscr{T}_i \hookrightarrow \mathscr{T}$ admits both right and left adjoints.
(2) $\mathrm{Hom}_{\mathscr{T}}(t_k, t_\ell) = 0$ for all $t_k \in \mathscr{T}_k$ and $t_\ell \in \mathscr{T}_\ell$, if $k > \ell$, and
(3) $\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n$ generates $\mathscr{T}$, *i.e.,* the smallest triangulated category containing $\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n$ that is closed under shifting and taking cones.

The subcategory $\mathscr{T}_i$ satisfying the condition (1) is called *admissible*. A sequence $\mathscr{T}_1, \dots, \mathscr{T}_n$ satisfying the conditions (1) & (2) is called *a semiorthogonal collection*. When each $\mathscr{T}_i$ is generated by only one object E_i, the sequence $E_1, \dots, E_n$ is called *an exceptional collection*.

Suppose $\mathscr{T}'$ is an admissible subcategory of a triangulated category $\mathscr{T}$. Then denote

$$\mathscr{T}'^{\perp} := \{T \in \mathscr{T} \mid \mathrm{Hom}(\mathscr{T}', T) = 0\}, \qquad {}^{\perp}\mathscr{T}' := \{T \in \mathscr{T} \mid \mathrm{Hom}(T, \mathscr{A}) = 0\}$$

to be the *right* and respectively *left orthogonal* of $\mathscr{T}'$ inside $\mathscr{T}$. $\mathscr{T}'^{\perp}$ and ${}^{\perp}\mathscr{T}'$ are both admissible, and we have SOD $\mathscr{T} = \langle \mathscr{T}'^{\perp}, \mathscr{T}' \rangle = \langle \mathscr{T}', {}^{\perp}\mathscr{T}' \rangle$.

Starting with a SOD, one can obtain a whole collection of new decompositions by *mutations*. Let $\mathscr{T}'$ be an admissible subcategory of a triangulated category $\mathscr{T}$. Then the functor $\mathbb{L}_{\mathscr{T}'} := i_{\mathscr{T}'^{\perp}} i^*_{\mathscr{T}'^{\perp}} : \mathscr{T} \to \mathscr{T}$ (resp. $\mathbb{R}_{\mathscr{T}'} := i_{{}^{\perp}\mathscr{T}'} i^!_{{}^{\perp}\mathscr{T}'} : \mathscr{T} \to \mathscr{T}$) is called the *left* (resp. *right*) *mutation through* $\mathscr{T}'$, where $i^*_{\mathscr{T}'^{\perp}}$ (resp. $i^!_{{}^{\perp}\mathscr{T}'}$) is the left (resp. right) adjoint functor to the inclusion $i_{\mathscr{T}'^{\perp}} : \mathscr{T}'^{\perp} \hookrightarrow \mathscr{T}$. The following results are standard, see [14], [2] and [12].

Lemma A1. *Let $\mathscr{T}'$ and $\mathscr{T}_1, \dots, \mathscr{T}_n$ be admissible subcategories of a triangulated category $\mathscr{T}$ where $n \geq 2$ is an integer.*

(1) For any $b \in \mathscr{T}$, there are distinguished triangles

$$i_{\mathscr{T}'} i^!_{\mathscr{T}'}(b) \to b \to \mathbb{L}_{\mathscr{T}'} b \xrightarrow{[1]}, \qquad \mathbb{R}_{\mathscr{T}'} b \to b \to i_{\mathscr{T}'} i^*_{\mathscr{T}'}(b) \xrightarrow{[1]}.$$

(2) $(\mathbb{L}_{\mathscr{T}'})|_{\mathscr{T}'} = 0$ and $(\mathbb{R}_{\mathscr{T}'})|_{\mathscr{T}'} = 0$ are the zero functors, and $(\mathbb{L}_{\mathscr{T}'})|_{\mathscr{T}'^{\perp}} = \mathrm{Id}_{\mathscr{T}'^{\perp}} : \mathscr{T}'^{\perp} \to \mathscr{T}'^{\perp}$, $(\mathbb{R}_{\mathscr{T}'})|_{{}^{\perp}\mathscr{T}'} = \mathrm{Id}_{{}^{\perp}\mathscr{T}'} : {}^{\perp}\mathscr{T}' \to {}^{\perp}\mathscr{T}'$ are identity functors. Furthermore $(\mathbb{L}_{\mathscr{T}'})|_{{}^{\perp}\mathscr{T}'} : {}^{\perp}\mathscr{T}' \to \mathscr{T}'^{\perp}$ and $(\mathbb{R}_{\mathscr{T}'})|_{\mathscr{T}'^{\perp}} : \mathscr{T}'^{\perp} \to {}^{\perp}\mathscr{T}'$ are mutually inverse equivalences of categories.
(3) If $\mathscr{T}$ admits a Serre functor S, then $\mathbb{L}_{\mathscr{T}'}|_{{}^{\perp}\mathscr{T}'} = S_{\mathscr{T}}|_{{}^{\perp}\mathscr{T}'}$ and $\mathbb{R}_{\mathscr{T}'}|_{\mathscr{T}'^{\perp}} = S^{-1}_{\mathscr{T}}|_{\mathscr{T}'^{\perp}}$.
(4) If $\mathscr{T} = \langle \mathscr{T}_1, \dots, \mathscr{T}_{k-1}, \mathscr{T}_k, \mathscr{T}_{k+1}, \dots, \mathscr{T}_n \rangle$, then

$$\begin{aligned}\langle \mathscr{T}_1, \dots, \mathscr{T}_{k-1}, \mathscr{T}_k, \mathscr{T}_{k+1}, \dots, \mathscr{T}_n \rangle &= \langle \mathscr{T}_1, \dots, \mathscr{T}_{k-2}, \mathbb{L}_{\mathscr{T}_{k-1}}(\mathscr{T}_k), \mathscr{T}_{k-1}, \mathscr{T}_{k+1}, \dots, \mathscr{T}_n \rangle \\ &= \langle \mathscr{T}_1, \dots, \mathscr{T}_{k-1}, \mathscr{T}_k, \mathbb{R}_{\mathscr{T}_k}(\mathscr{T}_{k-1}), \mathscr{T}_{k-1}, \mathscr{T}_{k+1}, \dots, \mathscr{T}_n \rangle.\end{aligned}$$

In particular, if $\mathscr{T}'$ is generated by only one object E, then for $b \in \mathscr{T}$,

$$\mathbb{L}_E(b) = Cone(\mathbb{R}Hom(E,b) \otimes E \xrightarrow{ev} b) \tag{.2}$$

$$\mathbb{R}_E(b) = Cone(b \xrightarrow{ev^\vee} \mathbb{R}Hom(b,E)^\vee \otimes E^\vee)[-1]. \tag{.3}$$

A1. **Borel-Weil-Bott Theorem.** We will use the following special case of Borel-Weil-Bott (BWB) theorem repeatedly. Recall that for any non-increasing sequence of integers (a_1, a_2), one can associate the *Schur functor* Σ^{a_1,a_2}. The readers can refer to section 2 of [13] for the general statement of BWB and relevant background.

Theorem A2 (Special case of BWB)**.** *For any integers $a_1 \geq a_2$,*

$$H^\bullet(Gr(2,N), \Sigma^{a_1,a_2} U^\vee) = 0$$

if $1-N \leq a_1 \leq -2$ or $2-N \leq a_2 \leq -1$.

Also we will use the following projection formula frequently.

Lemma A3. *For any integer a,*

$$\mathbb{R}p_{2*}\mathscr{O}(ah) = \begin{cases} S^a U^\vee & \text{if } a \geq 0, \\ 0 & \text{if } a = -1, \\ S^{-a-2} U \otimes \mathscr{O}(-H)[-1] & \text{if } a \leq -2. \end{cases}$$

Proof. Only the third line need checking:

$$\mathscr{H}om(\mathbb{R}p_{2*}\mathscr{O}(ah), \mathscr{O}) = \mathbb{R}p_{2*}\mathscr{H}om(\mathscr{O}(ah), p_2^!\mathscr{O}) = \mathbb{R}p_{2*}\mathscr{H}om(\mathscr{O}(ah), \mathscr{O}(H-2h)[1])$$
$$= \mathbb{R}p_{2*}\mathscr{H}om(\mathscr{O}, \mathscr{O}(H+(-a-2)h)[1]) = \mathscr{O}(H) \otimes S^{-a-2}U^*[1].$$

□

A1. **Proof of Lemma 2.2.**

Proof of Lemma 2.2 (2): Follow the proof of (1), it is sufficient to show for $k+2 \leq a \leq n-1$ and $0 \leq b \leq k$,

$$\mathbb{R}Hom_E(S^a U^\vee, \mathscr{O}((k+1)h)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^a U^\vee, S^{k+1} U^\vee) = 0$$
$$\mathbb{R}Hom_E(S^a U^\vee(H), \mathscr{O}(kh)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^a U^\vee(H), S^k U^\vee) = 0$$
$$\mathbb{R}Hom_E(S^b U^\vee(H), \mathscr{O}((k+1)h)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^b U^\vee(H), S^{k+1} U^\vee) = 0$$
$$\mathbb{R}Hom_E(S^b U^\vee(2H), \mathscr{O}(kh)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^b U^\vee(2H), S^k U^\vee) = 0$$

These all hold by SOD of $D(Gr(2,N))$. □

Proof of Lemma 2.2 (3): It is sufficient to show that

$$\begin{aligned}
&\mathrm{Ext}^\bullet_E(\mathscr{O}(\ell H), S^{k-1}U^\vee(H-h)) = \mathrm{Ext}^{\bullet-1}_{Gr(2,N)}(S^{k-1}U(-H), S^{l-1}U(-H)) = 0;\\
&\mathrm{Ext}^\bullet_E(\mathscr{O}(\ell H), S^{k-1}U^\vee(-2h)) = \mathrm{Ext}^{\bullet-1}_{Gr(2,N)}(S^{k-1}U, S^{l}U) = 0;\\
&\mathrm{Ext}^\bullet_E(S^lU^\vee(H-2h), S^{k-1}U^\vee(H-h)) = \mathrm{Ext}^{\bullet}_{Gr(2,N)}(S^lU^\vee \otimes S^{k-1}U, p_{2*}\mathscr{O}(-h)) = 0;\\
&\mathrm{Ext}^\bullet_E(S^lU^\vee(2H-h), S^{k-1}U^\vee(H-h)) = \mathrm{Ext}^{\bullet}_{Gr(2,N)}(S^lU^\vee(H), S^{k-1}U^\vee) = 0.
\end{aligned}$$

The third line uses the projection formula (Lemma A3) and the rest use the SOD of $D(Gr(2,N))$.

□

Proof of Lemma 2.2 (4): It is sufficient to show for $0 \le a \le n-k-2$ and $0 \le b \le n-k-3$,

$$\begin{aligned}
&\mathbb{R}Hom_E(S^{n-2-k}U^\vee(H-h), \mathscr{O}(\ell H)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-2-k}U^\vee(H), S^{l+1}U^\vee) = 0;\\
&\mathbb{R}Hom_E(S^{n-2-k}U^\vee(2H), \mathscr{O}(\ell H)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-2-k}U^\vee(2H), S^{l}U^\vee) = 0;\\
&\mathbb{R}Hom_E(S^{n-k}U^\vee(H-h), S^aU^\vee(H)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-k}U^\vee(H), S^aU^\vee \otimes U^\vee) = 0;\\
&\mathbb{R}Hom_E(S^{n-k}U^\vee(2H), S^aU^\vee(H)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{n-k}U^\vee(H), S^aU^\vee) = 0;\\
&\mathbb{R}Hom_E(\mathscr{O}((n-k)(H-h)-h, S^bU^\vee(H)) = \mathrm{Ext}^\bullet_{Gr(2,N)}(\mathscr{O}((n-1-k)H), S^bU^\vee \otimes S^{n-k+1}U^\vee) = 0;\\
&\mathbb{R}Hom_E(\mathscr{O}((n-k)(H-h), S^bU^\vee) = \mathrm{Ext}^\bullet_{Gr(2,N)}(\mathscr{O}((n-k)H), S^bU^\vee \otimes S^{n-k}U^\vee) = 0.
\end{aligned}$$

All these can be shown by theorem A2 and SOD of $D(Gr(2,N))$. □

Proof of Lemma 2.2 (5): It is sufficient to show for $n-2\ell-1 \le a \le n-1$ and $0 \le b \le n-2k-2$,

$$\begin{aligned}
&\mathrm{Ext}^\bullet_E(S^aU^\vee((\ell-k)H), S^bU^\vee) = H^\bullet(Gr(2,N), S^aU \otimes S^bU^\vee((k-l)H)) = 0;\\
&\mathrm{Ext}^\bullet_E(S^aU^\vee((\ell-k+1)H), S^bU^\vee(-h)) = 0.
\end{aligned}$$

The second line uses the projection formula (Lemma A3). For the first line, by Littlewood-Richardson rule,

$$S^aU \otimes S^bU^\vee((k-\ell)H = \bigoplus_{t=0}^{b} \Sigma^{b-t+k-\ell,-a+t+k-\ell}U^\vee.$$

It is easy to check that $-a+t+k-\ell \in [2-N,-1]$ whenever $0 \le t \le b, n-2\ell-1 \le a \le n-1, 0 \le b \le n-2k-2, 0 \le \ell < k \le r$. □

Proof of Lemma 2.2 (6). It is sufficient to show that

$$\begin{aligned}
&\mathrm{Ext}^\bullet_E(\mathscr{O}(ah), \mathscr{O}(bH)) = H^{\bullet-1}(Gr(2,N), \Sigma^{b-1,b-a+1}U^\vee) = 0;\\
&\mathrm{Ext}^\bullet_E(\mathscr{O}((a+1)h), \mathscr{O}((b-1)H) = H^{\bullet-1}(Gr(2,N), \Sigma^{b-2,b-a-1}U^\vee) = 0.
\end{aligned}$$

These hold whenever condition (i) or (ii) by theorem A2. □

A1. **Proof of Lemma 2.3.**

Proof of Lemma 2.3: We will give the proof of (1) only. The others are very similar. We need to compute

$$\mathrm{Ext}^\bullet(S^{k-1}U^\vee(H-h), S^kU^\vee).$$

For that, we need to compute (i) $\mathrm{Ext}^\bullet_E(S^{k-1}U^\vee(2H), S^kU^\vee)$ and (ii) $\mathrm{Ext}^\bullet_E(S^{k-1}U^\vee(H-h), S^kU^\vee)$ by the distinguished triangle (2.2).

Actually, by SOD of $D(Gr(2,N))$

(i) $\mathrm{Ext}^\bullet_E(S^{k-1}U^\vee(2H), S^kU^\vee) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{k-1}U^\vee(2H), S^kU^\vee) = 0$.

(ii) $\mathrm{Ext}^\bullet_E(S^{k-1}U^\vee(H-h), S^kU^\vee) = \mathrm{Ext}^\bullet_{Gr(2,N)}(S^{k-1}U^\vee(H), S^kU^\vee \otimes U^\vee) =$
$\mathrm{Ext}^\bullet_{Gr(2,N)}(S^{k-1}U^\vee(H), S^{k+1}U^\vee \oplus S^{k-1}U^\vee(H)) = \mathbb{C}[0]$.

By (.2),

$$\begin{aligned}
&\mathbb{L}_{S^{k-1}U^\vee((k-1)(H-h))}S^kU^\vee \\
&= Cone(\mathbb{R}Hom(S^{k-1}U^\vee((k-1)(H-h)), S^kU^\vee) \otimes S^{k-1}U^\vee((k-1)(H-h)) \longrightarrow S^kU^\vee) \\
&= Cone(S^{k-1}U^\vee((k-1)(H-h)) \longrightarrow S^kU^\vee) = \mathscr{O}(kh).
\end{aligned}$$

The last equality uses the short exact sequence (2.3). □

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