Wave Character of Metrics and Hyperbolic Geometric Flow

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In this letter, we illustrate the wave character of the metrics and curvatures of manifolds, and introduce a new understanding tool - the hyperbolic geometric flow. This kind of flow is new and very natural to understand certain wave phenomena in the nature as well as the geometry of manifolds. It possesses many interesting properties from both mathematics and physics. Several applications of this method have been found.

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1. Introduction. Let us observe the water in the beautiful Westlake in Hangzhou. If there is no wind, the water surface can be regarded as a plane with a flat Riemannian metric $\delta_{ij}(i,j=1,2)$. When wind blows over, the water wave propagates from one side to another side. In this case, the metric of the water surface is not flat globally, and changes along the time. There exists a front, called *wave front*, such that the metric is not flat after the front and is still flat before the front. We would like to call this phenomenon *the wave character of the metric*. Motivated by such wave character of metric as well as the work of Ricci flow, we introduce and study the following evolution equation which we would like to call the *hyperbolic geometric flow*: let $\mathscr{M}$ be n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}, the Levi-Civita connection is given by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left\{\frac{\partial g_{il}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right\},$$

where g^{ij} is the inverse of g_{ij}. The Riemannian curvature tensors read

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp}R^p_{ijl}.$$

The Ricci tensor is the contraction

$$R_{ik} = g^{jl}R_{ijkl}$$

and the scalar curvature is

$$R = g^{ij}R_{ij}.$$

The hyperbolic geometric flow considered here is the evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}. \tag{1}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$. (1) is a nonlinear system of second order partial differential equations on the metric g_{ij}. The hyperbolic geometric flow (1) is only weakly hyperbolic, since the symbol of the derivative of $E = E(g_{ij}) \overset{\triangle}{=} -2R_{ij}$ has zero eigenvectors. However, using DeTurck's technique (see [4]), instead of considering the system (1) we only need to consider a modified evolution system which is strictly hyperbolic (see [2] for the details).

The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics and wave phenomenon of the curvatures. We will prove that it has many surprisingly good properties, which have essential and fundamental differences from the Einstein field equations (see [1]) and the Ricci flow (see [7]). More applications of hyperbolic geometric flow to both mathematics and physics can be expected.

The elliptic and parabolic partial differential equations have been successfully applied to differential geometry and physics (see [8]). Typical examples are Hamilton's Ricci flow and Schoen-Yau's solution of the positive mass conjecture (see [7], [9]). A natural and important question is if we can apply the well-developed theory of hyperbolic differential equations to solve problems in differential geometry and theoretical physics. This letter is an attempt to apply the hyperbolic equation techniques to study some geometrical problems and physical problems. We have already found interesting results in these directions (see [3]). The method may be more important than the results presented in this letter. Our results show that the hyperbolic geometric flow is a natural and powerful tool to study some problems arising form differential geometry such as singularities, existence and regularity.

2. Hyperbolic geometric flow. Hyperbolic geometric flow considered here is the evolution equation (1), it describes the wave character of the Riemannian metrics $g_{ij}(t)$ on an n-dimensional complete Riemannian manifold $\mathscr{M}$. The version (1) of the hyperbolic geometric flow is the unnormalized evolution equation. We next consider the normalized version of hyperbolic geometric flow (1), which preserves the volume of the flow.

The hyperbolic geometric flow and the normalized hyperbolic geometric flow differ only by a change of scale in space and a change of parametrization in time. We now derive the normalized version of (1). Assume that $g_{ij}(t)$ is a solution of the (unnormalized) hyperbolic geometric flow (1) and choose the normalization factor $\varphi = \varphi(t)$

such that

$$\widetilde{g}_{ij} = \varphi^2 g_{ij} \quad \text{and} \quad \int_M d\widetilde{V} = 1. \tag{2}$$

Next we choose a new time scale

$$\widetilde{t} = \int \varphi(t) dt. \tag{3}$$

Then, for the normalized metric $\widetilde{g}_{ij}$, we have

$$\widetilde{R}_{ij} = R_{ij}, \quad \widetilde{R} = \frac{1}{\varphi^2} R, \quad \widetilde{r} = \frac{1}{\varphi^2} r, \tag{4}$$

where $r = \int_M R dV / \int_N dV$ is the average scalar curvature. Noting the second equation in (2) gives

$$\int_M dV = \varphi^{-n}. \tag{5}$$

Then

$$\begin{aligned}
\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} &= \varphi \frac{\partial g_{ij}}{\partial t} + 2 \frac{d\varphi}{dt} g_{ij}, \\
\frac{\partial^2 \widetilde{g}_{ij}}{\partial \widetilde{t}^2} &= \frac{\partial^2 g_{ij}}{\partial t^2} + 3 \left(\frac{d}{dt} \log \varphi \right) \frac{\partial g_{ij}}{\partial t} + \\
& \quad 2 \left(\frac{d}{dt} \log \varphi \right) \left(\frac{d}{dt} \log \frac{d\varphi}{dt} \right) g_{ij} \\
&= -2 \widetilde{R}_{ij} + 3 \varphi^{-1} \left(\frac{d}{dt} \log \varphi \right) \frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + \\
& \quad 2 \varphi^{-2} \left(\frac{d}{dt} \log \varphi \right) \left\{ \frac{d}{dt} \log \frac{d\varphi}{dt} - 3 \frac{d}{dt} \log \varphi \right\} \widetilde{g}_{ij} \\
&\triangleq -2 \widetilde{R}_{ij} + a \frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + b \widetilde{g}_{ij}.
\end{aligned}$$

By (5) and calculations, we observe that a and b are certain functions of t. The following evolution equation

$$\frac{\partial^2 \widetilde{g}_{ij}}{\partial \widetilde{t}^2} = -2 \widetilde{R}_{ij} + a \frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + b \widetilde{g}_{ij} \tag{6}$$

is called the *normalized version* of the hyperbolic geometric flow (1). Thus, studying the behavior of the hyperbolic geometric flow near the maximal existence time is equivalent to studying the long-time behavior of normalized hyperbolic geometric flow.

Motivated by (6), we may consider the following more general evolution equations

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2 R_{ij} + \mathscr{F}_{ij} \left(g, \frac{\partial g}{\partial t} \right) = 0, \tag{7}$$

where $\mathscr{F}_{ij}$ are some given smooth functions of the Riemannian metric g and its first order derivative with respect to t. We name the evolution equations (7) as *general version of hyperbolic geometric flow*. Obviously, when $\mathscr{F}_{ij} \equiv 0$, the system (7) goes back to the standard hyperbolic geometric flow (1).

In particular, consider the space-time $\mathbb{R} \times \mathscr{M}$ with the Lorentzian metric

$$ds^2 = -dt^2 + g_{ij}(x,t) dx^i dx^j. \tag{8}$$

The Einstein equations in the vacuum, which correspond to the metric (8), read

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2 R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t} = 0. \tag{9}$$

Clearly, the system (9) is a special case of (7) in which

$$\mathscr{F}_{ij} = \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t}.$$

(9) is called *Einstein's hyperbolic geometric flow*. We will study the geometric and physical meanings and applications of the Einstein's hyperbolic geometric flow later.

Remark 1. Neglecting the lower order terms in (9) leads to the hyperbolic geometric flow (1). In this sense, (1) can be regarded as the resulting equations taking leading terms of the Einstein equations in the vacuum with respect to the metric (8). We will see that the intrinsically defined hyperbolic geometric flow equations have many interesting new features. □

Remark 2. Notice that the the equations for Einstein manifolds, i.e.,

$$R_{ij} = \kappa g_{ij} \quad (\kappa \text{ is a constant}) \tag{10}$$

are elliptic, the Ricci flow equations

$$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij} \tag{11}$$

are parabolic, and the hyperbolic geometric flow (see (1)) are hyperbolic. In some sense, the above three kinds of equations can be regarded as the generalization on manifolds of the famous Laplace equation, heat equation and wave equation, respectively. □

We may also consider the following field equations

$$\alpha_{ij} \frac{\partial^2 g_{ij}}{\partial t^2} + \beta_{ij} \frac{\partial g_{ij}}{\partial t} + \gamma_{ij} g_{ij} + 2 R_{ij} = 0, \tag{12}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are certain smooth functions on $\mathscr{M}$ which may depend on t. In particular, if $\alpha_{ij} = 1, \beta_{ij} = \gamma_{ij} = 0$, then (12) goes back to the hyperbolic geometric flow; if $\alpha_{ij} = 0, \beta_{ij} = 1, \gamma_{ij} = 0$, then (12) is nothing but the famous Ricci flow; if $\alpha_{ij} = 0, \beta_{ij} = 1, \gamma_{ij} = -\frac{2}{n} r$, then (12) is the normalized Ricci flow (see [7]). In this sense, we name the evolution equations (12) as *hyperbolic-parabolic geometric flow*.

At the end of this section, we remark that if the underlying manifold $\mathscr{M}$ is a complex manifold and the metric is Kähler, similar to (12) the following complex evolution equations are very natural to consider

$$a_{ij} \frac{\partial^2 g_{i\bar{j}}}{\partial t^2} + b_{ij} \frac{\partial g_{i\bar{j}}}{\partial t} + c_{ij} g_{i\bar{j}} + 2 R_{i\bar{j}} = 0, \tag{13}$$

where a_{ij}, b_{ij}, c_{ij} are certain smooth functions on $\mathscr{M}$ which may also depend on t. The evolution equations (13) are called the *complex hyperbolic-parabolic geometric flow.*

3. Exact solutions. This section is devoted to studying the exact solutions for the hyperbolic geometric flow (1). These exact solutions are useful to understand the basic features of the hyperbolic geometric flow. They will also be useful to understand the general Einstein equations. We believe that there is a correspondence between the solutions of the hyperbolic geometric flow and the general Einstein equations.

First recall that a Riemannian metric g_{ij} is called Einstein if $R_{ij} = \lambda g_{ij}$ for some constant λ. A smooth manifold $\mathscr{M}$ with an Einstein metric is called an Einstein manifold.

If the initial metric $g_{ij}(0, x)$ is Ricci flat, i.e., $R_{ij}(0, x) = 0$, then $g_{ij}(t, x) = g_{ij}(0, x)$ is obviously a solution to the evolution equation (1). Therefore, any Ricci flat metric is a stationary solution of the hyperbolic geometric flow (1).

If the initial metric is Einstein, that is, for some constant λ it holds

$$R_{ij}(0, x) = \lambda g_{ij}(0, x), \quad \forall\, x \in \mathscr{M}, \tag{14}$$

then the evolving metric under the hyperbolic geometric flow (1) will be steady state, or will expand homothetically for all time, or will shrink in a finite time.

Indeed, since the initial metric is Einstein, (14) holds for some constant λ. Let

$$g_{ij}(t, x) = \rho(t) g_{ij}(0, x). \tag{15}$$

By the defintion of the Ricci tensor, one obtains

$$R_{ij}(t, x) = R_{ij}(0, x) = \lambda g_{ij}(0, x). \tag{16}$$

In the present situation, the equation (1) becomes

$$\frac{\partial^2 (\rho(t) g_{ij}(0, x))}{\partial t^2} = -2\lambda g_{ij}(0, x). \tag{17}$$

This gives an ODE of second order

$$\frac{d^2 \rho(t)}{dt^2} = -2\lambda. \tag{18}$$

Obviously, one of the initial conditions for (18) is

$$\rho(0) = 1. \tag{19}$$

Another one is assumed as

$$\rho'(0) = v, \tag{20}$$

where v is a real number standing for the initial velocity. The solution of the initial value problem (18)-(20) is given by

$$\rho(t) = -\lambda t^2 + vt + 1. \tag{21}$$

A typical example of the Einstein metric is

$$ds^2 = \frac{1}{1 - \kappa r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \tag{22}$$

where κ is a constant taking its value -1, 0 or 1. We can prove that

$$ds^2 = R^2(t) \left\{ \frac{1}{1 - \kappa r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\} \tag{23}$$

is a solution of the hyperbolic geometric flow (1), where

$$R^2(t) = -2\kappa t^2 + c_1 t + c_2$$

in which c_1 and c_2 are two constants. The metric (23) plays an important role in cosmology.

More interesting examples are the exact solutions with axial symmetry. We are interested in the exact solutions with the following form for the hyperbolic geometric flow (1)

$$ds^2 = f(t, z) dz^2 + \frac{h(t)}{g(t, z)} \left[(dx - \mu(t, z) dy)^2 + g^2(t, z)) dy^2 \right], \tag{24}$$

where $f,\ h,\ g$ are smooth functions with respect to variables. Clearly,

$$(g_{ij}) = \begin{pmatrix} \frac{h}{g} & -\frac{\mu h}{g} & 0 \\ -\frac{\mu h}{g} & \frac{(\mu^2 + g^2) h}{g} & 0 \\ 0 & 0 & f \end{pmatrix} \tag{25}$$

and the inverse of (g_{ij}) is

$$(g_{ij})^{-1} = \begin{pmatrix} \frac{\mu^2 + g^2}{gh} & \frac{\mu}{gh} & 0 \\ \frac{\mu}{gh} & \frac{1}{gh} & 0 \\ 0 & 0 & f^{-1} \end{pmatrix}.$$

In order to guarantee that the metric g_{ij} is Riemannian, we assume

$$f(t, z) > 0, \quad \frac{h(t)}{g(t, z)} > 0. \tag{26}$$

Since the coordinates x and y do not appear in the preceding metric formula, the coordinate vector fields ∂_x and ∂_y are Killing vector fields. The flow ∂_x (resp. ∂_y) consists of the coordinate translations that send x to $x + \Delta x$ (resp. y to $y + \Delta y$), leaving the other coordinates fixed. Roughly speaking, these isometries express the x-invariance (resp. y-invariance) of the model. The x-invariance and y-invariance show that the model possesses the z-axial symmetry.

By a direct calculation, we obtain the Ricci curvature corresponding to the metric (24)

$$\begin{cases} R_{11} &= \frac{h}{4f^2g^3}\left[2f\mu_z^2+2fgg_{zz}-gg_zf_z-2fg_z^2\right], \\ R_{12} &= \frac{h}{4f^2g^3}\left[g\mu f_zg_z+2f\mu g_z^2+2fg^2\mu_{zz}\right. \\ & \left.\quad -g^2f_z\mu_z-4fgg_z\mu_z-2f\mu\mu_z^2-2fg\mu g_{zz}\right], \\ R_{13} &= 0, \\ R_{22} &= \frac{h}{4f^2g^3}\left[2g^2\mu f_z\mu_z+8fg\mu g_z\mu_z+2f(\mu^2-g^2)\mu_z^2\right. \\ & \quad +2fg(\mu^2-g^2)g_{zz}-g(\mu^2-g^2)f_zg_z \\ & \left.\quad -2f(\mu^2-g^2)g_z^2-4fg^2\mu\mu_{zz}\right], \\ R_{23} &= 0, \\ R_{33} &= -\frac{1}{2g^2}(g_z^2+\mu_z^2). \end{cases} \tag{27}$$

Noting (25) and (27), we obtain from (1) that

$$\begin{aligned} &\frac{1}{g^3}\left[g^2h_{tt}+2hg_t^2-ghg_{tt}-2gh_tg_t\right] \\ &\quad = -\frac{h}{2f^2g^3}\left[2f\mu_z^2+2fgg_{zz}-gg_zf_z-2fg_z^2\right], \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{1}{g^3}\left[gh\mu g_{tt}^2+2ghg_t\mu_t+2g\mu g_th_t-g^2h\mu_{tt}-\right. \\ &\left. g^2\mu h_{tt}-2g^2h_t\mu_t-2h\mu g_t^2\right] \\ &\quad = -\frac{h}{2f^2g^3}\left[g\mu f_zg_z+2f\mu g_z^2+2fg^2\mu_{zz}-g^2f_z\mu_z\right. \\ &\left.\qquad -4fgg_z\mu_z-2f\mu\mu_z^2-2fg\mu g_{zz}\right], \end{aligned} \tag{29}$$

$$\begin{aligned} &\frac{1}{g^3}\left[g^2(\mu^2+g^2)h_{tt}+2g^2h\mu\mu_{tt}+4g^2\mu h_t\mu_t+2h\mu^2g_t^2+\right. \\ &\left. 2g^2h\mu_t^2-gh(\mu^2-g^2)g_{tt}-2g(\mu^2-g^2)h_tg_t-4gh\mu g_t\mu_t\right] \\ &\quad = -\frac{h}{2f^2g^3}\left[2g^2\mu f_z\mu_z+8fg\mu g_z\mu_z+2f(\mu^2-g^2)\mu_z^2\right. \\ &\qquad +2fg(\mu^2-g^2)g_{zz}-g(\mu^2-g^2)f_zg_z \\ &\left.\qquad -2f(\mu^2-g^2)g_z^2-4fg^2\mu\mu_{zz}\right], \end{aligned} \tag{30}$$

and

$$f_{tt} = \frac{1}{g^2}(g_z^2+\mu_z^2). \tag{31}$$

Multiplying (28) by μ then summing (29) gives

$$\begin{aligned} &2ghg_t\mu_t-g^2h\mu_{tt}-2g^2h_t\mu_t \\ &\quad = -\frac{h}{2f^2}\left(2fg^2\mu_{zz}-4fgg_z\mu_z-g^2f_z\mu_z\right). \end{aligned} \tag{32}$$

Multiplying (28) by $(g^2-\mu^2)$, (32) by 2μ then summing these two resulting equations and (30) leads to

$$g^2h_{tt}+h\left(g_t^2+\mu_t^2\right)=0. \tag{33}$$

Consider the linear expanding of the rotation, i.e.,

$$h(t)=t.$$

In this case, it follows from (33) that

$$g_t=\mu_t=0. \tag{34}$$

This implies that g and μ are independent of t, that is, $g=g(z)$ and $\mu=\mu(z)$. Therefore, (28)-(31) reduce to

$$\begin{cases} 2fgg_{zz}+2f\mu_z^2-2fg_z^2-gf_zg_z=0, \\ 2fg\mu_{zz}-4fg_z\mu_z-gf_z\mu_z=0, \\ f_{tt}=\frac{1}{g^2}\left(g_z^2+\mu_z^2\right)\triangleq F(z). \end{cases} \tag{35}$$

By the third equation in (35), we have

$$f(t,z)=\frac{1}{2}F(z)t^2+c_1(z)t+c_2(z), \tag{36}$$

where $c_1(z)$ and $c_2(z)$ are two arbitrary smooth functions of z. Substituting (36) into the first equation in (35) yields a quadratic equation on t

$$A(z)t^2+B(z)t+C(z)=0, \tag{37}$$

where

$$\begin{cases} A(z)=\mu'(g\mu'g''+\mu'^3+g'^2\mu'-gg'\mu''), \\ B(z)=2c_1(gg''+\mu'^2-g'^2)-gg'c_1', \\ C(z)=2c_2(gg''+\mu'^2-g'^2)-gg'c_2', \end{cases} \tag{38}$$

where $\cdot'$ stands for the derivative of $\cdot$ with respect to z. Noting the arbitrariness of t gives the following system of ODEs

$$\begin{cases} \mu'(g\mu'g''+\mu'^3+g'^2\mu'-gg'\mu'')=0, \\ 2c_1(gg''+\mu'^2-g'^2)-gg'c_1'=0, \\ 2c_2(gg''+\mu'^2-g'^2)-gg'c_2'=0. \end{cases} \tag{39}$$

Similarly, substituting (36) into the second equation in (35) leads to

$$\begin{cases} g'(gg'\mu''-\mu'^3-g\mu'g''-g'^2\mu')=0, \\ 2c_1(g\mu''-2g'\mu')-g\mu'c_1'=0, \\ 2c_2(g\mu''-2g'\mu')-g\mu'c_2'=0. \end{cases} \tag{40}$$

Case I $\quad g'=0$

In this case, it follows from the first equation in (39) that $\mu'=0$. This implies that

$$g=a, \quad \mu=b,$$

where a and b are constants. Then the solution of (35) is

$$f=c_1(z)t+c_2(z), \quad g=a, \quad \mu=b. \tag{41}$$

Therefore, the desired solution reads

$$ds^2=(c_1(z)t+c_2(z))dz^2+\frac{t}{a}[(dx-bdy)^2+a^2dy^2],$$

where a is a positive constant, and $c_1(z)$, $c_2(z)$ are two positive smooth functions of z.

Case II $\mu' = 0$, i.e., $\mu = \mu_0 = const.$

In this case, (40) is always true, and (39) becomes

$$\begin{cases} 2c_1(gg'' - g'^2) = gg'c_1', \\ 2c_2(gg'' - g'^2) = gg'c_2'. \end{cases} \tag{42}$$

It follows from (42) that

$$c_1(z) = a_1 \left[(\ln g)'\right]^2, \quad c_1(z) = a_2 \left[(\ln g)'\right]^2,$$

where a_1, a_2 are two constants, and g is an arbitrary positive smooth function of z. Hence, the desired solution reads

$$ds^2 = \left[(\ln g)'\right]^2 (t^2/2 + a_1 t + a_2) dz^2 + \frac{t}{g}[(dx - \mu_0 dy)^2 + g^2 dy^2].$$

In order to guarantee the above metric is Riemannian, we assume that the constants a_1 and a_2 satisfy

$$a_1^2 < 2a_2$$

and g is an arbitrary positive smooth function satisfying $(\ln g)' \neq 0$.

Case III $g' \neq 0, \quad \mu' \neq 0$

Obviously, the first equation in (39) is equivalent to the first one in (40), and they are equivalent to

$$g\mu' g'' + \mu'^3 + g'^2 \mu' - gg'\mu'' = 0. \tag{43}$$

On the one hand, it follows from the second equation in (39) that

$$\frac{c_1'}{c_1} = 2\frac{gg'' + \mu'^2 - g'^2}{gg'}. \tag{44}$$

On the other hand, by the second equation in (40) we have

$$\frac{c_1'}{c_1} = 2\frac{g\mu'' - 2g'\mu'}{g\mu'}. \tag{45}$$

Noting (43), we observe that (44) is equivalent to (45). Thus, we have

$$\frac{c_1'}{c_1} = 2\left(\frac{\mu''}{\mu'} - \frac{2g'}{g}\right). \tag{46}$$

It follows from (46) that

$$c_1(z) = b_1 g^{-4}(\mu')^2, \tag{47}$$

where b_1 is a constant. Similarly, we get

$$c_2(z) = b_2 g^{-4}(\mu')^2, \tag{48}$$

where b_2 is a constant. Therefore, the desired metric reads

$$ds^2 = f(t,z)dz^2 + \frac{t}{g(z)}\left[(dx - \mu(z)dy)^2 + g^2(z))dy^2\right], \tag{49}$$

where

$$f(t,z) = \frac{g'^2 + \mu'^2}{2g^2} + \frac{\mu'^2}{g^4}(b_1 t + b_2),$$

g, μ are smooth functions satisfying (43). Moreover, in order to guarantee the metric (49) is Riemannian, we require $g > 0$, $b_1 \geq 0$ and $b_2 > 0$.

Remark 3. Recently, Shu and Shen prove that Birkhoff theorem is still true for the hyperbolic geometric flow (see [10]). □

4. Short-time existence and uniqueness. In this section we state the short-time existence and uniqueness result for the hyperbolic geometric flow (1) on a compact n-dimensional manifold $\mathscr{M}$. We can show that the hyperbolic geometric flow is a system of second order nonlinear weakly hyperbolic partial differential equations. The degeneracy of the system is caused by the diffeomorphism group of $\mathscr{M}$ which acts as the gauge group of the hyperbolic geometric flow. Because the hyperbolic geometric flow (1) is only weakly hyperbolic, the short-time existence and uniqueness result on a compact manifold does not come from the standard PDEs theory. However we can still prove the following short-time existence and uniqueness theorem.

Theorem 1. *Let $(\mathscr{M}, g_{ij}^0(x))$ be a compact Riemannian manifold. Then there exists a constant $\varepsilon > 0$ such that the initial value problem*

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2}(t,x) = -2R_{ij}(t,x), \\ g_{ij}(0,x) = g_{ij}^0(x), \ \dfrac{\partial g_{ij}}{\partial t}(0,x) = k_{ij}^0(x), \end{cases}$$

has a unique smooth solution $g_{ij}(t,x)$ on $\mathscr{M} \times [0,\varepsilon)$, where $k_{ij}^0(x)$ is a symmetric tensor on $\mathscr{M}$.

The above short-time existence and uniqueness theorem can be proved by the following two ways: (a) using the gauge fixing idea as in Ricci flow, we can derive a system of second order nonlinear strictly hyperbolic partial differential equations, thus Theorem 1 comes from the standard PDEs theory; (b) we reduce the hyperbolic geometric flow (1) to a first-order quasilinear symmetric hyperbolic system, then using the Friedrich's theory [6] of symmetric hyperbolic system (more exactly, the quasilinear version [5]) we can also prove Theorem 1. See Dai, Kong and Liu [2] for the details.

5. Wave property of curvatures. The hyperbolic geometric flow is an evolution equation on the metric $g_{ij}(t,x)$. The evolution for the metric implies a nonlinear wave equation for the Riemannian curvature tensor R_{ijkl}, the Ricci curvature tensor R_{ij} and the scalar curvature R.

Theorem 2. *Under the hyperbolic geometric flow (1), the curvature tensors satisfy the evolution equations*

$$\frac{\partial^2 R_{ijkl}}{\partial t^2} = \Delta R_{ijkl} + \textit{(lower order terms)}, \tag{50}$$

$$\frac{\partial^2 R_{ij}}{\partial t^2} = \Delta R_{ij} + \textit{(lower order terms)}, \tag{51}$$

$$\frac{\partial^2 R}{\partial t^2} = \Delta R + \textit{(lower order terms)}, \qquad (52)$$

where Δ is the Laplacian with respect to the evolving metric, the lower order terms only contain lower order derivatives of the curvatures.

By (1), the equations (50)-(52) come from direct calculations. See Dai, Kong and Liu [3] for the details and geometric applications. The equations (50)-(52) show that the curvatures possess interesting wave property.

6. Summary and discussion. Hyperbolic partial differential equations can be used to describe the wave phenomena in the nature. In this letter, the hyperbolic geometric flow is introduced to illustrate the wave character of the metrics, which also implies the wave property of the curvature. Note that the hyperbolic geometric flow possesses very interesting geometric properties and dynamical behavior. A direct application of Theorem 2 gives the stability of solutions to the hyperbolic geometric flow equation on the Euclidean spaces under metric perturbations (see [3]). More applications of this flow to differential geometry and physics can be expected.

So far there have been many successes of elliptic and parabolic equations applied to mathematics and physics, but by now very few results on the applications of hyperbolic PDEs are known (see [5]). We believe that the hyperbolic geometric flow is a new and powerful tool to study geometric problems. Moreover, its physical application has been observed (see [3]). In the future we will study several fundamental problems, for examples, long-time existence, formation of singularities, as well as the physical and geometrical applications.

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